Constrained Portfolio Selection Using Multiple Swarms

Ben Niu,, Lijing Tan, Bing Xue, Li Li, and Yujuan Chai

Abstract—**Markowitz mean-variance model is one of the best known models that has been heavily studied in modern world of finance. However, the model is considered to be too basic in practice, as it ignores many of the constraints that real-world investors have to face with. In this paper we focused on a complex constrained portfolio selection model with additional constraining factors including the transaction fee, the minimal transaction unit, the maximal transaction quantity of every assets and the minimum/maximum of the investment. When taken these complex constraints in to account, the process became a high-dimensional constrained optimization problem. In our study, based on the study of symbiosis phenomenon in natural ecosystem, a multiple-swarm approach (SMPSO) was proposed to solve the resulting model. A numerical experimental study of a portfolio selection problem was conducted to illustrate our proposed method. The simulation results demonstrated that our proposed method is more efficient than PSO based method in solving the complex constrained portfolio selection problem.**

I. Introduction

THE particle swarm optimization (PSO) is a heuristic technique proposed comparatively recently by Kennedy and Eberhart [1, 2]. It is inspired by natural phenomenon such as fish schooling, bird flocking and human social relations. Similar to genetic algorithms, PSO is also a population based algorithm, where each individual is regarded as a particle, and each particle is a potential solution to the problem. Unlike genetic algorithm (GA) in updating a population of particles with regard to their internal velocity and position, PSO is informed by the experiences of all the particles, which lends itself well to effectively tackling complex optimization problems. However, it does exhibit some disadvantages. It sometimes converges to undesired local of solution due to the decreasing of population diversity in the latter periodic of evolution. When approaching to the optimal solution, the algorithm stops optimizing, and thus the accuracy that the algorithm can achieve is limited.

Recently, a number of modifications have been proposed to improve the rate and reliability of the original PSO model [3, 4, 5, 6]. In this paper, we presented a symbiotic multi-swarm particle swarm optimizer (SMPSO), which included a new variant and employed a multi-swarm cooperative evolutionary strategy where each sub-swarm executed PSO (or its variants) independently to maintain the diversity of the particles. Meanwhile, each sub-swarm enhanced its particles based on its own knowledge as well as the knowledge of the particles in the other sub-swarms. In SMPSO, a center particle, whose position is updated with the average position of the global best particle found by each sub-swarms at every iteration, is used to monitor the information exchange between sub-swarms. By this way, the search information can be transferred among each sub-swarm. Consequently, the performance of the center particle could greatly influence the performance of the SMPSO algorithm.

To demonstrate the performance of our proposed algorithm, we applied it to portfolio optimization (PO) and compared the performance with original PSO. PO is consists of the portfolio selection problem in which we wanted to find the optimum way of investing a particular amount of money in a given set of securities or assets [7]. This problem is NP-hard and non-linear with many local optima. A number of different algorithmic approaches have been proposed for solving this problem, including GA [8], simulated annealing [9], neural networks [7] and others [10, 11]. However, most of the PO models used in these pioneer works are considered to be too basic, as they ignore many of the constrains, such as cardinality or bounding constraints, which restrict the number of assets as well as the upper and the lower bounds of proportion of each asset in the portfolio.

In this work, we used a modified PO model considering the transaction fee, the minimal transaction unit, the maximal transaction quantity of every assets and the minimum /maximum of the investment. The main motivation of this study was to employ symbiotic multi-swarm PSO for this modified PO model. The rest of the paper was organized as follows. Section 2 gave a review of PSO and a description of the proposed algorithm SMPSO. Section 3 described the portfolio optimization model and how our proposed SMPSO was applied to solve this problem. Section 4 presented the detailed experimental studies. Finally, conclusions were drawn in Section 5.

II. PSO and SMPSO

A. Particle Swarm Optimization (PSO)

The basic PSO is a population based optimization tool, where the system is initialized with a population of random solutions and the algorithm searches for optima by updating generations. In PSO, the potential solutions, called particles, fly in a D-dimension search space with a velocity which is dynamically adjusted according to its own experience and

Ben Niu Bing Xue and Li Li are with the College of Management, Shenzhen University, Shenzhen, Guangdong 518060, China (corresponding author email: drniuben@gmail.com)

Lijing Tan is with the Measurement Specialties Inc, Shenzhen University, Shenzhen, Guangdong 518107, China

Yujuan Chai is with the Faculty of Science, McMaster University, Hamilton, Ontario L8S4L8, Canada

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that of its neighbors.

The position of the ith particle is represented as $\vec{x}_i = (x_{i1}, x_{i2}, ..., x_{iD})$, where $x_{id} \in [l_d, u_d]$, $d \in [1, D]$, l_d, u_d are the lower and upper bounds for the dth dimension, respectively. The rate of velocity for particle i is represented as $\vec{v}_i = (v_{i1}, v_{i2}, ..., v_{iD})$, is clamped to a maximum velocity vector $\vec{v}_{\max}$, which is specified by the user. The best previous position of the ith particle is recorded and represented as $P_i = (P_{i1}, P_{i2}, ..., P_{iD})$, which is also called $pbest$. The index of the best particle among all the particles in the population is represented by the symbol g, and p_g is called $gbest$. At each iteration step, the particles are manipulated according to the following equations:

$$v_{id} = wv_{id} + R_1 c_1 (P_{id} - x_{id}) + R_2 c_2 (p_{gd} - x_{id}), \tag{1}$$

$$x_{id} = x_{id} + v_{id}. \tag{2}$$

where w is inertia weight; c_1 and c_2 are acceleration constants; and R_1, R_2 are random vectors with components uniformly distributed in [0, 1]. For Eq. (1), the portion of the adjustment to the velocity influenced by the individual's own $pbest$ position is considered as the cognition component, and the portion influenced by $gbest$ is the social component. After the velocity is updated, the new position of the ith particle in its dth dimension is recomputed. This process is repeated for each dimension of the ith particle and for all the particles in the swarm.

B. *Symbiotic multi-swarm particle swarm optimizer (SMPSO)*

The motivation for developing SMPSO derives from the phenomenon of symbiosis in natural ecosystems, where many species have developed cooperative interactions with other species to improve their survival. According to the different symbiotic interrelationships, symbiosis can be classified into three main categories: mutualism (both species benefit by the relationship), commensalism (one species benefits while the other species is not affected), and parasitism (one species benefits and the other is harmed) [12]. Among these relations, the commensalism model is suitable to be incorporated in the SMPSO. Inspired by this research, a symbiotic multi-swarm PSO is proposed in this paper.

The population in SMPSO consists of several sub-swarms with the same properties, i.e. they are both identical sub-swarms. Each sub-swarm can supply many new promising particles to other sub-swarm as the evolution proceeds. Each sub-swarm updates the particle states based on the best position discovered so far by all the particles in the other sub-swarms and its own. The interactions between the other sub-swarms and its own influence the balance between exploration and exploitation and maintain a suitable diversity in the population even when it is approaching the global solution. Thus, the risk of converging to local sub-optima is reduced.

The search information can be transformed among sub-swarms by a center communication mechanism that uses a center particle whose position is obtained by averaging the sub-swarms to guide the flight of particles in all the sub-swarms. During the flight each particle of the sub-swarm adjusts its trajectory according to its own experience, the experience of its neighbors, and the experience of the particles in other sub-swarms, making use of the best previous position encountered by itself, its neighbors and the center particle position. In this way, the search information can be transferred between sub-swarms to accelerate the convergence rate.

From other aspects, we now describe the SMPSO algorithm for evolving symbiotic co-adapted species. In SMPSO, we use a population of $N \times P$ individuals, or in symbiosis terminology, an ecosystem of $N \times P$ organisms. The whole population is divided into N colonies to modeling symbiosis in the context of the evolving ecosystems (for convenience, each species has the same population size P). As in nature, the colonies are separated breeding populations and evolve parallel, while interact with one another within each generation and have a symbiotic relationship.

To realize this mechanism, we proposed a modification to the original PSO velocity update equation. In each generation, particle i in species n will evolve according to the following equations:

$$\begin{aligned} v_i^n(t+1) &= wv_i^n(t) + R_1 c_1 (p_i^n - x_i^n(t)) + \\ & R_2 c_2 (p_g^n - x_i^n(t)) + R_3 c_3 (p_c^n - x_i^n(t)) \end{aligned} \tag{3}$$

where p_i^n and P_g^n are the best previous solution achieved so far by particle i and the species n, respectively. R_3 is a random value between 0 and 1. c_3 is acceleration constant; P_c^n represents the center position of the global best particle in all the sub-swarms. After N sub-swarms update their positions and best performed particle is found, a center particle is updated according to the following formula:

$$\begin{aligned} & P_c^n(t+1) = \frac{1}{N} \sum_{i=1}^{N} p_g^n(t), \\ & n = 1, 2, \cdots N, \\ & i = 1, 2, \cdots P. \end{aligned} \tag{4}$$

Unlike other particles, the center particle has no velocity, but it is involved in all operations the same as the ordinary particle, such as fitness evaluation, competition for the best particle, except for the velocity calculation.

III. SMPSO BASED PORTFOLIO OPTIMIZATION

A. Portfolio Optimization Problem

The portfolio optimization problem is one of the most important issues in asset management, which deals with how to form a satisfying portfolio. The original portfolio optimization model is usually called mean–variance model [13], firstly proposed by Markowitz. This model is often used to solve portfolio selection problem under a certain number of simplifying assumptions. Many other studies have been devoted to consider several key aspects of portfolio optimization when solving the real problem. In this paper, based on the original mean-variance model, we present an improved mean–variance model considering the transaction fee and other constraint conditions such as the minimal transaction unit, the maximal transaction quantity of every assets, the minimum and maximum of the investment. It assumes that an investor's wealth is allocated among n assets. Some notations are introduced as follows:

r_i : The yield of the i asset, $i = 1, \dots \dots, n$;

$R = (R_1 \dots \dots, R_n)$: $R_i = E(r_i)$ denoting the expected yield;

$\sigma_{ij} = \mathrm{cov}(r_i, r_j)$: the covariance of r_i and r_j ;

x_0 : the minimal transaction unit ;

$H = (h_1 \dots \dots, h_n)$: the maximal transaction quantity of the i asset;

$X = (x_1 \dots \dots, x_n)$: x_i is the quantity of the i asset that investor want to buy;

p_i : The current price of every share;

$c = (c_1 \dots \dots, c_n)$: $c_i = x_0 p_i$ is the price of the minimal transaction unit;

$\xi = (\xi_1 \dots \dots, \xi_n)$: ξ_i is the proportion of the money used in the i asset; $\xi^0 = (\xi_1^0 \dots \dots, \xi_n^0)$ is the proportion at the beginning of the period ;

$k = (k_1 \dots \dots, k_n)$: k_i is the transaction fee of the i asset;

c_0: $c_0 = \sum_{i=1}^{n} c_i x_i (1 + \sum_{i=1}^{n} k_i \left|\xi_i - \xi_i^0\right|)$ is the total investment of the investor (including the transaction fee);

$c_{\min}$: The minimum of the investment (in order to make the most of the wealth);

$c_{\max}$: The maximum of the investment;

λ : The risk factor, wihch distribute in [0, 1]. Larger λ represents the one who love the risk more.

TABLE 1
PSEUDO-CODE OF SMPSO FOR PORTFOLIO OPTIMIZATION

Begin
Create an initial population;
Loop
While $0 <= x_i <= h_i$ **;**
Evaluate the fitness of each particle based on calculating the $\xi = (\xi_1 \dots \dots, \xi_n)$:
$\xi_i = x_i / c_0$;
$R(m)$ is a penalty item in the fitness function to ensure the total cost is between the minimum and maximum of the investment:
If $F(x) > c_{\max}$ $R(m) = 1$;
Else if $F(x) < c_{\min}$ $R(m) = 1$;
Else $R(m) = 0$ **;**
End if
$F(x) = F(x) + R(m) * M$, and M is a very large number.
Update the population after finding the Pbest and the Gbest based on the fitness value according to the Eq.(3) and Eq.(4);
Repeating the computations until the maximum iterations;
End loop
End

Based on these defined variables, the function $f(x)$ and $g(x)$ denotes the revenue and risk in the portfolio optimization problem can be obtained as following:

$$f(x) = \sum_{i=1}^{n} R_i \xi_i - \sum_{i=1}^{n} k_i \left|\xi_i - \xi_i^0\right| , \tag{5}$$

$$g(x) = \sum_{i=1}^{n} \sum_{i=1}^{n} \sigma_{ij} x_i x_j \, . \tag{6}$$

Our improved portfolio optimization model can be formulated as:

$$\min F(x) = \min\{\lambda g(x) \text{-} (1\text{-}\lambda) f(x)\}$$

$$\begin{cases} c_{\min} \le c_0 = \sum_{i=1}^{n} c_i x_i \le c_{\max} ; \\ 0 < x_i < h_i \ge 0, i = 0, 1, 2, \dots\dots, n. \end{cases} \tag{7}$$

where $0 < x_i$ means that the short sale is not permitted.

B. SMPSO for portfolio optimization

There is an n demension search space denoting n kinds of sassets in the PSO algorithm, and the position of the particle $X = (x_1 \dots \dots, x_n)$ presents the quantity of every assesst. The position of the particle with the minimum fitness value is the best selection of portfolio optimization. The pseudo-code for our proposed method is shown in Table 1.

IV. ILLUSTRATIVE EXAMPLES

A. Parameters Setting

In order to test the effectiveness of SMPSO for portfolio optimization, we chose five assets from different industry and different places as the sample. Two of them from Shanghai A share are Nanjing Zhong Da (600074) and Zhujiang Industry（600684）, three assets are from Shenzhen A share are Beijing Chemical Industry (0728), Tianshui Stock (0965) and Huandao Industry (0691). The basic data about the assets were from July 17th, 2000 to Aug 18th, 2008 and we got the interrelated index value needed in the experiment based on them. Parts of the parameters setting of our proposed PO model are showed in Table 2. The others are listed below the table, where different kinds of investors are considered. There are different risk factors λ identify the different kinds inverstors. The parameters used in PSO and SMPSO are listed in Table 3. A total of 20 runs for each experimental settings are performed.

TABLE 2
PARAMETERS SETTING OF THE PORTFOLIO MODEL

Parameter	Value	Parameter	Value
n	5	$c_{\min}$	2 000 000 RMB
x_0	100	$c_{\max}$	2 005 000 RMB
h_i	3000	k_i	0.075%
ξ_i^0	0		

$\lambda \in (0.1, 0.3, 0.5, 0.7, 0.9)$

$R = (R_1, R_2, R_3, R_4, R_5) = (0.01675, 0.00859, 0.05146, 0.04227, 0.09462)$

$c = (c_1, c_2, c_3, c_4, c_5) = (378, 372, 327, 282, 210)$

σ = [0.01002, 0.00319, 0.01093, 0.00025, 0.01786;
0.00319, 0.00934, -0.00057, -0.01612, -0.01779;
0.01093, -0.00057, 0.02392, 0.01793, 0.04677;
0.00025, -0.01612, 0.01793, 0.05139, 0.07250;
0.01786, -0.01779, 0.04677, 0.07250, 0.15965;]

TABLE 3
PARAMETERS SETTING OF THE PORTFOLIO MODEL

Type	PSO	SMPSO
Inertia weight	0.9 to 0.4	0.9 to 0.6
c_1	2.0	1.367
c_2	2.0	2.367
c_3	—	1.367
Swarm size	80	20 (each sub-swarm)
Max iteration	50	50

B. Experimental Results and Analysis

Numerical results with different λ obtained by the standard PSO and the SMPSO are showed in the Tables 4-6. The final portfolio selection results are listed in Tables 7-9. Figures 1-5 present the mean relative performance using different λ generated by PSO and SMPSO.

The maximum value, the minimum value, the standard deviation and the mean value are summarized in Tables 4-6, where ξ_i represent the different proportion of each asset. It is shown that SMPSO has smaller standard deviation and mean value, which demonstrated that it outperforms PSO in terms of result robustness and solution quality. Based on the analysis of the fitness function, it is suggested that the fitness value should increase along with λ. As shown in Tables 4-6, with the increase of λ, SMPSO produces steady increase results, while PSO performs un-conspicuously, e.g. when $\lambda=0.1$, the mean value is 0.5634; However, when $\lambda=0.3$, the mean value is 0.3764.

In Tables 7-9, x_i repents the amount of every asset, and c denotes the utilization of the capital, the income percent and risk percent are also included. For almost of all the different risk preferences, the SMPSO produces the lager c, which leads to the best of the investor's capital. As we know, the income and the risk have the same change rule, and in Tables 7-9, all the risk percent increases along with the income percent respectively, it can be testified that the results from the two ways are reasonable. At the same time, in most situations the income percents obtained by PSO are larger, and the risk percents are larger, too. However, although the income percent got by SMPSO is lower, when considering the larger c, the result produced by SMPSO can provide more benefit to the investors with lower risk percents.

From Figures 1-5, it is found that SMPSO has quicker convergence rate in different situations compared with PSO. Furthermore, its convergence process is much steadier than that of PSO.

All the results presented in the tables and figures suggested that the SMPSO can be a more effective way for the investors to solve the portfolio optimizations problems.

TABLE 4
NUMERICAL RESULTS WITH DIFFERENT λ

	$\lambda = 0.1$		$\lambda = 0.3$	
	PSO	SMPSO	PSO	SMPSO
Max	9.9632	-0.0376	9.9739	-0.0227
Min	-0.0477	-0.0462	-0.0285	-0.0277
Mean	0.5634	-0.0412	0.3764	-0.0256
Stdev	2.3970	0.0020	1.9789	0.0011
ξ_1	0.0494	0.0991	0.0488	0.0751
ξ_2	0.0409	0.0503	0.0670	0.0875
ξ_3	0.3377	0.3467	0.4877	0.4464
ξ_4	0.2628	0.2094	0.1081	0.1252
ξ_5	0.3092	0.2945	0.2885	0.2659

TABLE 5
NUMERICAL RESULTS WITH DIFFERENT λ

	$\lambda=0.5$		$\lambda=0.7$	
	PSO	SMPSO	PSO	SMPSO
Max	-0.0071	-0.0115	9.9973	-0.0025
Min	-0.0132	-0.0130	-0.0044	-0.0041
Mean	-0.0108	-0.0123	0.7992	-0.0034
Stdev	0.0010	3.5809e -004	2.7396	4.2876e -004
ξ_1	0.0091	0.0087	0.0113	0.0564
ξ_2	0.3360	0.3957	0.5446	0.4921
ξ_3	0.4659	0.3995	0.1571	0.3066
ξ_4	0.0301	0.0653	0.2572	0.1194
ξ_5	0.1589	0.1307	0.0297	0.0256

TABLE 6
NUMERICAL RESULTS WITH DIFFERENT λ

	$\lambda = 0.9$	
	PSO	SMPSO
Max	10.0024	10.0005
Min	0.0017	-0.0011
Mean	0.2074	1.0027
Stdev	1.4135	3.0296
ξ_1	0.1921	0.1104
ξ_2	0.4822	0.5066
ξ_3	0.1217	0.1058
ξ_4	0.1771	0.2286
ξ_5	0.0269	0.0485

TABLE 7
PORTFOLIO SELECTION RESULTS WITH DIFFERENT λ

	$\lambda = 0.1$		$\lambda = 0.3$	
	PSO	SMPSO	PSO	SMPSO
x_1	261	524	258	398
x_2	220	270	360	471
x_3	2064	2119	2981	2734
x_4	1863	1484	766	889
x_5	2943	2803	2746	2536
c	2.0003e +006	2.0000e +006	2.0004e +006	2.0044e +006
Income percent	0.0582	0.0559	0.0576	0.0547
Risk percent	0.0464	0.0411	0.0393	0.0352

TABLE 8
PORTFOLIO SELECTION RESULTS WITH DIFFERENT λ

	$\lambda=0.5$		$\lambda=0.7$	
	PSO	SMPSO	PSO	SMPSO
x_1	48	46	60	298
x_2	1809	2126	2926	2644
x_3	2854	2442	960	1874
x_4	214	463	1823	846
x_5	1516	1244	283	244
c	2.0046e +006	2.0001e +006	2.0001e +006	2.0003e +006
Income percent	0.0426	0.0385	0.0259	0.0277
Risk percent	0.0162	0.0126	0.0048	0.0060

TABLE 9
PORTFOLIO SELECTION RESULTS WITH DIFFERENT λ

	$\lambda = 0.9$	
	PSO	SMPSO
x_1	48	46
x_2	1809	2126
x_3	2854	2442
x_4	214	463
x_5	1516	1244
c	2.0046e +006	2.0001e +006
Income percent	0.0426	0.0385
Risk percent	0.0162	0.0126

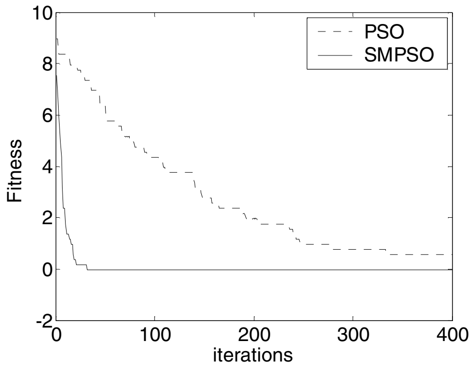

Fig. 1. $\lambda = 0.1$

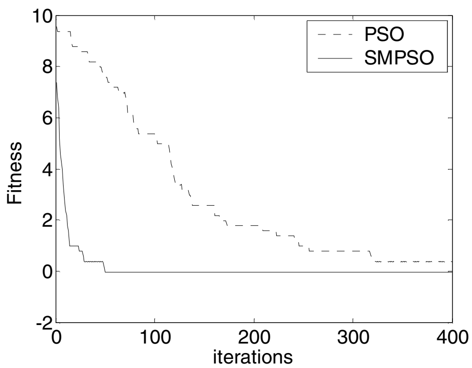

Fig. 2. $\lambda = 0.3$

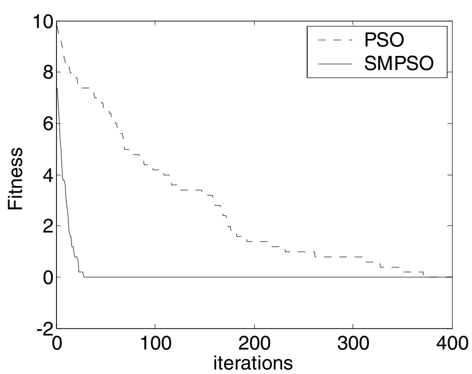

Fig. 3. $\lambda = 0.5$

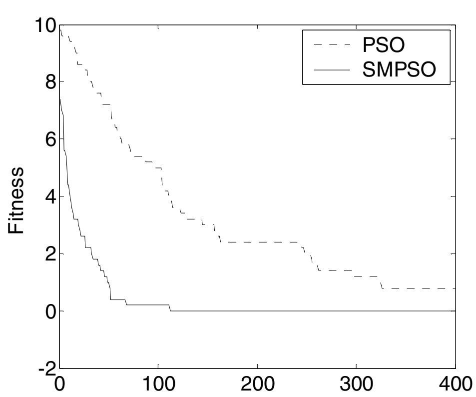

Fig. 4. $\lambda = 0.7$

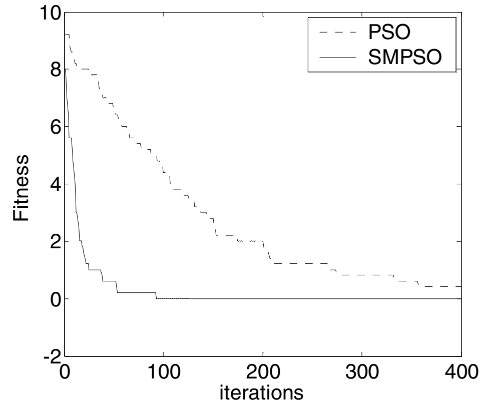

Fig. 5. $\lambda = 0.9$

V. CONCLUSIONS

In this paper, we proposed a new variant of original PSO that is inspired by the phenomenon of symbiosis in natural ecosystems. This modified symbiotic multi-swarm PSO (SMPSO) is based on a multiple swarm scheme, in which the whole population is divided into several sub-swarms. The particles in each sub-swarm are enhanced by the experience of its own and the other sub-swarms. By introducing the center communication mechanism, the search information can be transferred among sub-swarms, that helps to accelerate the convergence rate and avoid the particles be trapped into local minima.

In order to test our proposed algorithm, we established an improved Markowitz model considering four real-world constraints. The preliminary experimental results suggested that SMPSO have superior features, both in high quality of the solution and robustness of the results. Our proposed portfolio model and SMPSO are applicable and reliable in real markets with large number of stocks.

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REFERENCES

[1] Eberchart, R.C., Kennedy, J., "A New Optimizer Using Particle Swarm Theory," in *Proc.*6th *IEEE international symposium. Micro-machine and Human Science*, Piscataway, NJ, 1995, pp.39-43.

[2] Kennedy, J., Eberhart, R.C., "Particle Swarm Optimization," in *Proc.1995 IEEE Int. Conf. Neural Networks,* pp. 1942-1948.

[3] Niu, B., Zhu, Y.L., He, X.X., Wu, H., "MCPSO: A Multi-Swarm Cooperative Particle Swarm Optimizer," *Applied Mathematics and Computation*, vol. 185, pp.1050-1062, 2007.

[4] Parsopoulos K. E., Vrahatis M. N., "UPSO-A Unified Particle Swarm Optimization Scheme," *Lecture Series on Computational Sciences*, pp.868-873, 2004.

[5] Liang, J. J., Suganthan, P. N., Qin, A. K., Baskar, S., "Comprehensive Learning Particle Swarm Optimizer for Global Optimization of Multimodal Functions," *IEEE Trans. Evolutionary Computation 10*, pp.281-295, 2006.

[6] Peram, T., Veeramachaneni, K., Mohan, C. K., "Fitness-distance-ratio based particle swarm optimization," in *Proc. Swarm Intelligence Symposium*, IEEE Press, Indiana, USA 2003, pp.174-181.

[7] Fernandez, A., Gomez, S., "Portfolio Selection Using Neural Networks," *Computers & Operations Research 34*, pp.1177–1191, 2007.

[8] Chang, T.J., Meade, N., Beasley, J.E., Sharaiha, Y.M., "Heuristics for Cardinality Constrained Portfolio Optimization," *Computers & Operations Research 27*, pp.1271–1302, 2000.

[9] Crama, Y., Schyns, M., "Simulated Annealing for Complex Portfolio Selection Problems," *European Journal of Operational Research 150*, pp.546–571, 2003.

[10] Derigs, U., Nickel, N.H., "On a Local-search Heuristic for a Class of Tracking Error Minimization Problems in Portfolio Management," *Annals of Operations Research 131*, pp.45–77, 2004.

[11] Schlottmann, F., Seese, D., "A Hybrid Heuristic Approach to Discrete Multi-Objective Optimization of Credit Portfolios," *Computational Statistics & Data Analysis 47*, pp.373–399, 2004.

[12] Wiegand, R. P., "An Analysis of Cooperative Co-evolutionary Algorithms," PhD thesis, George Mason Univ, Fairfax, Virginia, 2004.

[13] Markowitz, H.W., "Foundations of Portfolio Theory," *Journal of Finance 46*, pp.469-477, 1991.