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Weak Maps and Stabilizers of Classes of Matroids

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Let $\mathbf{F}$ be a field and let N be a matroid in a class $\mathscr{N}$ of $\mathbf{F}$-representable matroids that is closed under minors and the taking of duals. Then N is an $\mathbf{F}$-stabilizer for $\mathscr{N}$ if every representation of a 3-connected member of $\mathscr{N}$ is determined up to elementary row operations and column scaling by a representation of any one of its N-minors. The study of stabilizers was initiated by Whittle. This paper extends that study by examining certain types of stabilizers and considering the connection with weak maps.

The notion of a universal stabilizer is introduced to identify the underlying matroid structure that guarantees that N will be an $\mathbf{F}'$-stabilizer for $\mathscr{N}$ for every field $\mathbf{F}'$ over which members of $\mathscr{N}$ are representable. It is shown that, just as with $\mathbf{F}$-stabilizers, one can establish whether or not N is a universal stabilizer for $\mathscr{N}$ by an elementary finite check. If N is a universal stabilizer for $\mathscr{N}$, we determine additional conditions on N and $\mathscr{N}$ that ensure that if N is not a strict rank-preserv-

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ing weak-map image of any matroid in $\mathcal{N}$, then no connected matroid in $\mathcal{N}$ with an N-minor is a strict rank-preserving weak-map image of any 3-connected matroid in $\mathcal{N}$.

Applications of the theory are given for quaternary matroids. For example, it is shown that $U_{2,5}$ is a universal stabilizer for the class of quaternary matroids with no $U_{3,6}$-minor. Moreover, if M_1 and M_2 are distinct quaternary matroids with $U_{2,5}$-minors but no $U_{3,6}$-minors and M_1 is connected while M_2 is 3-connected, then M_1 is not a rank-preserving weak-map image of M_2.

1. INTRODUCTION

Let $\mathcal{F}$ be a set of fields containing either $GF(2)$ or $GF(3)$, and let $\mathcal{M}(\mathcal{F})$ denote the class of matroids representable over all fields in $\mathcal{F}$. There are just seven possibilities for $\mathcal{M}(\mathcal{F})$. If $\mathcal{F}$ contains $GF(2)$, then $\mathcal{M}(\mathcal{F})$ is either the class of binary matroids or the class of regular matroids. Otherwise, $\mathcal{M}(\mathcal{F})$ is one of the classes of ternary, near-regular, dyadic, or $\sqrt[6]{1}$-matroids, or is the class obtained by taking direct sums and 2-sums of dyadic and $\sqrt[6]{1}$-matroids [23, 24]. These classes form fundamental subclasses of the classes of binary and ternary matroids, respectively. Moreover, to varying degrees, they are quite well understood. For example, excluded-minor characterizations are known for binary matroids [20], regular matroids [20], ternary matroids [1, 18], $\sqrt[6]{1}$-matroids [8], and near-regular matroids [7]. There is a very satisfactory decomposition theory for regular matroids [19], and a decomposition theory for $\sqrt[6]{1}$-matroids in terms of near-regular matroids [9].

In the light of the above and the recent excluded-minor characterization of quaternary matroids [8], it is natural to turn attention to sets of fields containing $GF(4)$ and to attempt to describe the classes of matroids that are representable over such sets of fields. One of the major motivations for this paper is that of obtaining techniques that would make such characterizations possible. Having said this, there is limited value in developing methods that are specific to $GF(4)$. Ideally, one would like to have a range of techniques in matroid representation theory that are quite general in their applicability. This is the other major motivation for the material in this paper. Sections 5, 6, and 7 are devoted to the development of methods of reasonable generality, while Section 8 gives applications of the theory to quaternary matroids. The main results of the paper are Theorems 6.1, 7.4, 8.3, and 8.4. A more specific description of the structure of the paper follows.

Let **F** be a field. A matroid N is an **F**-stabilizer for a class $\mathcal{N}$ of **F**-representable matroids if an **F**-representation of a 3-connected member M of $\mathcal{N}$ is determined, up to elementary row operations and column scalings, by a representation of any one of its N-minors. It is shown in [25] that the task of checking that a matroid is an $\mathbf{F}$-stabilizer can be reduced to an elementary finite check. The notion of an $\mathbf{F}$-stabilizer is somewhat coarse. For example, there is no guarantee that a representation of an N-minor will extend to a representation of M. In Section 3, we introduce the notion of a "strong" $\mathbf{F}$-stabilizer, which is an $\mathbf{F}$-stabilizer with such a guarantee. Another feature of stabilizers is that whether or not N is an $\mathbf{F}$-stabilizer for $\mathscr{N}$ is, at times, independent of the field, so that N is also an $\mathbf{F}'$-stabilizer for $\mathscr{N}$ for every field $\mathbf{F}'$ over which members of $\mathscr{N}$ are representable. When this occurs, the reason that N is a stabilizer is due to underlying matroid structure. We identify this structure via the notion of a "universal stabilizer." The definition and some basic properties of universal stabilizers are given in Section 5. In Section 6, it is shown in Theorem 6.1 that one can identify universal stabilizers via an elementary case check that is analogous to that given in [25] for $\mathbf{F}$-stabilizers.

In [15], it is shown that if M_1 is a connected ternary matroid with a $U_{2,4}$-minor and M_2 is a distinct 3-connected ternary matroid with a $U_{2,4}$-minor, then M_1 is not a rank-preserving weak-map image of M_2. This result turns out to be a useful tool in the characterizations of [23, 24]. Section 7 considers general results of this type. In other words, when does the presence of a certain minor in a matroid guarantee that, with sufficient connectivity, there are no nontrivial rank-preserving weak-map images of the matroid within a certain class? It is shown that there is an intimate connection between this question and the material on stabilizers developed in earlier sections. Specifically, it is proved in Theorem 7.4 that if N is a universal stabilizer for $\mathscr{N}$ that is also a strong $\mathbf{F}$-stabilizer for some field $\mathbf{F}$, then, with a natural extra condition, we are guaranteed a weak-map result of the above type.

The last section considers quaternary matroids. In Theorem 8.3, we identify types of stabilizers for certain classes of quaternary matroids and this enables us to give a weak-map result for these classes in Theorem 8.4. A problem with quaternary matroids is that, while 3-connected quaternary matroids are uniquely representable over $GF(4)$, they can have an unbounded number of inequivalent representations over other finite fields. For example, the quaternary matroids representable over $GF(9)$ have an unbounded number of inequivalent representations over $GF(9)$. It had been hoped to use the theory developed in this paper to characterize precisely the way in which such inequivalent representations arise. However, this turned out not to be possible as the example presented in Section 8 shows. The difficulties presented by this example are not insuperable, but their resolution requires a theory distinct from that of stabilizers. This theory is developed in [10].

2. PRELIMINARIES

Familiarity is assumed with the elements of matroid theory. In particular, we assume that the reader is familiar with the theory of matroid representations and matroid connectivity. Notation and terminology follow Oxley [13] with some small exceptions. We denote the simple and cosimple matroid canonically associated with a matroid M by $\mathrm{si}(M)$ and $\mathrm{co}(M)$, respectively; and we sometimes write $M(S)$ to denote that the matroid M has ground set S. We shall also write $X \sqcup Y$ to denote the union of *disjoint* sets X and Y.

Free Spikes

For an integer $k \geq 3$, a rank-k *spike* with *tip* p is a rank-k matroid with ground set $\{p, a_1, b_1, a_2, b_2, \dots, a_k, b_k\}$ such that

(i) $\{p, a_i, b_i\}$ is a triangle for all i in $\{1, 2, \dots, k\}$, and

(ii) $r(\bigcup_{j \in J}\{a_j, b_j\}) = |J| + 1$ for every proper subset J of $\{1, 2, \dots, k\}$.

Each pair $\{a_i, b_i\}$ is a *leg* of the spike. Consider nonspanning circuits of a rank-k spike. These include the above-mentioned triangles containing p. They also include sets of the form $\{a_i, b_i, a_j, b_j\}$ for all distinct i and j in $\{1, 2, \dots, k\}$. Otherwise the only nonspanning circuits have the form $\{z_1, z_2, \dots, z_k\}$ where $z_i \in \{a_i, b_i\}$. If all such sets $\{z_1, z_2, \dots, z_k\}$ are independent, then the spike obtained is called the *free* rank-k spike with tip p, and is denoted by Φ_k^+. The *tipless* free rank-k spike is the matroid $\Phi_k = \Phi_k^+ \setminus p$.

It appears that spikes play an important role in matroid structure theory. For example, spikes appear amongst the unavoidable minors of 3-connected matroids in the Ramsey-theoretic results of [5]. Moreover, free spikes have already played a role in matroid representation theory. It is shown in [14] that if $\mathbf{F}$ is a nonprime field, then Φ_k^+ is $\mathbf{F}$-representable for all $k \geq 3$. Furthermore, if the additive group of $\mathbf{F}$ has a proper subgroup of order at least 3, then Φ_k^+ has at least 2^{k-1} inequivalent $\mathbf{F}$-representations. Thus free spikes provide counterexamples to a conjecture of Kahn [11] that 3-connected matroids representable over a finite field $\mathbf{F}$ have a bounded number of inequivalent $\mathbf{F}$-representations. It follows that, for every nonprime field $\mathbf{F}$ other than $GF(4)$, the class of 3-connected $\mathbf{F}$-representable quaternary matroids has an unbounded number of inequivalent $\mathbf{F}$-representations.

Partial Fields

A partial field $\mathbf{F}$ is a structure that behaves very much like a field except that addition may be a partial operation. More precisely, Vertigan [22] has shown that every partial field $\mathbf{F}$ can be obtained from a commutative ring R and a multiplicative group G of units of R for which $-1 \in G$. The *partial field* $\mathbf{F}$ associated with the pair (G, R) has $G \cup \{0\}$ as its elements and has the binary operations of addition and multiplication restricted from R to $G \cup \{0\}$. Thus multiplication is a total binary operation, but addition is a partial binary operation; that is, if a and b are elements of $G \cup \{0\}$, then their product ab is always in $G \cup \{0\}$, but their sum $a + b$ need not be, in which case it is undefined. Partial fields were introduced in [17] where it was shown that one can develop a theory of matroid representation for them. Numerous properties of matroids representable over fields hold in the more general setting of partial fields and a number of natural classes of matroids can be characterized as classes of matroids representable over a fixed partial field. Many results in this paper are stated for partial fields. Readers whose sole interest is in fields should simply treat these as results for fields.

Stabilizers

In this paper, we are always interested in classes of matroids that are minor-closed, closed under isomorphism, and closed under duality. For convenience, we call such a class a *well-closed class*. Two matrix representations of a matroid over a partial field are *equivalent* if one can be obtained from the other via a sequence of the following operations: adding one row to another; permuting rows; permuting columns (along with their labels); multiplying a row or a column by a nonzero scalar; and applying an automorphism of $\mathbf{F}$ to the entries of the matrix. The two representations are *strongly equivalent* if one can be obtained from the other via the above matrix operations without applying a field automorphism.

Let $\mathbf{F}$ be a partial field, N be an $\mathbf{F}$-representable matroid, and M be an $\mathbf{F}$-representable matroid with an N-minor. Then N *stabilizes* M over $\mathbf{F}$ if an $\mathbf{F}$-representation of M is determined up to strong equivalence by an $\mathbf{F}$-representation of any one of its N-minors. In other words, if we are given an $\mathbf{F}$-representation of N that does extend to a representation of M, and we obtain two representations of M by extending the given representation of N, then the two representations are strongly equivalent.

Now let $\mathcal{N}$ be a well-closed class of $\mathbf{F}$-representable matroids, and N be a matroid in $\mathcal{N}$. Then N is an $\mathbf{F}$-*stabilizer* for $\mathcal{N}$ if N stabilizes every 3-connected matroid in $\mathcal{N}$ with an N-minor. It would appear to be a potentially infinite task to check that N is an **F**-stabilizer, but the following result [25, Theorem 5.8] shows that the task is finite.

THEOREM 2.1. *Let $\mathcal{N}$ be a well-closed class of matroids representable over a partial field* **F** *and let N be a* 3*-connected matroid in $\mathcal{N}$. Then N is an* **F**-*stabilizer for $\mathcal{N}$ if and only if N stabilizes every* 3*-connected matroid M in $\mathcal{N}$ that has one of the following properties.*

(i) *M has an element x such that $M \backslash x = N$.*

(ii) *M has an element y such that $M/y = N$.*

(iii) *M has a pair of elements $\{x, y\}$ such that $M \backslash x/y = N$, and $M \backslash x$ and M/y are both* 3*-connected.*

3. STRONG STABILIZERS

Strong Stabilizers

Assume that N is an **F**-stabilizer for $\mathcal{N}$. Then, given a 3-connected matroid M in $\mathcal{N}$ with an N-minor, there is no guarantee that a given representation of N will extend to a representation of M. Loosely speaking, stabilizers guarantee that the number of representations does not increase, but they do not guarantee that this number does not decrease. Moreover, it is certainly of interest to know when a representation of N is guaranteed to extend to a representation of M.

The matroid N *strongly stabilizes* M over **F** if N stabilizes M, and every **F**-representation of N extends to an **F**-representation of M. The matroid N is a *strong* **F**-stabilizer for $\mathcal{N}$ if N is an **F**-stabilizer and N strongly stabilizes every matroid in $\mathcal{N}$ with an N-minor.

It is easily seen that if N is a strong **F**-stabilizer for $\mathcal{N}$, then every representation of N extends to a representation of any matroid M in $\mathcal{N}$ with an N-minor, 3-connected or otherwise. However, if M is not 3-connected, we lose the guarantee that the extension is unique. One way to be sure that the **F**-stabilizer N is also a strong **F**-stabilizer is for N to be uniquely representable over **F**. More generally, we have the following proposition, which is an immediate corollary of [25, Prop. 5.6].

PROPOSITION 3.1. *Let N be an* **F**-*stabilizer for the well-closed class $\mathcal{N}$, and let N' be a* 3*-connected matroid in $\mathcal{N}$ with an N- or N^*-minor. Then N' is an* **F**-*stabilizer for $\mathcal{N}$. Moreover, if N' is uniquely* **F**-*representable, then N' is a strong* **F**-*stabilizer.*

To illustrate these ideas, consider an example. It is easily checked, and shown in [25], that $U_{2,4}$ is a $GF(9)$-stabilizer for the class of ternary matroids. But it is not a strong $GF(9)$-stabilizer since not all $GF(9)$-representations of $U_{2,4}$ extend to representations of, for example, the non-Fano matroid F_7^-. However, by Proposition 3.1, F_7^- is a strong $GF(9)$-stabilizer for the class since it is easily checked that F_7^- is uniquely representable over $GF(9)$.

It appears that there is no finite-check theorem, analogous to Theorem 2.1, to determine whether a matroid is a strong **F**-stabilizer. In the above example, the nearest matroid to $U_{2,4}$ not strongly stabilized by $U_{2,4}$ has three more elements, and it is easy to construct examples where the distance is greater than that. We believe that this is a fundamental problem and is one that lies at the heart of the difficulty of finding excluded-minor characterizations for classes of representable matroids.

4. CLONES AND FIXED ELEMENTS

Stabilizers and strong stabilizers as defined above depend not only on the class of matroids, but on the choice of field. We wish to define a type of stabilizer that depends only on the structure of the matroids in the class. In this section, we develop some theory that enables us to do this.

Elements x and x' of a matroid M are *clones* if interchanging x and x' is an automorphism of M. Thus clones are elements of a matroid that are indistinguishable up to labeling. If $\{x, x'\}$ is a pair of loops, a pair of coloops, a parallel pair, or a series pair, then x and x' are clones. It is also immediate that x and x' are clones in M if and only if they are clones in M^*.

Let x be an element of the matroid M. The matroid M' is obtained by *cloning* x *with* x' if M' is a single-element extension of M by x', and x and x' are clones in M'. Dually, we have that M' is obtained by *cocloning* x *with* x' if M' is a single-element coextension of M by x' and $\{x, x'\}$ are clones in M'.

It is always possible to clone x with x'; if x is a loop, just add x' as a loop, while if x is not a loop, then add x' in parallel to x. However, it is not always possible to clone x and x' so that $\{x, x'\}$ is independent. In the case that x cannot be cloned with x' so that x and x' are independent, we say that x is *fixed* in M. Dually, x is *cofixed* in M if M has no coextension by x' such that x and x' are coindependent clones in this coextension. In other words, x is cofixed in M if and only if x is fixed in M^*.

If x is not fixed, then there exists a matroid M' obtained by cloning x with x' such that $\{x, x'\}$ is independent in M'. We say that M' is obtained by *independently* cloning x with x'. Dually, we refer to a matroid being obtained by *coindependently* cocloning x with x'. Note that knowing that

M' is obtained by independently cloning x with x' does not, in general, determine M' up to isomorphism. For example, if $x \in E(U_{3,4})$, then one can obtain both $U_{3,5}$ and $U_{2,4} \oplus_2 U_{2,3}$ by independently cloning x.

Though the terminology is new, the ideas described above are not. To see this, we begin by recalling the definition of a modular cut. Recall that the flats F_1 and F_2 of a matroid form a *modular pair* if $r(F_1 \cup F_2) + r(F_1 \cap F_2) = r(F_1) + r(F_2)$. A *modular cut* in a matroid M is a collection $\mathscr{F}$ of flats of M with the following properties: if F_1 and F_2 are a modular pair of flats in $\mathscr{F}$, then $F_1 \cap F_2$ is in $\mathscr{F}$; and if $F \in \mathscr{F}$, then any flat of M that contains F is also in $\mathscr{F}$. It is known [4] that modular cuts are in one-to-one correspondence with single-element extensions of M. The single-element extension M' defined by the modular cut $\mathscr{F}$ has ground set $E(M) \cup e$. The flats of M' fall into three disjoint classes (see, for example, Oxley [13, Corollary 7.2.4]):

(i) flats of M that are not in $\mathscr{F}$;

(ii) sets $F \cup e$ where F is a flat of M that is in $\mathscr{F}$; and

(iii) sets $F \cup e$ where F is a flat of M that is not in $\mathscr{F}$, and there is no flat G of M belonging to $\mathscr{F}$ such that $F \subseteq G$ and $r(G) = r(F) + 1$.

Cheung and Crapo [3] have defined the notion of the *degree* of a modular cut and Duke [6] has defined the notion of the *freedom* of an element in a matroid. It is shown in [6] that a modular cut has degree k if and only if the freedom of the element of extension in the single-element extension defined by the modular cut is k. Moreover, it follows easily from results in [6] that an element e is fixed in M if and only if it has freedom at most 1, or, equivalently, if and only if M is obtained from $M \backslash e$ by a modular cut of degree at most 1. The next result gives us a way of telling that an element is fixed in terms of modular cuts.

A flat of a matroid is *cyclic* if it is a union of circuits. When ordered by inclusion, the collection of modular cuts of a matroid forms a lattice. It follows that, given a set $\mathscr{F}$ of flats of a matroid, there is a unique minimal modular cut containing that set of flats. This is the modular cut *generated* by $\mathscr{F}$ and is denoted by $\langle \mathscr{F} \rangle$.

PROPOSITION 4.1 [6, Corollary 3.5]. *Let e be an element of the matroid M. Then e is fixed in M if and only if* $\mathrm{cl}(e)$ *is in the modular cut generated by the cyclic flats of M containing e.*

EXAMPLE 4.2. Consider $U_{3,6}$. Say that $E(U_{3,6}) = \{a_1, a_2, a_3, b_1, b_2, b_3\}$. Note that $U_{3,6}$ is the tipless rank-3 free spike Φ_3. Let M_1 and M_2 be obtained by extending $U_{3,6}$ by elements e_1 and e_2 via the modular cuts $\langle\{\{a_1, b_1\}\}\rangle$ and $\langle\{\{a_1, b_1\}, \{a_2, b_2\}\}\rangle$, respectively. Also, Φ_3^+ is obtained by extending $U_{3,6}$ by the element p via the modular cut $\langle\{\{a_1, b_1\}, \{a_2, b_2\}, \{a_3, b_3\}\}\rangle$. It is easy to check, using Theorem 4.1, that e_1 is not fixed in M_1, but e_2 and p are fixed in the matroids M_2 and Φ_3^+, respectively. Note that, in some sense, p is more fixed in Φ_3^+ than e_2 is in M_2. This is because $\langle\{a_1, b_1\}, \{a_2, b_2\}, \{a_3, b_3\}\rangle$ properly contains $\langle\{a_1, b_1\}, \{a_2, b_2\}\rangle$. Evidently, over any partial field, $U_{3,6}$ could not strongly stabilize any class of matroids that contained both M_2 and Φ_3^+.

Next we give some elementary equivalent conditions for x and x' to be clones in a matroid M.

PROPOSITION 4.3. *Let x and x' be elements of a matroid M. Then the following are equivalent.*

(i) *x and x' are clones in M.*

(ii) *Replacing x by x' and fixing every other element is an isomorphism from $M \backslash x'$ to $M \backslash x$.*

(iii) *$M/x \backslash x' = M/x' \backslash x$ and $r(\{x\}) = r(\{x'\})$.*

The next proposition is a straightforward consequence of the definitions. It is a useful way of showing that an element is not fixed in a minor. We omit the obvious dual of the proposition.

PROPOSITION 4.4. *Assume that x and x' are independent clones in the matroid M', and $M = M' \backslash x'$. If X and Y are disjoint subsets of $E(M') - \{x, x'\}$, then $\{x, x'\}$ are clones in $M' \backslash X/Y$. Moreover, if $\{x, x'\}$ is independent in $M' \backslash X/Y$, then x is not fixed in $M \backslash X/Y$.*

A point p of a matroid M is *freely placed* on a flat F if $p \in F$, and $\mathrm{cl}_M(C) \supseteq F$ for every circuit C of M containing p. The next proposition is a special case of [6, Prop. 3.1].

PROPOSITION 4.5. *If p is fixed in M, and F is a flat of M of rank greater than* 1, *then p is not freely placed on F.*

Note that the converse of Proposition 4.5 does not hold as examples given by Duke [6] show. To see this, take a matroid M with a 3-separation $\{X, Y\}$ where $r(X), r(Y) \geq 3$ and $\mathrm{cl}(X) \cap \mathrm{cl}(Y) = \emptyset$. Let M' be the extension of M obtained via the modular cut $\langle X, Y\rangle$. One readily checks that x is not fixed in M' and that x is not freely placed on any line of M.

The following corollary of Proposition 4.5 will prove useful in this paper.

COROLLARY 4.6. *Let M be a matroid, a be an element of $E(M)$ that is not a loop or a coloop, and b be an element of $E(M) - a$ that is not a loop and is not parallel to a. If a is fixed in M, then there is an independent subset I of $E(M) - \{a, b\}$ such that $\mathrm{cl}_M(I)$ contains a but not b.*

Proof. Assume that a is fixed in M. Consider the line $\mathrm{cl}(\{a, b\})$. Then, by Proposition 4.5, a is not freely placed on this line. Hence there is a circuit C of M containing a with the property that $\mathrm{cl}(C)$ does not contain $\mathrm{cl}(\{a, b\})$. Since a and b are not parallel, this means that $b \notin \mathrm{cl}(C)$. Hence $C - a$ is an independent set whose closure contains a but not b. ∎

The next proposition enables us to deduce that an element is fixed from the fact that it is fixed in certain minors.

PROPOSITION 4.7. *Let x be an element of the matroid M.*

(i) *If M has an element a such that x is fixed in $M \backslash a$, then x is fixed in M.*

(ii) *If M has distinct elements a and b such that $\{a, b, x\}$ is independent in M and x is fixed in both M/a and M/b, then x is fixed in M.*

Proof. Assume that x is not fixed in M. Let M' be a matroid obtained by independently cloning x with x'. If $a \in E(M) - x$, then, by Proposition 4.4, x is not fixed in $M \backslash a$. This proves part (i). Consider part (ii). Say that $\{x, a, b\}$ is independent in M. Then, in M', either $\{x, x', a\}$ is independent or $\{x, x', b\}$ is independent. Assume the former. By Proposition 4.4, x is not fixed in M/a. It follows that if x is fixed in both M/a and M/b, then x is fixed in M. ∎

By dualizing the last result, we immediately obtain the following:

COROLLARY 4.8. *Let x be an element of the matroid M.*

(i) *If M has an element a such that x is cofixed in M/a, then x is cofixed in M.*

(ii) *If M has distinct elements a and b such that $\{a, b, x\}$ is coindependent and x is cofixed in both $M \backslash a$ and $M \backslash b$, then x is cofixed in M.*

Evidently, if x and x' are independent clones in M, then x is not fixed in $M \backslash x'$. The next proposition extends this observation.

PROPOSITION 4.9. *If x and x' are independent clones in M, then x is fixed in neither M nor $M \backslash x'$. Dually, if x and x' are coindependent clones in M, then x is cofixed in neither M nor M/x'.*

Proof. Assume that x and x' are independent clones in M. By duality and the remarks preceding the proposition, it suffices to show that x is not fixed in M. Assume the contrary. Then x is not a coloop of M. By Corollary 4.6, since $\{x, x'\}$ is independent in M, there is an independent subset I of $E(M) - \{x, x'\}$ such that $\mathrm{cl}_M(I)$ contains x but not x'. This contradicts the fact that x and x' are clones in M. ∎

It follows that if x and x' are independent, coindependent clones, then x is neither fixed nor cofixed in M. However, it is quite possible for x to be fixed in M/x' and for x to be cofixed in $M \backslash x'$. To see this, consider the rank-r free spike Φ_r where $r \geq 3$. For any leg $\{a_i, b_i\}$ of Φ_r, the elements a_i and b_i are independent, coindependent clones. Moreover, it is easily checked that a_i is fixed in M/b_i and cofixed in $M \setminus b_i$.

The easy proof of the next proposition is omitted.

LEMMA 4.10. *Let x be an element of the matroid M.*

(i) *If x is parallel to some other element of M, then x is fixed in M.*

(ii) *If x is in series with another element of M, then x is cofixed in M.*

(iii) *If x is not in a series class and is not a coloop, then x is fixed (respectively, cofixed) in M if and only if x is fixed (respectively, cofixed) in* $\mathrm{co}(M)$.

(iv) *If x is not in a parallel class and is not a loop, then x is fixed (respectively, cofixed) in M if and only if x is fixed (respectively, cofixed) in* $\mathrm{si}(M)$.

5. UNIVERSAL STABILIZERS

Let $\mathscr{N}$ be a well-closed class of matroids and let N be a 3-connected matroid in $\mathscr{N}$. Then N is a *universal stabilizer* for $\mathscr{N}$ if the following holds: whenever M and $M \setminus x$ are 3-connected matroids in $\mathscr{N}$ for which $M \setminus x$ has an N-minor, the element x is fixed in M; and, whenever M and M/x are 3-connected matroids in $\mathscr{N}$ for which M/x has an N-minor, the element x is cofixed in M.

In the next section, we show that, just as is the case for stabilizers, it can be decided if N is a universal stabilizer for $\mathscr{N}$ by an elementary finite case check. In this section, we establish some properties of universal stabilizers. For a partial field $\mathbf{F}$, the class of $\mathbf{F}$-representable matroids is denoted $\mathscr{M}(\mathbf{F})$. We begin by proving the following:

THEOREM 5.1. *Let N be a 3-connected matroid that is a universal stabilizer for the well-closed class of matroids $\mathscr{N}$ and let $\mathbf{F}$ be a partial field over which N is representable. Then N is an $\mathbf{F}$-stabilizer for the class $\mathscr{N} \cap \mathscr{M}(\mathbf{F})$.*

Theorem 5.1 justifies the use of the term "universal." The proof is elementary if $\mathbf{F}$ is a field. The proof for partial fields is a little more technical. In what follows, we assume that the reader is familiar with the basic theory of partial fields as set forth in [17]. We always assume that representations of matroids are in *standard form*, that is, of the form $[I|Z]$ where I is an identity matrix. If $[A|\mathbf{x}]$ and $[A|\mathbf{y}]$ both represent matroids over a given field, then obviously $[A|\mathbf{x}, \mathbf{y}]$ represents a matroid over $\mathbf{F}$. This is not true in general for partial fields. The difficulties caused by this are dealt with in the next lemma. We first fix some notation.

Let Q be an $r \times n$ matrix over a partial field and suppose that the columns of Q are labeled by members of the set S. For an r-subset T of S, let $\mathbf{T}$ denote the $r \times r$ submatrix consisting of those columns of Q labeled by members of T. Over a partial field, $\det(\mathbf{T})$ may be zero, nonzero, or undefined. Define the collection $\mathscr{B}(Q)$ of subsets T of S as follows: T is in $\mathscr{B}(Q)$ if $|T| = r$, and $\det(\mathbf{T})$ is either nonzero or is undefined. In general, there is no guarantee that $\mathscr{B}(Q)$ is the collection of bases of a matroid.

LEMMA 5.2. *Let $\mathbf{F}$ be a partial field and let M_x and M_y be rank-r $\mathbf{F}$-representable matroids on the ground sets $E \cup x$ and $E \cup y$, respectively, with the property that $M_x \backslash x = M_y \backslash y$. Assume that some $\mathbf{F}$-representation A of this common matroid extends to $\mathbf{F}$-representations $[A|\mathbf{x}]$ and $[A|\mathbf{y}]$ of M_x and M_y, respectively. Then $\mathscr{B}([A|\mathbf{x}, \mathbf{y}])$ is the collection of bases of a matroid M on $E \cup \{x, y\}$.*

Proof. We write D for $[A|\mathbf{x}, \mathbf{y}]$. It suffices to prove that $\mathscr{B}(D)$ is the collection of bases of a matroid.

Just as with matrices over fields, one can pivot on an entry of a matrix over a partial field, although such an operation is not always defined. The next two observations follow easily from results in [17]. We omit the proofs.

5.2.1. *Consider $D = [A|\mathbf{x}, \mathbf{y}]$. A pivot on a nonzero entry of A is always defined. If D' is obtained from D by such a pivot, or by interchanging rows, then $\mathscr{B}(D') = \mathscr{B}(D)$.*

5.2.2. *Let t label the column $[1, 0, \dots, 0]^T$ of D. Let D' denote the matrix obtained by deleting the first row and column of D. Then an $(r-1)$-subset X' of $(E \cup \{x, y\}) - t$ is a member of $\mathscr{B}(D')$ if and only if $X' \cup t$ is a member of $\mathscr{B}(D)$.*

Assume that $\mathscr{B}(D)$ is not the collection of bases of a matroid, and assume further that, among all counterexamples to the lemma, $\mathscr{B}(D)$ is one for which r is minimal. Certainly $\mathscr{B}(D) \neq \emptyset$. Thus, since $\mathscr{B}(D)$ is not the collection of bases of a matroid, there is a pair $B_1, B_2 \in \mathscr{B}(D)$, and an element z of $B_1 - B_2$ having the property that there is no element u of $B_2 - B_1$ such that $(B_1 - z) \cup u$ is a member of $\mathscr{B}(D)$.

5.2.3. $B_1 \cap B_2 \subseteq \{x, y\}$.

Proof. Say $t \in (B_1 \cap B_2) - \{x, y\}$. By (5.2.1), we lose no generality in assuming that t labels the column $[1, 0, \dots, 0]^T$ of D. Let D' denote the matrix obtained from D by deleting the first row and column from D. By the minimality assumption, $\mathscr{B}(D')$ is the collection of bases of a matroid.

By (5.2.2), $B_1 - t$ and $B_2 - t$ are both members of $\mathscr{B}(D')$. Since $z \in (B_1 - t) - (B_2 - t)$, there is an element v of $(B_2 - t) - (B_1 - t)$ such that $((B_1 - t) - z) \cup v \in \mathscr{B}(D')$. But then, $(B_1 - z) \cup v \in \mathscr{B}(D)$, contradicting the assumption that $\mathscr{B}(D)$ has no member of this form. ∎

5.2.4. $B_1 \subseteq \{x, y, z\}$.

Proof. Assume $B_1 \not\subseteq \{x, y, z\}$; say $t \in B_1 - \{x, y, z\}$. By (5.2.3), $t \in B_1 - B_2$. As before, we lose no generality in assuming that t labels $[1, 0, \dots, 0]^T$, and we let D' denote the matrix obtained by deleting the first row and column of D. Again, by minimality, $\mathscr{B}(D')$ is the collection of bases of a matroid on $(E \cup \{x, y\}) - t$. By (5.2.2), $B_1 - t$ is a member of $\mathscr{B}(D')$. Consider B_2. It labels an $(r-1) \times r$-submatrix $\mathbf{B}_2'$ of D'. If the determinant of every $(r-1) \times (r-1)$-submatrix of $\mathbf{B}_2'$ is 0, then an easy argument using elementary facts about determinants in partial fields shows that $\det(\mathbf{B}_2) = 0$, contradicting the fact that B_2 is a member of $\mathscr{B}(D)$. Thus there is an $(r-1)$-subset B_2' of B_2 that belongs to $\mathscr{B}(D')$. Since $\mathscr{B}(D')$ is the collection of bases of a matroid, there is an element v of $B_2' - (B_1 - t)$ for which $((B_1 - t) - z) \cup v$ is a member of $\mathscr{B}(D')$. But now, $(B_1 - z) \cup v$ is in $\mathscr{B}(D)$, and $v \in B_2 - B_1$, contradicting the assumption that B_2 has no element with this property. ∎

An easy case check now shows that there is no counterexample satisfying the properties of (5.2.3) and (5.2.4) and it follows that the lemma holds. ∎

We can now complete the proof of Theorem 5.1.

Proof of Theorem 5.1. Let M be a 3-connected matroid in $\mathscr{N} \cap \mathscr{M}(\mathbf{F})$ with an element x such that $M \backslash x$ is either equal to N or is a 3-connected single-element coextension of N. We show that $M \backslash x$ stabilizes M over $\mathbf{F}$. Consider an $\mathbf{F}$-representation A of $M \backslash x$ that extends to an $\mathbf{F}$-representation of M. Let $\mathbf{x}_1$ and $\mathbf{x}_2$ be vectors having the property that $[A|\mathbf{x}_1]$ and $[A|\mathbf{x}_2]$ both represent M. By Lemma 5.2, there is a matroid M' defined on the set of columns of $[A|\mathbf{x}_1, \mathbf{x}_2]$ having the property that $M' \backslash \mathbf{x}_2 = M[A|\mathbf{x}_1]$, and $M' \backslash \mathbf{x}_1 = M[A|\mathbf{x}_2]$. Evidently interchanging $\mathbf{x}_1$ and $\mathbf{x}_2$ is an automorphism of M', so $\mathbf{x}_1$ and $\mathbf{x}_2$ are clones in M'. It follows from the definition of a universal stabilizer that $\mathbf{x}_1$ is fixed in $M' \backslash \mathbf{x}_2$. Hence $\mathbf{x}_1$ and $\mathbf{x}_2$ are parallel in M'.

Suppose that $[A|\mathbf{x}_1, \mathbf{x}_2]$ has a submatrix with an undefined subdeterminant. By adjoining columns from the identity matrix if necessary, we deduce that $[A|\mathbf{x}_1, \mathbf{x}_2]$ has an $(r(M) \times r(M))$-submatrix with an undefined subdeterminant. This submatrix must use both columns $\mathbf{x}_1$ and $\mathbf{x}_2$. But the columns of the submatrix form a basis of M'. Thus there is a basis of M' using $\mathbf{x}_1$ and $\mathbf{x}_2$ contradicting the fact that these are parallel in M'. This shows that $M[A|\mathbf{x}_1, \mathbf{x}_2]$ is well defined and that $M' = M[A|\mathbf{x}_1, \mathbf{x}_2]$. We can now deduce from this and the fact that $\mathbf{x}_1$ and $\mathbf{x}_2$ are parallel in M' that $\mathbf{x}_1$ and $\mathbf{x}_2$ are scalar multiples of each other. It follows that the representations $[A|\mathbf{x}_1]$ and $[A|\mathbf{x}_2]$ are strongly equivalent.

The above argument can be dualized to deal with coextensions and we can conclude that N $\mathbf{F}$-stabilizes all 3-connected matroids in $\mathcal{N} \cap \mathcal{M}(\mathbf{F})$ that are 3-connected single-element extensions or coextensions of N, or are 3-connected single-element extensions of 3-connected single-element coextensions of N. It now follows by Theorem 2.1 that N is an $\mathbf{F}$-stabilizer for $\mathcal{N} \cap \mathcal{M}(\mathbf{F})$. ∎

The next theorem shows that the property of being a universal stabilizer is equivalent to an apparently much stronger property.

THEOREM 5.3. *Let N be a 3-connected matroid in the well-closed class $\mathcal{N}$. Then N is a universal stabilizer for $\mathcal{N}$ if and only if the following condition holds for every 3-connected matroid M in $\mathcal{M}$ and every x in $E(M)$:*

If $M \backslash x$ is connected with an N-minor, then x is fixed in M, and if M/x is connected with an N-minor, then x is cofixed in M.

If M is 3-connected, then, for every x in $E(M)$, one would expect $M \backslash x$ to be connected, and thus suspect that the statement of Theorem 5.3 has a redundant condition. There is, however, a single exception and that occurs if M is $U_{2,3}$. A dual comment applies for contraction.

We need to prepare the ground for the proof of Theorem 5.3. We begin by fixing some terminology. Assume that M has an exact 2-separation $\{S_1, S_2\}$. Then there are matroids $M_1(S_1 \sqcup p)$ and $M_2(S \sqcap p)$ such that $M = M_1 \oplus_2 M_2$. Say $i \in \{1, 2\}$. We call S_i an *N-part* if M_i has an N-minor. We also denote the closure operator of M^* by cl_M^* or, if no danger of ambiguity exists, by cl*. The elementary proof of the next lemma is omitted.

LEMMA 5.4. *Let M be a matroid with an exact 2-separation $\{S_1, S_2\}$, and let N be a matroid such that S_1 is an N-part of M.*

(i) $r_M(\mathrm{cl}(S_1) \cap S_2) \leq 1$. *In particular, if M is loopless, then* $\mathrm{cl}(S_1) \cap S_2$ *is either empty, a single element, or a set of parallel elements of M.*

(ii) *If $z \in S_2 - \mathrm{cl}(S_1)$ and $|S_2| > 2$, then $\{S_1, S_2 - z\}$ is a 2-separation of M/z. Moreover, S_1 is an N-part of M/z.*

(iii) $r_M^*(\mathrm{cl}^*(S_1) \cap S_2) \leq 1$. *In particular, if M is coloop-free, then* $\mathrm{cl}^*(S_1) \cap S_2$ *is either empty, a single element, or a set of series elements of M.*

(iv) *If $z \in S_2 - \mathrm{cl}^*(S_1)$ and $|S_2| > 2$, then $\{S_1, S_2 - z\}$ is a 2-separation of $M \backslash z$. Moreover, S_1 is an N-part of $M \backslash z$.*

Typically, a 3-connected matroid M is both simple and cosimple. The only exceptions occur for $|E(M)| \leq 3$. In this case, $M \in \{U_{2,3}, U_{1,3}, U_{1,2}, U_{1,1}, U_{0,1}, U_{0,0}\}$. Apart from the trivial matroid $U_{0,0}$, members of this set are either not simple or not cosimple. At times, it helps to distinguish these matroids from other 3-connected matroids. We say that a 3-connected matroid is *small* if its ground set has no more than 3 elements.

LEMMA 5.5. *Let N be a small 3-connected matroid in the well-closed class $\mathscr{N}$. If N is a universal stabilizer for $\mathscr{N}$, then all members of $\mathscr{N}$ are binary.*

Proof. Assume that $\mathscr{N}$ contains a nonbinary matroid. Then, by the excluded-minor characterization of binary matroids and the fact that $\mathscr{N}$ is minor-closed, $U_{2,4} \in \mathscr{N}$. Choose $x \in E(U_{2,4})$. Then x is neither fixed nor cofixed in $U_{2,4}$. Moreover, both $U_{2,4} \backslash x$ and $U_{2,4}/x$ are 3-connected, and every small 3-connected matroid is a minor of one of $U_{2,4} \backslash x$ or $U_{2,4}/x$. It follows that N is not a universal stabilizer for $\mathscr{N}$. ∎

It is not hard for an element of a binary matroid to be fixed or cofixed.

LEMMA 5.6. *Let M be a connected binary matroid M having at least two points and let x be an element of M. If $M \backslash x$ is connected, then x is fixed in M, while if M/x is connected, then x is cofixed in M.*

Proof. Suppose $M \backslash x$ is connected. We show that x is fixed in M. This is clearly the case if $|E(M)| = 2$ and, furthermore, is easily checked to be the case if $r(M) \leq 2$. Assume that $r(M) \geq 3$ and, for induction, that the result holds for every matroid satisfying the hypotheses whose ground set has cardinality less than $|E(M)|$. Assume that there is an element y of $E(M \backslash x)$ such that $M \backslash x, y$ is connected. One readily checks that x is not a coloop of $M \backslash y$ so that $M \backslash y$ is connected. Hence, by the inductive hypothesis and Proposition 4.7(i), x is fixed in M. If there is no element y of $E(M \backslash x)$ such that $M \backslash x, y$ is connected, then, by [13, Theorem 4.3.1], $M \backslash x/z$ is connected for all $z \in E(M \backslash x)$. Since $r(M) > 2$ we can choose distinct elements z_1 and z_2 of $E(M \backslash x)$ such that $\{x, z_1, z_2\}$ is independent in M. Then M/z_1 and M/z_2 are both connected. Thus, by the inductive hypothesis, x is fixed in both M/z_1 and M/z_2. It now follows by Proposition 4.7(ii) that, in this case too, x is fixed in M. The remainder of the lemma follows by duality. ∎

The matroid $U_{0,0}$ is not a universal stabilizer for any nontrivial well-closed class of matroids. To see this, note that such a class contains $U_{1,1}$, and, if $E(U_{1,1}) = \{x\}$, then x is not fixed in $U_{1,1}$; but, of course, $U_{1,1} \backslash x$ is 3-connected with a $U_{0,0}$-minor. For other 3-connected binary matroids, we have the following immediate consequence of Lemma 5.6.

PROPOSITION 5.7. *Let $\mathcal{N}$ be a class of binary matroids and let N be a* 3-*connected matroid in $\mathcal{N}$. If $N \not\cong U_{0,0}$, then N is a universal stabilizer for $\mathcal{N}$.*

An elementary argument proves the following:

LEMMA 5.8. *Let N be a* 3-*connected matroid and let M be a matroid with an N-minor. If N is not small, then both* $\operatorname{co}(M)$ *and* $\operatorname{si}(M)$ *have N-minors.*

Finally, we note a result of Bixby [2] (see Oxley [13, Prop. 8.4.6]).

LEMMA 5.9. *Let M be a* 3-*connected matroid and let e be an element of M. Then either* $\operatorname{co}(M \setminus e)$ *or* $\operatorname{si}(M/e)$ *is* 3-*connected.*

At last we can tackle the proof of Theorem 5.3.

Proof of Theorem 5.3. Let N be a universal stabilizer for $\mathcal{N}$. The theorem follows provided we can prove that N satisfies the specified condition. If N is a small 3-connected matroid, then, by Lemma 5.5, $\mathcal{N}$ is a class of binary matroids, and the result then holds by Lemma 5.6. For the remainder of the proof, we assume that N is not small.

Let M be a 3-connected matroid in $\mathcal{N}$ and let x be in $E(M)$. Assume that $M \setminus x$ has an N-minor. We prove, by induction on $|E(M)|$, that x is fixed in M. If $|E(M)| = |E(N) + 1$, then $M \setminus x \cong N$, so $M \setminus x$ is 3-connected, and it follows from the definition of a universal stabilizer that x is fixed in M. Now let $|E(M)| > |E(N)| + 1$, and assume that if M' is a 3-connected matroid in $\mathcal{N}$ with $|E(M')| < |E(M)|$ and x' is an element of $E(M')$ such that $M' \setminus x'$ has an N-minor, then x' is fixed in M'.

5.3.1. *If there is an element z of $E(M) - x$ with the properties that* $\operatorname{co}(M \setminus z)$ *is* 3-*connected, $M \setminus x, z$ has an N-minor, and x is not in a series pair of $M \setminus z$, then x is fixed in M.*

Proof. Since x is not in a series class of $M \setminus z$, the element x is unambiguously in the ground set of $\operatorname{co}(M \setminus z)$. Consider $M \setminus x, z$. This matroid has an N-minor, so by Lemma 5.8, $\operatorname{co}(M \setminus x, z)$ has an N-minor. It is easily seen that $\operatorname{co}(M \setminus x, z)$ is a minor of $\operatorname{co}(M \setminus z) \setminus x$. Hence $\operatorname{co}(M \setminus z) \setminus x$ has an N-minor. It now follows from the fact that $\operatorname{co}(M \setminus z)$ is 3-connected and the induction assumption that x is fixed in $\operatorname{co}(M \setminus z)$. By Lemma 4.10(iii), x is fixed in $M \setminus z$ and, by Lemma 4.7(i), x is fixed in M. ∎

5.3.2. *If there are distinct elements z_1 and z_2 of $E(M) - x$ with the properties that $\{z_1, z_2, x\}$ is independent and, for each i in $\{1, 2\}$, the simplification of M/z_i is* 3-*connected and $M/z_i \setminus x$ has an N-minor, then x is fixed in M.*

Proof. Consider M/z_i. If x is in a parallel class of this matroid, then x is fixed in M/z_i by Lemma 4.10(i). Otherwise, x is unambiguously an element of $\operatorname{si}(M/z_i)$. Evidently $\operatorname{si}(M/z_i) \setminus x = \operatorname{si}(M/z_i \setminus x)$. Moreover,

$M/z_i \backslash x$ has an N-minor. It now follows from the induction hypothesis that x is fixed in $\mathrm{si}(M/z_i)$. Hence, by Lemma 4.10(ii), x is fixed in M/z_i. Since $\{x, z_1, z_2\}$ is independent in M, it follows from Proposition 4.7(ii) that x is fixed in M. ∎

We now turn to the main task of the proof. Consider $M \backslash x$. If $M \backslash x$ is 3-connected, then it follows from the definition of a universal stabilizer that x is fixed in M. From now on, we assume that $M \backslash x$ is not 3-connected. Hence $M \backslash x$ has an exact 2-separation $\{X, Y\}$. Evidently either X or Y is an N-part of this 2-separation. Assume, without loss of generality, that X is an N-part.

5.3.3. *If no series pair of $M \backslash x$ is contained in Y, then x is fixed in M.*

Proof. Assume that Y contains no series pair of $M \backslash x$. Evidently $M \backslash x$ has no parallel pairs, so, by Lemma 5.4(i), $|\mathrm{cl}_{M \backslash x}(X) \cap Y| \leq 1$. Thus $|Y - \mathrm{cl}_{M \backslash x}(X)| \geq 1$. Again, since $M \backslash x$ has no parallel pairs, $|Y - \mathrm{cl}_{M \backslash x}(X)| \geq 2$. If $|Y - \mathrm{cl}_{M \backslash x}(X)| = 2$, then $Y - \mathrm{cl}_{M \backslash x}(X)$ is a series pair, contradicting the assumption that Y has no such series pairs. Hence $|Y - \mathrm{cl}_{M \backslash x}(X)| > 2$. Since Y contains no series pairs, by Lemma 5.4(iii), $|Y \cap \mathrm{cl}^*_{M \backslash x}(X)| \leq 1$. It follows that there is a pair y_1, y_2 of distinct elements of Y that are in neither $\mathrm{cl}^*_{M \backslash x}(X)$ nor $\mathrm{cl}_{M \backslash x}(X)$. By Lemma 5.4(ii) and (iv), for each i in $\{1, 2\}$, both $M \backslash x \backslash y_i$ and $M \backslash x / y_i$ have N-minors. Assume, for some i in $\{1, 2\}$, that $\mathrm{co}(M \backslash y_i)$ is 3-connected. If y_i were in a series pair $\{y_i, z\}$ of $M \backslash x$, then, since Y contains no series pairs, $z \in X$. But then $y_i \in \mathrm{cl}^*_{M \backslash x}(X)$. This contradiction shows that y_i is not in a series pair of $M \backslash x$, so x is not in a series pair of $M \backslash y_i$. It follows by (5.3.1) that x is fixed in M.

We may now assume that neither $\mathrm{co}(M \backslash y_i)$ nor $\mathrm{co}(M \backslash y_2)$ is 3-connected. In this case, by Lemma 5.9, both $\mathrm{si}(M/y_1)$ and $\mathrm{si}(M/y_2)$ are 3-connected. If $\{x, y_1, y_2\}$ is dependent, then $x \in \mathrm{cl}_M(\{y_1, y_2\})$. It is then easily checked that $\{X, Y \cup x\}$ is a 2-separation of M, contradicting the fact that this matroid is 3-connected. Therefore $\{x, y_1, y_2\}$ is independent. We can now conclude by (5.3.2) that x is fixed in M. This establishes (5.3.3). ∎

From now on, we may assume that $M \backslash x$ has at least one series pair.

5.3.4. *If $r(M) \leq 3$, then x is fixed in M.*

Proof. Since N is not small, $r(M) \geq 2$. If $r(M) = 2$ and $M \backslash x$ has series pairs, then $M \backslash x \cong U_{2,3}$, contradicting the assumption that N is not small. Thus we may suppose that $r(M) = 3$. Let $\{s_1, s_2\}$ be a series pair of $M \backslash x$. Certainly $\{x, s_1, s_2\}$ is independent. Moreover, $(\{s_1, s_2\}, E(M \backslash x) - \{s_1, s_2\})$ is an exact 2-separation of $M \backslash x$. As N is not small, $E(M \backslash x) -$

$\{s_1, s_2\}$ is an N-part of $M\backslash x$. Thus, for each i in $\{1, 2\}$, the matroid $M\backslash x/s_i$ has an N-minor. Since $\mathrm{si}(M/s_i)$ is easily seen to be 3-connected, (5.3.2) implies that x is fixed in M. ∎

From now on, we may assume that $r(M) \geq 4$.

5.3.5. *If $M\backslash x$ has a series pair that is contained in a triangle, then x is fixed in M.*

Proof. Assume that $\{s_1, s_2\}$ is a series pair of $M\backslash x$ and that $\{s_1, s_2, p\}$ is a triangle. Since $\{s_1, s_2, x\}$ is a triad of M and $r(M) \geq 4$, it follows without difficulty (see, for example, [25, Lemma 3.5]) that $\mathrm{si}(M/p)$ is not 3-connected. Hence, by Lemma 5.9, $\mathrm{co}(M\backslash p)$ is 3-connected. It is easily checked that $M\backslash p, x$ has an N-minor. Suppose that x is in a series pair of $M\backslash p$. Then there is a triad of M containing x and p. This triad must contain another element of $\{x, s_1, s_2\}$. Without loss of generality, assume that $\{x, p, s_1\}$ is a triad. Using cocircuit elimination on the triads $\{x, s_1, s_2\}$ and $\{x, p, s_1\}$, we deduce that $\{p, s_1, s_2\}$ is a triad. But the only 3-connected matroid in which a triad is also a triangle is $U_{2,4}$. This contradicts the fact that M has rank at least 4. Thus x is not in a series pair of $M\backslash p$. It now follows by the induction assumption and Lemma 4.10(iii) and Proposition 4.7(i) that x is fixed in M. ∎

We may now assume that no series pair of $M\backslash x$ is contained in a triangle.

5.3.6. *If $\{s_1, s_2\}$ is a series pair of $M\backslash x$, then at least one of* $\mathrm{si}(M/s_1)$ *and* $\mathrm{si}(M/s_2)$ *is* 3-*connected.*

Proof. Evidently $\{x, s_1, s_2\}$ is a triad of M. Moreover, we may assume that neither M/s_1 nor M/s_2 is 3-connected. By Tutte's triangle lemma [21] (see Oxley [13, Corollary 8.4.8]), there is a triangle of M using s_1 and exactly one of s_2 and x. But there is no triangle using s_1 and s_2. Hence there is an element z such that $\{s_1, x, z\}$ is a triangle. Moreover, since $\{s_1, s_2, x\}$ is independent in M, it follows that $\{s_1, x, z\}$ is also a triangle of M/s_2. It is now not difficult to see that $\mathrm{co}(M\backslash s_2)$ is not 3-connected so, by Lemma 5.9, $\mathrm{si}(M/s_2)$ is 3-connected. ∎

5.3.7. *If $M\backslash x$ has more than one series pair, then x is fixed in M.*

Proof. Assume that M has a series class S of size greater than 2. Then, by two applications of (5.3.6), we deduce that there is a series pair $\{s_1, s_2\} \subseteq S$ such that both $\mathrm{si}(M/s_1)$ and $\mathrm{si}(M/s_2)$ are 3-connected. By Lemma 5.8, $\mathrm{co}(M\backslash x)$ has an N-minor. One readily deduces from this that $M\backslash x/s_1$ and $M\backslash x/s_2$ both have N-minors. Moreover, $\{s_1, s_2, x\}$ is independent, otherwise M is not 3-connected. It now follows by (5.3.2) that x is fixed in M.

We may now assume that $M\backslash x$ has disjoint series pairs $\{t_1, t_2\}$ and $\{s_1, s_2\}$. By (5.3.6), we may also assume, without loss of generality, that $\mathrm{si}(M/s_1)$ and $\mathrm{si}(M/t_1)$ are 3-connected. Again we note that both $M\backslash x/s_1$ and $M\backslash x/t_1$ have N-minors.

Consider $M\backslash x/s_1$. This matroid has a series pair $\{t_1, t_2\}$. Thus, if $A = E(M) - \{t_1, t_2, x, s_1\}$, then $\{A, \{t_1, t_2\}\}$ is a 2-separation of $M\backslash x/s_1$. If $\{s_1, x, t_1\}$ is a triangle of M, then $\{A, \{t_1, t_2, x\}\}$ is a 2-separation of M/s_1, where neither A nor $\{t_1, t_2, x\}$ is a parallel class, contradicting the fact that $\mathrm{si}(M/s_1)$ is 3-connected. Thus $\{s_1, t_1, x\}$ is independent in M. It now follows by (5.3.2) that x is fixed in M in this case too. ∎

One case remains.

5.3.8. *If $M\backslash x$ has exactly one series pair, then x is fixed in M.*

Proof. Let $\{s_1, s_2\}$ be the unique series pair of $M\backslash x$. By (5.3.6), we may assume that $\mathrm{si}(M/s_1)$ is 3-connected. If $\mathrm{si}(M/s_2)$ is also 3-connected, then we can deduce from the induction assumption and Proposition 4.7(ii) that x is fixed in M. Thus we may assume that $\mathrm{si}(M/s_2)$ is not 3-connected. If $M\backslash x/s_2$ is 3-connected, then $\mathrm{si}(M/s_2)$ is 3-connected, so $M\backslash x/s_2$ is not 3-connected. If $M\backslash x/s_2$ has a parallel pair, then $M\backslash x$ has a triangle containing s_2. Such a triangle must also contain s_1, contradicting the assumption that no series pair of $M\backslash x$ is in a triangle. Thus we may assume that $M\backslash x/s_2$ has no parallel pairs. Then $M\backslash x/s_2$ has a 2-separation $\{X', Y'\}$, where $r_{M/s_2}(X'), r_{M/s_2}(Y') \geq 2$. Assume that X' is an N-part of this 2-separation. There are two cases to consider.

Assume that $s_1 \in X'$. Then $\{X' \cup s_2, Y'\}$ is a 2-separation of $M\backslash x$, where $X' \cup s_2$ is an N-part. Moreover, Y' contains no series pairs of $M\backslash x$, otherwise $M\backslash x$ has more than one series pair. Thus, by (5.3.3), x is fixed in M.

Assume that $s_1 \in Y'$. We now show that $|Y' - \mathrm{cl}_{M\backslash x/s_2}(X')| \geq 3$. Since $M\backslash x/s_2$ is connected, $|Y' - \mathrm{cl}_{M\backslash x/s_2}(X')| \neq 1$. If $|Y' - \mathrm{cl}_{M\backslash x/s_2}(X')| = 2$, then $Y' - \mathrm{cl}_{M\backslash x/s_2}(X')$ is a series pair of $M\backslash x/s_2$, and hence of $M\backslash x$, contradicting the fact that $\{s_1, s_2\}$ is the only series pair of $M\backslash x$. Thus $|Y' - \mathrm{cl}_{M\backslash x/s_2}(X') \geq 3$. Now consider $\{X', Y' \cup s_2\}$. This is clearly a 2-separation of $M\backslash x$. Moreover, since $s_1 \notin X'$, it follows that $s_2 \notin \mathrm{cl}_{M\backslash x}(X')$. Thus $\mathrm{cl}_{M\backslash x/s_2}(X') \supseteq \mathrm{cl}_{M\backslash x}(X')$. Hence $|(Y' \cup s_2) - \mathrm{cl}_{M\backslash x}(X')| \geq 4$. Now, using Lemma 5.4(iii) and the fact that $\{s_1, s_2\}$ is the only series pair of $M\backslash x$, we deduce that there is a pair $\{y_1, y_2\}$ of distinct elements of $Y' \cup s_2$ such that $\{y_1, y_2\}$ avoids both $\mathrm{cl}^*_{M\backslash x}(X')$ and $\mathrm{cl}_{M\backslash x}(X')$. From this point, we deduce that x is fixed in M using an argument that is identical to that of (5.3.3). ∎

All cases have been covered and we deduce that x is indeed fixed in M. Dualizing the above argument shows that if M/x has an N-minor, then x is cofixed in M. ∎

A matroid is *stable* if it cannot be expressed as the direct sum or 2-sum of nonbinary matroids. This notion plays an important role in the proof of the excluded-minor characterization of quaternary matroids [8]. We omit the routine proof of the next corollary.

COROLLARY 5.10. *Let N be a nonbinary matroid that is a universal stabilizer for a well-closed class $\mathcal{N}$, let M be a connected stable matroid in $\mathcal{N}$, and let $x \in E(M)$. If $M \backslash x$ is connected with an N-minor, then x is fixed in M, and if M/x is connected with an N-minor, then x is cofixed in M.*

It is an easy consequence of the splitter theorem that if M is a 3-connected minor of the 3-connected matroid M', then there is a sequence

$$M \cong M_0, M_1, M_2, \ldots, M_{k-1}, M_k = M'$$

of connected stable matroids such that M_i is a single-element extension or single-element coextension of M_{i-1} for all i in $\{1, 2, \ldots, k\}$. The next result, which follows easily from Corollary 5.10, puts a slightly different perspective on the notion of a universal stabilizer.

PROPOSITION 5.11. *Let N be a 3-connected nonbinary matroid in a well-closed class $\mathcal{N}$ of matroids. Then N is a universal stabilizer for $\mathcal{N}$ if and only if the following property holds for every sequence:*

$$N \cong M_0, M_1, M_2, \ldots, M_{k-1}, M_k$$

of connected stable matroids in $\mathcal{N}$ such that M_i is a single-element extension or single-element coextension of M_{i-1} for all i in $\{1, 2, \ldots, k\}$.

For $1 \le i \le k$, if M_i is a single-element extension of M_{i-1} by x_i, then x_i is fixed in M_i, and if M_i is a single-element coextension of M_{i-1} by y_i, then y_i is cofixed in M_i.

6. A FINITE CASE CHECK THEOREM

The next result shows that the task of checking that a matroid is a universal stabilizer for a class is finite. Indeed, the check is analogous to that used to check that a matroid is an **F**-stabilizer for a class.

THEOREM 6.1. *Let N be a 3-connected matroid in a well-closed class $\mathcal{N}$ and suppose that $|E(N)| \ge 2$. Then N is a universal stabilizer for $\mathcal{N}$ if and only if the following three conditions hold.*

(i) *If M is a* 3-*connected member of* $\mathscr{N}$ *with an element x such that* $M \backslash x = N$, *then x is fixed in M.*

(ii) *If M is a* 3-*connected member of* $\mathscr{N}$ *with an element y such that* $M/y = N$, *then y is cofixed in M.*

(iii) *If M is a* 3-*connected member of* $\mathscr{N}$ *with a pair of elements x and y such that* $M \backslash x/y = N$, *and* $M \backslash x$ *is* 3-*connected, then x is fixed in M.*

Proof. It is clear that if N universally stabilizes $\mathscr{N}$, then properties (i), (ii), and (iii) hold. For the converse, assume that (i), (ii), and (iii) hold. For $k \geq |E(N)|$, let $\mathscr{N}_k$ denote the class consisting of all matroids in $\mathscr{N}$ with ground sets having cardinality at most k. Clearly, $\mathscr{N}_k$ is a well-closed class. We prove by induction that N is a universal stabilizer for $\mathscr{N}_k$ for all $k \geq |E(N)|$. From this, it will follow immediately that N is a universal stabilizer for $\mathscr{N}$.

If $k = |E(N)|$, the result is trivial. Say $k = |E(N)| + 1$. Then, it follows from (i), (ii), and the fact that $\mathscr{N}$ is closed under isomorphism that N is a universal stabilizer for $\mathscr{N}_k$. Assume that $k \geq |E(N)| + 2$, and assume that N is a universal stabilizer for $\mathscr{N}_{k-1}$. Let M be a matroid in $\mathscr{N}_k$ with an element x such that $M \backslash x$ is 3-connected with an N-minor. We consider two cases. For the first, assume that $r^*(M) \geq r^*(N) + 2$. In this case, it is an easy consequence of the splitter theorem that there is an element y of $E(M \backslash x)$ such that $\mathrm{co}(M \backslash x \backslash y)$ is 3-connected with an N-minor. It is not difficult to check that $\mathrm{co}(M \backslash y)$ is also 3-connected. Since $M \backslash x$ is 3-connected, x is not in a triad of M. Thus x is not in a series pair of $M \backslash y$. Hence x is an element of $\mathrm{co}(M \backslash y)$. As $\mathrm{co}(M \backslash y) \backslash x$ is connected having an N-minor, the induction assumption and Theorem 5.3 imply that x is fixed in $\mathrm{co}(M \backslash y)$. Thus, by Lemma 4.10(iii), x is fixed in $M \backslash y$. Hence, by Corollary 4.7(i), x is fixed in M. For the second case, assume that $r^*(M) = r^*(N) + 1$. If also $r(M) = r(N) + 1$, then it follows by condition (iii) of the theorem that x is fixed in M. Thus we may assume that $r(M) \geq r(N) + 2$. Since $r^*(M \backslash x) = r^*(N)$, there is an independent set I of $E(M \backslash x)$ such that $M \backslash x/I \cong N$. Then $|I| \geq 2$. Moreover, it is easily checked, by thinking of the dual, that $M \backslash x/I'$ is 3-connected for all $I' \subseteq I$. If $y \in I$, then either x is in a parallel class of M/y, or M/y is 3-connected. It follows, either from Lemma 4.10(i) or the induction assumption, that x is fixed in M/y. Thus, if I has a pair $\{y_i, y_j\}$ for which $\{x, y_i, y_j\}$ is independent, then it can be deduced, by Proposition 4.7(ii), that x is fixed in M.

If $|I| \geq 3$, then such a pair can certainly be found. Thus we have shown that x is fixed in M except in the case that $|I| = 2$ and $I \cup x$ is a triangle. Consider this case letting $I = \{y_1, y_2\}$. Assume that x is not fixed in M. Then there is a matroid M' obtained from M by independently cloning x. Evidently $M'|\{x, x', y_1, y_2\} \cong U_{2,4}$ and M' is 3-connected. Moreover, it is easily checked that every 2-element subset S of $\{x, x', y_1, y_2\}$ has the property that $M'/S \setminus (\{x, x', y_1, y_2\} - S) = N$.

Consider $M' \setminus y_1, y_2$. If $\{x, x'\}$ is not coindependent in $M' \setminus y_1, y_2$, then it is easily checked, since $|E(N)| \geq 2$, that $\{\{x, x', y_1, y_2\}, E(M') - \{x, x', y_1, y_2\}\}$ is a 2-separation of M'. This contradiction to the fact that M' is 3-connected implies that $\{x, x'\}$ is coindependent in $M' \setminus y_1, y_2$. Thus $M' \setminus y_1, y_2$ is a coindependent cocloning of $M' \setminus y_1, y_2/x'$. Hence x is not cofixed in $M \setminus y_1, y_2$. But $(M' \setminus y_1, y_2/x')/x = N$, so $M' \setminus y_1, y_2/x'$ is either a 3-connected single-element coextension of N by x, or x is in a series class of $M' \setminus y_1, y_2/x'$. In the latter case, since $M' \setminus y_1, y_2/x' = M \setminus y_1, y_2$, it follows that x is cofixed in $M \setminus y_1, y_2$, a contradiction. In the former case, the fact that $M' \setminus y_1, y_2/x' = M \setminus y_1, y_2$ implies that $M' \setminus y_1, y_2/x' \in \mathscr{N}$. Then part (ii) of the hypothesis implies the contradiction that x is cofixed in $M \setminus y_1, y_2$. Thus the assumption that x is not fixed in M leads to a contradiction and we conclude that, in all cases, x is fixed in M.

By dualizing the above arguments, we obtain that if M and M/x are 3-connected with an N-minor, then x is cofixed in M. We conclude that N is a universal stabilizer for $\mathscr{N}$. ∎

If M and M' are matroids on a common ground set, then M' is *freer* than M if M is a rank-preserving weak-map image of M'. If M' is freer than M and $M' \neq M$, then M' is *strictly freer* than M. When N is a matroid in a well-closed class $\mathscr{N}$ and there is no matroid in $\mathscr{N}$ that is strictly freer than N, the check that N is a universal stabilizer for $\mathscr{N}$ can be simplified somewhat.

COROLLARY 6.2. *Let N be a* 3*-connected matroid in a well-closed class $\mathscr{N}$ such that no matroid in $\mathscr{N}$ is strictly freer than N. Then N is a universal stabilizer for $\mathscr{N}$ if and only if the following three conditions hold.*

(i) *If M is a* 3*-connected member of $\mathscr{N}$ with an element x such that $M \setminus x = N$, then x is fixed in M.*

(ii) *If M is a* 3*-connected member of $\mathscr{N}$ with an element y such that $M/y = N$, then y is cofixed in M.*

(iii) *If M is a* 3*-connected member of $\mathscr{N}$ with a pair of elements x and y such that $M \setminus x/y = N$ and $M \setminus x$ is* 3*-connected, then x and y are not clones.*

Proof. It is clear that if N is a universal stabilizer for $\mathscr{N}$, then (i) and (ii) hold. Moreover, it follows, by Theorem 5.3 and Proposition 4.9, that (iii) holds. For the converse, assume that (i), (ii), and (iii) are satisfied and suppose that N is not a universal stabilizer for $\mathscr{N}$. It is not difficult to check that $|E(N)| \geq 2$. Then Theorem 6.1(iii) fails. This means that there is a 3-connected matroid $M \in \mathscr{N}$ with a pair of elements x and y such that $M \setminus x/y = N$ and $M \setminus x$ is 3-connected, but x is not fixed in M.

Since x is not fixed in M, we can extend M by an element x' to obtain a matroid M' in which x and x' are independent clones. But either M/y is a 3-connected single-element extension of N, or x is in a parallel class of M/y. In the former case, it follows from (i) that x is fixed in M/y, and the same is true in the latter case by Lemma 4.10(i). If $\{x, x', y\}$ is not a triangle in M', then x and x' are independent clones in M'/y, and so x is not fixed in M/y, a contradiction. Hence $\{x, x', y\}$ is a triangle of M'. It follows from this that x is freely placed on the line $\mathrm{cl}_M(\{x, y\})$. We deduce that, if $I \subseteq E - \{x, y\}$ and $I \cup y$ is independent, then $I \cup x$ is independent. Hence $M/x \backslash y$ is freer than $M \backslash x/y$. But y is not freely placed on $\mathrm{cl}_M(\{x, y\})$, otherwise $M/x \backslash y = M \backslash x/y$ and so, by Proposition 4.3, x and y are clones, a contradiction to the assumption that part (iii) of this corollary holds. Therefore there is a subset I' of $E - \{x, y\}$ for which $I' \cup y$ is a circuit and $I' \cup x$ is independent. Hence I' is independent in $M/x \backslash y$ and dependent in $M/y \backslash x$. Thus $M/x \backslash y$ is strictly freer than $M/y \backslash x$. But $N = M/y \backslash x$, and $M/x \backslash y \in \mathscr{N}$, so we have contradicted the assumption that no matroid in $\mathscr{N}$ is strictly freer than N. We deduce that Theorem 6.1(iii) holds and the corollary is proved. ∎

7. STRONG UNIVERSAL STABILIZERS AND WEAK MAPS

Let N be a matroid in a well-closed class $\mathscr{N}$. We are interested in knowing under what circumstances we can be sure that, if M and M' are matroids in $\mathscr{N}$ with N-minors having appropriate connectivity, then neither M nor M' is strictly freer than the other. Lucas [12] proved a result of this type, showing that no proper rank-preserving weak-map image of a binary matroid is connected. In other words:

THEOREM 7.1. *If M is a connected binary matroid and M' is a binary matroid that is freer than M, then $M = M'$.*

It is also shown in [15] that if M is a connected nonbinary ternary matroid and M' is a 3-connected nonbinary ternary matroid that is freer than M, then $M = M'$. In other words:

THEOREM 7.2. *Let M and M' be ternary matroids both having $U_{2,4}$-minors where M' is freer than M. If M is connected and M' is 3-connected, then $M = M'$.*

We seek a general technique that would enable us to get other results of this type. We begin by defining yet another type of stabilizer and then we show that, with a mild extra condition, Theorem 7.2 can be extended with such a stabilizer playing the role of $U_{2,4}$ in the original theorem.

Let $\mathcal{N}$ be a well-closed class of matroids and let N be a 3-connected matroid in $\mathcal{N}$. Then N is a *strong universal stabilizer* for $\mathcal{N}$ if N is a universal stabilizer for $\mathcal{N}$ and there is a partial field $\mathbf{F}$ such that $\mathcal{N} \subseteq \mathcal{M}(\mathbf{F})$ and N is a strong $\mathbf{F}$-stabilizer for $\mathcal{N}$. One might be tempted to conjecture that if N is a strong universal stabilizer for $\mathcal{N}$, then N is a strong stabilizer for $\mathcal{N} \cap \mathcal{M}(\mathbf{F}')$ for every partial field $\mathbf{F}'$ over which N is representable. The next example shows that this conjecture is false.

EXAMPLE 7.3. We show in Theorem 8.3 that $U_{3,6}$ is a strong universal stabilizer for the class of quaternary matroids with no Φ_4-minor. However, $U_{3,6}$ is not a strong stabilizer for this class over, say $GF(16)$, since not all $GF(16)$-representations of $U_{3,6}$ extend to a representation of Φ_3^+, a matroid that is certainly in the class.

The main task of this section is to prove the following theorem, which connects strong universal stabilizers with weak maps.

THEOREM 7.4. *Let N be a strong universal stabilizer for a well-closed class $\mathcal{N}$ and assume that no matroid in $\mathcal{N}$ is strictly freer than N. Let M be a connected matroid in $\mathcal{N}$ with an N-minor and let M' be a 3-connected matroid in $\mathcal{N}$. If M' is freer than M, then $M' = M$.*

Theorem 7.4 will follow from a sequence of lemmas. Throughout, N is assumed to be a strong universal stabilizer for the well-closed class $\mathcal{N}$. We first note an obvious fact.

LEMMA 7.5. *N is a strong universal stabilizer for $\mathcal{N}$ if and only if N^* is.*

Suppose that N is small. Then, by Lemma 5.5, $\mathcal{N}$ is a class of binary matroids. It follows from Theorem 7.1 that $M = M'$. From now on, we shall assume that N is not small.

In Lemma 7.7, we establish the theorem in the case that M is 3-connected. The proof of the lemma will use the following:

LEMMA 7.6. *If M and N are both wheels or are both whirls, then $M' = M$.*

Proof. Choose a counterexample to the lemma in which $|E(M)| - |E(N)|$ is as small as possible. Then M' is strictly freer than M and hence $|E(M)| - |E(N)| \neq 0$. Assume first that $r(M) = 3$. Then $N \cong U_{2,4}$ and M is the rank-3 whirl. It follows that M' is obtained from M by relaxing some nonempty set of lines. Thus M' is isomorphic to Q_6, P_6, or $U_{3,6}$, and so M' has a $U_{2,5}$-minor. This is a contradiction since $U_{2,4}$ does not universally stabilize any class of matroids that contains $U_{2,5}$. Hence $r(M) > 3$.

Now M' has a basis B that is dependent in M. Certainly B is not equal to the set of spokes of M, so B contains at least one rim element, say x, of M. Let y and z be spokes of M such that $\{x, y, z\}$ is a triangle. If one of y

and z, say y, is not in B, then $M'/x \setminus y$ is strictly freer than $M/x \setminus y$ and the latter is a wheel or whirl that has rank 1 less than M and has an N-minor. This contradiction to the choice of M and N implies that both y and z are in B.

Let C be the rim of M. Then $(C - x) \cup y$ is a basis of M and hence of M'. Thus M' has a basis B' that contains $\{x, y, z\}$ and is contained in $[(C - x) \cup y] \cup \{x, y, z\}$. Since $r(M) > 3$, there is a rim element u in B' such that $u \neq x$. Let v be a spoke of M such that $\{u, v\}$ is in a triangle and $v \notin B'$. Then $M'/u \setminus v$ is strictly freer than $M/u \setminus v$ and the latter is a wheel or whirl that has rank 1 less than M and has an N-minor. Again the choice of M and N is contradicted, and the lemma follows. ∎

LEMMA 7.7. *Let M and M' be 3-connected members of $\mathcal{N}$ each with N-minors. If M' is freer than M, then $M = M'$.*

Proof. The proof is by induction on $|E(M)|$. The result certainly holds if $|E(M)| = |E(N)|$, since then $M = M' = N$. Assume that $|E(M)| > |E(N)|$ and that the result holds for all 3-connected matroids in $\mathcal{N}$ that have N-minors and have cardinality less than $|E(M)|$. By the splitter theorem, M has an element e such that either $M \setminus e$ or M/e is 3-connected with an N-minor unless M, and hence N, is a wheel or a whirl. But, in the exceptional case, the last lemma implies that $M' = M$.

We may now assume that M has an element e such that $M \setminus e$ or M/e is 3-connected with an N-minor. It is well known (see, for example, [13, Corollary 7.3.13]) that M' is freer than M if and only if $(M')^*$ is freer than M^*. Also, by Lemma 7.5, N is a strong universal stabilizer for $\mathcal{N}$ if and only if N^* is. Hence, by dualizing if necessary, we may assume that $M \setminus e$ is 3-connected with an N-minor. Since $M' \setminus e$ is freer than $M \setminus e$, the former is also 3-connected.

We now show that $M \setminus e$ and $M' \setminus e$ have a common N-minor. There is an independent set I of $M \setminus e$ with $|I| = r(M \setminus e) - r(N)$, and a coindependent set J such that $N \cong (M \setminus e) \setminus J/I$. It is easily checked that the choice of I and J guarantees that $(M' \setminus e) \setminus J/I$ is freer than $(M \setminus e) \setminus J/I$. It now follows by the hypotheses of the theorem that $(M' \setminus e) \setminus J/I \cong N$; that is, $M' \setminus e$ and $M \setminus e$ have a common N-minor.

Let $\mathbf{F}$ be a partial field for which $\mathcal{N} \subseteq \mathcal{M}(\mathbf{F})$ and for which N is a strong $\mathbf{F}$-stabilizer. By the induction assumption, $M \setminus e = M' \setminus e$. Let A be an $\mathbf{F}$-representation of this matroid. This matrix is obtained by extending some representation of an N-minor. Since N is a strong $\mathbf{F}$-stabilizer for the class, this representation extends uniquely to the representation A of $M \setminus e$. The fact that N is a strong $\mathbf{F}$-stabilizer also implies that the representation A of $M \setminus e$ extends to representations of both M and M'. It follows that there are vectors $\mathbf{x}$ and $\mathbf{x}'$ such that $[A|\mathbf{x}]$ and $[A|\mathbf{x}']$ are representations of M and M', respectively. By Lemma 5.2, there is a matroid M'' on the columns of $[A|\mathbf{x}, \mathbf{x}']$ with the property that $M'' \setminus \mathbf{x} = M[A|\mathbf{x}']$ and $M'' \setminus \mathbf{x}' = M[A|\mathbf{x}]$.

Assume that $\mathbf{x}$ and $\mathbf{x}'$ are not parallel in M''. Consider $M'' \setminus \mathbf{x}$. This is a representation of M', where $\mathbf{x}'$ corresponds to the element e. By the definition of a universal stabilizer, e is fixed in M'. Hence $\mathbf{x}'$ is fixed in $M'' \setminus \mathbf{x}$. Thus, by Proposition 4.7, $\mathbf{x}'$ is fixed in M''. Hence, by Corollary 4.6, there is an independent set I of M'' such that $\mathrm{cl}_{M''}(I)$ contains $\mathbf{x}'$ but not $\mathbf{x}$. Now I is a subset of the elements of $M \setminus e$. We can now deduce that $I \cup e$ is dependent in M' and independent in M, contradicting the fact that M' is freer than M. From this contradiction, we deduce that $\mathbf{x}$ and $\mathbf{x}'$ are parallel in M'' and hence that $M = M'$. ▮

Most of the work in the proof of Theorem 7.4 is involved in proving the case when M is not 3-connected. Much of the argument is a straightforward modification of argument from the proof of Theorem 7.2, which is given in [15]. For the sake of presenting a complete argument in one place, we give full details here.

We first extend our terminology associated with 2-separations. Assume that M is the 2-sum of matroids M_1 and M_2 associated with the exact 2-separation $\{S_1, S_2\}$. Recall from Section 5 that S_i is an *N-part* of M if M_i has an N-minor. Note that $M|S_i$ may not have an N-minor, even when S_i is an N-part of M. Following [15], we say that an N-part S_1 of M is a *minimal* N-part of M if every 2-separation $\{T_1, T_2\}$ of M for which T_1 is a proper subset of S_1 has the property that T_1 is not an N-part of M.

LEMMA 7.8. *Suppose that M and M' are matroids on E such that M' is freer than M and that M is a connected matroid in $\mathcal{N}$ with an N-minor. Assume that $\{a, b\}$ is a circuit of M that is independent in M' and that $E - \{a,b\}$ is a minimal N-part of M. Then M' is not in $\mathcal{N}$.*

The proof of this lemma will use the following two lemmas, which are proved in [15].

LEMMA 7.9. *Let $\{X, E - X\}$ and $\{Y, E - Y\}$ be 2-separations of a connected matroid M and suppose that $Y \subseteq X$. Then there are connected matroids M_1, M_2, M_3, and M_4 such that $M = M_1 \oplus_2 M_2 = M_3 \oplus_2 M_4$ where the basepoints of both 2-sums are labeled by p, the ground sets of M_1 and M_3 are $X \sqcup p$ and $Y \sqcup p$, and M_3 is a minor of M_1.*

LEMMA 7.10. *Let $\{X_1, Y_1\}$ and $\{X_2, Y_2\}$ be 2-separations of a connected matroid M. If both $X_1 \cap X_2$ and $Y_1 \cap Y_2$ are nonempty, then $r(X_1 \cap X_2) + r(Y_1 \cup Y_2) = r(M) + 1$ and $r(Y_1 \cap Y_2) + r(X_1 \cup X_2) = r(M) + 1$. Moreover, $\{X_1 \cap X_2, Y_1 \cup Y_2\}$ is a 2-separation of M provided that $|X_1 \cap X_2| \geq 2$.*

Proof of Lemma 7.8. Assume that the lemma fails and take a pair of matroids M' and M satisfying the hypotheses for which M' is in $\mathcal{N}$ and $|E|$

is as small as possible. We first show that

7.7.1. *$M \setminus a$ is not 3-connected.*

Proof. Assume that contrary. Then both $M \setminus a$ and $M \setminus b$ are 3-connected and, hence, so too are $M' \setminus a$ and $M' \setminus b$. Thus, by Lemma 7.7, $M \setminus a = M' \setminus a$ and $M \setminus b = M' \setminus b$. Now M' is 3-connected since M' is freer than M and does not have $\{a, b\}$ as a circuit. Since $M' \setminus a$ is 3-connected with an N-minor, a is fixed in M' by the definition of a universal stabilizer. By Corollary 4.6, $M' \setminus a, b$ has an independent set I such that $I \cup a$ is a circuit of M' and $I \cup b$ is independent in M'. Now M' is freer than M, so $I \cup a$ contains a circuit of M containing a. Therefore, since $\{a, b\}$ is a circuit of M, then set $I \cup b$ is dependent in M. Hence $I \cup b$ is dependent in $M \setminus a$ and independent in $M' \setminus a$, contradicting the fact that these two matroids are equal. ∎

From the minimality assumption, we can immediately deduce that

7.7.2. *$M \setminus a$ is simple.*

Next we show:

7.7.3. *If $\{T_1, T_2\}$ is a 2-separation of M such that T_2 is an N-part and $\{a, b\} \subseteq T_2$, then $a \notin \mathrm{cl}_M(T_1)$.*

Proof. Now M is the 2-sum of two matroids M_1 and M_2 having ground sets $T_1 \cup p$ and $T_2 \cup p$, respectively. Suppose that $a \in \mathrm{cl}_M(T_1)$. Then a and b are parallel to p in M_2, so $\{T_1 \cup \{a, b\}, T_2 - \{a, b\}\}$ is a 2-separation of M having $T_2 - \{a, b\}$ as an N-part. This contradicts the choice of $E - \{a, b\}$. Hence (7.7.3) holds. ∎

We already know that M has no parallel pairs other than $\{a, b\}$. We now show that

7.7.4. *M has no series pairs.*

Proof. Suppose that $\{u, v\}$ is a cocircuit of M. Then $\{\{u, v\}, E - \{u, v\}\}$ is a 2-separation of M, and $\{a, b\} \subseteq E - \{u, v\}$. By (7.7.3), $a \notin \mathrm{cl}_M(\{u, v\})$, so $\{u, v, a\}$ is independent in M and hence in M'. Thus, as $\{a, b\}$ is independent in M', one of $\{a, b, u\}$ and $\{a, b, v\}$ is independent in M'. Without loss of generality, assume the former. Evidently $\{a, b\}$ is a circuit of M/u and an independent set of M'/u. Moreover, if $\{T_1, T_2\}$ is a 2-separation of M/u such that T_1 is an N-part and T_2 properly contains $\{a, b\}$, then one easily checks that $\{T_i \cup u, T_j\}$ is a 2-separation of M where $\{i, j\} = \{1, 2\}$ and $v \in T_i$. This contradicts the choice of $E - \{a, b\}$. Hence $E(M/u) - \{a, b\}$ is a minimal N-part of M/u and it follows by the choice of the pair (M', M) that M'/u is not in $\mathcal{N}$. This contradiction to the fact that $M' \in \mathcal{N}$ implies that (7.7.4) holds. ∎

By (7.7.1), $M \setminus a$ is not 3-connected. Hence $M \setminus a$ has a 2-separation $\{S_1, S_2\}$, where $b \in S_2$. Thus $\{S_1, S_2 \cup a\}$ is a 2-separation of M which, by the choice of $E - \{a, b\}$, has the property that S_1 is not an N-part. Thus $S_2 \cup a$ is an N-part. Therefore M is the 2-sum of two matroids M_1 and M_2 having ground sets $S_1 \cup p$ and $S_2 \cup a \cup p$, respectively. Since M has no series pairs or parallel pairs other than $\{a, b\}$, it follows that

7.7.5. $|S_1| \geq 3$.

We show next that

7.7.6. *S_1 contains an element x for which $M \backslash x$ is connected.*

Proof. Suppose that S_1 does not contain such an element. Then, for every element y of $E(M_1) - p$, the matroid $M_1 \backslash y$ is not connected. Thus, by [13, Lemma 10.2.1], S_1 contains a series pair of M_1 and so contains a series pair of M, a contradiction. ∎

Now $M \backslash x$ is in $\mathcal{N}$, has an N-minor, and is connected, and $M' \backslash x$ is freer than $M \backslash x$. Moreover, $\{a, b\}$ is a circuit of $M \backslash x$ and an independent set of $M' \backslash x$. Indeed, $E - \{a, b, x\}$ is an N-part of $M \backslash x$. If $E - \{a, b, x\}$ is a minimal N-part of $M \backslash x$, then, by the choice of (M', M), it follows that $M' \backslash x$ is not in $\mathcal{N}$. This implies the contradiction that M' is not in $\mathcal{N}$. Thus $M \backslash x$ has a 2-separation $\{T_1, T_2\}$ where T_1 is an N-part of $M \backslash x$ and T_2 properly contains $\{a, b\}$. Thus

$$r(T_1) + r(T_2) = r(M \backslash x) + 1. \tag{7.1}$$

Moreover, as neither $T_1 \cup x$ nor T_1 is an N-part of M,

$$r(T_1 \cup x) = r(T_1) + 1 \tag{7.2}$$

and

$$r(T_2 \cup x) = r(T_2) + 1. \tag{7.3}$$

Since $r(S_1) + r(S_2 \cup a) = r(M) + 1$ and $M \backslash x$ is connected,

$$r(S_1 - x) + r(S_2 \cup a) = r(M \backslash x) + 1 \tag{7.4}$$

and so

$$r(S_1 - x) = r(S_1). \tag{7.5}$$

Thus, by (7.4) and (7.7.5), $\{S_1 - x, S_2 \cup a\}$ is a 2-separation of $M \backslash x$ for which $S_1 - x$ is not an N-part and so $S_2 \cup a$ is an N-part. By (7.2), (7.3), and (7.5), neither T_1 nor T_2 contains $S_1 - x$. Hence

$$T_2 \cap (S_1 - x) \neq \emptyset \tag{7.6}$$

and

$$T_1 \cap (S_1 - x) \neq \emptyset. \tag{7.7}$$

Moreover, T_2 contains $\{a, b\}$, so

$$T_2 \cap (S_2 \cup a) \neq \emptyset. \tag{7.8}$$

Finally,

$$T_1 \cap (S_2 \cup a) \neq \emptyset, \tag{7.9}$$

otherwise $T_1 \subseteq S_1 - x$, which contradicts Lemma 7.9 since T_1 is an N-part of $M \backslash x$, but $S_1 - x$ is not an N-part of $M \backslash x$.

Statements (7.6)–(7.9) enable us to apply Lemma 7.10 twice to the 2-separations $\{T_1, T_2\}$ and $\{S_1 - x, S_2 \cup a\}$ of $M \backslash x$. This yields four equations, two of which are

$$r\big(T_2 \cap (S_2 \cup a)\big) + r\big(T_1 \cup (S_1 - x)\big) = r(M \backslash x) + 1 \tag{7.10}$$

and

$$r\big(T_1 \cap (S_2 \cup a)\big) + r\big(T_2 \cup (S_1 - x)\big) = r(M \backslash x) + 1. \tag{7.11}$$

As $\{T_2 \cap (S_2 \cup a), T_1 \cup (S_1 - x)\}$ is a partition of $E(M \backslash x)$ and $|T_2 \cap (S_2 \cup a)| \geq 2$, this partition is a 2-separation of $M \backslash x$. By (7.5), it follows that $\{T_2 \cap (S_2 \cup a), T_1 \cup S_1\}$ is a 2-separation of M. Since T_1 is an N-part of $M \backslash x$, Lemma 7.10 implies that $T_1 \cup S_1$ is an N-part of M. Since $T_2 \cap (S_2 \cup a) \supseteq \{a, b\}$, the choice of $E - \{a, b\}$ means that equality must hold here. Thus $T_1 \cap (S_2 \cup a) = S_2 - b$ and $T_2 \cup (S_1 - x) = (S_1 - x) \cup \{a, b\}$. As $S_2 \cup a$ is an N-part of M and $\{a, b\}$ is a 2-circuit of M, we have that $|S_2 - b| \geq 2$. Hence, by (7.5) and (7.11), $\{S_2 - b, S_1 \cup \{a, b\})$ is a partition of $E(M)$ that is a 2-separation of M.

We conclude that both $\{S_2 - b, S_1 \cup b\}$ and $\{S_2, S_1\}$ are 2-separations of $M \backslash a$. Moreover, S_2 is an N-part of $M \backslash a$ and, by the choice of $E - \{a, b\}$, the set $S_2 - b$ is not an N-part of M and hence is not an N-part of $M \backslash a$.

Now, apply (7.7.3) to the 2-separation $\{S_2 - b, S_1 \cup \{a, b\}\}$ of M to obtain that $a \notin \mathrm{cl}_M(S_2 - b)$, so $b \notin \mathrm{cl}_M(S_2 - b)$. Hence

$$r(S_2 - b) = r(S_2) - 1. \tag{7.12}$$

By Lemma 7.9, there are connected matroids M_1, M_2, M_3, and M_4 such that $M \backslash a = M_1 \oplus_2 M_2 = M_3 \oplus_2 M_4$ where M_1 and M_3 have ground sets $S_2 \sqcup p$ and $(S_2 - b) \sqcup p$, and M_3 is a minor of M_1. From above, M_1 has an N-minor and M_3 does not. Since $M_1 \backslash p = (M \backslash a)|S_2$ and $M_3 \backslash p = (M \backslash a)|(S_2 - b)$, and both M_1 and M_3 are connected, $r(M_1) = r(M_1 \backslash p) = r(S_2)$ and $r(M_3) = r(M_3 \backslash p) = r(S_2 - b)$. Thus, by (7.12), $r(M_1) = r(M_3) + 1$. But

$$M_1 \backslash p, b = \big[(M \backslash a)|S_2\big] \backslash b = (M \backslash a)|(S_2 - b) = M_3 \backslash p.$$

Hence $\{p, b\}$ is a cocircuit of M_1. Since M_3 is a minor of M_1 and $E(M_1) - E(M_3) = \{b\}$, it follows that $M_3 = M_1/b$. As M_1 has an N-minor, so too does M_3. This contradiction completes the proof of Lemma 7.8. ∎

Lemma 7.8 establishes the base case for the inductive argument which proves the following:

LEMMA 7.11. *Let M be a connected matroid that is in $\mathcal{N}$ and has an N-minor. Suppose that $\{S_1, S_2\}$ is a 2-separation of M, and S_1 is a minimal N-part of M. Let M' be freer than M, and assume that $r_{M'}(S_2) > r_M(S_2)$. Then M' is not in $\mathcal{N}$.*

Again we use some subsidiary lemmas. We omit the straightforward proof of the first of these as it is a routine generalization of the proof of [15, Lemma 4.6].

LEMMA 7.12. *Let $\{S_1, S_2\}$ be a 2-separation of the connected matroid M and suppose that S_1 is a minimal N-part of M and $|S_2| > 2$. Let M_1 be a connected minor of the form $M \backslash x$ or M/x for some x in S_2. Then $\{S_1, S_2 - x\}$ is a 2-separation of M_1, and S_1 is a minimal N-part of M_1.*

The next lemma is [15, Lemma 4.7].

LEMMA 7.13. *Let $Q'(E \sqcup p)$ be a connected matroid and let $T(E)$ be a matroid. Let $Q = Q' \backslash p$. Assume that $|E| \geq 3$, that $r(Q) < r(T)$, and that Q is a weak-map image of T. Then E has an element x such that either $Q' \backslash x$ is connected and $r(Q \backslash x) < r(T \backslash x)$, or Q'/x is connected and $r(Q/x) < r(T/x)$.*

Proof of Lemma 7.11. The proof is by induction on the cardinality of S_2. Assume that $|S_2| = 2$, say $S_2 = \{a, b\}$. Then $\{a, b\}$ is a circuit in M and an independent set in M', and it follows from Lemma 7.8 that M' is not in $\mathcal{N}$.

Now suppose that $|S_2| > 2$. Assume that the lemma holds for all pairs of matroids (M, M') satisfying the conditions of the lemma and having $|S_2| < n$, and let (M, M') be such a pair with $|S_2| = n$. Since $\{S_1, S_2\}$ is a 2-separation of M, this matroid is the 2-sum of matroids $M_1(S_1 \sqcup p)$ and $M_2(S_2 \sqcup p)$. Now $M_2 \backslash p = M|S_2$, and $M|S_2$ is a weak-map image of $M'|S_2$ with $r(M|S_2) < r(M'|S_2)$. By Lemma 7.13, S_2 contains an element x such that either (i) $M_2 \backslash x$ is connected and $r((M|S_2) \backslash x) < r((M'|S_2) \backslash x)$ or (ii) M_2/x is connected and $r((M|S_2)/x) < r((M'|S_2)/x)$. In case (i), $M \backslash x = M_1 \oplus_2 (M_2 \backslash x)$. Since $M \backslash x$ is the 2-sum of connected matroids, $M \backslash x$ is connected. Moreover, by Lemma 7.11, S_1 is a minimal N-part of $M \backslash x$. Finally, $r_{M \backslash x}(S_2 - x) < r_{M' \backslash x}(S_2 - x)$. It now follows by the induction assumption that $M' \backslash x$, and hence M', is not in $\mathcal{N}$. The argument for case (ii) is entirely similar. ∎

The easy proof of the next lemma is given in [15, Lemma 5.8].

LEMMA 7.14. *Let $M'(E)$ be freer than $M(E)$ and let A be a subset of E. Then $r_{M'}(A) = r_M(A)$ if and only if $r_{(M')^*}(E - A) = r_{M^*}(E - A)$.*

At last, we can complete the proof of Theorem 7.4.

Proof of Theorem 7.4. Let M' be a 3-connected matroid in $\mathscr{N}$ that is freer than M, where M is a connected matroid in $\mathscr{N}$ with an N-minor. Assume that M is not 3-connected. Then there is a 2-separation $\{S_1, S_2\}$ of M for which S_1 is a minimal N-part. Now M' is 3-connected, so $\{S_1, S_2\}$ is not a 2-separation of M'. Hence either $r_M(S_1) < r_{M'}(S_1)$ or $r_M(S_2) < r_{M'}(S_2)$. It then follows from Lemma 7.14 that either $r_M(S_2) < r_{M'}(S_2)$ or $r_{M^*}(S_2) < r_{(M')^*}(S_2)$. But it is easily checked that S_1 is a minimal N^*-part of M^*. By Lemma 7.5, N is a strong universal stabilizer for $\mathscr{N}$ if and only if N^* is. All other properties of M' and M relevant to the conditions of the theorem are preserved under duality. It follows that we may assume, without loss of generality, that $r_M(S_2) < r_{M'}(S_2)$. But then, by Lemma 7.11, M' is not in $\mathscr{N}$, a contradiction. Hence M is 3-connected. It then follows by Lemma 7.7 that $M' = M$. ∎

8. QUATERNARY STRONG UNIVERSAL STABILIZERS

We begin by developing some elementary properties of free spikes. It is shown in [16] that to characterize a matroid it suffices to know its rank and nonspanning circuits. The next lemma follows easily from this fact.

LEMMA 8.1. *If $k = 3$, then $\Phi_k \cong U_{3,6}$, and Φ_k has no nonspanning circuits. However, if $k > 3$, then Φ_k is the unique rank-k matroid on $\{a_1, b_1, a_2, b_2, \dots, a_k, b_k\}$ whose collection of nonspanning circuits is $\{\{a_i, b_i, a_j, b_j\}\colon 1 \le i < j \le k\}$. Moreover, for $k > 3$, the legs of Φ_k are precisely the 2-element subsets of $E(\Phi_k)$ that are in more than one 4-circuit.*

Next we summarize some basic properties of free spikes.

LEMMA 8.2. *Let k be an integer greater than 2.*

(i) $\Phi_k^* = \Phi_k$.

(ii) *If $k > 3$ and $e \in \{a_i, b_i\}$, then $\Phi_k/e \cong \Phi_{k-1}^+$, and the unique member of $\{a_i, b_i\} - e$ is the tip of this Φ_{k-1}^+-minor.*

(iii) $\Phi_k \setminus a_i/b_i = \Phi_k \setminus b_i/a_i \cong \Phi_{k-1}$ *for all i in $\{1, 2, \dots, k\}$.*

(iv) Φ_k^+ *and Φ_k are 3-connected.*

(v) Φ_k^+ *is quaternary.*

(vi) Φ_3 *has a unique 3-connected quaternary extension, namely Φ_3^+.*

The proof of Lemma 8.2 is elementary and is omitted. Observe that part (vi) reveals an interesting symmetry of the quaternary projective plane. Consider an embedding of $U_{3,6}$ in $PG(2,4)$. There are $(4^2 + 4 + 1) - 6 = 15$ distinct elements that can be chosen to produce a 3-connected quaternary extension of $U_{3,6}$. There are also 15 ways to partition a 6-element set into three 2-element subsets, that is, 15 distinct ways to extend $U_{3,6}$ to Φ_3^+. This reflects the fact that, when $k = 3$, the legs of Φ_k are not canonical although, as noted in Lemma 8.1, when $k > 3$, they are. Hence, for the latter such k, there is a unique modular cut that extends Φ_k to Φ_k^+.

THEOREM 8.3.

(i) *$U_{2,3}$ is a strong universal stabilizer for the class of binary matroids; that is, $U_{2,3}$ is a strong universal stabilizer for the class of quaternary matroids with no $U_{2,4}$-minor.*

(ii) *$U_{2,4}$ is a strong universal stabilizer for the class of quaternary matroids with no $U_{2,5}$- or $U_{3,5}$-minor.*

(iii) *$U_{2,5}$ is a strong universal stabilizer for the class of quaternary matroids with no $U_{3,6}$-minor.*

(iv) *$U_{3,6}$ is a strong universal stabilizer for the class of quaternary matroids with no Φ_4-minor.*

Proof. First note that all of the above classes are certainly well-closed. Note, also, that if N is a 3-connected nonbinary $GF(4)$-stabilizer for a class of quaternary matroids, then, by the unique representability of 3-connected quaternary matroids over $GF(4)$ [11] and Proposition 3.1, N is a strong $GF(4)$-stabilizer for the class. Thus the theorem will follow if it can be shown that the relevant matroids are universal stabilizers for their classes. No matroid is freer than a uniform matroid. Thus, by Corollary 6.2, to show that a 3-connected uniform matroid N is a universal stabilizer for a well-closed class $\mathcal{N}$, it suffices to consider 3-connected matroids M in $\mathcal{N}$ such that

(a) $M \backslash x \cong N$;

(b) $M/y \cong N$; or

(c) $M \backslash x/y = N$ and $M \backslash x$ is 3-connected.

Matroids M of the types just described will be called *majors* of N of types (a), (b), and (c). Moreover, we will say that N *universally stabilizes* such a major if, for type (a), x is fixed in M; for type (b), y is cofixed in M; and, for type (c), x and y are not clones in M.

Consider parts (i) and (ii). These follow immediately since $U_{2,3}$ has no major of type (a), (b), or (c) in the class of binary matroids, and $U_{2,4}$ has no major of type (a), (b), or (c) in the class of quaternary matroids with no $U_{2,5}$- or $U_{3,5}$-minor.

Now consider part (iii). This check is essentially identical to that given in [25, Lemma 6.1(iv)], but is included here for the sake of completeness. Clearly, $U_{2,5}$ has no type-(a) majors. Evidently M is a type-(b) major of $U_{2,5}$ if and only if M^* is a type-(a) major of $U_{3,5}$. It is easily checked that the only 3-connected quaternary single-element extension of $U_{3,5}$ other than the excluded $U_{3,6}$ is the matroid Q_6 that is obtained by placing a point on the intersection of two lines of $U_{3,5}$. Clearly, this added point is fixed in Q_6. Hence, by duality, $U_{2,5}$ universally stabilizes type-(b) majors in the class. Moreover, Q_6 is the only 3-connected quaternary single-element coextension of $U_{2,5}$ other than $U_{3,6}$. Thus, if M is a type-(c) major of $U_{2,5}$, then $M \backslash x \cong Q_6$ and y is the freely placed point in this matroid. If x and y are clones in M, then it is not difficult to check that M has $U_{3,6}$, P_6, or $U_{2,6}$ as a minor, a contradiction since M is quaternary with no $U_{3,6}$-minor. Thus $U_{2,5}$ universally stabilizes its type-(c) majors in the class. It follows by Corollary 6.2 that $U_{2,5}$ is a universal stabilizer for the class of quaternary matroids with no $U_{3,6}$-minor. This completes the proof of (iii) of the theorem.

To prove (iv), we first note that it follows from Lemma 8.2(vi) and the fact that $U_{3,6}$ is self-dual that $U_{3,6}$ universally stabilizes its type-(a) and type-(b) quaternary majors. Consider type-(c) quaternary majors letting M be a 3-connected quaternary matroid with a pair of elements $\{x, y\}$ such that $M \backslash x$ is a 3-connected coextension of $U_{3,6}$ by the element y. Assume that x and y are clones in M. Since no contraction of $(M \backslash x)^*$ is isomorphic to $U_{2,6}$, every element of $(M \backslash x)^*$ is on a 3-point line with y, so $(M \backslash x)^* \cong \Phi_3^+$ with tip y. Label the legs of $(M \backslash x)^*$ by $\{a_1, b_1\}, \{a_2, b_2\}, \{a_3, b_3\}$. As x is not fixed in M, there is an extension M' of M by x' obtained by cloning x. Then interchanging x and x' is an automorphism of M' and hence is an automorphism of M'/y. But $M'/y \backslash x, x' \cong U_{3,6}$ and $M'/y \backslash x'$ equals M/y and is a quaternary extension of $U_{3,6}$. Thus either $M/y \cong \Phi_3^+$ or M/y is a parallel extension of $U_{3,6}$. In both cases, x is fixed in M/y. Hence $\{x, x'\}$ is a circuit of M'/y, and so $\{x, x', y\}$ is a circuit of M'. By relabeling if necessary, we may assume that no member of $\{a_1, a_2, a_3\}$ is in $\mathrm{cl}_{M'}(\{x, x', y\})$. To see this, observe that if $\{a_i, b_i\} \subseteq \mathrm{cl}_{M'}(\{x, x', y\})$, then $\{\{a_i, b_i, x, y\}, E(M) - \{a_i, b_i, x, y\}\}$ is a 2-separation of the 3-connected matroid M.

Now, for each i in $\{1, 2, 3\}$, since $\{x, x'\}$ is independent in M'/a_i, it follows by Proposition 4.4 that x is not fixed in M/a_i. But, by the properties of free spikes, $M \backslash x/a_i \cong Q_6$ where b_i is on the intersection of the two 3-point lines of Q_6 and y is on neither of these 3-point lines. Thus, as x is not fixed in M/a_i, it follows that x is in no 2-circuits of M/a_i and so M/a_i is 3-connected. As $M \backslash x/a_i/y \cong U_{2,5}$, both $M/a_i \backslash x$ and M/a_i are 3-connected quaternary matroids with a $U_{2,5}$-minor. But x is not fixed in M/a_i, so, by (iii) of this theorem, since $U_{2,5}$ is a strong universal stabilizer for the class of quaternary matroids with no $U_{3,6}$-minor, it follows that $U_{3,6}$ is a minor of M/a_i. Thus $M/a_i \cong \Phi_3^+$ and, as b_i is on two 3-point lines in $M \backslash x/a_i$, it follows that b_i is the tip of M/a_i and so $\{b_i, x, y\}$ is a circuit of M/a_i. Thus either $\{a_i, b_i, x, y\}$ or $\{b_i, x, y\}$ is a circuit of M. But if, for some i in $\{1, 2, 3\}$, the set $\{b_i, x, y\}$ is a circuit, then, by circuit elimination, for all $j \neq i$, the set $\{a_j, b_j, b_i, y\}$ contains a circuit of M, and so $\{a_j, b_j, b_i\}$ contains a circuit of $M \backslash x/y$. This is a contradiction since the last matroid is isomorphic to $U_{3,6}$. Thus, for all i in $\{1, 2, 3\}$, the set $\{a_i, b_i, x, y\}$ is a circuit of M. Furthermore, all sets of the form $\{a_i, b_i, a_j, b_j\}$ are circuits of M since $M \backslash x \cong (\Phi_3^+)^*$. We now show that these are the only nonspanning circuits of M. First we note that, as $M \backslash x \cong (\Phi_3^+)^*$, there is no triangle of M avoiding x; and, since the only triangle of M/a_i containing x is properly contained in a circuit of M, we deduce that M has no triangles. Now suppose that C is a 4-circuit of M different from those already noted. If C contains an element z of $\{a_1, a_2, a_3\}$, then $C - z$ is a triangle of M/z. But every such triangle comes from a known 4-circuit of M. Thus $C \subseteq \{b_1, b_2, b_3, x, y\}$ and, since x and y are clones, $\{x, y\} \subseteq C$. Hence $C = \{x, y, b_i, b_j\}$ for some i and j. It follows that $\{x, y, b_i, b_j, a_i, a_j\}$ is a hyperplane of M, so M has a 2-circuit, a contradiction. We conclude that $M \cong \Phi_4$. This contradiction completes the proof of (iv) and thereby finishes the proof of the theorem. ∎

Part (iv) of the last theorem establishes that, for $k = 3$, the matroid Φ_k is a strong universal stabilizer for the class of quaternary matroids with no Φ_{k+1}-minor. Indeed, it is tempting to conjecture that this result remains true for all $k \geq 3$. However, although Φ_k universally stabilizes its type-(a) and type-(b) quaternary majors, the conjecture fails as we now show.

To construct a counterexample to the last conjecture, we proceed as follows. Suppose that $k \geq 4$ and take a copy of Φ_k^+ embedded as a restriction of $PG(k-1, 4)$. Let this free spike have tip p and legs $\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_{k-1}, b_{k-1}\}$, and $\{c, d\}$, and let e be a point of the projective space that is on the line spanned by $\{p, c, d\}$ but is distinct from these three points. Let M_1 be the extension of Φ_k^+ by e. Now take a copy of the Fano matroid with ground set $\{e, c, d, a_k, b_k, x, x'\}$ such that the lines through e are $\{e, c, d\}$, $\{e, x, x'\}$, and $\{e, a_k, b_k\}$. Let M be obtained by deleting $\{e, c, d\}$ from the generalized parallel connection of F_7 and M_1 across the triangle $\{e, c, d\}$. Then each of M and $M \backslash x$ is 3-connected, and $M \backslash x/x'$ is isomorphic to Φ_k^+ with tip p and legs $\{a_1, b_1\}, \{a_2, b_2\}, \ldots, \{a_k, b_k\}$. But, from considering its potential legs, we deduce that $M \not\cong \Phi_k^+$. We conclude, by Theorem 6.1, that M is indeed a counterexample to the conjecture because x and x' are clones in M and therefore x is not fixed in M.

THEOREM 8.4. *Let M and M' be quaternary matroids with the properties that M is connected, M' is 3-connected, and M' is strictly freer than M.*

(i) *If M' has a $U_{2,4}$-minor but no $U_{2,5}$- or $U_{3,5}$-minor, then M is binary.*

(ii) *If M' has a $U_{2,5}$- or $U_{3,5}$-minor but no $U_{3,6}$-minor, then M has no $U_{2,5}$- or $U_{3,5}$-minor.*

(iii) *If M' has a $U_{3,6}$-minor but no Φ_4-minor, then M has no $U_{3,6}$-minor.*

Proof. The proofs of all three parts are similar, so we shall give the details only for parts (ii) and (iii). For (ii), suppose first that M has no $U_{3,6}$-minor and let $\mathscr{N}$ be the class of quaternary matroids with no $U_{3,6}$-minor. By duality, we may assume that M' has a $U_{2,5}$-minor. By Theorem 8.3(iii), $U_{2,5}$ is a strong universal stabilizer for $\mathscr{N}$. Clearly, no matroid in $\mathscr{N}$ is strictly freer than $U_{2,5}$. Thus, by Theorem 7.4, M has no $U_{2,5}$-minor. Moreover, if $r(M) = 2$, then M certainly has no $U_{3,5}$-minor. If $r(M) > 2$, then $r(M') > 2$ and so, by [13, Prop. 11.2.16], since M' has $U_{2,5}$ as a minor, M' has $U_{3,5}$ as a minor. Then, by dualizing the argument above, we deduce that M has no $U_{3,5}$-minor.

To complete the proof of (ii), it remains to consider the case when M has $U_{3,6}$ as a minor. Then $M \backslash X/Y \cong U_{3,6}$ where Y is independent and X is coindependent in M. Since M is a rank-preserving weak-map image of M', it follows that Y is independent and X is coindependent in M'. Thus $M' \backslash X/Y$ has the same rank as $M \backslash X/Y$ but is freer. Hence $M' \backslash X/Y \cong U_{3,6}$. This contradiction finishes the proof of (ii).

To prove (iii), suppose first that M has no Φ_4-minor and let $\mathscr{N}$ be the class of quaternary matroids with no Φ_4-minor. In this case, we argue as in the first paragraph, using Theorem 8.3(iv) and Theorem 7.4, to deduce that M has no $U_{3,6}$-minor. We may now assume that M has Φ_4 as a minor, say $M \backslash X/Y \cong \Phi_4$ where Y is independent and X is coindependent in M. Then, as in the second paragraph, we deduce that $M' \backslash X/Y$ is a rank-4 quaternary matroid that is freer than Φ_4. Let the latter matroid have legs $\{a_1, b_1\}$, $\{a_2, b_2\}$, $\{a_3, b_3\}$, and $\{a_4, b_4\}$. Since $(M' \backslash X/Y) \backslash \{a_4, b_4\}$ is quaternary and so has no $U_{4,6}$-minor, at least one of $\{a_1, b_1, a_2, b_2\}$, $\{a_1, b_1, a_3, b_3\}$, and $\{a_2, b_2, a_3, b_3\}$ is a circuit of this matroid. Assume, without loss of generality, that the first of these sets is a circuit. Then each of $\{a_1, b_1, a_3, b_3\}$ and $\{a_1, b_1, a_4, b_4\}$ is a circuit of $M' \backslash X/Y$, otherwise $(M' \backslash X/Y)/a_1$ has a P_6-minor. By considering both $(M' \backslash X/Y)/a_2$ and $(M' \backslash X/Y)/a_3$, we deduce that every 4-circuit of Φ_4 is a circuit of $M' \backslash X/Y$, so $M' \backslash X/Y \cong \Phi_4$, a contradiction. ∎

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