q-LIFTS OF TANGENTIAL k-BLOCKS

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ABSTRACT

A tangential k-block over $GF(q)$ is a geometry representable over $GF(q)$ with critical exponent $k+1$ for which every proper loopless minor has critical exponent at most k. We define what is meant by a q-lift of a matroid representable over $GF(q)$ and show that, if M is a tangential k-block over $GF(q)$ and M' is a q-lift of M, then M' is a tangential $k+1$-block over $GF(q)$. This enables us to extend the class of known tangential k-blocks and to answer some natural questions concerning the existence of modular hyperplanes in tangential k-blocks.

1. *Introduction*

This paper continues the study of tangential k-blocks over $GF(q)$ begun in [**9**, **10**, **11**]. To some extent it completes a picture. We define what is meant by a q-lift of a matroid representable over $GF(q)$, and show that, if M is a tangential k-block over $GF(q)$ and M' is a q-lift of M, then M' is also a tangential k-block over $GF(q)$. While the construction is general, interest is focused on tangential k-blocks over $GF(2)$; in particular those obtained by successively taking q-lifts of $M^*(P_{10})$ (the cocycle matroid of the Petersen graph).

We assume that the reader is familiar with the basic concepts of matroid theory, particulárly those of the characteristic polynomial $P(M;\lambda)$ of a matroid and the critical exponent $c(M;q)$ of a matroid representable over $GF(q)$. Welsh [**7**, Chapter 16] provides a good introduction to these topics; free use will be made of results from this chapter. The terminology used here for matroids will in general follow Welsh [**7**]. Some differences follow. A geometry is a matroid without loops or parallel elements. If M is a matroid with ground set E and $S \subseteq E$, the restriction of M to $E\backslash S$ will be denoted by $M|(E\backslash S)$ or by $M\backslash S$ according to convenience and the contraction of M to $E\backslash S$ will be denoted by M/S. The closure and rank of S in M will be denoted by $\mathrm{cl}_M(S)$ and $\mathrm{r}_M(S)$, respectively or if no danger of ambiguity exists by $\mathrm{cl}(S)$ and $\mathrm{r}(S)$, respectively. The geometry whose lattice of flats is isomorphic to that of M will be denoted by $\bar{M}$.

Some familiarity with properties of modular flats of matroids is assumed. Modular flats are studied extensively in [**1**].

A tangential k-block over $GF(q)$ is a geometry representable over $GF(q)$ with critical exponent $k+1$ for which every proper loopless minor has critical exponent less than $k+1$. If M is a geometry representable over $GF(q)$ with critical exponent $k+1$, then it is straightforward to show that M is a tangential k-block over $GF(q)$ if $c(M/F;q) \leqslant k$ whenever F is a proper non-empty flat of M.

Let S be a set of points of $PG(r-1,q)$. To avoid unwieldy notation we denote $\mathrm{cl}_{PG(r-1,q)}(S)$ by $\mathrm{cl}_P(S)$.

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2. *q-lifts of tangential k-blocks*

Let M be a rank r matroid representable over $\mathrm{GF}(q)$; then the geometry M' is a *q-lift* of M with *base* E and *apex* p if the following conditions hold:

(i) E is a set of points of $\mathrm{PG}(r,q)$ with $\mathrm{PG}(r,q)\,|\,E \cong \overline{M}$;
(ii) p is a point of $\mathrm{PG}(r,q)$ not contained in $\mathrm{cl}_{\mathrm{P}}(E)$;
(iii) $M' = \mathrm{PG}(r,q)\,|\,\{\bigcup_{x\in E} \mathrm{cl}_{\mathrm{P}}(\{p,x\})\}$.

In other words, one embeds $\overline{M}$ in a hyperplane H of $\mathrm{PG}(r,q)$ and then obtains M' by restricting $\mathrm{PG}(r,q)$ to the set of points on lines joining points of the embedding of $\overline{M}$ to a fixed point of $\mathrm{PG}(r,q)$ not on H.

The object of this section is to prove the following result.

THEOREM 2.1. *If M is a tangential k-block over* $\mathrm{GF}(q)$ *and M' is a q-lift of M then M' is a tangential $k+1$-block over* $\mathrm{GF}(q)$.

Before proving Theorem 2.1 we need some lemmas on the structure of q-lifts. Throughout this section it is assumed that M is representable over $\mathrm{GF}(q)$ and that M' is a q-lift of M with base E and apex p. The ground set of M' is denoted by S.

The following lemma is straightforward.

LEMMA 2.2. (i) *Any line of M' containing p is isomorphic to* $\mathrm{PG}(1,q)$.
(ii) $\overline{M'/p} \cong \overline{M}$.

LEMMA 2.3. (i) *If H is a hyperplane of* $\mathrm{PG}(r,q)$ *disjoint from p, then $H\cap S$ is a hyperplane of M'.*
(ii) *If H' is a hyperplane of M' disjoint from p, then* $M'\,|\,H' \cong \overline{M}$.
(iii) *If H' is a hyperplane of M' disjoint from p, then M' is also a q-lift of M with base H' and apex p.*

Proof. We first show that, if H is a hyperplane of $\mathrm{PG}(r,q)$ not containing p, then $M'\,|\,(H\cap S) \cong \overline{M}$. Let H be such a hyperplane. Since $p \notin H$, no line of M' containing p is contained in H. By Lemma 2.2, each of these lines is isomorphic to $\mathrm{PG}(1,q)$, so none of these lines is disjoint from H. Therefore H meets each line of M' containing p in a single point, so $(M'/p)\,|\,(H\cap S) \cong \overline{M'/p}$. But p is a coloop of $M'\,|\,((H\cap S)\cup p)$, so $(M'/p)\,|\,(H\cap S) \cong M'\,|\,(H\cap S)$ and, by Lemma 2.2, $\overline{M'/p} \cong \overline{M}$ so $M'\,|\,(H\cap S) \cong \overline{M}$.

Part (i) of the lemma follows since $\mathrm{r}(M') = \mathrm{r}(M)+1$. Part (ii) follows from the observation that every hyperplane H' of M' disjoint from p is of the form $H\cap S$ for some hyperplane H of $\mathrm{PG}(r,q)$ disjoint from p. Part (iii) follows since $S = \{\bigcup_{x\in H\cap S} \mathrm{cl}(\{p,x\})\}$.

LEMMA 2.4. *If x is an element of S distinct from p, then $\overline{M'/x}$ is isomorphic to a q-lift of a proper minor of M.*

Proof. Let H' be a hyperplane of M' containing x but disjoint from p; then, by Lemma 2.3(iii), M' is a q-lift of M with base H' and apex p. Therefore we may assume without loss of generality that $x \in E$. Let H be a hyperplane of $\mathrm{PG}(r,q)$ containing p but missing x. Let

$$S_1 = \mathrm{PG}(r,q) \mid \{\bigcup_{y \in S} \{\mathrm{cl_P}(\{x,y\}) \cap H\}\}$$

and

$$S_2 = \mathrm{PG}(r,q) \mid \{\bigcup_{y \in E} \{\mathrm{cl_P}(\{x,y\}) \cap H\}\}.$$

Clearly $\mathrm{PG}(r,q) \mid S_1 \cong \overline{M'/x}$ and $\mathrm{PG}(r,q) \mid S_2 \cong \overline{(M' \mid E)/x}$. Therefore $\mathrm{PG}(r,q) \mid S_2$ is isomorphic to a proper minor of M. Since $p \in H$ the theorem is proved if we can show that $S_1 = \{\bigcup_{y \in S_2} \mathrm{cl_P}(\{p,y\})\}$. Assume that $a \in S_1$; then there exists $b \in S\backslash x$ with the property that $\mathrm{cl_P}(\{x,b\}) \cap H = a$. If $b = p$ or $b \in E$, then $a \in \{\bigcup_{y \in S_2} \mathrm{cl_P}(\{p,y\})\}$, so we assume that $b \in S\backslash\{p \cup E\}$. In this case, $b \in \mathrm{cl_P}(\{p,e\})$ for some point $e \in E$. But then $a \in \mathrm{cl_P}(\{p, \mathrm{cl_P}(\{x,e\}) \cap H\})$ and therefore $a \in \{\bigcup_{y \in S_2} \mathrm{cl_P}(\{p,y\})\}$. Hence $S_1 \subseteq \{\bigcup_{y \in S_2} \mathrm{cl_P}(\{p,y\})\}$. Assume that $a \in \{\bigcup_{y \in S_2} \mathrm{cl_P}(\{p,y\})\}$. If $a \in S_2$ or $a = p$, then $a \in S_1$; so we assume that $a \notin \{S_2 \cup p\}$. In this case there exists $e' \in S_2$ with the property that $a \in \mathrm{cl_P}(\{p,e'\})$. But $e' = \mathrm{cl_P}(\{x,e\}) \cap H$ for some point $e \in E$ and there exists a point $b \in \mathrm{cl_P}(\{p,e\})$ with the property that $\mathrm{cl_P}(\{x,b\}) \cap H = a$. But $b \in S$ so $a \in S_1$. Hence $\{\bigcup_{y \in S_2} \mathrm{cl_P}(\{p,y\})\} \subseteq S_1$. The two sets are therefore equal, and the lemma is proved.

LEMMA 2.5. *Let F be a non-empty flat of M'.*
(i) *If $p \in F$, then $\overline{M'/F}$ is isomorphic to a minor of M.*
(ii) *If $p \notin F$, then $\overline{M'/F}$ is isomorphic to a q-lift of a proper minor of M.*

Proof. Parts (i) and (ii) are routine consequences of Lemmas 2.2 and 2.4 respectively.

LEMMA 2.6. *If M is loopless and $c(M;q) = k$, then $c(M';q) = k+1$.*

Proof. Say $c(M';q) = j$ and let $(H_1, \ldots, H_j)$ be a j-tuple of hyperplanes of $\mathrm{PG}(r,q)$ which distinguishes S. At least one of these hyperplanes is disjoint from p, so assume without loss of generality that H_j misses p. Then

$$j-1 = c(\mathrm{PG}(r,q) \mid (H_j \cap S); q).$$

But, by Lemma 2.3, $\mathrm{PG}(r,q) \mid (H_j \cap S) \cong \overline{M}$, and since M is loopless, it follows that $c(\overline{M};q) = c(M;q)$; hence $j-1 = c(M;q) = k$. Therefore $j = k+1$ and $c(M';q) = k+1$.

We are now in a position to prove our main theorem.

Proof of Theorem 2.1. Certainly M' is a geometry and is representable over $\mathrm{GF}(q)$. Since M is a tangential k-block over $\mathrm{GF}(q)$ it follows that $c(M;q) = k+1$. Therefore, by Lemma 2.6, $c(M';q) = k+2$. Let F be a non-empty flat of M'. If $p \in F$, then, by Lemma 2.5(i), $\overline{M'/F}$ is isomorphic to a minor M'' of M. Therefore, since M is a tangential k-block, $c(\overline{M'/F};q) < k+2$ and also $c(M'/F;q) < k+2$. If $p \notin F$, then, by Lemma 2.5(ii), $\overline{M'/F}$ is isomorphic to a q-lift of a proper minor M'' of M. We may assume that M'' is loopless; then, since M'' is a proper loopless minor of a tangential k-block over $\mathrm{GF}(q)$ it follows that $c(M'';q) < k+1$. Therefore, by Lemma 2.6, $c(M'/F;q) = c(M'';q)+1 < k+2$.

We conclude that M' is a geometry representable over GF(q) with critical exponent $k+2$ and, whenever F is a non-empty flat of M', $c(M'/F) < k+2$; hence M' is a tangential $k+1$-block over GF(q).

By repeatedly taking q-lifts, it is possible, given a tangential k-block over GF(q), to construct tangential k'-blocks over GF(q) for all $k' > k$. The extent to which this construction is interesting depends, in part, on the properties inherited by q-lifts. It is to this question that we now turn our attention.

3. *Modularity in q-lifts*

In **[9]** it is shown that tangential k-blocks with modular hyperplanes form a well-behaved class. It is therefore of interest to know to what extent modularity is preserved in q-lifts. The following lemma is a routine consequence of results in **[1]**. It is proved in **[8]**.

LEMMA 3.1. *Let M be a loopless rank r matroid and let F be a rank k flat of M; then F is modular in M if and only if $F \cap F' \neq \emptyset$ for every flat F' of M with* $\mathrm{r}(F') = r-k+1$.

As before, we assume that M' is a q-lift of M with base E and apex p, and that the ground set of M' is S. Since our interest is in tangential k-blocks, we lose no generality in assuming that M is the geometry $\mathrm{PG}(r-1,q)\,|\,E$.

LEMMA 3.2. *Let F be a flat of M; then F is modular in M if and only if* $\mathrm{cl}_{M'}(F \cup p)$ *is modular in M'.*

Proof. Let F be a rank k flat. Assume that $\mathrm{cl}_{M'}(F \cup p)$ is modular in M'; then if F' is a flat of M with $\mathrm{r}(F') = r-k+1$, we have that $F' \cap \mathrm{cl}_{M'}(F \cup p) \neq \emptyset$. But $F' \subseteq E$ and $\mathrm{cl}_{M'}(F \cup p) \cap E = F$; so $F' \cap F \neq \emptyset$. Therefore F is modular in M.

Assume that F is modular in M. Now $\mathrm{r}(\mathrm{cl}_{M'}(F \cup p)) = k+1$, so let F' be a flat of M' with $\mathrm{r}(F_1) = r+1-(k+1)+1 = r-k+1$. If $p \in F'$, then $F' \cap \mathrm{cl}_{M'}(F \cup p) \neq \emptyset$; so we assume that $p \notin F'$. Let H be a hyperplane of M' containing F' but missing p. By Lemma 2.3(ii), $\mathrm{PG}(r,q)\,|\,H \cong M$. Furthermore, it is easily seen that the function $\psi(x) = \mathrm{cl}_{\mathrm{P}}(\{p,x\}) \cap E$ is an isomorphism from $\mathrm{PG}(r,q)\,|\,H$ to M. Now F is modular in M and $\mathrm{r}(F') = r-k+1$ so $\psi(F') \cap F \neq \emptyset$, and therefore $F' \cap \psi^{-1}(F) \neq \emptyset$. But $\psi^{-1}(F) \subseteq \mathrm{cl}_{M'}(F \cup p)$, so $\mathrm{cl}_{M'}(F \cup p)$ is modular in M', and the result follows.

A matroid M is supersolvable if it contains a saturated chain of modular flats; that is, a chain of flats $F_0 \subset F_1 \subset \ldots \subset F_{r-1} \subset F_r$ with the property that for $0 \leqslant i \leqslant r$, F_i is a rank i modular flat of M. Stanley **[3, 4]** shows that if M is a supersolvable geometry then

$$P(M;\lambda) = \prod_{i=1}^{r} (\lambda - |F_i \backslash F_{i-1}|).$$

THEOREM 3.3. *If M is supersolvable with saturated chain of modular flats*

$$F_0 \subset F_1 \subset \ldots \subset F_{r-1} \subset F_r,$$

then M' is supersolvable and

$$P(M';\lambda) = (\lambda-1)\prod_{i=2}^{r+1}(\lambda - q\,|F_{i-1}\backslash F_{i-2}|).$$

Proof. Let $F'_0 = \emptyset$ and for $1 \leqslant i \leqslant r+1$ let $F'_i = \mathrm{cl}_{M'}(\{F_{i-1} \cup p\})$. Then, using Lemma 3.2, we obtain a saturated chain $F'_0 \subset F'_1 \subset \ldots \subset F'_r \subset F'_{r+1}$ of modular flats of M'; hence M' is supersolvable. A routine argument then shows that for $i > 1$, $|F'_i\backslash F'_{i-1}| = q|F_{i-1}\backslash F_{i-2}|$. The result follows from this observation.

'Nice' tangential k-blocks are ones which are supersolvable or at least contain modular hyperplanes. Lemma 3.2 and Theorem 3.3 show that these properties are inherited by q-lifts. 'Nasty' tangential k-blocks are ones which contain no modular hyperplanes. I do not hide the fact that my initial interest in q-lifts was due to the hope that q-lifts of tangential k-blocks would always contain modular hyperplanes. For fixed k and q this would enable bounds to be placed on the rank of tangential k-blocks over GF(q). In dealing with the critical problem naive hopes such as these are invariably futile.

THEOREM 3.4. *M contains no modular hyperplanes if and only if M' contains no modular hyperplanes.*

Proof. If M contains a modular hyperplane then, by Lemma 3.2, M' contains a modular hyperplane so assume that M contains no modular hyperplanes. If H is a hyperplane of M' containing p, then $H = \mathrm{cl}_{M'}(F \cup p)$ for some flat F of M. But since $\mathrm{r}(\mathrm{cl}_{M'}(F \cup p)) = \mathrm{r}(F)+1$, F is a hyperplane of M. But F is not modular in M so, by Lemma 3.2, H is not modular in M'. If H is disjoint from p then, by Lemma 2.3, $M'\,|\,H \cong M$ and M' is also a q-lift of M with base H and apex p. Therefore we may assume without loss of generality that $H = E$. Let $C \subseteq E$ be a coline of M'. Then C is a hyperplane of M and since C is not modular in M there exists $\{x, y\} \subseteq E$ with $\mathrm{cl}_M(\{x,y\}) \cap C = \emptyset$. One routinely shows that

$$\mathrm{r}(\mathrm{cl}_{\mathrm{P}}(\{x,y\}) \cap \mathrm{cl}_{\mathrm{P}}(C)) = 1,$$

so let $a = \mathrm{cl}_{\mathrm{P}}(\{x,y\}) \cap \mathrm{cl}_{\mathrm{P}}(C)$. Let b be a point of $\mathrm{cl}_{\mathrm{P}}(\{p,x\})\backslash\{p,x\}$ and let $c = \mathrm{cl}_{\mathrm{P}}(\{a,b\}) \cap \mathrm{cl}_{\mathrm{P}}(\{p,y\})$. Since $\{x,y\} \subseteq E, \{b,c\} \subseteq S$. Now $\mathrm{cl}_{\mathrm{P}}(\{b,c\}) \cap \mathrm{cl}_{\mathrm{P}}(E) = a$ and $a \notin E$ so $\mathrm{cl}_{M'}(\{b,c\}) \cap E = \emptyset$. A modular hyperplane of a geometry must meet every line of that geometry so E is not modular in M' and the theorem is proved.

By the preceding theorem, q-lifts of nasty tangential k-blocks are also nasty. Now $M^*(\mathrm{P}_{10})$, is a tangential 2-block over GF(2) containing no modular hyperplanes. By taking successive q-lifts of $M^*(\mathrm{P}_{10})$ one obtains (in the light of Theorems 2.1 and 3.4) tangential k-blocks over GF(2) containing no modular hyperplanes for all $k \geqslant 2$. In **[11]** it is shown that whenever q is a prime power exceeding two, then there exist tangential k-blocks over GF(q) containing no modular hyperplanes for all positive integers k except in the special case when both $q = 3$ and $k = 1$. In **[5]** it is shown that the only tangential 1-block over GF(2) is the three point line $\mathrm{U}_{2,3}$, and in **[6]** it is shown that the only tangential 1-blocks over GF(3) are the four point line $U_{2,4}$ and the cycle matroid of K_4, $M(K_4)$. Therefore there exist tangential k-blocks over GF(q) containing no modular hyperplanes for all prime powers q and positive integers k except in the case when $k = 1$ and $q = 2$ or 3. That is, nasty tangential k-blocks abound.

It is of interest to note that $k = 1$ and $q = 2$ or $q = 3$ are the only values of k and q for which all tangential k-blocks over GF(q) are known (or at least known to be known). The existence of tangential k-blocks containing no modular hyperplanes would seem to lie at the heart of the difficulty of the critical problem.

We conclude with an example.

4. *Tangential* 3-*blocks over* GF(2)

Tutte's tangential 2-block conjecture states that the only tangential 2-blocks over GF(2) are PG(2, 2), $M(K_5)$ and $M^*(P_{10})$. As yet this conjecture is unresolved (but see [2]). A similar conjecture for tangential 3-blocks over GF(2) requires a purported complete list of such blocks.

In [**10**] a number of tangential 3-blocks over GF(2) is given with their matrix representations over GF(2). These blocks are all supersolvable and have the characteristic polynomials shown in Table 1.

TABLE 1

Rank	Characteristic polynomial	
8	$(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-5)(\lambda-6)(\lambda-7)(\lambda-8)$	$M(K_9)$
7	$(\lambda-1)(\lambda-2)(\lambda-4)(\lambda-5)(\lambda-6)(\lambda-7)(\lambda-8)$	
6	$(\lambda-1)(\lambda-2)(\lambda-4)(\lambda-6)(\lambda-7)(\lambda-8)$	
5	$(\lambda-1)(\lambda-2)(\lambda-4)(\lambda-7)(\lambda-8)$	
4	$(\lambda-1)(\lambda-2)(\lambda-4)(\lambda-8)$	PG(3, 2)

Any 2-lift of a tangential 2-block over GF(2) is a tangential 3-block over GF(2). A 2-lift of PG(2, 2) is just PG(3, 2) which is already in the above list. By Theorem 3.3, a 2-lift of $M(K_5)$ is supersolvable and has characteristic polynomial $(\lambda-1)(\lambda-2)(\lambda-4)(\lambda-6)(\lambda-8)$, so there exist at least two non-isomorphic supersolvable rank 5 tangential 3-blocks over GF(2). Since $M^*(P_{10})$ has rank 6 a 2-lift of $M^*(P_{10})$ has rank 7. This geometry is not supersolvable, so there exist two non-isomorphic rank 7 tangential 3-blocks over GF(2). This gives a list of seven tangential 3-blocks over GF(2). I have no idea whether or not this list is likely to be complete.

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