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On Maximum-Sized Near-Regular and $\sqrt[6]{1}$-Matroids

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Abstract. The classes of near-regular and $\sqrt[6]{1}$-matroids arise in the study of matroids representable over $GF(3)$ and other fields. For example, a matroid is representable over all fields except possibly $GF(2)$ if and only if it is near-regular, and a matroid is representable over $GF(3)$ and $GF(4)$ if and only if it is a $\sqrt[6]{1}$-matroid. This paper determines the maximum sizes of a simple rank-r near-regular and a simple rank-r $\sqrt[6]{1}$-matroid and determines all such matroids having these sizes.

1. Introduction

It is well known that if $\mathscr{F}$ is a set of fields containing $GF(2)$, then unless every member of $\mathscr{F}$ has characteristic two, the class of matroids representable over all fields in $\mathscr{F}$ is the class of regular matroids. In [9, 10], analogous results are given for sets of fields containing $GF(3)$. It is shown there that essentially three new classes arise. These classes are defined in a way that is analogous to that of regular matroids, that is, they are defined as matroids representable by certain types of matrices over certain fields. We begin by reviewing their definitions.

A matrix over the rationals is *dyadic* if all its non-zero subdeterminants are in $\{\pm 2^i : i \in \mathbf{Z}\}$. A *dyadic matroid* is one that can be represented over the rationals by a dyadic matrix. A matrix over the complex numbers is a $\sqrt[6]{1}$-*matrix* if all its non-zero subdeterminants are complex sixth roots of unity. A $\sqrt[6]{1}$-*matroid* is one that can be represented over the complex numbers by a $\sqrt[6]{1}$-matrix. Let $\mathbf{Q}(\alpha)$ denote the field obtained by extending the rationals by the transcendental α. A matrix over $\mathbf{Q}(\alpha)$ is *near-unimodular* if all its non-zero subdeterminants are in $\{\pm \alpha^i(\alpha - 1)^j : i, j \in \mathbf{Z}\}$. A *near-regular* matroid is one that can be represented over $\mathbf{Q}(\alpha)$ by a near-unimodular matrix.

Let $\mathscr{F}$ be a set of fields containing $GF(3)$. Assume that $\mathscr{F}$ does not contain $GF(2)$ and contains at least one field of characteristic other than three. It is shown in [9, 10] that the class of matroids representable over all fields in $\mathscr{F}$ is either the class of near-regular matroids, the class of dyadic matroids, the class of $\sqrt[6]{1}$-matroids, or the class of matroids obtained by taking direct sums and 2-sums of dyadic matroids and $\sqrt[6]{1}$-matroids. For a flavour of the results of [9, 10] note that a matroid is representable over $GF(3)$ and the rationals if and only if it is dyadic; a matroid is representable over all fields except possibly $GF(2)$ if and only if it is near-regular; and a matroid is representable over $GF(3)$ and $GF(4)$ if and only if it is a $\sqrt[6]{1}$-matroid.

Evidently near-regular, dyadic, and $\sqrt[6]{1}$-matroids play similar roles within the class of ternary matroids to that played by regular matroids within the class of binary matroids. Regular matroids are well understood and they have many attractive properties. It is certainly of interest to know which properties of regular matroids have analogues for near-regular, dyadic, and $\sqrt[6]{1}$-matroids. This paper focuses on one such property.

A simple rank-r matroid is *maximum-sized* in a class if its ground set has maximum size amongst all other simple rank-r matroids in the class. The maximum-sized regular matroids are known to be the cycle matroids of complete graphs [2]. Moreover, it follows from results of Kung [3] and Kung and Oxley [4] that a maximum-sized rank-r dyadic matroid has r^2 points and is the ternary Dowling geometry $Q_r(GF(3)^*)$. This paper determines all maximum-sized members of the classes of near-regular and $\sqrt[6]{1}$-matroids. It turns out that, for all ranks r other than three, both these classes have a single maximum-sized rank-r member and this matroid is the same for the two classes. Geometrically, this common maximum-sized matroid is obtained by adding a point freely on a 3-point line of $M(K_{r+2})$, contracting that point, and simplifying the resulting matroid.

The deeper motivation for this paper is the desire to obtain analogues of Seymour's decomposition theory for regular matroids [7]. The results of [3, 4] and this paper present the tantalising hope that such analogues are feasible. In such a development, minors of the maximum-sized dyadic, near-regular, and $\sqrt[6]{1}$-matroids would be expected to play a role similar to that played by graphic matroids in Seymour's decomposition for regular matroids.

We assume familiarity with matroid theory as set forth in [5]. Terminology follows [5] with the single exception that we denote the simple and cosimple matroids canonically associated with a matroid M by $\mathrm{si}(M)$ and $\mathrm{co}(M)$, respectively. We frequently use the fact that the class of near-regular matroids is contained within the class of $\sqrt[6]{1}$-matroids.

2. The Main Results

For all $r \geq 2$, let D_r denote the $r \times \binom{r}{2}$ matrix whose columns consist of all r-tuples with exactly two non-zero entries, the first equal to 1 and the second equal to -1.

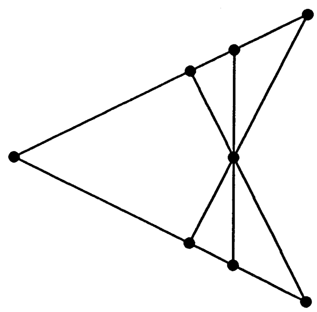

Fig. 1. The matroid T_3

Now, for all $r \geq 3$, let A_r be the following matrix over $\mathbf{Q}(\alpha)$:

$$\left[\begin{array}{c|c|c|c|c} 1 & 0\ 0 \cdots 0 & 1\ 1 \cdots 1 & \alpha\ \alpha \cdots \alpha & 0\ 0 \cdots 0 \\ \hline 0 & & & & \\ 0 & & & & \\ & I_{r-1} & I_{r-1} & I_{r-1} & D_{r-1} \\ \vdots & & & & \\ 0 & & & & \end{array}\right].$$

Let $A_1 = [1]$ and $A_2 = \begin{bmatrix} 1 & 0 & 1 & \alpha \\ 0 & 1 & 1 & 1 \end{bmatrix}$. Now define T_r to be the rank-r matroid that is represented over $\mathbf{Q}(\alpha)$ by A_r. Thus $T_1 \cong U_{1,1}$ and $T_2 \cong U_{2,4}$. A geometric representation for T_3 is shown in Fig. 1.

The following are the main results of this paper.

Theorem 2.1. *Let M be a simple $\sqrt[6]{1}$-matroid of rank r. Then*

$$|E(M)| \leq \begin{cases} \binom{r+2}{2} - 2 & \text{if } r \neq 3; \\ 9 & \text{if } r = 3. \end{cases}$$

Moreover, equality is attained in this bound if and only if $M \cong T_r$ when $r \neq 3$, or $M \cong AG(2,3)$ when $r = 3$.

Corollary 2.2. *Let M be a simple near-regular matroid of rank r. Then*

$$|E(M)| \leq \binom{r+2}{2} - 2.$$

Moreover, equality is attained in this bound if and only if $M \cong T_r$.

The proofs of these results will require some preliminaries. We begin by showing that T_r is, indeed, near-regular.

3. Near-Regularity of T_r

Lemma 3.1. *For all r, the matrix A_r is near-unimodular.*

Proof. We argue by induction on r. Recall that to show that A_r is near-unimodular, we need to show that all non-zero subdeterminants of this matrix are in the set $\{\pm\alpha^i(\alpha-1)^j : i,j \in \mathbf{Z}\}$. Evidently A_1 and A_2 are near-unimodular. Now assume that $r > 2$ and that A_{r-k} is near-unimodular for all k in $\{1,2,\ldots,r-2\}$. Let X be an $n \times n$ submatrix of A_r. Since the matrix $[I_{r-1}|D_{r-1}]$ is a totally unimodular representation of $M(K_r)$, we may assume that X meets row 1 of A_r. Moreover, if X avoids some row of A_r, then either X has two columns such that one is a scalar multiple of the other, or X can be obtained from a submatrix of A_{r-1} by multiplying some columns by α or -1. In either case, we deduce that $\det(X)$ is in the desired set. Thus we may assume that $n = r$. Suppose X has a row with at most one non-zero entry. Then $\det(X)$ is obtained by multiplying the determinant of a submatrix of A_{r-1} by some member of $\{0,1,-1,\alpha,-\alpha\}$. Thus $\det(X)$ is in the desired set. Hence we may assume that every row of X has at least two non-zero entries. But every column of X has at most two non-zero entries. Thus X has exactly $2n$ non-zero entries, two per row and two per column. Then, after permuting some rows and columns and multiplying some rows or columns by -1, we can get the matrix

$$\begin{bmatrix} x & 0 & 0 & & 0 & \alpha \\ 1 & 1 & 0 & & 0 & 0 \\ 0 & -1 & 1 & & 0 & 0 \\ 0 & 0 & -1 & & 0 & 0 \\ & & & \ddots & & \\ 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & & -1 & 1 \end{bmatrix},$$

where $x \in \{1,\alpha\}$. The determinant of this matrix is $x + (-1)^{1+n}\alpha(-1)^{n-2}$, that is, $x - \alpha$. Hence $\det(X)$ is in the desired set and we conclude, by induction, that A_r is indeed near-unimodular. □

By using the last lemma and examining the matrix A_r, it is straightforward to deduce the following result.

Corollary 3.2. *The matroid T_r is near-regular and has exactly $r-1$ four-point lines all of which share a common point.*

Geometrically, T_r can be obtained from $M(K_{r+2})$ by freely adding a point on a 3-point line of this matroid, and then simplifying the resulting matroid; that is, T_r is the simplification of the principal truncation $T_{M(K_3)}(M(K_{r+2}))$. To see this, take a totally unimodular representation of $M(K_{r+2})$ over $\mathbf{Q}(\alpha)$ of the form $[I_{r+1}|D_{r+1}]$, where D_{r+1} is as defined at the start of Section 2. Adjoin the column $[1,-\alpha,0,\ldots,0]^t$

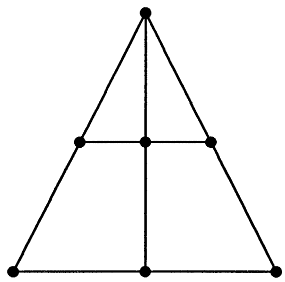

Fig. 2. The matroid P_7

to this matrix. Evidently this places a point freely on the line

$$\{[1,0,0,\dots,0]', [0,1,0,\dots,0]', [1,-1,0,\dots,0]'\}.$$

It is now easily seen that A_r represents the matroid obtained by contracting the point represented by $[1,-\alpha,0,\dots,0]'$ and simplifying.

4. Some Structural Properties

In this section we establish structural properties of near-regular and $\sqrt[6]{1}$-matroids that will be needed for the proof of Theorem 2.1. We first consider matroids of low rank. The following result is proved in [9, Theorem 5.1].

Lemma 4.1. *Let M be a simple $\sqrt[6]{1}$-matroid of rank three. Then M is a restriction of $AG(2,3)$ or T_3. Moreover, if M is near-regular, it is a restriction of T_3.*

A *long line* of a matroid is a line that contains at least three points. The matroid that is obtained from T_3 by deleting the point that is on both 4-point lines is called P_7 (see Fig. 2). We call the point of P_7 that is on three 3-point lines its *tip*. We observe that T_3 has a unique P_7-restriction and a unique element that is on more than two long lines.

The matroid $AG(2,3)$ has all of its single-element deletions isomorphic. We denote such a deletion by $AG(2,3)-e$. All two-element deletions from $AG(2,3)$ are also isomorphic, the resulting matroid in this case being P_7. The next result follows from these observations and the last lemma.

Corollary 4.2. *Let M be a simple $\sqrt[6]{1}$-matroid of rank 3 having a point that is incident with at least three long lines. Then each of these lines has exactly three points, and M is isomorphic to $AG(2,3)$, $AG(2,3)-e$, T_3, or P_7. Moreover, if M is near-regular, then M is isomorphic to T_3 or P_7.*

The last result dealt with three concurrent long lines in a $\sqrt[6]{1}$-matroid of rank three. Next we consider four concurrent long lines in a $\sqrt[6]{1}$-matroid of rank four.

Lemma 4.3. *If a rank-4 matroid M has four concurrent long lines no three of which are coplanar, then M is not a $\sqrt[6]{1}$-matroid.*

Proof. Assume that M is a $\sqrt[6]{1}$-matroid and that p is the point of concurrency of four long lines. Let S be the union of these four lines and consider $M|S$. Evidently if $x \in S - p$, then $\mathrm{si}((M|S)/x)$ has rank three and is the union of three concurrent long lines. Thus, by Corollary 4.2, $\mathrm{si}((M|S)/x) \cong P_7$. It follows, by the Scum Theorem, that $M|S$ is a ternary matroid with no $M(K_4)$-minor. Moreover, $M|S$ is easily seen to be 3-connected. All 3-connected ternary matroids with no $M(K_4)$-minor were determined in [6]. It follows from that result that $M|S$ has a P_8-restriction where P_8 is an 8-element rank-4 excluded minor for $GF(4)$-representability. But this is a contradiction since M is a $\sqrt[6]{1}$-matroid and such matroids are quaternary. □

Next we prove two more structural results for $\sqrt[6]{1}$-matroids that will be needed in the proof of the main theorem.

Lemma 4.4. *Let M be a* 3-*connected $\sqrt[6]{1}$-matroid. Then M does not have as a restriction the parallel connection of P_7 and $U_{2,4}$ in which the basepoint of the parallel connection is the tip of P_7.*

Proof. Assume, to the contrary, that M does have such a parallel connection as a restriction. Let L be the 4-point line, p be the basepoint of the parallel connection, and T be the ground set of the P_7-restriction. As M is 3-connected, for each element e of $E(M) - \mathrm{cl}_M(L \cup T)$, we have, by Bixby [1] (see also [5, Proposition 8.4.6]), that $\mathrm{co}(M\backslash e)$ or $\mathrm{si}(M/e)$ is 3-connected. It follows that, by repeated application of this result, we can obtain a 3-connected rank-4 minor N_2 of M having $M|(L \cup T)$ as a restriction. Evidently N_2 has an element f that is in neither the closure of L nor the closure of T. Then N_2/f has rank 3 and has at least three long lines through p, at least one of which contains four points. This contradiction to Corollary 4.2 completes the proof of the lemma. □

Lemma 4.5. *Let M be a 3-connected $\sqrt[6]{1}$-matroid. Suppose that X and Y are subsets of $E(M)$ such that $M|X \cong P_7 \cong M|Y$ and $r(X \cup Y) \geq 4$. Then the tip of $M|Y$ is not in X.*

Proof. Let p_X and p_Y be the tips of $M|X$ and $M|Y$, repectively, and assume that $p_Y \in X$. Since $r(X \cap Y) \geq 1$, the matroid $M|(X \cup Y)$ has rank at most five.

Suppose that $r(X \cup Y) = 5$. Then $M|(X \cup Y)$ is the parallel connection of two copies of P_7. As M is 3-connected, we may apply Bixby's lemma, as in the proof of the last result, to deduce that M has a 3-connected minor N of rank 5 having $M|(X \cup Y)$ as a restriction. Evidently N has an element f such that $f \notin \mathrm{cl}_N(X) \cup \mathrm{cl}_N(Y)$. Thus $\mathrm{si}(N/f)$ is a 3-connected $\sqrt[6]{1}$-matroid of rank 4 having two distinct P_7-restrictions such that the tip of one is an element of the other. We conclude that it suffices to prove that $r(X \cup Y) \neq 4$.

Assume the contrary. If $p_X = p_Y$, then $M|(X \cup Y)$ is a rank-4 matroid having at least five long lines through the common tip of the two P_7-restrictions. Evidently some four of these lines have the property that no three are coplanar. By Lemma 4.3, this yields a contradiction. Hence we may suppose that $p_Y \neq p_X$. Now take $y \in Y - \mathrm{cl}_M(X)$. Then either (a) $\mathrm{cl}_M(X) \cap \mathrm{cl}_M(Y)$ contains a long line L_X of $M|X$; or (b) $\mathrm{si}([M|(X \cup Y)]/y)$ has three long lines through p_Y at least one of which has four points. By Corollary 4.2, the second possibility does not occur. Thus we may assume that (a) holds. Take $x \in X - (L_X \cup p_X)$. Then $\mathrm{si}([M|(X \cup Y)]/y)$ has three long lines through p_Y including one with four points. This contradiction to Corollary 4.2 completes the proof of Lemma 4.5. □

5. Proof of Theorem 2.1

We are now ready to prove Theorem 2.1. Recall that a $\sqrt[6]{1}$-matroid M is *maximum-sized* if M is simple and has the maximum number of elements among all $\sqrt[6]{1}$-matroids of its rank.

Proof of Theorem 2.1. We argue by induction on r to simultaneously prove the bound and the characterization of the matroids that attain equality in this bound. The result is clear for $r < 3$. Moreover, when $r = 3$, it follows by Lemma 4.1.

Let M be a maximum-sized $\sqrt[6]{1}$-matroid of rank r, where $r \geq 4$, and assume that the theorem holds for all smaller ranks. Then

$$|E(M)| \geq |E(T_r)| = \binom{r+2}{2} - 2. \tag{5.1}$$

Recall that, for a positive integer k, a matroid N is *vertically k-separated* if there is a partition $\{X, Y\}$ of $E(N)$ such that $\min\{r(X), r(Y)\} \geq k$, and $r(X) + r(Y) - r(M) \leq k - 1$. A vertical k-separation is *exact* if it has the property that $r(X) + r(Y) - r(M) = k - 1$. The matroid N is *vertically n-connected* if, for all $k < n$, it has no vertical k-separation.

Lemma 5.1. *M is vertically 4-connected.*

Proof. The argument that M does not have an exact vertical 1- or 2-separation is similar to, but simpler than, the argument that M has no exact vertical 3-separation. We shall present only the latter.

Assume that M is 3-connected but has an exact vertical 3-separation $\{X_1, X_2\}$. View M as a restriction of $PG(r-1, 3)$. Then, since $r(X_1) + r(X_2) - r = 2$, the closures of X_1 and X_2 in $PG(r-1, 3)$ meet in a line L of that matroid. Let $r_i = r(X_i)$.

We consider $|L \cap E(M)|$, noting that it is at most four. The strategy of the proof is to consider, for each $i \in \{1, 2\}$, a simple rank-r_i minor M_i of M that is spanned by X_i, contains $(X_1 \cup X_2) \cap L$, and has the maximum number of points among such minors. Clearly we may view M_i as a restriction of $PG(r-1, 3)|(L \cup X_i)$.

Now

$$\begin{aligned}|E(M)| &= |X_1| + |X_2| \\ &= (|E(M_1)| - |(E(M_1) \cap L) - X_1|) \\ &\quad + (|E(M_2)| - |(E(M_2) \cap L) - X_2|).\end{aligned}$$

By Heller [2], if M_i is binary, then $|E(M_i)| \le \binom{r_i+1}{2}$. Moreover, in general, by the induction assumption, $|E(M_i)| \le \binom{r_i+2}{2} - 2$ unless $M_i \cong AG(2,3)$, in which case, $|E(M_i)| \le \binom{r_i+2}{2} - 1$. Thus

$$\begin{aligned}|E(M)| \le{}& \binom{r_1+2}{2} + \binom{r_2+2}{2} - 2 - \delta_1 - \delta_2 \\ &- [|(E(M_1) \cap L) - X_1| + |(E(M_2) \cap L) - X_2|]\end{aligned}$$

where $\delta_i = \binom{r_i+2}{2} - 1 - |E(M_i)|$. Thus $\delta_i = 0$ if $M_i \cong AG(2,3)$; $\delta_i \ge 1$ if M_i is non-binary and $M_i \not\cong AG(2,3)$; and $\delta_i \ge r_i$ if M_i is binary. But

$$|E(M)| \ge \binom{(r_1+r_2-2)+2}{2} - 2.$$

So

$$\begin{aligned}\tfrac{1}{2}(r_1+r_2)(r_1+r_2-1) \le{}& \tfrac{1}{2}[(r_1+2)(r_1+1) + (r_2+2)(r_2+1)] - \delta_1 - \delta_2 \\ &- [|(E(M_1) \cap L) - X_1| + |(E(M_2) \cap L) - X_2|].\end{aligned}$$

Thus

$$(r_1-2)(r_2-2) \le 6 - \delta_1 - \delta_2 - [|(E(M_1) \cap L) - X_1| + |(E(M_2) \cap L) - X_2|].$$

Hence

$$\begin{aligned}(r_1-2)(r_2-2) \le{}& 6 - \delta_1 - \delta_2 - [|E(M_1) \cap L| \\ &- |X_1 \cap L| + |E(M_2) \cap L| - |X_2 \cap L|].\end{aligned} \tag{5.2}$$

But

$$\begin{aligned}|E(M_i) \cap L| &\ge |(X_1 \cup X_2) \cap L| \\ &= |X_1 \cap L| + |X_2 \cap L|,\end{aligned} \tag{5.3}$$

so, for each $i \in \{1,2\}$,

$$(r_1-2)(r_2-2) \le 6 - \delta_1 - \delta_2 - |E(M_i) \cap L|. \tag{5.4}$$

Next, we take a basis B_1 for X_1 and extend it to a basis B for M. Then $|B - B_1| = r(M) - r(X_1) = r(X_2) - 2$. It follows that $r_{M/(B-B_1)}(X_2 - B) = 2$. This means that M can certainly be assumed to satisfy

$$|E(M_1) \cap L| \ge 2. \tag{5.5}$$

Similarly,

$$|E(M_2) \cap L| \geq 2. \tag{5.6}$$

Combining (5.5) and (5.6) with (5.4), we get

$$(r_1 - 2)(r_2 - 2) \leq 4 - \delta_1 - \delta_2. \tag{5.7}$$

If r_1 and r_2 are both at least four, then δ_1 and δ_2 are both positive and (5.7) yields a contradiction. Thus we may assume that $r_1 = 3$. Hence (5.7) becomes $r_2 \leq 6 - \delta_1 - \delta_2$. But $r_2 \geq 3$. Let $\{i, j\} = \{1, 2\}$. If M_i is binary, then $\delta_i \geq r_i \geq 3$, so $\delta_j = 0$. Hence $M_j \cong AG(2,3)$, and so, contracting any element of $X_j - L$ gives a 4-point line minor of $M|X_j$ implying that $|E(M_i) \cap L| = 4$. This contradicts (5.4). Hence M_1 and M_2 are both non-binary.

Now suppose $|(X_1 \cup X_2) \cap L| \geq 3$. Let $M_1' = M|[X_1 \cup (X_2 \cap L)]$. If $\{Y_1, Y_2\}$ is a k-separation of M_1' for some $k \leq 2$, then $r(Y_1) + r(Y_2) - r(X_1) \leq k - 1$, so

$$r(Y_1) + r(Y_2) - r(M) + r(X_2) - 2 \leq k - 1.$$

Without loss of generality, we may assume that $|Y_1 \cap L| \geq 2$. Then

$$\begin{aligned} r(Y_1 \cup X_2) &\leq r(Y_1) + r(X_2) - r(\mathrm{cl}(Y_1) \cap \mathrm{cl}(X_2)) \\ &= r(Y_1) + r(X_2) - 2. \end{aligned}$$

Hence $r(Y_2) + r(Y_1 \cup X_2) - r(M) \leq k - 1$, so $\{Y_2, (Y_1 \cup X_2) - Y_2\}$ is a k-separation of M; a contradiction. Thus M_1' is 3-connected and so, therefore, is M_1. Similarly, M_2 is 3-connected. By Seymour [8] (see also [5, Proposition 11.3.8]), for $\{i, j\} = \{1, 2\}$, the matroid M_i has a 4-point line minor using $(X_1 \cup X_2) \cap L$ and so $|E(M_j) \cap L| = 4$. Thus $\delta_j = 1$ for each j in $\{1, 2\}$ and so (5.2) becomes

$$r_2 \leq 8 - \delta_1 - \delta_2 - 8 + |X_i \cap L| + |X_j \cap L|.$$

Thus $3 \leq r_2 \leq |(X_1 \cup X_2) \cap L| - 2 \leq 2$; a contradiction.

We may now assume that $|(X_1 \cup X_2) \cap L| \leq 2$. Thus, by (5.2),

$$3 \leq r_2 \leq 8 - \delta_1 - \delta_2 - |E(M_1) \cap L| - |E(M_2) \cap L| + 2. \tag{5.8}$$

If $\{i, j\} = \{1, 2\}$ and $M_i \cong AG(2,3)$, then $|E(M_j) \cap L| = 4$ and $|E(M_i) \cap L| = 3$. Thus $\delta_j \geq 1$ and so we have a contradiction. Hence neither M_1 nor M_2 is $AG(2,3)$, so both δ_1 and δ_2 are positive. Therefore, by (5.8), $|E(M_1) \cap L| \leq 3$ and $|E(M_1) \cap L| + |E(M_2) \cap L| \leq 5$. But $r_1 = 3$ and δ_1 is the difference between $|E(AG(2,3))|$ and $|E(M_1)|$. Thus $|E(M_1)| \geq 7$ otherwise $\delta_1 \geq 3$ and (5.8) is contradicted. Hence M_1 has P_7 as a restriction. Thus $|E(M_2) \cap L| \geq 3$ so, by (5.8), $|E(M_1) \cap L| = 2$ and $\delta_1 = 1$. Therefore M_1 is $AG(2,3) - e$ or T_3 and one easily checks, since $|E(M_1) \cap L| = 2$, that $|E(M_2) \cap L| = 4$; a contradiction. □

Lemma 5.2. *If $p \in E(M)$, then p is on at most $r + 1$ long lines. Moreover, for each $k \in \{0, 1\}$, if the point p is on exactly $r + k$ long lines, then*

(i) *all long lines through p have size three; and,*
(ii) *the restriction of $\mathrm{si}(M/p)$ to the points corresponding to these lines is the direct sum of $U_{2,3+k}$ and a set of coloops.*

Proof. Consider the restriction $M|S$ of M to the long lines through p, assuming there are at least r such lines. Each rank-1 flat of $(M|S)/p$ has either two or three elements. Call a point of $\mathrm{si}((M|S)/p)$ *fat* if the associated parallel class of $(M|S)/p$ contains three elements. Evidently the fat points of $\mathrm{si}((M|S)/p)$ correspond to the 4-point lines of M through p.

Now $\mathrm{si}((M|S)/p)$ has rank at most $r-1$ and has at least r points. It therefore has a circuit. Suppose $\mathrm{si}((M|S)/p)$ has a circuit containing a fat point. Then $\mathrm{si}((M|S)/p)$ has a minor in which the fat point is in a 3-circuit. Therefore $M|S$ has a simple rank-3 minor in which there are three long lines through p at least one of which contains four points. This contradicts Corollary 4.2. We conclude that no circuit of $\mathrm{si}((M|S)/p)$ contains a fat point. Hence all fat points of $\mathrm{si}((M|S)/p)$ are coloops of this matroid.

Next we show

5.2.1. *Every circuit of* $\mathrm{si}((M|S)/p)$ *has three elements.*

Proof. To see this, suppose that $\mathrm{si}((M|S)/p)$ has a circuit of size at least four. Then $\mathrm{si}((M|S)/p)$ has a 4-circuit as a minor. Thus $M|S$ has as a minor a simple rank-4 matroid consisting of four 3-point lines through p where no three of these lines are coplanar. This contradiction to Lemma 4.3 verifies (5.2.1). □

By Lemma 4.1, corresponding to every 3-circuit of $\mathrm{si}((M|S)/p)$, there is a P_7-restriction of $M|S$ in which p is the tip. It now follows, by Lemma 4.4, that M has no 4-point lines through p.

We are now ready to prove that M has at most $r+1$ long lines through p. Assume the contrary. Then, since $\mathrm{si}((M|S)/p)$ has no circuits of size exceeding three and has no lines with more than four points, $\mathrm{si}((M|S)/p)$ has the direct sum of two 3-circuits as a restriction. Therefore, in $M|S$, there are two copies of P_7, each with tip p, such that their parallel connection at p is a restriction of $M|S$. This contradiction to Lemma 4.5 completes the proof that M has at most $r+1$ long lines through p. Since none of these lines has four points, (i) holds. Moreover, by (5.2.1), (ii) also holds. □

Lemma 5.3. *If* $p \in E(M)$ *and* p *is on at least two* 4*-point lines, then* M/p *is regular.*

Proof. Let L_1 and L_2 be 4-point lines containing p and assume that M/p is non-regular. Then M/p is non-binary. As M is vertically 4-connected, $\mathrm{si}(M/p)$ is 3-connected. Let x_1 and x_2 be the points of $\mathrm{si}(M/p)$ corresponding to L_1 and L_2, respectively. Then, by Seymour [8] (see also [5, Proposition 11.3.8]), $\mathrm{si}(M/p)$ has a 4-point line minor using $\{x_1, x_2\}$. Thus M has, as a minor, a rank-3 matroid containing three 4-point lines; a contradiction to Lemma 4.1. □

Lemma 5.4. *M has no point p for which* $\mathrm{si}(M/p) \cong AG(2,3)$.

Proof. Assume that M has such a point p. Then $r(M) = 4$; moreover, by inequality (5.1), $|E(M)| \geq 13$.

We now show that

5.4.1. *M does not have* $AG(2,3) - e$ *as a restriction.*

Proof. Assume that M has a subset Z for which $M|Z \cong AG(2,3) - e$. Then M has no 4-point lines, otherwise, by contracting an element that avoids both Z and such a line, we obtain a rank-3 minor of M that contains both $U_{2,4}$ and $AG(2,3) - e$ as restrictions; a contradiction. Take a point x of $E(M) - \mathrm{cl}(Z)$. Since $|E(M)| \geq 13$, but $|\mathrm{cl}(Z)| \leq 9$, there are at least three long lines containing x. If there are three coplanar long lines through x, then the restriction of M to the union of these lines is isomorphic to P_7. Thus, by contracting an element on such a line that is not in $\mathrm{cl}(Z) \cup x$, we obtain a rank-3 minor of M having both $U_{2,4}$ and $AG(2,3) - e$ as restrictions; a contradiction. Hence we may assume that there are three long lines through x that are not coplanar. Let these lines be L_1, L_2, and L_3. Then, for some pair of these lines, say L_1 and L_2, we have $|\mathrm{cl}(L_1 \cup L_2) \cap Z| = 3$. Then $M|\mathrm{cl}(L_1 \cup L_2) \cong M(K_4)$, otherwise by contracting an element from $L_1 - (x \cup Z)$, we get a minor of M with $U_{2,4}$ and $AG(2,3) - e$ as restrictions; a contradiction. If we now contract an element of M that is in neither $\mathrm{cl}(Z)$ nor $\mathrm{cl}(L_1 \cup L_2)$, we get a minor of M with both $M(K_4)$ and $AG(2,3) - e$ as restrictions. This minor is not a restriction of $AG(2,3)$. This contradiction completes the proof of (5.4.1). □

Now every 4-point line of M must pass through p and, by Lemma 5.3, there is at most one such line. Next we prove the following:

5.4.2. *There is a 3-point line $\{p, x_1, x_2\}$ such that $|E(\mathrm{si}(M/x_1))| \leq 8$ and $|E(\mathrm{si}(M/x_2))| \leq 8$.*

Proof. If there is a 4-point line through p, then, since $|E(M)| \geq 13$, there is also a 3-point line $\{p, x_1, x_2\}$ through p. Since $\mathrm{si}(M/x_i)$ has a 4-point line for each i in $\{1, 2\}$, we deduce that (5.4.2) holds. Thus we may assume that there is no 4-point line through p. Then there are at least three long lines through p. If M has three coplanar long lines through p, then M has a P_7-restriction with tip p. Let $\{p, x_1, x_2\}$ be one of the lines of this restriction. Then $\mathrm{si}(M/x_1)$ and $\mathrm{si}(M/x_2)$ both have 4-point lines. Hence, by Lemma 4.1, each has at most 8 elements, and (5.4.2) holds. Thus M does not have three coplanar long lines through p.

We may now assume that there are three long lines L_1, L_2, and L_3 through p such that the rank of their union is four. Consider $M|\mathrm{cl}(L_1 \cup L_2)$. This matroid has either $M(K_4)$ or $\mathcal{W}^3$ as a restriction. In the first case, let $\{p, x_1, x_2\}$ be L_3. Then $\mathrm{si}(M/x_i)$ has an $M(K_4)$-minor and so has at most eight elements, and (5.4.2) holds. Thus we may assume that $M|\mathrm{cl}(L_j \cup L_k)$ has no $M(K_4)$-minor for all $j \neq k$. In that case, there is an element y_1 of L_2 such that $\mathrm{si}(M/y_1)$ has a 4-point line. Let $L_2 = \{p, y_1, y_2\}$. If $\mathrm{si}(M/y_2)$ has a 4-point line, then we can take $L_2 = \{p, x_1, x_2\}$. Thus we may assume that $\mathrm{si}(M/y_2)$ has no 4-point line. Now M/p has no 4-point lines and $|E(\mathrm{si}(M/y_1)| \leq 8$. Thus M has at least four long lines through y_1. By Lemma 4.3, some three of these must be coplanar. Hence, if the union of these three lines is V, then $M|V$ is isomorphic to P_7 and has y_1 as tip. Moreover, as $\mathrm{si}(M/p)$ has no 4-point line, p is not an element of this P_7-restriction. In $\mathrm{si}(M/p)$, there are exactly four long lines through y_1. Two of these are the images of $\mathrm{cl}_M(L_1 \cup L_2)$ and $\mathrm{cl}_M(L_2 \cup L_3)$. In $M|V$, the element y_1 is on three long lines. These three long lines remain distinct lines through y_1 in M/p. Thus, in M, one of these lines must be in $\mathrm{cl}_M(L_1 \cup L_2)$ or $\mathrm{cl}_M(L_2 \cup L_3)$, otherwise there are five long lines through y_1 in $\mathrm{si}(M/p)$. Since $\mathrm{cl}_M(L_1 \cup L_2)$ has no $M(K_4)$-restriction, it follows that M/y_2 has a 4-point line. This contradiction completes the proof of (5.4.2). □

Now use the line $\{p, x_1, x_2\}$ whose existence has just been established. For each $i \in \{1, 2\}$, as $|E(\mathrm{si}(M/x_i))| \leq 8$, it follows, as above, that M has a P_7-restriction $M|X_i$ with tip x_i. By Lemma 4.5 and the fact that M has no $(AG(2,3) - e)$-restriction, it follows that neither $M|X_1$ nor $M|X_2$ contains p and $r(X_1 \cup X_2) = 4$. In M/p, each of the lines of $M|X_1$ through x_1 becomes a line through the rank-one flat containing $\{x_1, x_2\}$. But $\mathrm{si}(M/p) \cong AG(2,3)$ and so has exactly four such lines. Thus M has long lines L_1 and L_2 contained in X_1 and X_2 respectively, such that $r(p \cup L_1 \cup L_2) = 3$ and $x_i \in L_i$. Now, if $L_1 \cap L_2 = \emptyset$, then $M|(p \cup L_1 \cup L_2) \cong P_7$. But this P_7-restriction contains x_1, a contradiction to Lemma 4.5. Thus $L_1 \cap L_2$ contains one common element and $M|(p \cup L_1 \cup L_2) \cong M(K_4)$, otherwise $\mathrm{si}(M/p)$ has a 4-point line.

Take an element f of $X_1 - (p \cup L_1 \cup L_2)$. Then $\mathrm{si}(M/f)$ has $M(K_4)$ as a minor, so $|E(\mathrm{si}(M/f)| \leq 8$. Thus M has four long lines through f and so has a P_7-restriction with tip f; a contradiction to Lemma 4.5. □

The last few lemmas enable us to verify the bound on $|E(M)|$ and to begin the determination of the matroids that attain this bound. Recall that we have assumed that $r \geq 4$.

Lemma 5.5. $|E(M)| = \binom{r+2}{2} - 2$. *Moreover, every point p of M obeys one of the following:*

(i) *p is on exactly r long lines each of which has exactly three points, and p is the tip of a unique P_7-restriction of M;*
(ii) *p is on exactly* $r - 1$ *long lines, exactly one of which has four points;*
(iii) *p is on exactly* $r - 1$ *long lines, each of which has exactly four points, and* $\mathrm{si}(M/p) \cong M(K_r)$.

Proof. Suppose that M has a point p_1 that is on $r + 1$ long lines. Then, by Lemma 5.2, M has an $AG(2,3)$-restriction containing p_1 and has a long line L_1 that meets this restriction in p_1. Let $L_1 = \{p_1, p_2, p_3\}$. Then, by Lemma 5.1, $\mathrm{si}(M/p_3)$ is 3-connected. Thus, by Lemma 4.4, M has no 4-point lines through p_2. Hence

$$|E(M)| \leq 1 + |E(\mathrm{si}(M/p_2))| + n_2$$

where n_2 is the number of long lines through p_2. Thus

$$n_2 \geq |E(M)| - |E(\mathrm{si}(M/p_2))| - 1. \quad (5.9)$$

Now $|E(M)| \geq \binom{r+2}{2} - 2$. By the induction assumption and Lemma 5.4, it follows that $|E(\mathrm{si}(M/p_2))| \leq \binom{r+1}{2} - 2$. Hence $n_2 \geq r$. It follows, by Lemma 5.2, that M has a P_7-restriction with tip p_2. By Lemma 4.5, this P_7-restriction has no element in common with the $AG(2,3)$-restriction. Moreover, the union of these two restrictions has rank exceeding four otherwise we get a contradiction to Lemma 4.4 or Lemma 4.1. Thus $\mathrm{si}(M/p_3)$ has two P_7-restrictions with tip p_1 such that their union has rank at least four. This contradicts Lemma 4.5. Thus M has no point that is on $r+1$ long lines.

Next suppose that M has a point p that is on r long lines. Then, by Lemma 5.2(i), each of these lines has exactly three points. Thus

$$|E(M)| = 1 + r + |E(\mathrm{si}(M/p))|.$$

By the induction assumption and Lemma 5.4, $|E(\mathrm{si}(M/p))| \le \binom{r+1}{2} - 2$. Hence $|E(M)| \le \binom{r+2}{2} - 2$ and so, by (5.1), equality holds here.

Now take a point p of M that is on at most $r-1$ long lines. Then either

(a) p is on at least two 4-point lines; or
(b) p is on at most one 4-point line.

In case (a), each of the at most $r-1$ long lines through p has at most four points. Thus

(5.10) $$|E(M)| \le 1 + 2(r-1) + |E(\mathrm{si}(M/p))|.$$

Moreover, by Lemma 5.3, $\mathrm{si}(M/p)$ is regular, so, by [2],

(5.11) $$|E(\mathrm{si}(M/p))| \le \binom{r}{2}.$$

Hence $|E(M)| \le 1 + 2(r-1) + \binom{r}{2}$, so

(5.12) $$|E(M)| \le \binom{r+2}{2} - 2.$$

Thus equality holds in (5.12) and hence also holds in (5.11) and (5.10). Therefore, in case (a), by [2], $\mathrm{si}(M/p) \cong M(K_r)$ and p is on exactly $r-1$ long lines each having exactly four points.

In case (b), p is on at most one 4-point line and on at most $r-2$ three-point lines, so

(5.13) $$|E(M)| \le 1 + 2 + (r-2) + |E(\mathrm{si}(M/p))|.$$

By the induction assumption and Lemma 5.4, $|E(\mathrm{si}(M/p))| \le \binom{r+1}{2} - 2$. Thus

$$|E(M)| \le 1 + 2 + (r-2) + \binom{r+1}{2} - 2,$$

so

(5.14) $$|E(M)| \le \binom{r+2}{2} - 2.$$

Therefore equality holds in (5.14) and so also holds in (5.13). Hence p is on exactly one 4-point line and exactly $r-2$ three-point lines, and the lemma is verified. □

We shall classify each point p of M as being of type (i), (ii), or (iii) depending on which of (i)–(iii) of Lemma 5.5 the point satisfies.

Lemma 5.6. *M has a point of type* (i) *or* (iii).

Proof. Suppose every point of M is of type (ii). Then every point of M is on a unique 4-point line. Thus, letting l_4 be the number of 4-point lines of M, we have

$$4l_4 = |E(M)| = \binom{r+2}{2} - 2.$$

Hence

$$l_4 = \tfrac{1}{8}\,(r^2 + 3r - 2).$$

Now take a point p of M. Then $|E(\mathrm{si}(M/p))| = \binom{r+1}{2} - 2$, so, by Lemma 5.4, $\mathrm{si}(M/p)$ is a maximum-sized $\sqrt[6]{1}$-matroid of rank $r-1$. Hence $\mathrm{si}(M/p) \cong T_{r-1}$, so $\mathrm{si}(M/p)$ has exactly $r-2$ four-point lines. But $\mathrm{si}(M/p)$ has at least $l_4 - 1$ four-point lines. Thus

$$r - 2 \geq \tfrac{1}{8}\,(r^2 + 3r - 2) - 1.$$

Hence $0 \geq (r-2)(r-3)$. But $r \geq 4$, so we have a contradiction. □

Lemma 5.7. *M has a point of type* (iii).

Proof. Assume that every point of M is of type (i) or (ii). Then, by the last lemma, M has a point p_1 of type (i). Consider the P_7-restriction N_1 of M having p_1 as tip.

Suppose that some point of N_1 other than p_1 has type (i). Then M has a P_7-restriction N_2 having p_2 as tip. By Lemma 4.5, $r(E(N_1) \cup E(N_2)) \leq 3$. Thus $M|[E(N_1) \cup E(N_2)]$ has $AG(2,3) - e$ as a restriction. Let $\{p_1, q_2, q_3\}$ be a line of M not in $\mathrm{cl}_M(E(N_1))$. If q_2 is of type (ii), then $\mathrm{si}(M/q_3)$ is 3-connected and has as a restriction either three coplanar copunctual long lines including one with four points, or the parallel connection of P_7 and $U_{2,4}$ with basepoint equal to the tip of the P_7. Since these possibilities contradict Lemma 4.1 and Corollary 4.2, we deduce that q_2 has type (i). Let N_3 be a P_7-restriction of M with tip q_2. Then, since p_1 is the tip of a unique P_7-restriction of $\mathrm{si}(M/q_3)$, it follows that $r(E(N_1) \cup E(N_2)) = 4$. Thus, if $x \in E(N_3) - [\mathrm{cl}(E(N_1)) \cup \{q_2\}]$, then, in $\mathrm{si}(M/x)$, the closure of $E(N_1)$ contains copies of both $AG(2,3) - e$ and $U_{2,4}$; a contradiction. We deduce that every point of $E(N_1) - p_1$ has type (ii). Hence every such point is incident with a unique 4-point line of M and, since M has no type (iii) points, no such line is contained in $\mathrm{cl}_M(E(N_1))$.

Now $|E(\mathrm{si}(M/p_1))| = \binom{r+1}{2} - 2$. Hence $\mathrm{si}(M/p_1)$ is a maximum-sized $\sqrt[6]{1}$-matroid of rank $r-1$ and so, by the induction assumption, has all of its 4-point lines copunctual. Let $\{p_1, x_2, x_3\}$ be a line of N_1. Both x_2 and x_3 have type (ii) and so meet unique 4-point lines. Thus, in $\mathrm{si}(M/p_1)$, since M has no type (iii) points, the point x_2 meets at least two 4-point lines. Thus every 4-point line of $\mathrm{si}(M/p_1)$ passes through x_2. But, if $x_4 \in E(N_1) - \{p_1, x_1, x_2\}$, the unique 4-point line through x_4 misses x_2 in $\mathrm{si}(M/p_1)$; a contradiction. □

Corollary 5.8. *M has a unique point p_0 of type* (iii).

Proof. By Lemma 5.7, M has a point p_0 of type (iii). By Lemma 5.3, M/p_0 is regular. Thus every 4-point line of M meets p_0 and it follows that p_0 is the only point of type (iii). □

Lemma 5.9. *Every element of M is in a plane spanned by two* 4*-point lines through* p_0.

Proof. Suppose M has an element e that is in none of the planes spanned by two 4-point lines through p_0. Then $\mathrm{si}(M/e)$ is 3-connected, has $r-1$ four-point lines through p_0, and has rank $r-1$. But e is of type (i) and $\mathrm{si}(M/e)$ is a maximum-sized $\sqrt[6]{1}$-matroid of rank $r-1$. Hence, by the induction assumption, $\mathrm{si}(M/e)$ has only $r-2$ four-point lines; a contradiction. □

Lemma 5.10. $M \cong T_r$.

Proof. Every plane of M that contains two 4-point lines through p_0 must be isomorphic to a restriction of T_3. Hence every such plane contains at most one point other than those on the 4-point lines. Thus, as there are $\binom{r-1}{2}$ such planes,

$$|E(M)| \leq 1 + 3(r-1) + \binom{r-1}{2}. \tag{5.15}$$

But $|E(M)| = \binom{r+2}{2} - 2$, which equals the right-hand side of (5.15). Therefore every plane that contains two 4-point lines through p_0 contains exactly one additional point and is therefore isomorphic to T_3.

Label the 4-point lines through p_0 by $L_1, L_2, \ldots, L_{r-1}$ and, for each $i < j$, let w_{ij} be the unique point of M in $\mathrm{cl}(L_i \cup L_j) - (L_i \cup L_j)$. Label the points of $L_1 - \{p_0\}$ arbitrarily by x_1,y_1, and z_1. Then, for each i in$\{2, 3, \ldots, r-1\}$, let x_i, y_i, and z_i be the points of intersection of L_i with $\mathrm{cl}(\{x_1, w_{1i}\})$, $\mathrm{cl}(\{y_1, w_{1i}\})$ and $\mathrm{cl}(\{z_1, w_{1i}\})$, respectively. Evidently $\{p_0, x_1, x_2, \ldots, x_{r-1}\}$ is a basis B for M. Since M is a $\sqrt[6]{1}$-matroid, there is a $\sqrt[6]{1}$-matrix X over $\mathbf{C}$ representing M where, for $t = r-1$, the matrix X has the following form:

$$\begin{array}{c|c|c|c|ccccccc}
p_0 & x_1 \ldots x_t & y_1 \ldots y_t & z_1 \ldots z_t & w_{12} & w_{13} & \ldots & w_{1t} & w_{23} & \ldots & w_{(t-1)t} \\
1 & 0 \ldots 0 & a_1 \ldots a_t & b_1 \ldots b_t & 0 & 0 & \ldots & 0 & d_{23} & \ldots & d_{(t-1)t} \\
\hline
0 & & & & 1 & 1 & \ldots & 1 & 0 & \ldots & 0 \\
0 & & & & c_2 & & & 1 & & & \\
\vdots & & & & & c_3 & & e_{23} & & & \\
 & I_t & I_t & I_t & & & & & & & \\
\vdots & & & & & & \ddots & & & & \\
0 & & & & & & & & & & 1 \\
0 & & & & & & & c_t & & & e_{(t-1)t}
\end{array}.$$

Here, all entries of the form a_i, b_j, c_k, or e_{pq} are non-zero, but the d_{vw} may be zero. By scaling row 1 and then column 1, we may assume that $a_1 = 1$. Let $b_1 = \beta$. By scaling rows $3, 4, \ldots, r$ and then columns

$$x_2, x_3, \ldots, x_{r-1}, y_2, y_3, \ldots, y_{r-1}, z_2, z_3, \ldots, z_{r-1},$$

we may assume that $c_2 = c_3 = \cdots = c_{r-1} = -1$. As $\{y_1, w_{1i}, y_i\}$ and $\{z_1, w_{1i}, z_i\}$ are lines of M for all i in $\{2, 3, \ldots, r-1\}$, we deduce that, for all such i, we have $a_i = a_1 = 1$ and $b_i = b_1 = \beta$.

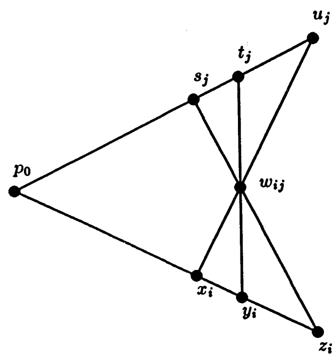

Fig. 3. T_3 with labels

Next, using the fact that $\mathrm{si}(M/p_0) \cong M(K_r)$, we have that, for all i and j in $\{2, 3, \ldots, r-1\}$ with $i < j$, the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ c_i & 0 & 1 \\ 0 & c_j & e_{ij} \end{bmatrix}$$

has zero determinant. Since $c_i = c_j = -1$, it follows that $e_{ij} = -1$.

Finally, we show that $d_{ij} = 0$ for all i and j in $\{2, 3, \ldots, r-1\}$ with $i < j$. Consider $M|\mathrm{cl}(L_i \cup L_j)$. We know that this matroid is isomorphic to T_3. Assume that it is labelled as in Fig. 3. If $u_j = x_j$, or $t_j = y_j$, or $s_j = z_j$, then we deduce immediately that $x_j = 0$. Thus we may assume that $u_j \neq x_j$, that $t_j \neq y_j$, and that $s_j \neq z_j$. Then clearly $u_j = y_j$, or $u_j = z_j$. In the first case, it follows that $t_j = z_j$ and $s_j = x_j$. In the second case, $t_j = x_j$ and $s_j = y_j$. In the first case, the lines $\{x_i, y_j, w_{ij}\}$ and $\{y_i, z_j, w_{ij}\}$ imply that the matrices

$$\begin{bmatrix} 0 & 1 & d_{ij} \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & \beta & d_{ij} \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$$

both have zero determinant. Hence $d_{ij} = -1$ and $d_{ij} = 1 - \beta$, so $-1 = 1 - \beta$, that is, $\beta = 2$; a contradiction. By symmetry, a similar contradiction arises in the second case. We conclude that $d_{ij} = 0$ for all i and j. Thus $M \cong T_r$. This establishes Lemma 5.10 and hence Theorem 2.1. □

Finally, we note that, since T_r is near-regular but neither $AG(2,3) - e$ nor $AG(2,3)$ is, Corollary 2.2 follows immediately from the theorem.

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