Growth rates of minor-closed classes of matroids✩

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For a minor-closed class \( \mathcal{M} \) of matroids, let \( h(k) \) denote the maximum number of elements in a simple rank-\( k \) matroid in \( \mathcal{M} \). We prove that, if \( \mathcal{M} \) does not contain all simple rank-2 matroids, then \( h(k) \) is finite and is either linear, quadratic, or exponential. © 2008 Elsevier Inc. All rights reserved.

1. Introduction

In this paper we consider classes of matroids that are closed both under taking minors and under isomorphism; for convenience we shall simply refer to such classes as minor-closed. Our main result, combined with earlier results of Geelen and Whittle and of Geelen and Kabell, yields the following theorem, conjectured by Kung [4, Conjecture 4.9].

Theorem 1.1 (Growth rate theorem). If \( \mathcal{M} \) is a minor-closed class of matroids, then either

1. there exists \( c \in \mathbb{R} \) such that \( |E(M)| \leq cr(M) \) for all simple matroids \( M \in \mathcal{M} \),
2. \( \mathcal{M} \) contains all graphic matroids and there exists \( c \in \mathbb{R} \) such that \( |E(M)| \leq c(r(M))^2 \) for all simple matroids \( M \in \mathcal{M} \),
3. there is a prime-power \( q \) and \( c \in \mathbb{R} \) such that \( \mathcal{M} \) contains all GF\((q)\)-representable matroids and \( |E(M)| \leq cq^{r(M)} \) for all simple matroids \( M \in \mathcal{M} \), or
4. \( \mathcal{M} \) contains all simple rank-2 matroids.

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We follow the notation of Oxley [5]. A rank-1 flat is referred to as a point and a rank-2 flat is referred to as a line. The number of points in M is denoted by $\epsilon(M)$. For a class $\mathcal{M}$ of matroids and integer $k \geq 0$, we let $h(\mathcal{M}, k)$ be the maximum of $\epsilon(M)$ among all rank-$k$ matroids $M \in \mathcal{M}$. Thus, if $G$ is the set of graphic matroids, then $h(G, k) = \binom{k+1}{2}$ and, for a prime-power $q$, if $L(q)$ is the set of all GF$(q)$-representable matroids, then $h(L(q), k) = \frac{q^{k+1} - 1}{q-1}$.

We begin by recounting two significant partial results towards the growth rate theorem. The first was proved by Geelen and Whittle [2].

**Theorem 1.2.** If $\mathcal{M}$ is a minor-closed class of matroids, then either

1. there exists $c \in \mathbb{R}$ such that, $h(\mathcal{M}, k) \leq ck$ for all $k$,
2. $\mathcal{M}$ contains all graphic matroids, or
3. $\mathcal{M}$ contains all simple rank-2 matroids.

The second result was proved by Geelen and Kabell [1] and in part, by Kung [4, Theorem 6.6].

**Theorem 1.3.** If $\mathcal{M}$ is a minor-closed class of matroids, then either

1. there exists a polynomial $p(k)$ such that, $h(\mathcal{M}, k) \leq p(k)$ for all $k$,
2. there is a prime-power $q$ and $c \in \mathbb{R}$ such that $\mathcal{M}$ contains all GF$(q)$-representable matroids and $h(\mathcal{M}, k) \leq cq^k$ for all $k$, or
3. $\mathcal{M}$ contains all simple rank-2 matroids.

In this paper, we bridge the gap by proving the following theorem.

**Theorem 1.4.** If $\mathcal{M}$ is a minor-closed class of matroids, then either

1. there exists $c \in \mathbb{R}$ such that, $h(\mathcal{M}, k) \leq ck^2$ for all $k$,
2. $h(\mathcal{M}, k) \geq 2^k - 1$ for all $k$, or
3. $\mathcal{M}$ contains all simple rank-2 matroids.

We conclude the introduction with two interesting corollaries of the growth rate theorem. The second of these was already known; see Kung [3].

**Corollary 1.5.** Let $q$ be a power of a prime $p$ and let $\mathcal{M}$ be a minor-closed class of GF$(q)$-representable matroids. If $\mathcal{M}$ does not contain all GF$(p)$-representable matroids, then there exists a constant $c \in \mathbb{R}$ such that $h(\mathcal{M}, k) \leq ck^2$ for all $k$.

**Corollary 1.6.** Let $\mathcal{M}$ be a minor-closed class of GF-representable matroids. If $\mathcal{M}$ does not contain all simple rank-2 matroids, then there exists a constant $c \in \mathbb{R}$ such that $h(\mathcal{M}, k) \leq ck^2$ for all $k$.

2. Excluding a line

Kung [4] proved the following theorem.

**Theorem 2.1.** For any integer $l \geq 2$, if $M$ is a matroid with no $U_{2,l+2}$-minor, then $\epsilon(M) \leq \frac{p^{(l)}}{l-1}$.  

Let $\mathcal{U}(l)$ denote the set of all matroids with no $U_{2,l+2}$-minor. Thus $h(\mathcal{U}(l), k) \leq \frac{p^{(l)}}{l-1}$. Note that, when $l$ is a prime-power, this bound is tight since $\mathcal{L}(l) \subseteq \mathcal{U}(l)$. However, when $l$ is not a prime-power, the growth rate theorem gives an asymptotically tighter bound of $cq^k$, where $q$ is the largest prime-power less than or equal to $l$.

We remark that Kung [4] has made a stronger conjecture.
Conjecture 2.2. If \( l \geq 2 \) is an integer and \( q \) is the largest prime-power less than or equal to \( l \), then \( h(U(l), k) = \frac{q^{l-1}}{q-1} \) for all sufficiently large \( k \).

Conjecture 2.2 is the case of Conjecture 4.9(a) in [4] when the set of excluded minors is empty. The general form of Conjecture 4.9(a) can be restated as follows. Let \( \mathcal{M} \) be a minor-closed class not containing all rank-2 simple matroids. If \( \mathcal{L}(q) \subseteq \mathcal{M} \) for some prime power \( q \) and \( q \) is maximum with this property, then \( h(\mathcal{M}, k) = \frac{q^{l-1}}{q-1} \) for sufficiently large \( k \). This conjecture is too good to be true. We construct a counterexample \( \mathcal{M} \) (using \( q \)-lifts or \( q \)-cones). Let \( q \) be a prime-power, let \( n \geq 2 \) be an integer, and let \( \mathcal{F} \) be the set of all pairs \((M, e)\) consisting of a \( GF(q^k) \)-representable matroid \( M \) and an element \( e \in E(M) \) such that \( M/e \) is \( GF(q) \)-representable. Now let \( \mathcal{M} \) be the set of all matroids \( M \setminus e \) where \((M, e) \in \mathcal{F}\). It is straightforward to verify that every extremal rank-\( k \) matroid \( M' \in \mathcal{M} \) contains a hyperplane \( H \) and an element \( e' \notin H \) such that \( M'/H \cong PG(k-2, q) \) and, for each \( f \in H \), the pair \((e', f)\) spans a \((q^k + 1)\)-point line in \( M' \). By adding an element \( e \) in parallel with \( e' \), we obtain associated pair \((M, e) \in \mathcal{F}\). Therefore,

\[
h(\mathcal{M}, k) = q^n q^{k-1} - 1 + 1.
\]

Our proof of the growth rate theorem requires a bound on the number of hyperplanes in a rank-\( k \) matroid in \( U(l) \). If \( M \) is \( GF(q) \)-representable, then, by considering \( PG(r-1, q) \), we see that \( M \) has at most \( q^{r-1} \) hyperplanes. On the other hand, when \( M \in U(l) \), we cannot prove a comparable bound, so we settle for the following crude upper bound from [2]; we include the short proof for completeness.

Lemma 2.3. Let \( k \geq 1 \) and \( l \geq 2 \) be integers and let \( M \in U(l) \) be a simple rank-\( k \) matroid. Then, \( M \) has at most \( \frac{l^k}{k-1} \) hyperplanes.

Proof. Let \( n = |E(M)| \); thus \( n \leq \frac{l^k-1}{k-1} \leq l^k \). Each hyperplane is spanned by a set of \( k-1 \) points, so the number of hyperplanes is at most \( \binom{n}{k-1} \leq n^{k-1} \leq \frac{l^k}{k-1} \). \( \Box \)

3. Local connectivity

Let \( M \) be a matroid and let \( A, B \subseteq E(M) \). We define \( \cap_M(A, B) = r_M(A) + r_M(B) - r_M(A \cup B) \); this is the local connectivity between \( A \) and \( B \). This definition is motivated by geometry. Suppose that \( M \) is a restriction of \( PG(k, q) \) and let \( F_A \) and \( F_B \) be the flats of \( PG(k, q) \) that are spanned by \( A \) and \( B \), respectively. Then \( F_A \cap F_B \) has rank \( \cap_M(A, B) \).

The following properties are intuitively obvious for representable matroids, and follow by elementary rank calculations for arbitrary matroids.

1. If \( A, B \subseteq E(M) \) and \( A' \subseteq A \), then \( \cap_M(A', B) \leq \cap_M(A, B) \).
2. If \( A \) and \( C \) are disjoint subsets of \( E(M) \), then \( r_M(C) = r_M(A) - \cap_M(A, C) \).
3. If \( A, B, \) and \( C \) are disjoint subsets of \( E(M) \), then \( \cap_M(C, A, B) = \cap_M(A, B) - \cap_M(A, C) \).

We say that two sets \( A, B \subseteq E(M) \) are skew if \( \cap_M(A, B) = 0 \). More generally, the sets \( A_1, \ldots, A_k \subseteq E(M) \) are skew if \( r_M(A_1) + \cdots + r_M(A_k) = r_M(A_1 \cup \cdots \cup A_k) \).

4. Books and dense minors

A line is long if it has at least 3 points. For sets \( A \) and \( B \) we let \( A \times B \) denote \{\((a, b)\): \( a \in A, b \in B\)\}. We use the following lemma to identify a dense minor.

Lemma 4.1. Let \( k \geq 1 \) be an integer and let \( n = k2^k \). Let \( F_1 \) and \( F_2 \) be skew flats in a matroid \( M \) such that \( M|F_1 \) is isomorphic to \( M(K_q) \), \( r(F_2) = k \), and each pair of points in \( F_1 \times F_2 \) spans a long line. Then \( M \) has a rank-\( k \) minor \( N \) with \( e(N) \geq 2^k - 1 \).
Proof. We may assume that $M$ is simple and that $r(M) = r_M(F_1 \cup F_2)$. We may also assume that $F_2$ is a $k$-element independent set in $M$ and that $M|F_1 = M(G)$, where $G$ is isomorphic to $K_\lambda$. Let $C$ denote the set of all subsets of $F_2$ with at least two elements. Since $n \geq |C|$, there exists a collection $(P_X: X \in C)$ of vertex-disjoint paths in $G$ each path $P_X$ has length $|X|$. For each $X \in C$, let $e_X$ be the edge of $G$ that connects the ends of $P_X$, and let $\phi_X: X \rightarrow E(P_X)$ be an arbitrary bijection. For each $x \in X$, let $f_x \in E(M) - (F_1 \cup F_2)$ be an element spanned by $\{x, \phi_X(x)\}$, and let $S_X = \{f_x: x \in X\}$. Finally, let $S$ denote the union of the sets $(S_X: X \in C)$ and let $N$ be the restriction of $M/S$ to the flat spanned by $F_2$. Note that the sets $F_2$ and $(P_X: X \in C)$ are skew and, for each $X \in C$, the set $S_X$ is contained in the flat of $M$ that is spanned by $F_2 \cup P_X$. Moreover, $F_2$ is independent in $N$ and, for each $X \in C$ and each $x \in X$, the elements $x$ and $\phi_X(x)$ are in parallel in $N$. Therefore, for each $X \in C$, the set $X \cup \{e_X\}$ is a circuit of $N$. Hence $\epsilon(N) \geq |F_2| + |C| = 2^k - 1$, as required. \(\square\)

We call a matroid $M$ round if each cocircuit of $M$ is spanning. Equivalently, $M$ is round if and only if $E(M)$ cannot be written as the union of two proper flats. The following properties are straightforward to check:

1. If $M$ is a round matroid and $e \in E(M)$ then $M/e$ is round.
2. If $N$ is a spanning minor of $M$ and $N$ is round, then $M$ is round.

Let $M$ be a matroid. A flat $F$ of $M$ is called round if the restriction of $M$ to $F$ is round. Each rank-one flat is round. Moreover, a line is round if and only if it is long. A sequence $(F_0, F_1, \ldots, F_t)$ is called a $k$-book if $F_0$ is a rank-$k$ flat of $M$ and $F_1, \ldots, F_t$ are distinct round rank-$(k+1)$ flats of $M$ each containing $F_0$.

The following lemma is the main result of the section.

Lemma 4.2. There exists a function $f_1: \mathbb{Z}^2 \rightarrow \mathbb{Z}$ such that, for integers $l, k \geq 2$, if $(F_0, F_1, \ldots, F_t)$ is a $(k+1)$-book in a matroid $M \in \U(l)$ and $t \geq f_1(l, k)r(M)$, then $M$ has a rank-$k$ minor $N$ with $\epsilon(N) = 2^k - 1$.

Proof. By Ramsey’s Theorem, there exists a function $R: \mathbb{Z}^2 \rightarrow \mathbb{Z}$ such that, for integers $n, c \geq 1$, we colour the edges of a clique on $(n, c)$ vertices with $c$ colours, then there is a monochromatic clique on $n$ vertices. By Theorem 1.2, there exists a function $\lambda: \mathbb{Z}^2 \rightarrow \mathbb{Z}$ such that, for integers $l, n \geq 2$, if $M \in \U(l)$ is a matroid with $\epsilon(M) > \lambda(l, n)r(M)$, then $M$ has an $M(K_{n,1})$-minor.

Let $l, k \geq 2$ be integers. Now let $n_1 = k2^k$, $n_2 = n_3 + 3$, $n_2 = R(n_3, 2^k + 1) + 1$, and $n_1 = l2^k + R(n_2, \lfloor l2^k \rfloor)$. Finally we let $f_1(l, k) = \lambda(l, n_1)$.

Now consider a matroid $M \in \U(l)$ containing a $(k+1)$-book $(F_0, F_1, \ldots, F_t)$ with $t \geq f_1(l, k)r(M)$. By way of contradiction, we assume that, for each rank-$k$ minor $N$ of $M$, we have $\epsilon(N) < 2^k - 1$. If follows easily that, for each rank-$(k+1)$ minor $N$ of $M$, we have $\epsilon(N) < l(2^k - 1) + 1 \leq l2^k$.

4.2.1. There is a minor $M_1$ of $M$ and a set $X_1 \subseteq E(M_1)$ such that

1. $F_0 \subseteq E(M_1)$ and $r_{M_1}(F_0) = k + 1$,
2. $(M_1/F_0)|X_1 \cong M(K_{n_1})$, and
3. for each $e \in X_1$, the flat of $M_1$ that is spanned by $F_0 \cup \{e\}$ is round.

Proof of 4.2.1. For each $i \in \{1, \ldots, t\}$, choose $x_i \in F_i - F_0$. Now let $X = \{x_1, \ldots, x_t\}$ and let $N = (M/F_0)|X$. Note that $\epsilon(N) \geq \lambda(l, n_1)r(N)$. Therefore there is a minor, say $N \setminus D/C$, of $N$ that is isomorphic to $M(K_{n_1})$. The claim follows by taking $M_1 := M/C$ and $X_1 := E(N \setminus D/C)$. \(\square\)

4.2.2. There is a minor $M_2$ of $M_1$, a set $X_2 \subseteq E(M_2)$, and a $(k+1)$-element independent set $Y_2$ of $M_2$ such that

1. $(M_2/Y_2)|X_2 \cong M(K_{n_2})$, and
2. each pair of elements in $X_2 \times Y_2$ spans a long line in $M_2$. 


Proof of 4.2.2. Let \( n' = R(n_2, (\frac{I^k}{k+1})) \), thus \( n_1 = I^k + n' \). Note that \( F_0 \) has rank \((k + 1)\) and, hence, it spans at most \( I^k \) points. We begin by repeatedly contracting elements from \( X_1 \) if doing so increases the number of points spanned by \( F_0 \); the number of points that we contract will be at most \( I^k \). Therefore, there is a minor \( M_2 \) of \( M_1 \) and a set \( X' \subseteq X_1 \) such that:

1. \( F_0 \subseteq E(M_2) \) and \( r_{M_2}(F_0) = k + 1 \),
2. \( (M_2/F_0))X' \cong M(K_{n'}) \),
3. for each \( e \in X' \), the flat of \( M_2 \) that is spanned by \( F_0 \cup \{ e \} \) is round, and
4. for each element \( a \in X' \) and each element \( b \in \text{cl}_{M_2}(F_0 \cup \{ a \}) - \text{cl}_{M_2}(F_0) \) that is not in parallel with \( a \), there is an element \( c \in \text{cl}_{M_2}(F_0) \) such that \( \{ a, b, c \} \) is a circuit of \( M_2 \).

Let \( F' = \text{cl}_{M_2}(F_0) \). We may assume, for notational convenience, that \( M_2 \) is simple. Thus \( |F'| \leq I^k \). For each element \( a \in X' \), let \( B_a \) be a basis of the flat spanned by \( \{ a \} \cup F' \) with \( \{ a \} \subseteq B_a \) and with \( B_a \cap F' = \emptyset \) (such a basis exists since the flat is round). By the last property of \( M_2 \) above, there is a basis \( B'_a \) of \( F' \) such that, for each \( b \in B_a - \{ a \} \), there is an element \( c \in F' \) such that \( \{ a, b, c \} \) is a circuit.

Note that \( B'_a \) is a \((k + 1)\)-element subset of \( F' \) and the number of such subsets is at most \( \binom{I^k}{k+1} \).

Therefore, by Ramsey’s Theorem, there is a basis \( Y_2 \) of \( F' \) and a set \( X_2 \subseteq X' \) such that \( (M_2/F_0))X_2 \cong M(K_{n'}) \) and, for each \( e \in X_2 \), we have \( B'_e = Y_2 \). Thus \( M_2, X_2, \) and \( Y_2 \) satisfy the claim. \( \Box \)

4.2.3. There is a set \( X'_2 \subseteq X_2 \) such that

1. \( (M_2/Y_2)|X'_2 \cong M(K_{n'_2}) \), and
2. \( r_{M_2}(X'_2, Y_2) \leq 1 \).

Proof of 4.2.3. Recall that \( (M_2/Y_2)|X_2 = M(G) \) where \( G \) is a graph that is isomorphic to \( K_{n_2} \). Let \( v \in V(G) \) and let \( C \) be the set of edges of \( G \) that are incident with \( v \). Note that \( Y_2 \cup C \) spans \( X_2 \) in \( M_2 \). Define a partition \((S_0, S_1, \ldots, S_m)\) of \( X_2 \) such that \( S_0 = \text{cl}_{M_2}(C) \cap X_2 \) and \( (S_1, \ldots, S_m) \) are the parallel classes of \( (M_2/X_2)/S_0 \). The flat spanned by \( Y_2 \) in \( M_2/C \) has rank \( k + 1 \) and at least \( m \) points, so \( m \leq I^k \). By definition, \( n'_2 = R(n'_2, I^k + 1) \). So, by Ramsey’s Theorem, there is a set \( X'_2 \subseteq E(G - v) \) and an element \( j \in \{ 0, \ldots, m \} \) such that \( (M_2/Y_2)|X'_2 \cong M(K_{n'_2}) \) and \( X'_2 \subseteq S_j \). Applying identities from the previous section, we get

\[
\begin{align*}
\cap_{M_2}(X'_2, Y_2) &\leq \cap_{M_2}(S_j \cup C, Y_2) \\
&\leq \cap_{M_2/C}(S_j, Y_2) + \cap_{M_2}(C, Y_2) \\
&= \cap_{M_2/C}(S_j, Y_2) \\
&\leq r_{M_2/C}(S_j) \\
&\leq 1,
\end{align*}
\]

as required. \( \Box \)

4.2.4. There is a minor \( M_3 \) of \( M_2 \), a set \( X_3 \subseteq E(M_3) \), and a \( k \)-element independent set \( Y_3 \) of \( M_3 \) such that

1. \( M_3|X_3 \cong M(K_{n_3}) \),
2. each pair of elements in \( X_3 \times Y_3 \) spans a long line in \( M_3 \), and
3. \( X_3 \) and \( Y_3 \) are skew in \( M_3 \).

Proof of 4.2.4. Recall that \( (M_2/Y_2)|X'_2 = M(G) \) where \( G \) is a graph that is isomorphic to \( K_{n'_2} \). Moreover, \( \cap_{M_2}(X'_2, Y_2) \leq 1 \). We may assume that \( \cap_{M_2}(X'_2, Y_2) = 1 \) otherwise the claim holds. It follows that \( r_{M_2}(X'_2, Y_2) = r_{M_2/Y_2}(X'_2) + 1 \). Now it is routine to show that there is a triangle \( T \) of \( G \) that is independent in \( M_2 \). Let \( a, b, c \in V(G) \) be the three vertices in \( G \) that are incident with edges in \( T \), let \( X_3 := E(G - \{ a, b, c \}) \), and let \( M_3 = M_2/T \). Now \( \lambda_{M_3}(T, Y_2) = r_{M_2}(T) - r_{M_2/Y_2}(T) = 1 \) and, hence,
\[\cap_{M_3}(X_3, Y_2) \subseteq \cap_{M_2/T}(X'_2 - T, Y_2) = \cap_{M_2}(X'_2, Y_2) - \cap_{M_2}(T, Y_2) = 0.\]

Therefore \(X_3\) is skew to \(Y_2\) in \(M_3\). Moreover, \(Y_2\) has rank \(k\) in \(M_3\); let \(Y_3 \subseteq Y_2\) be a maximal independent set in \(M_3\). Then \(M_3, X_3,\) and \(Y_3\) satisfy the claim. \(\square\)

The result now follows by Lemma 4.1. \(\square\)

5. Building a book

In order to build an appropriate book, we use the methods of [2]; in fact, this section is taken almost verbatim from that paper.

**Lemma 5.1.** For integers \(\alpha \geq 1\) and \(l \geq 2\), if \(M \in \mathcal{U}(l)\) is a matroid with \(\epsilon(M) > \alpha\left(\binom{r(M)+1}{2}\right)\), then there is a minor \(N\) of \(M\) that contains \(> \alpha\left(\frac{r(M)}{l+1}\right)\) \(\alpha\left(\binom{r(M)+1}{2}\right)\) long lines.

**Proof.** We may assume that \(M\) is simple. For each \(v \in E\), let \(N_v = M/v\). Inductively, we may assume that \(\epsilon(N_v) \leq \alpha\left(\binom{r(N_v)}{2}\right)\) for each \(v \in E\). Note that, \(r(N_v) = r(M) - 1\) and \(\binom{r(M)+1}{2} = \binom{r(M)}{2} + r(M)\). So \(\epsilon(M) - \epsilon(N_v) \geq \alpha r(M) + 1\). Since \(M \in \mathcal{U}(l)\), each long line in \(M\) has at most \(l+1\) points; so when we contract an element the parallel classes contain at most \(l\) elements. Thus \(v\) is on at least \(\frac{\alpha r(M)}{l}\) long lines. So the number of long lines is at least \(\frac{\alpha r(M)}{l+1}\)\(\alpha\left(\binom{r(M)+1}{2}\right)\). \(\square\)

The following lemma is proved in [2].

**Lemma 5.2.** Let \(M\) be a matroid, let \(F_1\) and \(F_2\) be round flats of \(M\) such that \(r_M(F_1) = r_M(F_2) = k\) and \(r_M(F_1 \cup F_2) = k + 1\), and let \(F\) be the flat of \(M\) spanned by \(F_1 \cup F_2\). If \(F \neq F_1 \cup F_2\) then \(F\) is round.

Let \(\mathcal{F}\) be a set of round flats in a matroid \(M\). A rank-\(k\) flat \(F\) is called \(\mathcal{F}\)-constructed if there exist two rank-\((k-1)\) flats \(F_1, F_2 \in \mathcal{F}\) such that \(F = \text{cl}_M(F_1 \cup F_2)\) and \(F \neq F_1 \cup F_2\). Thus, the \(\mathcal{F}\)-constructed flats are round. We let \(\mathcal{F}^+\) denote the set of \(\mathcal{F}\)-constructed flats.

Most of the remaining work is in the proof of the following technical lemma.

**Lemma 5.3.** There exists an integer-valued function \(f_2(k, \alpha, l)\) such that, for all integers \(k \geq 2\), \(\alpha \geq 1\), and \(l \geq 2\), if \(M \in \mathcal{U}(l)\) is a matroid with \(\epsilon(M) > f_2(k, \alpha, l)\left(\binom{r(M)+1}{2}\right)\), then there exists a minor \(N\) of \(M\) and a set \(\mathcal{F}\) of round rank-\((k-1)\) flats of \(N\) such that \(|\mathcal{F}^+| > \alpha r(N)|\mathcal{F}|\).

**Proof.** Let \(f_2(2, \alpha, l) = \alpha(l+1)^2\), and, for \(k \geq 2\), we recursively define \(f_2(k+1, \alpha, l) = f_2(k, l^{(k+1)^2} + l^k, l)\).

The proof is by induction on \(k\). Consider the case that \(k = 2\). Now, let \(M \in \mathcal{U}(l)\) be a simple matroid with \(|E(M)| > f_2(2, \alpha, l)\left(\binom{r(M)+1}{2}\right)\). By Lemma 5.1, there exists a simple minor \(N\) of \(M\) with more than \(\alpha r(N)|\mathcal{F}|\) long lines. Now, if \(\mathcal{F}\) is the set of points of \(N\), then \(\mathcal{F}^+\) is the set of long lines of \(N\) and \(|\mathcal{F}^+| > \alpha r(N)|\mathcal{F}|\), as required.

Suppose that the result holds for \(k = n\) and consider the case that \(k = n+1\). Now let \(M \in \mathcal{U}(l)\) be a simple matroid with \(\epsilon(M) > \beta(n+1, \alpha, l)\left(\binom{r(M)+1}{2}\right)\). We let \(\alpha' = l^{(n+1)^2} + l^n\). By the induction hypothesis there exists a minor \(N\) of \(M\) and a set \(\mathcal{F}\) of round rank-\((n-1)\) flats of \(N\) such that \(|\mathcal{F}^+| > \alpha' r(N)|\mathcal{F}|\). We may assume that no proper minor of \(N\) contains such a collection of flats. We may also assume that \(N\) is simple. We will prove that \(|\mathcal{F}^+| > \alpha r(N)|\mathcal{F}|\).

Now, for each \(v \in E(N)\), let \(N_v = N/v\). Let \(\mathcal{F}_v\) denote the set of rank-\((n-1)\) flats in \(N_v\) corresponding to the set of flats in \(\mathcal{F}\) in \(N\). That is, if \(F \in \mathcal{F}\) and \(v \notin F\), then \(\text{cl}_{N_v}(F) \in \mathcal{F}_v\). By our choice of \(N\),
|F^+| > \alpha' r(N)|F|, and, by the minimality of N, |F^+| \leq \alpha' r(N_v)|F_v| \leq \alpha' r(N)|F_v| for all v \in E(N). Thus,
\[(|F^+| - |(F_v)^+|) > \alpha' r(N)(|F| - |F_v|)).\]
Let
\[\Delta = \sum(|F - |F_v| : v \in E(N)) \quad \text{and} \quad \Delta^+ = \sum(|F^+| - |(F_v)^+| : v \in E(N)).\]
This proves:

5.3.1. \(\Delta^+ > \alpha' r(N)\Delta.\)

Consider a flat \(F \in F^+.\) By definition there exist flats \(F_1, F_2 \in \mathcal{F}\) such that \(F = cl_N(F_1 \cup F_2)\) and there exists an element \(v \in F - (F_1 \cup F_2).\) Now \(cl_{N_v}(F_1) = cl_{N_v}(F_2),\) so these two flats in \(\mathcal{F}\) are reduced to a single flat in \(\mathcal{F}_v.\) This proves:

5.3.2. \(\Delta \geq |F^+|.\)

Now, for some \(v \in E(N),\) compare \(F^+\) with \((F_v)^+.\) There are two ways to lose constructed flats; we can either contract an element in a flat or we contract two flats onto each other. Firstly, suppose \(F \in F^+\) and \(v \in F.\) Note that \(F - \{v\}\) only has rank \(n - 1\) in \(N/v,\) so it will not determine a flat in \((F_v)^+.\) Now \(F\) has rank \(n\) and, by Theorem 2.1, a rank-\(n\) flat contains at most \(\frac{p-1}{r} < p\) points; we destroy \(F\) if we contract any one of these points. Secondly, consider two flats \(F_1, F_2 \in F^+\) that are contracted onto each other in \(N_v.\) Let \(F\) be the flat of \(N\) spanned by \(F_1 \cup F_2\) in \(N.\) Since \(F_1\) and \(F_2\) are contracted onto a common rank-\(k\) flat in \(N_v,\) we see that \(F\) has rank \(k + 1\) and \(v \in F - (F_1 \cup F_2).\) Thus, \(F \in (F^+)^+.\) Now, \(F\) has rank \(n + 1,\) so it has at most \(l^{n+1}\) points. Moreover, by Lemma 2.3, in a flat of rank \(n + 1\) there are at most \(l^{(n+1)n}\) rank-\(n\) flats avoiding a given element. Thus, \(F - \{v\}\) contains at most \(l^{(n+1)n}\) flats of \(F;\) these flats will be contracted to a single flat in \((F_v)^+.\) This proves:

5.3.3. \(\Delta^+ \leq l^n|F^+| + l^{(n+1)^2}|(F^+)^+.\)

Now, combining 5.3.1–5.3.3, we get
\[l^{(n+1)^2}|(F^+)^+| \geq \Delta^+ - l^n|F^+| > \alpha' r(N)\Delta - l^n|F^+|\]
\[\geq (\alpha' r(N) - l^n)|F^+| \geq (\alpha' - l^n)r(N)|F^+|\]
\[= l^{(n+1)^2}\alpha r(N)|F^+|.\]
Therefore \(|(F^+)^+| > \alpha|F^+|;\) as required. \(\square\)

We are now ready to prove Theorem 1.4, which we restate here in a more convenient form.

**Theorem 5.4.** For all integers \(l \geq 2\) and \(k \geq 1,\) there is an integer \(c\) such that, if \(M \in \mathcal{U}(l)\) is a matroid with \(e(M) > c\left(\binom{M}{2}\right),\) then \(M\) has a rank-\(k\) minor \(N\) such that \(e(N) = 2^k - 1.\)

**Proof.** Let \(\alpha = l^{k+2}(k+1)f_1(l, k)\) and let \(c = f_2(k+2, \alpha, l).\) Now, let \(M \in \mathcal{U}(l)\) be a matroid with \(e(M) > c\left(\binom{M}{2}\right).\) By Lemma 5.3, there is a minor \(N\) of \(M\) and a collection \(\mathcal{F}\) of round rank-(\(k+1\) flats of \(N\) such that \(|\mathcal{F}^+| > \alpha r(N)|\mathcal{F}|.\) By Lemma 2.3, each flat in \(\mathcal{F}^+\) contains at most \(l^{(k+2)(k+1)}\) flats from \(\mathcal{F}.\) Let \(t = f_1(l, k)r(N).\) Therefore, there is a flat \(F_0 \in \mathcal{F}\) that is contained in \(t\) flats in \(\mathcal{F}^+;\) let \(F_1, \ldots, F_t \in \mathcal{F}^+\) be flats containing \(F_0.\) Then \((F_0, F_1, \ldots, F_t)\) is a \((k+1)\)-book and, hence, the theorem follows by Lemma 4.2. \(\square\)
References

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