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Growth rates of minor-closed classes of matroids ☆

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ABSTRACT

For a minor-closed class $\mathcal{M}$ of matroids, let $h(k)$ denote the maximum number of elements in a simple rank-k matroid in $\mathcal{M}$. We prove that, if $\mathcal{M}$ does not contain all simple rank-2 matroids, then $h(k)$ is finite and is either linear, quadratic, or exponential.

1. Introduction

In this paper we consider classes of matroids that are closed both under taking minors and under isomorphism; for convenience we shall simply refer to such classes as *minor-closed*. Our main result, combined with earlier results of Geelen and Whittle and of Geelen and Kabell, yields the following theorem, conjectured by Kung [4, Conjecture 4.9].

Theorem 1.1 *(Growth rate theorem). If $\mathcal{M}$ is a minor-closed class of matroids, then either*

(1) *there exists $c \in \mathbb{R}$ such that $|E(M)| \leqslant cr(M)$ for all simple matroids $M \in \mathcal{M}$,*
(2) *$\mathcal{M}$ contains all graphic matroids and there exists $c \in \mathbb{R}$ such that $|E(M)| \leqslant c(r(M))^2$ for all simple matroids $M \in \mathcal{M}$,*
(3) *there is a prime-power q and $c \in \mathbb{R}$ such that $\mathcal{M}$ contains all $\mathrm{GF}(q)$-representable matroids and $|E(M)| \leqslant cq^{r(M)}$ for all simple matroids $M \in \mathcal{M}$, or*
(4) *$\mathcal{M}$ contains all simple rank-2 matroids.*

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We follow the notation of Oxley [5]. A rank-1 flat is referred to as a *point* and a rank-2 flat is referred to as a *line*. The number of points in M is denoted by $\epsilon(M)$. For a class $\mathcal{M}$ of matroids and integer $k \geqslant 0$, we let $h(\mathcal{M}, k)$ be the maximum of $\epsilon(M)$ among all rank-k matroids $M \in \mathcal{M}$. Thus, if $\mathcal{G}$ is the set of graphic matroids, then $h(\mathcal{G}, k) = \binom{k+1}{2}$ and, for a prime-power q, if $\mathcal{L}(q)$ is the set of all GF(q)-representable matroids, then $h(\mathcal{L}(q), k) = \frac{q^k-1}{q-1}$.

We begin by recounting two significant partial results towards the growth rate theorem. The first was proved by Geelen and Whittle [2].

Theorem 1.2. *If $\mathcal{M}$ is a minor-closed class of matroids, then either*

(1) *there exists $c \in \mathbb{R}$ such that, $h(\mathcal{M}, k) \leqslant ck$ for all k,*
(2) *$\mathcal{M}$ contains all graphic matroids, or*
(3) *$\mathcal{M}$ contains all simple rank-2 matroids.*

The second result was proved by Geelen and Kabell [1] and in part, by Kung [4, Theorem 6.6].

Theorem 1.3. *If $\mathcal{M}$ is a minor-closed class of matroids, then either*

(1) *there exists a polynomial $p(k)$ such that, $h(\mathcal{M}, k) \leqslant p(k)$ for all k,*
(2) *there is a prime-power q and $c \in \mathbb{R}$ such that $\mathcal{M}$ contains all GF(q)-representable matroids and $h(\mathcal{M}, k) \leqslant cq^k$ for all k, or*
(3) *$\mathcal{M}$ contains all simple rank-2 matroids.*

In this paper, we bridge the gap by proving the following theorem.

Theorem 1.4. *If $\mathcal{M}$ is a minor-closed class of matroids, then either*

(1) *there exists $c \in \mathbb{R}$ such that, $h(\mathcal{M}, k) \leqslant ck^2$ for all k,*
(2) *$h(\mathcal{M}, k) \geqslant 2^k - 1$ for all k, or*
(3) *$\mathcal{M}$ contains all simple rank-2 matroids.*

We conclude the introduction with two interesting corollaries of the growth rate theorem. The second of these was already known; see Kung [3].

Corollary 1.5. *Let q be a power of a prime p and let $\mathcal{M}$ be a minor-closed class of GF(q)-representable matroids. If $\mathcal{M}$ does not contain all GF(p)-representable matroids, then there exists a constant $c \in \mathbb{R}$ such that $h(\mathcal{M}, k) \leqslant ck^2$ for all k.*

Corollary 1.6. *Let $\mathcal{M}$ be a minor-closed class of $\mathbb{R}$-representable matroids. If $\mathcal{M}$ does not contain all simple rank-2 matroids, then there exists a constant $c \in \mathbb{R}$ such that $h(\mathcal{M}, k) \leqslant ck^2$ for all k.*

2. Excluding a line

Kung [4] proved the following theorem.

Theorem 2.1. *For any integer $l \geqslant 2$, if M is a matroid with no $U_{2,l+2}$-minor, then $\epsilon(M) \leqslant \frac{l^{r(M)}-1}{l-1}$.*

Let $\mathcal{U}(l)$ denote the set of all matroids with no $U_{2,l+2}$-minor. Thus $h(\mathcal{U}(l), k) \leqslant \frac{l^k-1}{l-1}$. Note that, when l is a prime-power, this bound is tight since $\mathcal{L}(l) \subseteq \mathcal{U}(l)$. However, when l is not a prime-power, the growth rate theorem gives an asymptotically tighter bound of cq^k, where q is the largest prime-power less than or equal to l.

We remark that Kung [4] has made a stronger conjecture.

Conjecture 2.2. *If $l \geqslant 2$ is an integer and q is the largest prime-power less than or equal to l, then $h(\mathcal{U}(l), k) = \frac{q^k-1}{q-1}$ for all sufficiently large k.*

Conjecture 2.2 is the case of Conjecture 4.9(a) in [4] when the set of excluded minors is empty. The general form of Conjecture 4.9(a) can be restated as follows. Let $\mathcal{M}$ be a minor-closed class not containing all rank-2 simple matroids. If $\mathcal{L}(q) \subseteq \mathcal{M}$ for some prime power q and q is maximum with this property, then $h(\mathcal{M}, k) = \frac{q^k-1}{q-1}$ for sufficiently large k. This conjecture is too good to be true. We construct a counterexample $\mathcal{M}$ (using q-lifts or q-cones). Let q be a prime-power, let $n \geqslant 2$ be an integer, and let $\mathcal{F}$ be the set of all pairs (M, e) consisting of a GF(q^n)-representable matroid M and an element $e \in E(M)$ such that M/e is GF(q)-representable. Now let $\mathcal{M}$ be the set of all matroids $M \setminus e$ where $(M, e) \in \mathcal{F}$. It is straightforward to verify that every extremal rank-k matroid $M' \in \mathcal{M}$ contains a hyperplane H and an element $e' \notin H$ such that $M'|H \cong \mathrm{PG}(k-2, q)$ and, for each $f \in H$, the pair (e', f) spans a (q^n+1)-point line in M'. By adding an element e in parallel with e', we obtain associated pair $(M, e) \in \mathcal{F}$. Therefore,

$$h(\mathcal{M}, k) = q^n \frac{q^{k-1}-1}{q-1} + 1.$$

Our proof of the growth rate theorem requires a bound on the number of hyperplanes in a rank-k matroid in $\mathcal{U}(l)$. If M is GF(q)-representable, then, by considering $\mathrm{PG}(r-1, q)$, we see that M has at most $\frac{q^k-1}{q-1}$ hyperplanes. On the other hand, when $M \in \mathcal{U}(l)$, we cannot prove a comparable bound, so we settle for the following crude upper bound from [2]; we include the short proof for completeness.

Lemma 2.3. *Let $k \geqslant 1$ and $l \geqslant 2$ be integers and let $M \in \mathcal{U}(l)$ be a simple rank-k matroid. Then, M has at most $l^{k(k-1)}$ hyperplanes.*

Proof. Let $n = |E(M)|$; thus $n \leqslant \frac{l^k-1}{l-1} \leqslant l^k$. Each hyperplane is spanned by a set of $k-1$ points, so the number of hyperplanes is at most $\binom{n}{k-1} \leqslant n^{k-1} \leqslant l^{k(k-1)}$. $\square$

3. Local connectivity

Let M be a matroid and let $A, B \subseteq E(M)$. We define $\sqcap_M(A, B) = r_M(A) + r_M(B) - r_M(A \cup B)$; this is the *local connectivity* between A and B. This definition is motivated by geometry. Suppose that M is a restriction of $\mathrm{PG}(k, q)$ and let F_A and F_B be the flats of $\mathrm{PG}(k, q)$ that are spanned by A and B, respectively. Then $F_A \cap F_B$ has rank $\sqcap_M(A, B)$.

The following properties are intuitively obvious for representable matroids, and follow by elementary rank calculations for arbitrary matroids.

(1) If $A, B \subseteq E(M)$ and $A' \subseteq A$, then $\sqcap_M(A', B) \leqslant \sqcap_M(A, B)$.
(2) If A and C are disjoint subsets of $E(M)$, then $r_{M/C}(A) = r_M(A) - \sqcap_M(A, C)$.
(3) If A, B, and C are disjoint subsets of $E(M)$, then $\sqcap_{M/C}(A, B) = \sqcap_M(A, B) - \sqcap_M(A, C)$.

We say that two sets $A, B \subseteq E(M)$ are *skew* if $\sqcap_M(A, B) = 0$. More generally, the sets $A_1, \dots, A_l \subseteq E(M)$ are *skew* if $r_M(A_1) + \cdots + r_M(A_k) = r_M(A_1 \cup \cdots \cup A_k)$.

4. Books and dense minors

A line is *long* if it has at least 3 points. For sets A and B we let $A \times B$ denote $\{(a, b)\colon\ a \in A,\ b \in B\}$. We use the following lemma to identify a dense minor.

Lemma 4.1. *Let $k \geqslant 1$ be an integer and let $n = k2^k$. Let F_1 and F_2 be skew flats in a matroid M such that $M|F_1$ is isomorphic to $M(K_n)$, $r(F_2) = k$, and each pair of points in $F_1 \times F_2$ spans a long line. Then M has a rank-k minor N with $\epsilon(N) \geqslant 2^k - 1$.*

Proof. We may assume that M is simple and that $r(M) = r_M(F_1 \cup F_2)$. We may also assume that F_2 is a k-element independent set in M and that $M|F_1 = M(G)$, where G is isomorphic to K_n. Let $\mathcal{C}$ denote the set of all subsets of F_2 with at least two elements. Since $n \geqslant k|\mathcal{C}|$, there exists a collection $(P_X\colon\ X \in \mathcal{C})$ of vertex-disjoint paths in G where each path P_X has length $|X|$. For each $X \in \mathcal{C}$, let e_X be the edge of G that connects the ends of P_X, and let $\phi_X : X \to E(P_X)$ be an arbitrary bijection. For each $x \in X$, let $f_x \in E(M) - (F_1 \cup F_2)$ be an element spanned by $\{x, \phi_X(x)\}$, and let $S_X = \{f_x\colon\ x \in X\}$. Finally, let S denote the union of the sets $(S_X\colon\ X \in \mathcal{C})$ and let N be the restriction of M/S to the flat spanned by F_2. Note that the sets F_2 and $(P_X\colon\ X \in \mathcal{C})$ are skew and, for each $X \in \mathcal{C}$, the set S_X is contained in the flat of M that is spanned by $F_2 \cup P_X$. Moreover, F_2 is independent in N and, for each $X \in \mathcal{C}$ and each $x \in X$, the elements x and $\phi_X(x)$ are in parallel in N. Therefore, for each $X \in \mathcal{C}$, the set $X \cup \{e_X\}$ is a circuit of N. Hence $\epsilon(N) \geqslant |F_2| + |\mathcal{C}| = 2^k - 1$, as required. □

We call a matroid M *round* if each cocircuit of M is spanning. Equivalently, M is round if and only if $E(M)$ cannot be written as the union of two proper flats. The following properties are straightforward to check:

1. If M is a round matroid and $e \in E(M)$ then M/e is round.
2. If N is a spanning minor of M and N is round, then M is round.

Let M be a matroid. A flat F of M is called *round* if the restriction of M to F is round. Each rank-one flat is round. Moreover, a line is round if and only if it is long. A sequence $(F_0, F_1, \ldots, F_t)$ is called a *k-book* if F_0 is a rank-k flat of M and $F_1, \ldots, F_t$ are distinct round rank-$(k+1)$ flats of M each containing F_0.

The following lemma is the main result of the section.

Lemma 4.2. *There exists a function $f_1 : \mathbb{Z}^2 \to \mathbb{Z}$ such that, for integers $l, k \geqslant 2$, if $(F_0, F_1, \ldots, F_t)$ is a $(k+1)$-book in a matroid $M \in \mathcal{U}(l)$ and $t \geqslant f_1(l,k)r(M)$, then M has a rank-k minor N with $\epsilon(N) = 2^k - 1$.*

Proof. By Ramsey's Theorem, there exists a function $R : \mathbb{Z}^2 \to \mathbb{Z}$ such that, for integers $n, c \geqslant 1$, if we colour the edges of a clique on $R(n,c)$ vertices with c colours, then there is a monochromatic clique on n vertices. By Theorem 1.2, there exists a function $\lambda : \mathbb{Z}^2 \to \mathbb{Z}$ such that, for integers $l, n \geqslant 2$, if $M \in \mathcal{U}(l)$ is a matroid with $\epsilon(M) > \lambda(l,n)r(M)$, then M has an $M(K_n)$-minor.

Let $l, k \geqslant 2$ be integers. Now let $n_3 = k2^k$, $n_2' = n_3 + 3$, $n_2 = R(n_2', l2^k + 1) + 1$, and $n_1 = l2^k + R(n_2, \binom{l2^k}{k+1})$. Finally we let $f_1(l,k) = \lambda(l, n_1)$.

Now consider a matroid $M \in \mathcal{U}(l)$ containing a $(k+1)$-book $(F_0, F_1, \ldots, F_t)$ with $t \geqslant f_1(l,k)r(M)$. By way of contradiction, we assume that, for each rank-k minor N of M, we have $\epsilon(N) < 2^k - 1$. If follows easily that, for each rank-$(k+1)$ minor N of M, we have $\epsilon(N) < l(2^k - 1) + 1 \leqslant l2^k$.

4.2.1. *There is a minor M_1 of M and a set $X_1 \subseteq E(M_1)$ such that*

(1) $F_0 \subseteq E(M_1)$ *and* $r_{M_1}(F_0) = k + 1$,
(2) $(M_1/F_0)|X_1 \cong M(K_{n_1})$, *and*
(3) *for each $e \in X_1$, the flat of M_1 that is spanned by $F_0 \cup \{e\}$ is round.*

Proof of 4.2.1. For each $i \in \{1, \ldots, t\}$, choose $x_i \in F_i - F_0$. Now let $X = \{x_1, \ldots, x_t\}$ and let $N = (M/F_0)|X$. Note that $\epsilon(N) \geqslant \lambda(l, n_1)r(N)$. Therefore there is a minor, say $N \setminus D/C$, of N that is isomorphic to $M(K_{n_1})$. The claim follows by taking $M_1 := M/C$ and $X_1 := E(N \setminus D/C)$. □

4.2.2. *There is a minor M_2 of M_1, a set $X_2 \subseteq E(M_2)$, and a $(k+1)$-element independent set Y_2 of M_2 such that*

(1) $(M_2/Y_2)|X_2 \cong M(K_{n_2})$, *and*
(2) *each pair of elements in $X_2 \times Y_2$ spans a long line in M_2.*

Proof of 4.2.2. Let $n' = R(n_2, \binom{l2^k}{k+1})$, thus $n_1 = l2^k + n'$. Note that F_0 has rank-$(k+1)$ and, hence, it spans at most $l2^k$ points. We begin by repeatedly contracting elements from X_1 if doing so increases the number of points spanned by F_0; the number of points that we contract will be at most $l2^k$. Therefore, there is a minor M_2 of M_1 and a set $X' \subseteq X_1$ such that:

(1) $F_0 \subseteq E(M_2)$ and $r_{M_2}(F_0) = k+1$,
(2) $(M_2/F_0)|X' \cong M(K_{n'})$,
(3) for each $e \in X'$, the flat of M_2 that is spanned by $F_0 \cup \{e\}$ is round, and
(4) for each element $a \in X'$ and each element $b \in \mathrm{cl}_{M_2}(F_0 \cup \{a\}) - \mathrm{cl}_{M_2}(F_0)$ that is not in parallel with a, there is an element $c \in \mathrm{cl}_{M_2}(F_0)$ such that $\{a, b, c\}$ is a circuit of M_2.

Let $F' = \mathrm{cl}_{M_2}(F_0)$. We may assume, for notational convenience, that M_2 is simple. Thus $|F'| \leqslant l2^k$. For each element $a \in X'$, let B_a be a basis of the flat spanned by $\{a\} \cup F'$ with $\{a\} \subseteq B_a$ and with $B_a \cap F' = \emptyset$ (such a basis exists since the flat is round). By the last property of M_2 above, there is a basis B'_a of F' such that, for each $b \in B_a - \{a\}$, there is an element $c \in F'$ such that $\{a, b, c\}$ is a circuit. Note that B'_a is a $(k+1)$-element subset of F' and the number of such subsets is at most $\binom{l2^k}{k+1}$. Therefore, by Ramsey's Theorem, there is a basis Y_2 of F' and a set $X_2 \subseteq X'$ such that $(M_2/F_0)|X_2 \cong M(K_{n_2})$ and, for each $e \in X_2$, we have $B'_e = Y_2$. Thus M_2, X_2, and Y_2 satisfy the claim. □

4.2.3. *There is a set $X'_2 \subseteq X_2$ such that*

(1) $(M_2/Y_2)|X'_2 \cong M(K_{n'_2})$, *and*
(2) $\sqcap_{M_2}(X'_2, Y_2) \leqslant 1$.

Proof of 4.2.3. Recall that $(M_2/Y_2)|X_2 = M(G)$ where G is a graph that is isomorphic to K_{n_2}. Let $v \in V(G)$ and let C be the set of edges of G that are incident with v. Note that $Y_2 \cup C$ spans X_2 in M_2. Define a partition $(S_0, S_1, \ldots, S_m)$ of X_2 such that $S_0 = \mathrm{cl}_{M_2}(C) \cap X_2$ and $(S_1, \ldots, S_m)$ are the parallel classes of $(M_2|X_2)/S_0$. The flat spanned by Y_2 in M_2/C has rank $k+1$ and at least m points, so $m \leqslant l2^k$. By definition, $n_2 = R(n'_2, l2^k + 1) + 1$. So, by Ramsey's Theorem, there is a set $X'_2 \subseteq E(G - v)$ and an element $j \in \{0, \ldots, m\}$ such that $(M_2/Y_2)|X'_2 \cong M(K_{n'_2})$ and $X'_2 \subseteq S_j$. Applying identities from the previous section, we get

$$\begin{aligned}\sqcap_{M_2}\big(X'_2, Y_2\big) &\leqslant \sqcap_{M_2}(S_j \cup C, Y_2)\\ &\leqslant \sqcap_{M_2/C}(S_j, Y_2) + \sqcap_{M_2}(C, Y_2)\\ &= \sqcap_{M_2/C}(S_j, Y_2)\\ &\leqslant r_{M_2/C}(S_j)\\ &\leqslant 1,\end{aligned}$$

as required. □

4.2.4. *There is a minor M_3 of M_2, a set $X_3 \subseteq E(M_3)$, and a k-element independent set Y_3 of M_3 such that*

(1) $M_3|X_3 \cong M(K_{n_3})$,
(2) *each pair of elements in $X_3 \times Y_3$ spans a long line in M_3, and*
(3) *X_3 and Y_3 are skew in M_3.*

Proof of 4.2.4. Recall that $(M_2/Y_2)|X'_2 = M(G)$ where G is a graph that is isomorphic to $K_{n'_2}$. Moreover, $\sqcap_{M_2}(X'_2, Y_2) \leqslant 1$. We may assume that $\sqcap_{M_2}(X'_2, Y_2) = 1$ otherwise the claim holds. It follows that $r_{M_2}(X'_2) = r_{M_2/Y_2}(X'_2) + 1$. Now it is routine to show that there is a triangle T of G that is independent in M_2. Let $a, b, c \in V(G)$ be the three vertices in G that are incident with edges in T, let $X_3 := E(G - \{a, b, c\})$, and let $M_3 = M_2/T$. Now $\lambda_{M_2}(T, Y_2) = r_{M_2}(T) - r_{M_2/Y_2}(T) = 1$ and, hence,

$$\begin{aligned}\sqcap_{M_3}(X_3, Y_2) &\leqslant \sqcap_{M_2/T}\big(X_2' - T, Y_2\big)\\ &= \sqcap_{M_2}\big(X_2', Y_2\big) - \sqcap_{M_2}(T, Y_2)\\ &= 0.\end{aligned}$$

Therefore X_3 is skew to Y_2 in M_3. Moreover, Y_2 has rank k in M_3; let $Y_3 \subset Y_2$ be a maximal independent set in M_3. Then M_3, X_3, and Y_3 satisfy the claim. □

The result now follows by Lemma 4.1. □

5. Building a book

In order to build an appropriate book, we use the methods of [2]; in fact, this section is taken almost verbatim from that paper.

Lemma 5.1. *For integers $\alpha \geqslant 1$ and $l \geqslant 2$, if $M \in \mathcal{U}(l)$ is a matroid with $\epsilon(M) > \alpha\binom{r(M)+1}{2}$, then there is a minor N of M that contains $> \frac{\alpha}{(l+1)^2} r(N)\epsilon(N)$ long lines.*

Proof. We may assume that M is simple. For each $v \in E$, let $N_v = M/v$. Inductively, we may assume that $\epsilon(N_v) \leqslant \alpha\binom{r(N_v)}{2}$ for each $v \in E$. Note that, $r(N_v) = r(M) - 1$ and $\binom{r(M)+1}{2} = \binom{r(M)}{2} + r(M)$. So $\epsilon(M) - \epsilon(N_v) \geqslant \alpha r(M) + 1$. Since $M \in \mathcal{U}(l)$, each long line in M has at most $l + 1$ points; so when we contract an element the parallel classes contain at most l elements. Thus v is on at least $\frac{\alpha r(M)}{l}$ long lines. So the number of long lines is at least $\frac{\alpha r(M)}{l(l+1)}\epsilon(M)$. □

The following lemma is proved in [2].

Lemma 5.2. *Let M be a matroid, let F_1 and F_2 be round flats of M such that $r_M(F_1) = r_M(F_2) = k$ and $r_M(F_1 \cup F_2) = k + 1$, and let F be the flat of M spanned by $F_1 \cup F_2$. If $F \neq F_1 \cup F_2$ then F is round.*

Let $\mathcal{F}$ be a set of round flats in a matroid M. A rank-k flat F is called *$\mathcal{F}$-constructed* if there exist two rank-$(k-1)$ flats $F_1, F_2 \in \mathcal{F}$ such that $F = \mathrm{cl}_M(F_1 \cup F_2)$ and $F \neq F_1 \cup F_2$. Thus, the $\mathcal{F}$-constructed flats are round. We let $\mathcal{F}^+$ denote the set of $\mathcal{F}$-constructed flats.

Most of the remaining work is in the proof of the following technical lemma.

Lemma 5.3. *There exists an integer-valued function $f_2(k, \alpha, l)$ such that, for all integers $k \geqslant 2$, $\alpha \geqslant 1$, and $l \geqslant 2$, if $M \in \mathcal{U}(l)$ is a matroid with $\epsilon(M) > f_2(k, \alpha, l)\binom{r(M)+1}{2}$, then there exists a minor N of M and a set $\mathcal{F}$ of round rank-$(k-1)$ flats of N such that $|\mathcal{F}^+| > \alpha r(N)|\mathcal{F}|$.*

Proof. Let $f_2(2, \alpha, l) = \alpha(l + 1)^2$, and, for $k \geqslant 2$, we recursively define

$$f_2(k + 1, \alpha, l) = f_2\big(k, l^{(k+1)^2}\alpha + l^k, l\big).$$

The proof is by induction on k. Consider the case that $k = 2$. Now, let $M \in \mathcal{U}(l)$ be a simple matroid with $|E(M)| > f_2(2, \alpha, l)\binom{r(M)+1}{2}$. By Lemma 5.1, there exists a simple minor N of M with more than $\alpha r(N)\epsilon(N)$ long lines. Now, if $\mathcal{F}$ is the set of points of N, then $\mathcal{F}^+$ is the set of long lines of N and $|\mathcal{F}^+| > \alpha r(N)|\mathcal{F}|$, as required.

Suppose that the result holds for $k = n$ and consider the case that $k = n + 1$. Now let $M \in \mathcal{U}(l)$ be a simple matroid with $\epsilon(M) > \beta(n + 1, \alpha, l)\binom{r(M)+1}{2}$. We let $\alpha' = l^{(n+1)^2}\alpha + l^n$. By the induction hypothesis there exists a minor N of M and a set $\mathcal{F}$ of round rank-$(n-1)$ flats of N such that $|\mathcal{F}^+| > \alpha' r(N)|\mathcal{F}|$. We may assume that no proper minor of N contains such a collection of flats. We may also assume that N is simple. We will prove that $|(\mathcal{F}^+)^+| \geqslant \alpha r(N)|\mathcal{F}^+|$.

Now, for each $v \in E(N)$, let $N_v = N/v$. Let $\mathcal{F}_v$ denote the set of rank-$(n-1)$ flats in N_v corresponding to the set of flats in $\mathcal{F}$ in N. That is, if $F \in \mathcal{F}$ and $v \notin F$, then $\mathrm{cl}_{N_v}(F) \in \mathcal{F}_v$. By our choice of N,

$|\mathcal{F}^+| > \alpha' r(N)|\mathcal{F}|$, and, by the minimality of N, $|\mathcal{F}_v^+| \leqslant \alpha' r(N_v)|\mathcal{F}_v| \leqslant \alpha' r(N)|\mathcal{F}_v|$ for all $v \in E(N)$. Thus,

$$\left(\left|\mathcal{F}^+\right| - \left|(\mathcal{F}_v)^+\right|\right) > \alpha' r(N)\left(|\mathcal{F}| - |\mathcal{F}_v|\right).$$

Let

$$\Delta = \sum\left(|\mathcal{F}| - |\mathcal{F}_v| : v \in E(N)\right) \quad \text{and} \quad \Delta^+ = \sum\left(\left|\mathcal{F}^+\right| - \left|(\mathcal{F}_v)^+\right| : v \in E(N)\right).$$

This proves:

5.3.1. $\Delta^+ > \alpha' r(N)\Delta$.

Consider a flat $F \in \mathcal{F}^+$. By definition there exist flats $F_1, F_2 \in \mathcal{F}$ such that $F = \mathrm{cl}_N(F_1 \cup F_2)$ and there exists an element $v \in F - (F_1 \cup F_2)$. Now $\mathrm{cl}_{N_v}(F_1) = \mathrm{cl}_{N_v}(F_2)$, so these two flats in $\mathcal{F}$ are reduced to a single flat in $\mathcal{F}_v$. This proves:

5.3.2. $\Delta \geqslant |\mathcal{F}^+|$.

Now, for some $v \in E(N)$, compare $\mathcal{F}^+$ with $(\mathcal{F}_v)^+$. There are two ways to lose constructed flats; we can either contract an element in a flat or we contract two flats onto each other. Firstly, suppose $F \in \mathcal{F}^+$ and $v \in F$. Note that $F - \{v\}$ only has rank $n-1$ in N/v, so it will not determine a flat in $(\mathcal{F}_v)^+$. Now F has rank n and, by Theorem 2.1, a rank-n flat contains at most $\frac{l^n-1}{l-1} < l^n$ points; we destroy F if we contract any one of these points. Secondly, consider two flats $F_1, F_2 \in \mathcal{F}^+$ that are contracted onto each other in N_v. Let F be the flat of N spanned by $F_1 \cup F_2$ in N. Since F_1 and F_2 are contracted onto a common rank-k flat in N_v, we see that F has rank $k+1$ and $v \in F - (F_1 \cup F_2)$. Thus, $F \in (\mathcal{F}^+)^+$. Now, F has rank $n+1$, so it has at most l^{n+1} points. Moreover, by Lemma 2.3, in a flat of rank $n+1$ there are at most $l^{(n+1)n}$ rank-n flats avoiding a given element. Thus, $F - \{v\}$ contains at most $l^{(n+1)n}$ flats of $\mathcal{F}$; these flats will be contracted to a single flat in $(\mathcal{F}_v)^+$. This proves:

5.3.3. $\Delta^+ \leqslant l^n|\mathcal{F}^+| + l^{(n+1)^2}|(\mathcal{F}^+)^+|$.

Now, combining 5.3.1–5.3.3, we get

$$\begin{aligned} l^{(n+1)^2}\left|\left(\mathcal{F}^+\right)^+\right| &\geqslant \Delta^+ - l^n\left|\mathcal{F}^+\right| > \alpha' r(N)\Delta - l^n\left|\mathcal{F}^+\right| \\ &\geqslant \left(\alpha' r(N) - l^n\right)\left|\mathcal{F}^+\right| \geqslant \left(\alpha' - l^n\right)r(N)\left|\mathcal{F}^+\right| \\ &= l^{(n+1)^2}\alpha r(N)\left|\mathcal{F}^+\right|. \end{aligned}$$

Therefore $|(\mathcal{F}^+)^+| > \alpha|\mathcal{F}^+|$; as required. □

We are now ready to prove Theorem 1.4, which we restate here in a more convenient form.

Theorem 5.4. *For all integers $l \geqslant 2$ and $k \geqslant 1$, there is an integer c such that, if $M \in \mathcal{U}(l)$ is a matroid with $\epsilon(M) > c\binom{r(M)+1}{2}$, then M has a rank-k minor N such that $\epsilon(N) = 2^k - 1$.*

Proof. Let $\alpha = l^{(k+2)(k+1)} f_1(l,k)$ and let $c = f_2(k+2, \alpha, l)$. Now, let $M \in \mathcal{U}(l)$ be a matroid with $\epsilon(M) > c\binom{r(M)+1}{2}$. By Lemma 5.3, there is a minor N of M and a collection $\mathcal{F}$ of round rank-$(k+1)$ flats of N such that $|\mathcal{F}^+| > \alpha r(N)|\mathcal{F}|$. By Lemma 2.3, each flat in $\mathcal{F}^+$ contains at most $l^{(k+2)(k+1)}$ flats from $\mathcal{F}$. Let $t = f_1(l,k)r(N)$. Therefore, there is a flat $F_0 \in \mathcal{F}$ that is contained in t flats in $\mathcal{F}^+$; let $F_1, \dots, F_t \in \mathcal{F}^+$ be flats containing F_0. Then $(F_0, F_1, \dots, F_t)$ is a $(k+1)$-book and, hence, the theorem follows by Lemma 4.2. □

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