THE CRITICAL PROBLEM FOR POLYMATROIDS

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1. Introduction

LET S be a collection of subspaces of $V(r, q)$, the rank-r vector space over $GF(q)$. Of the subspaces of $V(r, q)$ which contain no member of S, let U be one with maximum rank. What is the rank of U?

In the special case that the subspaces in U all have rank less than or equal to 1, the above problem is well-studied. It is, in essence, the critical problem for matroids developed by Crapo and Rota [**2**]. There it is shown that the rank of U depends only on the matroid structure of S and that it is determined by an evaluation of the characteristic polynomial of this matroid. In this paper we show that a similar result holds in the more general case. In this case, the rank of U depends only on the polymatroid structure of U and is determined by an evaluation of the characteristic polynomial of this polymatroid.

The main results, presented in Section 3, are direct polymatroid-theoretic generalisations of the standard matroid-theoretic ones. With some polymatroid-theoretic preliminaries, established in Section 2, the proofs are very simple. Given this, the material in this paper perhaps needs some justification and the remainder of this introduction is devoted to this.

Firstly, the results presented in Section 3 subsume the standard matroid-theoretic results on the critical problem in unweakened form as special cases. Now these days no mathematician believes in generalising for the sake of generalising; nonetheless, it must always be true that ideally a result should be presented at the maximum level of generality possible without weakening the conclusion.

Secondly, one of the notable features of the critical problem for matroids is that it provides a unified setting for a large number of extremal (and enumerative) problems in combinatorics. In [**12**] it is shown that the critical problem for polymatroids generalises (weak) hypergraph colouring in just the same way that the critical problem for matroids generalises graph colouring. Many combinatorial problems have natural interpretations as hypergraph colouring problems so that the number of problems which can be interpreted as critical problems is considerably increased.

Thirdly, consider inequivalent representations. Some terms used here

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are not defined until later in this paper. It is an interesting fact that the critical exponent of a set of points is a matroid invariant. It is a consequence of results in this paper that the same is true in the more general case; the critical exponent of a collection of subspaces is a polymatroid invariant so that inequivalent representations of the same polymatroid have the same critical exponent. In general, a polymatroid has very sharply inequivalent representations. For example, consider the polymatroid on a 3-element set S which takes the value 2 on all proper non-empty subsets of S and takes the value 3 on S. Over any rank-3 vector space this polymatroid can either be represented affinely by three distinct concurrent lines or by three non-concurrent lines. Note also, in contrast to the matroid case that binary 2-polymatroids are not, in general, uniquely representable.

Finally consider polymatroid theory. The well-known fact that every polymatroid is embeddable as a collection of flats of some matroid may lead one to believe that the theory of polymatroids can, in some sense, be reduced to matroid theory. The critical exponent of a collection of subspaces is determined by an evaluation of the characteristic polynomial of a polymatroid, but, in general, this polynomial is not the characteristic polynomial of any associated matroid or, indeed, of any matroid at all.

2. Preliminaries

Let S be a finite set. A *polymatroid on* S is a function $f:2^S \rightarrow Z$ which is

(i) *normalised,* that is, $f(\emptyset)=0$,
(ii) *increasing,* that is, if $A \subseteq B \subseteq S$, then $f(A) \leqslant f(B)$, and is
(iii) *submodular,* that it, if $A \subseteq S$ and $B \subseteq S$, then $f(A \cup B)+f(A \cap B) \leqslant f(A)+f(B)$.

The set S is the *ground set* of f. Note that, if, in addition, the value of f on singletons never exceeds 1, then f is a *matroid.*

Let S and E be finite sets and consider a function ϕ: $S \rightarrow 2^E$. (If $A \subseteq S$, then we denote $\bigcup_{a \in A} \phi(a)$ by $\phi(A)$.) Now let f be a polymatroid on S and r be a matroid on E. Then ϕ is a *representation of* f *in* r if, for all subsets A of S, $f(A)=r(\phi(A))$. Moreover, f is *representable over* r if there exists a representation of f in r. Let q be a prime power, then f *is representable over* $GF(q)$ if, for some positive integer n, there exists a representation of f in $V(n, q)$ (equivalently $PG(n-1, q)$).

Polymatroid representation is just like matroid representation except that elements of the ground set of the polymatroid are represented by subsets rather than singletons. Let cl denote the closure operator of the matroid r. Then, if ϕ represents f in r, the function ϕ' defined by $\phi'(a)=\mathrm{cl}\,(\phi(a))$ also represents f in r so that one can think of the elements of the polymatroid as being represented by flats of the matroid. A central well-known fact in the theory of polymatroids is that every polymatroid is representable over some matroid (see for example [4, 7]). Conversely, if S is a collection of subsets of the ground set of the matroid r, then the function f on S defined for all subsets A of S by $f(A)=r\left(\bigcup_{a\in A} a\right)$ is a polymatroid on S.

If ϕ_1 and ϕ_2 both represent the polymatroid f on S in $V(r, q)$, then ϕ_1 and ϕ_2 are *equivalent* representations of f if there exists an automorphism α: $V(r, q)\to V(r, q)$ such that, for all x in S, $\alpha(\mathrm{cl}(\phi_1(x))=\mathrm{cl}(\phi_2(x))$. The use of closure in this definition avoids trivially inequivalent representations.

Let f be a polymatroid on S and X be a subset of S. Then the *deletion* of X from S, denoted $f\backslash X$, is the polymatroid on $S-X$ defined, for all $A\subseteq S-X$, by $f\backslash X(A)=f(A)$. The *contraction* of X from S, denoted f/X, is the polymatroid on $S-X$ defined for all $A\subseteq S-X$, by

$$f/X(A)=f(A\cup X)-f(A).$$

It is easily seen that, just as with matroids, deletion and contraction commute, both with themselves and each other. The polymatroid f' is a *minor* of f if $f'=f\backslash X/Y$ for some disjoint subsets X and Y of S.

The definition given above of deletion is uncontentious. To justify the definition of contraction, note first that it generalises that of contraction in matroids. Note also, that if ϕ represents f in the matroid r on E, and if $s\in S$, then the function ϕ': $S-s\to 2^{E-\phi(s)}$ defined, for all elements x of S, by $\phi'(x)=\phi(x)-\phi(s)$ is a representation of f/s in $r/\phi(s)$.

The *closure operator* of a polymatroid f on S is the function cl: $2^S\to 2^S$ defined, for all $A\subseteq S$, by

$$\mathrm{cl(A)}=\{x\in S: f(A\cup x)=f(A)\}.$$

It is easily seen (and well known, see for example [2, Ch. 7]) that cl is increasing and idempotent. A set A is a *flat* of f if $\mathrm{cl}(A)=A$. When ordered by inclusion, the flats of f form a lattice, but without additional structure this lattice carries little of the significant information of the polymatroid. Note that Dilworth [3, Ch. 7] has shown that every finite lattice can be obtained from some polymatroid in the above way.

A *weighted lattice* is a pair (L, g) where L is a finite lattice and g is a function from L into the integers with the properties that

(i) $g(x)\leq g(y)$ whenever x and y are elements of L with $x<y$, and
(ii) $g(\hat{0})=0$.

If (L, g) is a weighted lattice, then L and g are the *underlying lattice* and *underlying function* of (L, g) respectively. Now let f be a polymatroid on

S. Then associated with f is a weighted lattice which we denote by L_f. The underlying lattice of L_f is the lattice of flats of f ordered by inclusion. The underlying function of L_f is the restriction of f to the flats of f. By an abuse of notation, we denote this function by f also. The lattice just described is the *weighted lattice of flats* of f.

The weighted lattice of flats of a matroid is just the usual geometric lattice one associates with the matroid weighted by its rank function. Now lattices of flats of polymatroids are not, in general, graded lattices. Even when they are, the rank of an element determined from the grading will typically differ from its rank in the polymatroid. It is for this reason that we have, in general, to consider weighted lattices.

Let (L_1, f_1) and (L_2, f_2) be weighted lattices. Then (L_1, f_1) is *representable over* (L_2, f_2) if there exists an injective function ϕ: $L_1 \to L_2$ with the properties that, if x and y are elements of L_1, then $f_1(x) = f_2(\phi(x))$, and $\phi(x \vee y) = \phi(x) \vee \phi(y)$. An injective function ϕ with these properties is a *representation* of (L_1, f_1) in (L_2, f_2). Since ϕ is injective and preserves joins, it follows that $\phi(x) \leq \phi(y)$ if and only if $x \leq y$. It is then evident that the subset $\phi(L_1)$ of L_2 forms a lattice isomorphic to L_1. Note that while joins are always preserved by ϕ, meets are not, so that $\phi(L_2)$ is generally not a sublattice of L_1. The routine proof of the following proposition is omitted.

PROPOSITION 2.1. *Let f be a polymatroid and r be a matroid. Let L_f and L_r denote their weighted lattices of flats. Then f is representable over r if and only if L_f is representable over L_r.*

The *characteristic polynomial,* denoted $P(f; \lambda)$, of the polymatroid f on S is defined by

$$P(f; \lambda) = \sum_{A \subseteq S} (-1)^{|A|} \lambda^{f(S) - f(A)}.$$

Characteristic polynomials of polymatroids generalise those of matroids, where they are often called chromatic polynomials [9, Ch. 15]. As with matroids, characteristic polynomials of polymatroids satisfy a deletion-contraction recursion. A routine computation proves

PROPOSITION 2.2. *For all elements a of S,*

$$P(f; \lambda) = \lambda^{f(S) - f(S-a)} P(f \backslash a; \lambda) - P(f/a; \lambda).$$

A *loop* of the polymatroid f is an element a with $f(\{a\}) = 0$. It is easily seen that if f has a loop, then $P(f; \lambda) = 0$.

Let (L, f) be a weighted lattice and let μ denote the Möbius function of L. Then the *characteristic polynomial* of (L, f), denoted $P((L, f); \lambda)$, is defined by

$$P((L, f); \lambda) = \sum_{x \in L} \mu(\hat{0}, x) \lambda^{f(\hat{1}) - f(x)}.$$

Characteristic polynomials of weighted lattices are studied in a general setting in [**11**]. The following proposition is a special case of [**11**, Theorem 3.2].

PROPOSITION 2.3. *Let f be a polymatroid and L_f be its weighted lattice of flats. If f is loopless then*

$$P(f;\lambda) = P(L_f;\lambda).$$

3. Main results

Consider $V(r, q)$, the rank-r vector space over $GF(q)$. Let A be a collection of subsets of the ground set of $V(r, q)$. A k-tuple of linear functionals $(L_1, \ldots, L_k)$ on $V(r, q)$ is said to *distinguish* the members of A if no member of A is contained in $\bigcap_{i=1}^{k} \operatorname{Ker}(L_i)$. Recall that the polymatroid f is *representable* over $GF(q)$ if there exists a representation of f in $V(r, q)$. Let f be such a polymatroid and let ϕ' be a representation of f in $V(r, q)$. Let L_f denote the weighted lattice of flats of f. Define the function ϕ: $L_f \to V(r, q)$ by $\phi(F) = cl_{V(r,q)}(\phi'(F))$, for all flats F of f. Evidently, ϕ is a representation of L_f in the weighted lattice of flats of $V(r, q)$, which we also denote by $V(r, q)$. Also evident is the fact that $(L_1, \ldots, L_k)$ distinguishes the members of $\{\phi'(S): s \in S\}$ if and only if $(L_1, \ldots, L_k)$ distinguishes the members of $\{\phi(F): F \in L_f\}$.

THEOREM 3.1. *Let f be a polymatroid on S representable over $GF(q)$ and let ϕ' be a representation of f in $V(r, q)$. Then the number of k-tuples of linear functionals on $V(r, q)$ which distinguish the members of $\{\phi'(s): s \in S\}$ is given by $(q^k)^{r-f(S)}P(f; q^k)$.*

Proof. If f has a loop, the result is immediate, so assume that f is loopless. Let L_f denote the weighted lattice of flats of f and let ϕ be the associated representation of L_f in $V(r, q)$ (as discussed in the preamble to the theorem). For $x \in L_f$, let $\alpha(k, x)$ denote the number of k-tuples of linear functionals $(L_1, \ldots, L_k)$ for which $\phi(x) \le \bigcap_{i=1}^{k} \operatorname{Ker}(L_i)$. It is easily seen (and shown in [**13**, Theorem 7.6.1]) that $\alpha(k, x) = (q^k)^{r-r(\phi(x))}$. But $r(\phi(x)) = f(x)$, so $\alpha(k, x) = (q^k)^{r-f(x)}$. Since ϕ is a representation of L_f in $V(r, q)$, it follows that if u and v are members of $\phi(L_f)$, then $u \vee v$ is a member of $\phi(L_f)$. Hence any subspace of $V(r, q)$ contains a unique maximal member of $\phi(L_f)$. For x in L_f, let $\beta(k, x)$ denote the number of k-tuples of linear functionals $(L_1, \ldots, L_k)$ for which $\phi(x)$ is the maximal member of $\phi(L_f)$ contained in $\bigcap_{i=1}^{k} \operatorname{Ker}(L_i)$. Since $x \le y$ if and only if

$\phi(x) \leq \phi(y)$, it now follows that

$$\alpha(k, x) = \sum_{y \in L_f;\, x \leq y} \beta(k, y).$$

Let μ denote the Möbius function of L_f. Then applying Möbius inversion and setting $y = 0$ we see that

$$\begin{aligned}\beta(k, 0) &= \sum_{x \in L_f} \mu(0, x)\alpha(k, x) \\ &= \sum_{x \in L_f} \mu(0, x)(q^k)^{r-f(x)} \\ &= (q^k)^{r-f(S)} P(L_f; q^k).\end{aligned}$$

Since f is loopless, $\beta(k, 0)$ is equal to the number of k-tuples of linear functionals on $V(r, q)$ which distinguish the members of $\{\phi(x): x \in L_f\}$, and this is the number which distinguish the members of $\{\phi'(s): s \in S\}$. Also, since f is loopless, $P(L_f; \lambda) = P(f; \lambda)$, and the theorem is proved.

The above proof is similar to that of the special case for matroids given in Crapo and Rota [**2**, Theorem 16.1] (see also Zaslavsky [**13**, Theorem 7.6.1]). A proof could also be given using a deletion-contraction argument for the characteristic polynomial. This proof would generalise that of Welsh [**9**, Theorem 15.5.1]. Yet another proof could be given using the techniques of Kung [**5**].

Theorem 3.1 is also intimately related to a theorem of Kung, Murty and Rota [**6**, Theorem 10]. This theorem deals with collections of subgroups of a given finite abelian group. While Theorem 3.1 deals with linear functionals (homomorphisms into the underlying field) [**6**, Theorem 10] is concerned with enumerating k-tuples of distinguishing group characters (homomorphisms into the complex unit circle).

COROLLARY 3.2. *Let f be a polymatroid representable over $GF(q)$. Then the minimal number k for which there exists a k-tuple of linear functionals on $V(r, q)$ which distinguish the members of any representation of f in $V(r, q)$ is independent of the representation and is given by the least positive integer k for which $P(f; q^k) > 0$.*

As with matroids, the number k given in Corollary 3.2 is the *critical exponent* of f over q, denoted $c(f; q)$. Since the kernel of a linear functional is a hyperplane, we also have,

COROLLARY 3.3. *Let f be a polymatroid on S representable over $GF(q)$ and let ϕ be a representation of f in S. Then $c(f; q)$ is equal to the minimum number k for which there exists a k-tuple $(H_1, \ldots, H_k)$ of hyperplanes of $V(r, q)$ with the property that $\bigcap_{i=1}^{k} H_i$ contains no member of $\{\phi(s): s \in S\}$.*

We now return to the question asked at the beginning of this paper. Let S be a collection of subsets of $V(r, q)$, then the *polymatroid determined by* S is the polymatroid on S whose value on any subset A of S is equal to the rank in $V(r, q)$ of the span of A. An easy argument now proves

COROLLARY 3.4. *Let S be a collection of subspaces of $V(r, q)$ and let f be the polymatroid determined by S. Then the rank of a maximum ranked subspace U of $V(r, q)$ with the property that U contains no member of S is equal to $r-k$ where k is the least positive integer with the property that $P(f; q^k) > 0$.*

A less natural, but slightly more general, version of Corollary 3.4 can be obtained by letting S be a collection of subsets of $V(r, q)$ rather than a collection of subspaces. Clearly, a subspace U of $V(r, q)$ contains a given subset if and only if it contains the span of that subset.

4. Boolean polymatroids and Rédei functions

While the matroids that are representable over free matroids are trivial—their connected components have rank at most one—the polymatroids representable over free matroids form an interesting class. One can develop theory analogous to that of the previous section for this class.

In this section we blur the distinction between a free matroid and its ground set, and use the same symbol, typically X, to denote both. Since in a free matroid, cardinality is equal to rank, a polymatroid f on E is representable over the free matroid X if and only if there exists a function $\phi: E \to 2^X$ with the property that $f(A) = |\phi(A)|$ for all subsets A of E. Following **[8]**, we call a polymatroid *Boolean* if it is representable over some free matroid.

Assume then, that f is Boolean and is represented in X by ϕ, and that $|X| = r$. Of course, the flats of X are just the subsets of X and, in particular, the hyperplanes of X are just those subsets of cardinality $r-1$. We say that f *spans* X if $f(E) = |X|$. Naturally, the k-tuple of hyperplanes $(H_1, \ldots, H_k)$ *distinguishes* the members of $\{\phi(e): e \in E\}$ if $\bigcap_{i=1}^{k} H_i$ contains no member of $\{\phi(e): e \in E\}$. An elementary argument shows that if A is a subset of X, then the number of k-tuples of hyperplanes $(H_1, \ldots, H_k)$ of X with the property that A is contained in $\bigcap_{i=1}^{k} H_i$ is equal to $(r-|A|)^k$.

A straightforward application of Möbius inversion similar to that of Theorem 3.1 now proves that if f is loopless and spans X, then the