JOURNAL OF COMBINATORIAL THEORY, Series B 37, 94–95 (1984)

Note

On the Critical Exponent of Transversal Matroids

GEOFFREY P. WHITTLE

Mathematics Department, University of Tasmania, G. P. O. Box 252C, HOBART. Tasmania, 7001

Communicated by the Managing Editors

Received September 1, 1983

It is shown that loopless transversal matroids have critical exponent at most 2.

Brylawski [2] has shown that loopless principal transversal matroids have critical exponent at most 2. Welsh [4] asks if a similar result holds for all transversal matroids. We answer in the affirmative by proving that all loopless transversal matroids have critical exponent at most 2.

The terminology used here for matroids will in general follow Welsh [3]. A rank r transversal matroid M is representable over $GF(q)$ for some q. In this case the canonical simple matroid associated with M is isomorphic to a submatroid of $PG(r-1, q)$ and if M is loopless this submatroid and M have the same critical exponent $c(M; q)$. So without loss of generality we may consider only transversal submatroids of $PG(r-1, q)$. A cyclic flat is one which is a union of circuits.

LEMMA. *If a matroid M has a submatroid M' which meets all cyclic flats of M, then $c(M; q) \leqslant c(M'; q) + 1$.*

Proof. If M has a loop the result is immediate. If not we may assume that there exist sets E and $E' \subseteq E$ with $M = PG(r-1, q) \mid E$ and $M' = PG(r-1, q) \mid E'$.

If $c(M'; q) = k$ there are hyperplanes $H_1, \ldots, H_k$ of $PG(r-1, q)$ with $(\bigcap_{i=1}^{k} H_i) \cap E' = \varnothing$. If $(\bigcap_{i=1}^{k} H_i) \cap E$ contains a circuit C it contains $\mathrm{cl}_M(C)$, a cyclic flat and so contains a point of E'. So $(\bigcap_{i=1}^{k} H_i) \cap E$ is independent. It is routine to show that there is a hyperplane of $PG(r-1, q)$ missing an independent set. Say H_{k+1} is such a hyperplane, then $H_{k+1} \cap (\bigcap_{i=1}^{k} H_i \cap E) = (\bigcap_{i=1}^{k+1} H_i) \cap E = \varnothing$, and therefore $c(M; q) \leqslant k+1$.

0095-8956/84 $3.00

LEMMA. *If M is a transversal matroid of rank r defined on set E with presentation $(A_1, A_2, \ldots, A_r)$ then all proper cyclic flats of M can be obtained by taking intersections of members of the family $\{E - A_1, \ldots, E - A_r\}$.*

Proof. An immediate consequence of [2, Proposition 3.1 and Corollary 3.1].

THEOREM. *If M is a loopless transversal matroid then M has critical exponent at most 2.*

Proof. Let $M = PG(r-1, q) \mid E$ be a spanning subgeometry of $PG(r-1, q)$. M has a presentation $\{C_1, \ldots, C_r\}$ consisting of co-circuits [1, Theorem 1]. All proper cyclic flats of M are obtained by taking intersections of members of the set $\{E - C_1, \ldots, E - C_r\}$ of hyperplanes of M. Let H_i be the unique hyperplane of $PG(r-1, q)$ containing $E - C_i$ and consider $\bigcap_{i \in I} H_i$, where $I = \{1, \ldots, r\}$.

$$\text{If } \bigcap_{i \in I} H_i = \varnothing \text{ then for } j \in I \text{ let } v_j = \bigcap_{i \in I - j} H_i.$$

It is routine to show that v_j is a point of $PG(r-1, q)$ and that $V = \{v_1, \ldots, v_r\}$ spans and hence is a basis. If F is a cyclic flat of $PG(r-1, q) \mid (E \cup V)$ and $F \cap V = \varnothing$ then F is a proper cyclic flat of M and $F = \bigcap_{j \in J} (E - C_j) = E \cap (\bigcap_{j \in J} H_j)$ for some $J \subseteq I$. But $\mathrm{cl}_{(E \cup V)} F = (E \cup V) \cap (\bigcap_{j \in J} H_j)$ so that F contains $\{v_i \mid i \notin J\}$, which contradicts the assumption that F is a cyclic flat of $PG(r-1, q) \mid (E \cup V)$. So V meets all cyclic flats of $PG(r-1, q) \mid (E \cup V)$ and since $PG(r-1, q) \mid V$ is affine we have $c(PG(r-1, q) \mid (E \cup V); q) \leqslant 2$ and therefore $c(M; q) \leqslant 2$.

On the other hand if $\bigcap_{i \in I} H_i \neq \varnothing$ we let $v \in \bigcap_{i \in I} H_i$ and by a similar argument it can be shown that v meets all cyclic flats of $PG(r-1, q) \mid (E \cup V)$. In either case $c(M; q) \leqslant 2$.

REFERENCES

1. J. A. BONDY AND D. J. A. WELSH, Some results on transversal matroids and constructions for identically self-dual matroids, *Quart. J. Math. Oxford Ser.* (2) **22** (1971), 435–451.
2. T. H. BRYLAWSKI, An affine representation for transversal geometries, *Stud. Appl. Math.* **54** (1975), 143–160.
3. D. J. A. WELSH, "Matroid Theory," Academic Press, New York/London, 1976.
4. D. J. A. WELSH, Colouring problems and matroids, preprint.