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AN OBSTACLE TO A DECOMPOSITION THEOREM FOR NEAR-REGULAR MATROIDS*

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Abstract. Seymour's decomposition theorem [*J. Combin. Theory Ser. B*, 28 (1980), pp. 305–359] for regular matroids states that any matroid representable over both GF(2) and GF(3) can be obtained from matroids that are graphic, cographic, or isomorphic to R_{10} by 1-, 2-, and 3-sums. It is hoped that similar characterizations hold for other classes of matroids, notably for the class of near-regular matroids. Suppose that all near-regular matroids can be obtained from matroids that belong to a few basic classes through k-sums. Also suppose that these basic classes are such that, whenever a class contains all graphic matroids, it does not contain all cographic matroids. We show that, in that case, 3-sums will not suffice.

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1. Introduction.

A regular matroid is a matroid representable over every field. Much is known about this class, the deepest result being Seymour's decomposition theorem.

THEOREM 1.1 (Seymour [16]). *Let M be a regular matroid. Then M can be obtained from matroids that are graphic, cographic, or equal to R_{10} through* 1-*,* 2-*, and* 3-*sums.*

A class $\mathcal{C}$ of matroids is *polynomial-time recognizable* if there exists an algorithm that decides, for any matroid M, in time $f(|E(M)|, \tau)$ whether or not $M \in \mathcal{C}$, where τ is the time of one rank evaluation and $f(x, y)$ a polynomial. Seymour [17] showed that the class of graphic matroids is polynomial-time recognizable. Also every finite class is polynomial-time recognizable. Using these facts, Truemper [18] (see also Schrijver [14, Chapter 20]) showed the following.

THEOREM 1.2. *The class of regular matroids is polynomial-time recognizable.*

A *near-regular matroid* is a matroid representable over every field, except possibly GF(2). Near-regular matroids were introduced by Whittle [19, 20]. The following is one of his results.

THEOREM 1.3 (Whittle [20]). *Let M be a matroid. The following are equivalent:*

i. *M is representable over* GF(3)*,* GF(4)*, and* GF(5)*;*
ii. *M is representable over $\mathbb{Q}(\alpha)$ by a totally near-unimodular matrix;*
iii. *M is near-regular.*

In this theorem α is an indeterminate. A *totally near-unimodular matrix* is a matrix over $\mathbb{Q}(\alpha)$ such that the determinant of every square submatrix is either zero or

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equal to $(-1)^s\alpha^i(1-\alpha)^j$ for some $s, i, j \in \mathbb{Z}$. Whittle [20, 21] wondered if an analogue of Theorem 1.1 would hold for the class of near-regular matroids. The following conjecture was made.

CONJECTURE 1.4. *Let M be a near-regular matroid. Then M can be obtained from matroids that are signed-graphic, their duals, or members of some finite set through* 1-*,* 2-*, and* 3-*sums.*

A matroid is *signed-graphic* if it can be represented by a GF(3)-matrix with at most two nonzero entries in each column. (See Zaslavsky [22, 23] for more on these matroids.) One difference with the regular case is that not every signed-graphic matroid is near-regular.

Several people have made an effort to understand the structure of near-regular matroids. Oxley, Vertigan, and Whittle [8] studied maximum-sized near-regular matroids. Hliněný [5] and Pendavingh [10] have both written software to investigate all 3-connected near-regular matroids up to a certain size. Pagano [9] studied signed-graphic near-regular matroids, and Pendavingh and Van Zwam [11] studied a closely related class of matroids which they call near-regular-graphic.

Despite these efforts, an analogue to Theorem 1.1 is still not in sight. In this paper we record an obstacle we found that will have to be taken into account in any structure theorem. Our result is the following.

THEOREM 1.5. *Let G_1, G_2 be graphs. There exists an internally* 4*-connected near-regular matroid M having both $M(G_1)$ and $M(G_2)^*$ as minors.*

From this, and the fact that not all cographic matroids are signed-graphic, it follows that Conjecture 1.4 is false. More generally, suppose we want to find a decomposition theorem for near-regular matroids such that each basic class that contains all graphic matroids does not contain all cographic matroids. Theorem 1.5 implies that such a characterization must employ at least 4-sums.

The paper is organized as follows. In section 2 we give some preliminary definitions. In section 3 we prove a lemma that shows how generalized parallel connection can preserve representability over a partial field. In section 4 we prove Theorem 1.5. We conclude in section 5 with some updated conjectures.

Throughout this paper we assume familiarity with matroid theory as set out in Oxley [7].

2. Preliminaries.

In this section we give definitions and results on connectivity, partial fields, and the manipulation of partial-field matrices through pivoting. This section contains no new material but introduces notation that may be unfamiliar to the reader.

2.1. Connectivity.

In addition to the usual definitions of connectivity and separations (see Oxley [7, Chapter 8]), we say a partition (A, B) of the ground set of a matroid is *k-separating* if $\mathrm{rk}_M(A) + \mathrm{rk}_M(B) - \mathrm{rk}(M) < k$. Recall that (A, B) is a *k-separation* if it is k-separating and $\min\{|A|, |B|\} \geq k$.

DEFINITION 2.1. *A matroid is* internally 4-connected *if it is* 3*-connected and* $\min\{|X|, |Y|\} = 3$ *for every* 3*-separation* (X, Y).

This notion of connectivity is useful in our context. For instance, Theorem 1.1 can be rephrased as follows.

THEOREM 2.2. *Let M be an internally* 4*-connected regular matroid. Then M is graphic, cographic, or equal to R_{10}.*

Intuitively, separations (X, Y), where both $|X|$ and $|Y|$ are big, should give rise to a decomposition into smaller matroids.

DEFINITION 2.3. *Let M be a matroid and N a minor of M. Let (X', Y') be a k-separation of N. We say that (X', Y') is* induced *in M if M has a k-separation (X, Y) such that $X' \subseteq X$ and $Y' \subseteq Y$.*

At several points we will use the following easy fact.

LEMMA 2.4. *Let M be a matroid, let N be a minor of M, and let (A, B) be a k-separating partition of $E(M)$. Then $(A \cap E(N), B \cap E(N))$ is k-separating in N.*

Note that $(A \cap E(N), B \cap E(N))$ need not be *exactly* k-separating.

2.2. Partial fields. Our main tool in the proof of Theorem 1.5 is useful outside the scope of this paper. Hence we have stated it in the general framework of partial fields. For that purpose we need a few definitions. More on the theory of partial fields can be found in Semple and Whittle [15] and in Pendavingh and Van Zwam [13, 12].

DEFINITION 2.5. *A* partial field *is a pair (R, G), where R is a commutative ring with identity and G is a subgroup of the group of units of R such that $-1 \in G$.*

For example, the near-regular partial field is $(\mathbb{Q}(\alpha), \langle -1, \alpha, 1-\alpha \rangle)$, where $\langle S \rangle$ denotes the multiplicative group generated by S. For $\mathbb{P} = (R, G)$, we abbreviate $p \in G \cup \{0\}$ to $p \in \mathbb{P}$.

We will adopt the convention that matrices have labelled rows and columns, so an $X \times Y$ matrix A is a matrix whose rows are labelled by the (ordered) set X and whose columns are labelled by the (ordered) set Y. The identity matrix with rows and columns labelled by X will be denoted by I_X. We will omit the subscript if it can be deduced from the context.

Let A be an $X \times Y$ matrix. If $X' \subseteq X$ and $Y' \subseteq Y$, then we denote the submatrix of A indexed by X' and Y' by $A[X', Y']$. If $Z \subseteq X \cup Y$, then we write $A[Z] := A[X \cap Z, Y \cap Z]$. If A is an $X \times Y$ matrix, where $X \cap Y = \emptyset$, then we denote by $[I\ A]$ the $X \times (X \cup Y)$ matrix obtained from A by prepending the identity matrix I_X.

DEFINITION 2.6. *Let $\mathbb{P} := (R, G)$ be a partial field, and let A be a matrix with entries in R. Then A is a* $\mathbb{P}$-matrix *if, for every square submatrix A' of A, either $\det(A') = 0$ or $\det(A') \in G$.*

THEOREM 2.7. *Let $\mathbb{P}$ be a partial field, let A be an $X \times Y$ $\mathbb{P}$-matrix for disjoint sets X and Y, let $E := X \cup Y$, and let $A' := [I\ A]$. If $\mathcal{B} = \{B \subseteq E : |B| = |X|, \det(A'[X, B]) \neq 0\}$, then $\mathcal{B}$ is the set of bases of a matroid.*

We denote this matroid by $M[I\ A]$.

2.3. Pivoting. Let A be an $X \times Y$ $\mathbb{P}$-matrix. Then X is a basis of $M[I\ A]$. We say that X is the *displayed* basis. Pivoting in the matrix allows us to change the basis that is displayed. Roughly speaking, a pivot in A consists of row reduction applied to $[I\ A]$, followed by a column exchange. The precise definition is as follows.

DEFINITION 2.8. *Let A be an $X \times Y$ matrix over a ring R, and let $x \in X, y \in Y$ be such that $A_{xy} \in R^*$. Then A^{xy} is the $(X - x) \cup y \times (Y - y) \cup x$ matrix with entries*

$$(A^{xy})_{uv} = \begin{cases} (A_{xy})^{-1} & \text{if } uv = yx, \\ (A_{xy})^{-1} A_{xv} & \text{if } u = y, v \neq x, \\ -A_{uy}(A_{xy})^{-1} & \text{if } v = x, u \neq y, \\ A_{uv} - A_{uy}(A_{xy})^{-1} A_{xv} & \text{otherwise.} \end{cases}$$

We say that A^{xy} was obtained from A by *pivoting*. Slightly less opaquely, if

$$A = \begin{array}{c} \\ x \\ X' \end{array} \begin{array}{c} \begin{array}{cc} y & Y' \end{array} \\ \left[\begin{array}{cc} a & c \\ b & D \end{array} \right] \end{array},$$

then

$$A^{xy} = \begin{array}{c} \\ y \\ X' \end{array} \begin{array}{c} \begin{array}{cc} x & Y' \end{array} \\ \left[\begin{array}{cc} a^{-1} & a^{-1}c \\ -ba^{-1} & D - ba^{-1}c \end{array}\right] \end{array}.$$

As Semple and Whittle [15] proved, pivoting maps $\mathbb{P}$-matrices to $\mathbb{P}$-matrices.

Proposition 2.9. *Let A be an $X \times Y$ $\mathbb{P}$-matrix, and let $x \in X, y \in Y$ be such that $A_{xy} \neq 0$. Then A^{xy} is a $\mathbb{P}$-matrix, and $M[I\ A] = M[I\ A^{xy}]$.*

Semple and Whittle also showed that pivots can be used to compute determinants of $\mathbb{P}$-matrices.

Lemma 2.10. *Let $\mathbb{P}$ be a partial field, and let A be an $X \times Y$ $\mathbb{P}$-matrix with $|X| = |Y|$. If $x \in X, y \in Y$ is such that $A_{xy} \neq 0$, then*

$$\det(A) = (-1)^{x+y} A_{xy} \det(A^{xy}[X - x, Y - y]).$$

3. Generalized parallel connection. Recall the generalized parallel connection of two matroids M_1, M_2 along a common restriction N, denoted by $P_N(M_1, M_2)$. This construction was introduced by Brylawski [1] (see also Oxley [7, section 12.4]). Brylawski proved that representability over a field can be preserved under generalized parallel connection, provided that the representations of the common minor are identical. Lee [6] generalized Brylawski's result to matroids representable over a field such that all subdeterminants are in a multiplicatively closed set. We generalize Brylawski's result further to matroids representable over a partial field, as follows.

Theorem 3.1. *Suppose A_1, A_2 are $\mathbb{P}$-matrices with the following structure:*

$$A_1 = \begin{array}{c} \\ X_1 \\ X \end{array} \begin{array}{c} \begin{array}{cc} Y_1 & Y \end{array} \\ \left[\begin{array}{cc} D_1' & 0 \\ D_1 & D_X \end{array}\right] \end{array}, \qquad A_2 = \begin{array}{c} \\ X \\ X_2 \end{array} \begin{array}{c} \begin{array}{cc} Y & Y_2 \end{array} \\ \left[\begin{array}{cc} D_X & D_2 \\ 0 & D_2' \end{array}\right] \end{array},$$

where X, Y, X_1, Y_1, X_2, Y_2 are pairwise disjoint sets. If $X \cup Y$ is a modular flat of $M[I\ A_1]$, then

$$A := \begin{array}{c} \\ X_1 \\ X \\ X_2 \end{array} \begin{array}{c} \begin{array}{ccc} Y_1 & Y & Y_2 \end{array} \\ \left[\begin{array}{ccc} D_1' & 0 & 0 \\ D_1 & D_X & D_2 \\ 0 & 0 & D_2' \end{array}\right] \end{array}$$

is a $\mathbb{P}$-matrix. Moreover, if $M_1 = M[I\ A_1]$ and $M_2 = M[I\ A_2]$, then $M[I\ A] = P_N(M_1, M_2)$, where $N = M[I\ D_X]$.

The main difficulty is to show that A is a $\mathbb{P}$-matrix. To prove this we will use a result known as the modular short-circuit axiom [1, Theorem 3.11]. We use Oxley's formulation [7, Theorem 6.9.9], and refer to that book for a proof.

Lemma 3.2. *Let M be a matroid and $X \subseteq E$ nonempty. The following statements are equivalent:*

i. *X is a modular flat of M.*
ii. *For every circuit C such that $C - X \neq \emptyset$, there is an element $x \in X$ such that $(C - X) \cup x$ is dependent.*
iii. *For every circuit C, and for every $e \in C - X$, there are an $f \in X$ and a circuit C' such that $e \in C'$ and $C' \subseteq (C - X) \cup f$.*

The following is an extension of Proposition 4.1.2 in [1] to partial fields. Note that Brylawski proves an "if and only if" statement, whereas we state only the "only if" direction.

LEMMA 3.3. *Let $M = (E, \mathcal{I})$ be a matroid and X a modular flat of M. Suppose B_X is a basis for $M|X$ and $B \supseteq B_X$ a basis of M. Suppose A is a $B \times (E - B)$ $\mathbb{P}$-matrix such that $M = M[I\ A]$. Then every column of $A[B_X, E - (B \cup X)]$ is a $\mathbb{P}$-multiple of a column of $[I\ A[B_X, X - B]]$.*

Proof of Lemma 3.3. Let M, X, B_X, B, A be as in Lemma 3.3, so

$$A = \begin{array}{c} \\ {\scriptstyle B-B_X} \\ {\scriptstyle B_X} \end{array} \begin{array}{c} \begin{array}{cc} {\scriptstyle E-(B\cup X)} & {\scriptstyle X-B} \end{array} \\ \left[\begin{array}{cc} D' & 0 \\ D & D_{B_X} \end{array}\right] \end{array}.$$

Let $v \in E-(B\cup X)$, and let C be the B-fundamental circuit containing v. If $C\cap X = \emptyset$, then $D[B_X, v]$ is an all-zero vector and the result holds, so assume $B_X \cap C \neq \emptyset$. By Lemma 3.2(iii) there are an $x \in X$ and a circuit C' with $v \in C'$ and $C' \subseteq (C - X) \cup x$.

Let $M' := M/(B - B_X)$. Then $C' \cap E(M') = \{v, x\}$ is a circuit of M'. Hence all 2×2 subdeterminants of $[I\ A][B_X, \{v, x\}]$ have to be 0, which implies that $A[B_X, v]$ is the all-zero vector or parallel to $[I\ A][B_X, x]$. □

Proof of Theorem 3.1. Let A_1, A_2, A be as in the theorem, and define $E := X_1 \cup X_2 \cup X \cup Y_1 \cup Y_2 \cup Y$. Suppose there exists a $Z \subseteq E$ such that $A[Z]$ is square, yet $\det(A[Z]) \notin \mathbb{P}$. Assume A_1, A_2, A, Z were chosen so that $|Z|$ is as small as possible.

If $Z \subseteq X_i \cup Y_i \cup X \cup Y$ for some $i \in \{1, 2\}$, then $A[Z]$ is a submatrix of A_i, a contradiction. Therefore we may assume that Z meets both $X_1 \cup Y_1$ and $X_2 \cup Y_2$. We may also assume that $A[Z]$ contains no row or column with only zero entries, so either there are $x \in X_1 \cap Z$, $y \in Y_1 \cap Z$ with $A_{xy} \neq 0$ or $x \in X \cap Z$, $y \in Y_1 \cap Z$ with $A_{xy} \neq 0$.

In the former case, pivoting over xy leaves D_X, D_2, and D_2' unchanged, yet by Lemma 2.10 $\det(A[Z]) \in \mathbb{P}$ if and only if $\det(A^{xy}[Z - \{x, y\}]) \in \mathbb{P}$. This contradicts minimality of $|Z|$. Therefore $Z \cap X_1 = \emptyset$. Similarly $Z \cap X_2 = \emptyset$.

Define $X' := Z \cap X$. Now pick some $y \in Y_1$. Since $A[X', Y_1 \cup Y]$ is obtained from $A[X, Y_1 \cup Y]$ by deleting rows, it follows from Lemma 3.3, applied to $M[I\ A_1]$, that the column $A[X', y]$ is either a unit vector (i.e., a column of an identity matrix) or parallel to $A[X', y']$ for some $y' \in Y$. In the first case, Lemma 2.10 implies again that $\det(A[Z]) \in \mathbb{P}$ if and only if $\det(A[Z - \{x, y\}]) \in \mathbb{P}$, contradicting minimality of $|Z|$. In the second case, if $y' \in Z$, then $\det(A[Z]) = 0$. Otherwise we can replace y by y' without changing $\det(A[Z])$ (up to possible multiplication with some nonzero $p \in \mathbb{P}$). It follows that $\det(A[Z]) = p' \det(A[Z'])$, where $Z' \subseteq X \cup Y \cup Y_2$ and $p' \in \mathbb{P} - \{0\}$. But $\det(A[Z']) \in \mathbb{P}$, so also $\det(A[Z]) \in \mathbb{P}$, a contradiction.

It remains to prove that $M[I\ A] = P_N(M_1, M_2)$. Suppose $\mathbb{P} = (R, G)$, and let I be a maximal ideal of R. Let $\varphi : R \to R/I$ be the canonical ring homomorphism. For a square $\mathbb{P}$-matrix D, we have $\det(D) = 0$ if and only if $\det(\varphi(D)) = 0$. Hence $M[I\ A] = M[I\ \varphi(A)]$. But R/I is a field, so the result now follows directly from Brylawski's original theorem. □

The special cases $X = \emptyset$ and $X = \{p\}$ were previously proven by Semple and Whittle [15].

4. The need for 4-sums. The core of the proof of Theorem 1.5 will be a special matroid $M_{12} := M[I\ A_{12}]$, where

$$A_{12} = \begin{array}{c} \\ a \\ b \\ c \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccccc} d & e & f & 4 & 5 & 6 \end{array} \\ \left[\begin{array}{cccccc} 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & -1 & 1 & 1 & 0 & \alpha \\ 1 & 1 & 0 & 0 & \alpha & -\alpha \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right] \end{array}. \tag{4.1}$$

LEMMA 4.1. *The following hold:*
i. M_{12} *is near-regular;*
ii. M_{12} *is internally* 4*-connected;*
iii. M_{12} *is self-dual;*
iv. $M_{12} \backslash \{1,2,3,4,5,6\} \cong M(K_4)$*;*
v. $M_{12}/\{a,b,c,d,e,f\} \cong M(K_4)$*;*
vi. *no triad of* $M_{12}\backslash\{1,2,3,4,5,6\}$ *is a triad of* M_{12}*.*

We will omit the proofs, each of which boils down to a finite case check that is easily done on a computer and is not too onerous by hand. Specifically, for the first property, one can either verify that A_{12} is totally near-unimodular or that M_{12} contains none of the excluded minors for near-regular matroids (see Hall, Mayhew, and Van Zwam [4]). The latter approach is facilitated by observing that M_{12} is the signed-graphic matroid associated with the signed graph illustrated in Figure 4.1. That graph can also be used to verify (ii) by examining all edge-partitions (A, B) that meet in two or three vertices. The remaining properties are readily extracted from the matrix A_{12}.

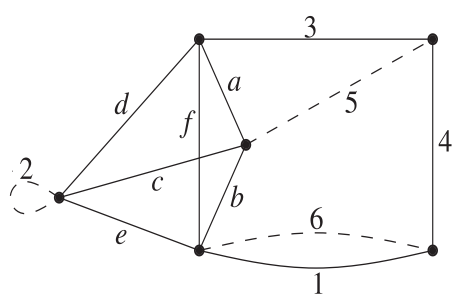

FIG. 4.1. *Signed-graphic representation of* M_{12}*. Negative edges are dashed; positive edges are solid.*

We will use the $M(K_4)$-restriction to create the generalized parallel connection of M_{12} with $M(K_n)$. The following is well known, but we will include the short proof.

LEMMA 4.2. *The matroid* $M(K_n)$ *is internally* 4*-connected.*

Proof. Fix an integer n, and suppose (A, B) is a 3-separation of $M(K_n)$ with $|A|, |B| \geq 4$. It follows that $n \geq 5$. Assume that $\text{rk}(A) \geq \text{rk}(B)$. Note that $\text{cl}(A)$ and $\text{cl}(B)$ induce complete subgraphs of K_n and that these subgraphs meet in at most three vertices. It follows that, for some vertex v of K_n, all edges incident with v are in A, or all edges are in B. Assume the former. Then $\text{cl}(A) = E(K_n)$, and therefore $\text{rk}(A) = n - 1$ and $\text{rk}(B) = 2$. But then B is a subset of a triangle of K_n, a contradiction. □

We need to show that, in forming the generalized parallel connection, we do not introduce unwanted 3-separations. The following lemma takes care of this.

LEMMA 4.3. *Let $M_1 = M(K_n)$ for some $n \geq 5$ and M_2 an internally 4-connected matroid such that there is a set $X = E(M_1) \cap E(M_2)$ with $N := M_1|X = M_2|X \cong M(K_4)$. Then $M := P_N(M_1, M_2)$ is a well-defined matroid. If no triad of N is a triad of M_2, then M is internally 4-connected.*

Proof. It is well known (see [7, page 236]) that N is a modular flat of M_1. Hence $M = P_N(M_1, M_2)$ is well defined. It remains to prove that M is internally 4-connected. Suppose not. M is obviously connected. Suppose (A, B) is a 2-separation of M. By relabelling we may assume $|A \cap E(M_1)| \geq |B \cap E(M_1)|$. By Lemma 2.4 we have that $(A \cap E(M_1), B \cap E(M_1))$ is 2-separating in M_1 (since M_1 is a restriction of M). But M_1 is 3-connected, so $|B \cap E(M_1)| \leq 1$. Similarly we have either $|A \cap E(M_2)| \leq 1$ or $|B \cap E(M_2)| \leq 1$. Since $|E(M_1) \cap E(M_2)| = 6$, the latter must hold. Hence $B = \{e, f\}$ for some $e \in E(M_1) - E(N)$ and $f \in E(M_2) - E(N)$. Since $E(M_1)$ and $E(M_2)$ are flats of M, we have $\mathrm{rk}_M(\{e, f\}) = 2$. Moreover $e \in \mathrm{cl}_M(E(M_1) - e)$ and $f \in \mathrm{cl}_M(E(M_2) - f)$, so $\{e, f\} \subseteq \mathrm{cl}_M(A)$. But then

$$\mathrm{rk}_M(A) + \mathrm{rk}_M(B) - \mathrm{rk}(M) = \mathrm{rk}_M(B) = 2, \tag{4.2}$$

contradicting the fact that (A, B) is a 2-separation.

Next suppose that (A, B) is a 3-separation of M with $|A| \geq 4$ and $|B| \geq 4$. By relabelling we may assume $|A \cap E(M_1)| \geq |B \cap E(M_1)|$. By Lemma 2.4 again, $(A \cap E(M_1), B \cap E(M_1))$ is 3-separating in M_1. Since M_1 is internally 4-connected, $|B \cap E(M_1)| \leq 3$. Define $T := B \cap E(M_1)$.

We will show that $T \subseteq \mathrm{cl}_M(B - T)$. Since M_1 has no cocircuits of size less than 4, we have $T \subseteq \mathrm{cl}_M(A)$. Therefore

$$\begin{aligned} \mathrm{rk}_M(A \cup T) + \mathrm{rk}_M(B - T) - \mathrm{rk}(M) &= \mathrm{rk}_M(A) + \mathrm{rk}_M(B - T) - \mathrm{rk}(M) \\ &\leq \mathrm{rk}_M(A) + \mathrm{rk}_M(B) - \mathrm{rk}(M) = 2. \end{aligned} \tag{4.3}$$

If $|B - T| \geq 2$, then it follows from 3-connectivity that equality holds in (4.3), so $\mathrm{rk}_M(B) = \mathrm{rk}_M(B - T)$. If $|B - T| = 1$, then $\mathrm{rk}_M(B - T) = 1$, and we must have $\mathrm{rk}_M(B) = 2$. In that case, T is a triangle of M_1, and some element $e \in E(M_2) - E(M_1)$ is in the closure of T. But no such element e exists since $E(M_1)$ is a flat of M.

Note that $B - T \subseteq E(M_2)$. Since $T \subseteq \mathrm{cl}_M(B - T)$ and $E(M_2)$ is a flat of M, we have that $T \subseteq E(M_2)$. Hence $T \subseteq E(N)$, and $B \cap E(M_2) = B$. Since $(A \cap E(M_2), B \cap E(M_2))$ is 3-separating and $|B \cap E(M_2)| = |B| \geq 4$, we have $|A \cap E(M_2)| \leq 3$. But $|B \cap E(M_1)| \leq 3$, and therefore $E(N) - B \subseteq A \cap E(M_2)$, from which it follows that $|A \cap E(M_2)| \geq 3$.

Since no triad of N is a triad of M_2, we must have that $A \cap E(M_2)$ is a triangle of M_2. Hence $B \cap E(N)$ is a triad of N. Now consider $(A \cap E(M_1), B \cap E(M_1))$ again. This partition of M_1 must be 3-separating, but $B \cap E(M_1)$ is not a triangle of M_1, and M_1 has no 3-element cocircuits. This contradiction completes the proof. ∎

Proof of Theorem 1.5. It suffices to prove the theorem for $G_1 = G_2 = K_n$, where $n \geq 5$. Label the edges of some K_4-restriction N_1 of G_1 by $\{a, b, c, d, e, f\}$, and define

$$M' := \left(P_{N_1}\left(M(G_1), M_{12}\right)\right)^*. \tag{4.4}$$

By Theorem 3.1, M' is near-regular, and by Lemma 4.3, M' is internally 4-connected.

Note that we still have $M'|\{1, 2, 3, 4, 5, 6\} \cong M(K_4)$. Label the edges of some K_4-restriction N_2 of G_2 by $\{1, 2, 3, 4, 5, 6\}$, and define

$$M := P_{N_2}\left(M(G_2), M'\right). \tag{4.5}$$

By Theorem 3.1, M is near-regular, and by Lemma 4.3, M is internally 4-connected. The result follows. □

Matroid M_{12} was found while studying the 3-separations of R_{12}. The unique 3-separation (X, Y) of R_{12} with $|X| = |Y| = 6$ is induced in the class of regular matroids. Pendavingh and Van Zwam had found, using a computer search for blocking sequences, that it is not induced in the class of near-regular matroids.

Unlike R_{10} and R_{12} in Seymour's work, the matroid M_{12} by itself is quite inconspicuous. A natural class of near-regular matroids is the class of near-regular signed-graphic matroids. As indicated earlier, M_{12} is a member of this class (see Figure 4.1). The K_4-restriction is readily identified. M_{12} is self-dual and has an automorphism group of size 6, generated by $(c, e)(d, f)(1, 5)(3, 6)$ and $(a, d)(b, e)$ $(1, 4)(2, 3)$.

5. Conjectures. While Theorem 1.5 is a bit of a setback, we remain hopeful that a satisfactory decomposition theory for near-regular matroids can be found. First of all, the construction in section 4 employs only graphic matroids. In fact, it seems difficult to extend the $M(G_1)$-restriction of the 4-sum to some strictly near-regular matroid. The proof of Theorem 1.5 suggests the following construction.

DEFINITION 5.1. *Let M_1, M_2 be matroids such that $E(M_1) \cap E(M_2) = X$, $N := M_1|X = M_2|X \cong M(K_k)$ and M_1 is graphic. Then the* graph k-clique sum *of M_1 and M_2 is $P_N(M_1, M_2)\backslash X$.*

Now we offer the following update of Conjecture 1.4.

CONJECTURE 5.2. *Let M be a near-regular matroid. Then M can be obtained from matroids that are signed-graphic, are the dual of a signed-graphic matroid, or are members of a finite set $\mathcal{C}$, by applying the following operations:*

i. 1-, 2-, *and* 3-*sums;*

ii. *graph k-clique sums and their duals, where $k \leq 4$.*

Note that the work of Geelen, Gerards, and Whittle [3], when finished, should imply a decomposition into parts that are bounded-rank perturbations of signed-graphic matroids and their duals. However, the bounds they require on connectivity are huge. Conjecture 5.2 expresses our hope that, for near-regular matroids, specialized methods will give much more refined results.

As noted in the Introduction, Seymour's decomposition theorem is not the only ingredient in the proof of Theorem 1.2. Another requirement is that the basic classes can be recognized in polynomial time. The following result suggests that this may not hold for the basic classes of near-regular matroids.

THEOREM 5.3. *Let M be a signed-graphic matroid. Let N be a matroid on $E(M)$ given by a rank oracle. It is not possible to decide if $M = N$ using a polynomial number of rank evaluations.*

A matroid is *dyadic* if it is representable over GF(p) for all primes $p > 2$. Since all signed-graphic matroids are dyadic (which was first observed by Dowling [2]), this in turn implies that dyadic matroids are not polynomial-time recognizable.

A proof of Theorem 5.3, analogous to the proof by Seymour [17] that binary matroids are not polynomial-time recognizable, was found by Geelen and independently by the first author. It involves ternary swirls, which have a number of circuit-hyperplanes that is exponential in the rank. To test if the matroid under consideration is really the ternary swirl, all these circuit-hyperplanes have to be examined since relaxing any one of them again yields a matroid.

However, this family of signed-graphic matroids is not near-regular for all ranks greater than 3. Hence the complexity of recognizing near-regular signed-graphic matroids is still open. The techniques used by Seymour [17] do not seem to extend, but perhaps some new idea can yield a proof of the following conjecture.

CONJECTURE 5.4. *Let $\mathcal{C}$ be the class of near-regular signed-graphic matroids. Then $\mathcal{C}$ is polynomial-time recognizable.*

In fact, we still have some hope for the following.

CONJECTURE 5.5. *The class of near-regular matroids is polynomial-time recognizable.*

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