Effective Action for Stochastic PDEs

Stochastic partial differential equations (SPDEs) are the basic tool for modeling systems where noise is important. We make use of a functional integral formalism to go from SPDEs to an effective action, and an effective potential that describe the dynamics and the "basic" states of the system, respectively. We set up a perturbation expansion to calculate these quantities, with the amplitude of the noise two-point function acting as the loop-counting parameter (analog of Planck's constant $\hbar$ in QFT).

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Outline of the Talk

1. Introduction.
2. Present the tools and ingredients that are needed to develop the functional integral formalism, starting from the stochastic partial differential equation.
3. Define the characteristic functional, the effective action, and the effective potential.
4. Carry out the perturbative expansion, and take a closer look at the one-loop effective action and effective potential.
5. Discuss the interpretation and physical meaning of the effective action and effective potential.
6. Example I: study the KPZ equation.
7. Example II: study the reaction-diffusion-decay system.
8. Summary and work in progress.

Introduction

1. SPDEs model systems where noise is relevant. They are used for models of many microscopic systems: turbulence, combustion, population dynamics in biology, pattern formation in chemistry, structure formation in the Universe, ocean dynamics, and mathematical finance.
2. The noise represents our ignorance about precise details in the dynamics of the system:
   (a) Represents the fluctuations intrinsic to the dynamics (Quantum Mechanics).
   (b) Represents the dynamics of short-scale degrees of freedom which have not completely decoupled from the macroscopic dynamics (*e.g.*, thermal or turbulent noise).
   (c) Implements our ignorance of the exact initial or boundary conditions in the system.
   (d) Summarizes the necessary truncation of the deterministic dynamics of a many-body system when we try to describe it via a finite set of variables (*e.g.*, a truncated BBGKY hierarchy).
3. The non-stochastic partial differential equation need not arise from a Lagrangian formalism (variational principle).
4. We start from an SPDE and by making use of functional integration techniques, we are able to define an effective action that has all the information regarding dynamics of the system. The effective action is a generalization of the Onsager-Machlup action. (The presence of noise leads to a generalized action principle for the SPDE).
5. We consider Gaussian additive noise, and show that the amplitude of the noise two-point function can be taken as the parameter of a perturbative expansion (analog of $\hbar$ in QFT).
6. For homogeneous and static field configurations one can define an effective potential which carries information about the "basic states" of the system.

Stochastic Partial Differential Equations

Given a field $\phi(t,\vec{x})$, we consider the class of SPDEs of the form

$$D\phi(t,\vec{x}) = F[\phi(t,\vec{x})] + \eta(t,\vec{x}) \ .$$

D is a linear differential operator, which does not involves the field $\phi(t,\vec{x})$. Some examples are

$$\begin{array}{lll} D = & \partial_t - \nu \vec{\nabla}\cdot\vec{\nabla} & \text{diffusion operator ,} \\ D = & (\partial_t)^2 - \vec{\nabla}\cdot\vec{\nabla} & \text{wave operator ,} \\ D = & \partial_t & \text{Langevin operator .} \end{array}$$

The function $F[\phi]$ is the forcing term, generally non-linear in the field $\phi(t,\vec{x})$. Some examples are

$$\begin{array}{lll} F[\phi] = & \frac{\lambda}{2}(\vec{\nabla}\phi)\cdot(\vec{\nabla}\phi) & \text{Kardar–Parisi–Zhang (KPZ) ,} \\ F[\phi] = & P[\phi] \text{ or } Q[\phi] & \text{reaction-diffusion ,} \\ F[\phi] = & -\frac{\delta H[\phi]}{\delta\phi} & \text{“purely dissipative” SPDE .} \end{array}$$

$\eta(t,\vec{x})$ is the stochastic variable which describes the noise present in the system. We assume the (additive) noise to be Gaussian, and write its two-point function $G_\eta(x,y)$ as

$$\langle \eta(x)\eta(y)\rangle \stackrel{\text{def}}{=} G_\eta(x,y) = \mathcal{A}\ g_2(x,y) \ ,$$

with $\mathcal{A}$ a constant amplitude and $g_2(x,y)$ a shape function. The noise probability distribution is normalized according to

$$\int (\mathcal{D}\eta)\ \mathcal{P}[\eta] = 1 \ .$$

Tools and Ingredients

1. We assume that for a given realization of the noise the differential equation has a unique solution, given by $\phi_{\text{solution}}(t, \vec{x}|\eta)$.
2. For any function $Q(\phi)$ its ensemble average is defined as

$$\langle Q(\phi)\rangle \stackrel{\text{def}}{=} \int (\mathcal{D}\eta)\ \mathcal{P}[\eta]\ Q(\phi_{\text{solution}}(t, \vec{x}|\eta)) \ ,$$

with $\mathcal{P}[\eta]$ the probability density functional of the noise.

3. We make use of the delta functional identity

$$\begin{aligned} \phi_{\text{solution}}(t, \vec{x}|\eta) &= \int (\mathcal{D}\phi)\ \phi\ \delta[\phi - \phi_{\text{solution}}(\vec{x}, t|\eta)] \\ &= \int (\mathcal{D}\phi)\ \phi\ \delta[D\phi - F[\phi] - \eta]\ \sqrt{\mathcal{J}\mathcal{J}^\dagger} \ , \end{aligned}$$

based on the change of variables

$$\begin{aligned} \phi &\to \Psi[\phi] = D\phi - F[\phi] - \eta \ , \\ \mathcal{J}[\phi] &\stackrel{\text{def}}{=} \det\left(\frac{\delta\Psi[\phi]}{\delta\phi}\right) = \det\left(D - \frac{\delta F}{\delta\phi}\right) \ . \end{aligned}$$

4. We can rewrite the stochastic ensemble average of $Q(\phi)$ as

$$\langle Q(\phi)\rangle = \int (\mathcal{D}\phi)\ Q(\phi)\ \mathcal{P}[D\phi - F[\phi]]\ \sqrt{\mathcal{J}\mathcal{J}^\dagger} \ .$$

5. We can read off the probability distribution for the field ϕ

$$\langle Q(\phi)\rangle \stackrel{\text{def}}{=} \int (\mathcal{D}\phi)\ Q(\phi)\ \mathbf{P}[\phi] \ \Rightarrow\ \mathbf{P}[\phi] = \mathcal{P}[D\phi - F[\phi]]\ \sqrt{\mathcal{J}\mathcal{J}^\dagger} \ .$$

Characteristic Functional

1. Define the characteristic functional as the ensemble average of $Q(\phi) = \exp \int \mathrm{d}x J(x) \phi(x)$

$$Z[J] \stackrel{\text{def}}{=} \left\langle \exp \left(\int \mathrm{d}x \; J(x) \; \phi(x) \right) \right\rangle = \int (\mathcal{D}\phi) \; \exp\left(J \phi\right) \; \mathcal{P}[D\phi - F[\phi]] \; \sqrt{\mathcal{J}\mathcal{J}^{\dagger}} \; .$$

2. Assume the noise is Gaussian ($\langle \eta(x_1) \dots \eta(x_n) \rangle_c = 0, \forall n \geq 3$)

$$\mathcal{P}[\eta] = \frac{1}{\sqrt{\det(2\pi G_\eta)}} \exp\left(-\tfrac{1}{2} \int \int \mathrm{d}x \; \mathrm{d}y \; \eta(x) \; G_\eta^{-1}(x,y) \; \eta(y) \right) \; .$$

3. For Gaussian noise the characteristic functional can be written as

$$Z[J] = \frac{1}{\sqrt{\det(2\pi G_\eta)}} \int (\mathcal{D}\phi) \exp\left(\int J\phi \right) \sqrt{\mathcal{J}\mathcal{J}^{\dagger}} \exp\left[-\tfrac{1}{2} \int (D\phi - F) G_\eta^{-1} (D\phi - F) \right] \; .$$

4. The classical action (module Jacobian determinants) is given by

$$\mathcal{S}_{\text{classical}} = \frac{1}{2} \int \int \mathrm{d}x \; \mathrm{d}y \; (D\phi - F[\phi])_x \; G_\eta^{-1}(x,y) \; (D\phi - F[\phi])_y \; .$$

This “classical action” is a generalization of the action introduced by Onsager and Machlup in Physical Review, **91**, 1505-1515 (1953). [See G. Eyink in Physical Review E, **54**, 3419-3435 (1996) for strong noise amplitude analysis.]

(One-loop) Effective Action: Review

1. We start by writing the noise two-point function as

$$G_\eta(x,y) \stackrel{\text{def}}{=} \mathcal{A}\; g_2(x,y) \;,$$

with $\mathcal{A}$ a constant amplitude, and $g_2(x,y)$ a shape function.

2. The generating functional of connected correlated n-point functions (Helmholtz free energy) can be defined as

$$W[J] \stackrel{\text{def}}{=} \mathcal{A}(\log Z[J] - \log Z[0]) \;, \text{ such that } \frac{\delta W[J]}{\delta J} \stackrel{\text{def}}{=} \bar{\phi} \;.$$

3. The effective action $\Gamma_{\phi_0}[\bar{\phi}]$ (Gibbs free energy) is defined as the Legendre transform of $W[J]$, with J and $\bar{\phi}$ conjugated variables. It is given by

$$\Gamma_{\phi_0}[\bar{\phi}] \stackrel{\text{def}}{=} -W[J] + \int J\bar{\phi} \;, \text{ such that } \frac{\delta \Gamma_{\phi_0}[\bar{\phi}]}{\delta \bar{\phi}} = J \text{ and } \langle \phi \rangle_J = \bar{\phi} \;.$$

The effective action yields a variational principle. ϕ_0 is a reference (fiducial) field such that $\phi_0 = \langle \phi \rangle_{J=0}$.

4. Given a characteristic functional $Z[J]$

$$Z[J] = \int (\mathcal{D}\phi) \exp\left(\frac{-\mathcal{S}[\phi] + \int \, \mathrm{d}x \; J(x)\; \phi(x)}{a} \right) \;,$$

with a the parameter that characterizes the fluctuations, the one-loop effective action (first order in a) is given by

$$\Gamma_{\phi_0}[\bar{\phi}] = \mathcal{S}[\bar{\phi}] + \tfrac{1}{2} a \log\det(\mathcal{S}_2[\bar{\phi}]) - (\bar{\phi} \to \phi_0) + O(a^2) \;,$$

with $\mathcal{S}_2(x,y) \stackrel{\text{def}}{=} \delta^2 \mathcal{S}[\phi]/\delta\phi(x)\delta\phi(y)$ the Jacobi operator of the classical action $\mathcal{S}[\phi]$.

One-loop Effective Action for SPDEs

1. In quantum field theory the expansion (loop-counting) parameter is Planck's constant ($\hbar$). For field theories based on SPDEs the expansion parameter is the noise amplitude ($\mathcal{A}$).
2. The "classical action" (tree-level) is the sum of the Onsager-Machlup action and the Jacobian determinants $\mathcal{J}$ and $\mathcal{J}^\dagger$

$$\mathcal{S}[\phi] = \mathcal{S}_{\text{classical}}[\phi] - \tfrac{1}{2}\mathcal{A}\ \left(\log \mathcal{J}[\phi] + \log \mathcal{J}^\dagger[\phi]\right)\ .$$

3. The one-loop effective action for an SPDE with Gaussian noise can be written as

$$\begin{aligned}\Gamma_{\phi_0}[\bar\phi] &= \mathcal{S}_{\text{classical}}[\bar\phi] + \frac{\mathcal{A}}{2}\left\{\log\det(\mathcal{S}_2[\bar\phi]) - \log(\mathcal{J}[\bar\phi]\mathcal{J}^\dagger[\bar\phi])\right\} \\ &- (\bar\phi \to \phi_0) + O(\mathcal{A}^2)\ .\end{aligned}$$

4. The Jacobi operator is given by

$$\mathcal{S}_2[\bar\phi] = \left(D^\dagger - \frac{\delta F^\dagger}{\delta\bar\phi}\right) g_2^{-1}\left(D - \frac{\delta F}{\delta\bar\phi}\right) - (D\bar\phi - F[\bar\phi])g_2^{-1}\frac{\delta^2 F}{\delta\bar\phi\,\delta\bar\phi}\ .$$

5. $\mathcal{J}$ is the Jacobian of the transformation

$$\begin{aligned}\phi \to \Psi[\phi] &= D\phi - F[\phi] - \eta\ , \\ \mathcal{J} &= \det\left(\frac{\delta\Psi[\phi]}{\delta\phi}\right) = \det\left(D - \frac{\delta F}{\delta\phi}\right)\ .\end{aligned}$$

6. The one-loop effective action has two contributions: the generalized Onsager-Machlup (classical) term, and the perturbative correction which is proportional to $\mathcal{A}$. It seems natural to expect that this one-loop term may change the nature of the dynamical equation for the field $\phi(t, \vec{x})$.

One-loop Effective Potential for SPDEs

1. We now consider field configurations that are homogeneous and static, and define

$$\mathcal{V}_{\phi_0}[\bar\phi] \stackrel{\text{def}}{=} \frac{\Gamma_{\phi_0}[\bar\phi]}{\Omega} ,$$

   with Ω the space-time volume.

2. The one-loop effective potential can be written as

$$\begin{aligned}\mathcal{V}_{\phi_0}[\bar\phi] \quad &= \quad \tfrac{1}{2}F^2[\bar\phi] + \tfrac{1}{2}\mathcal{A}\int \frac{d^d\vec{k}\, d\omega}{(2\pi)^{d+1}} \log\left[1 + \frac{\tilde g_2(\vec{k},\omega)F[\bar\phi]\frac{\delta^2 F}{\delta\bar\phi\,\delta\bar\phi}}{\left(D^\dagger(\vec{k},\omega) - \frac{\delta F^\dagger}{\delta\bar\phi}\right)\left(D(\vec{k},\omega) - \frac{\delta F}{\delta\bar\phi}\right)}\right] \\ &- \quad (\bar\phi \to \phi_0) + O(\mathcal{A}^2) \ .\end{aligned}$$

3. This equation is analogous to the equation for the one-loop effective potential in QFT, where one has

$$\mathcal{V}^{QFT}_{\phi_0}[\bar\phi] \quad = \quad V[\bar\phi] + \tfrac{1}{2}\hbar\int \frac{d^d\vec{k}\, d\omega}{(2\pi)^{d+1}} \log\left[1 + \frac{\frac{\delta^2 V}{\delta\bar\phi\,\delta\bar\phi}}{\omega^2 + \vec{k}^2 + m^2}\right] - (\bar\phi \to \phi_0) + O(\hbar^2) \ .$$

4. The structure of the one-loop effective potential shows that noise induced fluctuations can modify the tree-level (zero-loop) "classical" potential $\mathcal{V}^{(0)}_{\phi_0}[\bar\phi] = \frac{1}{2}F^2[\bar\phi]$.
5. We perform the previous integrals for white, cut-off noise, that is $\tilde g_2(\vec{k},\omega) = \delta(\omega)\theta(|\vec{k}| - \Lambda)$. We assume independence of the scale determined by Λ, and treat it as an effective theory.

Physical Interpretation

In QFT the effective action and the effective potential yield information about the dynamics of the field and the ground state of the system and its fluctuations, respectively. What can we say about the effective action and the effective potential so defined for an arbitrary SPDE?

1. The stationary points of $\Gamma_{\phi_0}[\bar{\phi}]$ correspond to stochastic expectation values of the field in the absence of an external current. This is a non-perturbative result.

$$\frac{\delta \Gamma_{\phi_0}[\bar{\phi}]}{\delta \bar{\phi}} = 0 \quad \Longleftrightarrow \quad \bar{\phi} = \langle \phi[J=0] \rangle \ .$$

2. What is the probability that an initial field configuration $\phi_i(t_i, \vec{x})$ evolves into a final one $\phi_f(t_f, \vec{x})$? By means of the saddle point approximation one can show that

$$\text{Prob}[\phi_f(t_f, \vec{x}) | \phi_i(t_i, \vec{x})] \approx \exp\left[-\frac{\Gamma_{\phi_0}[\bar{\phi}_{\text{int}}]}{\mathcal{A}} \right] \ ,$$

with $\phi_{\text{int}}(t, \vec{x})$ a field that minimizes $\mathcal{S}[\phi]$ and interpolates from $\phi_i(t_i, \vec{x})$ to $\phi_f(t_f, \vec{x})$.

3. The effective potential governs the probability distribution of the space-time average of the fluctuating field: minima of the effective potential correspond to maxima of the probability density of the space-time averaged field.

$$\text{Prob}\left(\int_{\Omega_s \times T} \mathrm{d}^d \vec{x} \ \mathrm{d}t \ \phi(t, \vec{x}) = \bar{\phi} \ \Omega_s T \right) \propto \exp\left(-\Omega_s T \frac{\mathcal{V}_{\phi_0}[\bar{\phi}]}{\mathcal{A}} \right) \ .$$

Example I: KPZ equation

The Kardar-Parisi-Zhang (KPZ) equation is given by

$$\left(\frac{\partial}{\partial t} - \nu \vec{\nabla} \cdot \vec{\nabla}\right) \phi = F_0 + \frac{\lambda}{2}(\vec{\nabla}\phi) \cdot (\vec{\nabla}\phi) + \eta \; .$$

It is a non-linear extension of the Edwards-Wilkinson equation. This equation models surface growth and is equivalent to vorticity-free Burgers equation if the fluid velocity is taken to be $\vec{v} = -\vec{\nabla}\phi$. This equation has the following symmetries

1. Galilean invariance I (displacement of the coordinate system with speed $\frac{\mathrm{d}L(t)}{\mathrm{d}t}$) for arbitrary noise:

$$\begin{aligned} \phi(t, \vec{x}) \quad &\rightarrow \quad \phi(t, \vec{x}) + L(t) \; , \\ F_0 \quad &\rightarrow \quad F_0 + \frac{\mathrm{d}L(t)}{\mathrm{d}t} \; . \end{aligned}$$

2. Galilean invariance II (coordinate system that is tilted an angle to the vertical with $\tan\theta = \|\ \vec{\epsilon}\ \|$) for translation-invariant temporal white noise:

$$\begin{aligned} \vec{x} \quad &\rightarrow \quad \vec{x}\,' = \vec{x} - \lambda\, \vec{\epsilon}\, t \; , \qquad \text{and} \qquad t \rightarrow t' = t \; , \\ \phi(t, \vec{x}) \quad &\rightarrow \quad \phi'(t', \vec{x}\,') = \phi(t, \vec{x}) - \vec{\epsilon} \cdot \vec{x} \; . \end{aligned}$$

We calculate the effective potential for fields such that $\phi = -\vec{v} \cdot \vec{x}$ and $\phi_0 = -\vec{v}_0 \cdot \vec{x}$. We consider white Gaussian noise, and calculate the finite effective potential (renormalized). After carrying out the regularization, the symmetries allow us to set both F_0 and $\vec{v}_0$ equal to zero. The zero-loop (classical) effective potential is given by

$$\mathcal{V}^{(0)}_{\vec{v}_0}[\vec{v}] = \frac{1}{8}\lambda^2 |\vec{v}|^4 \; .$$

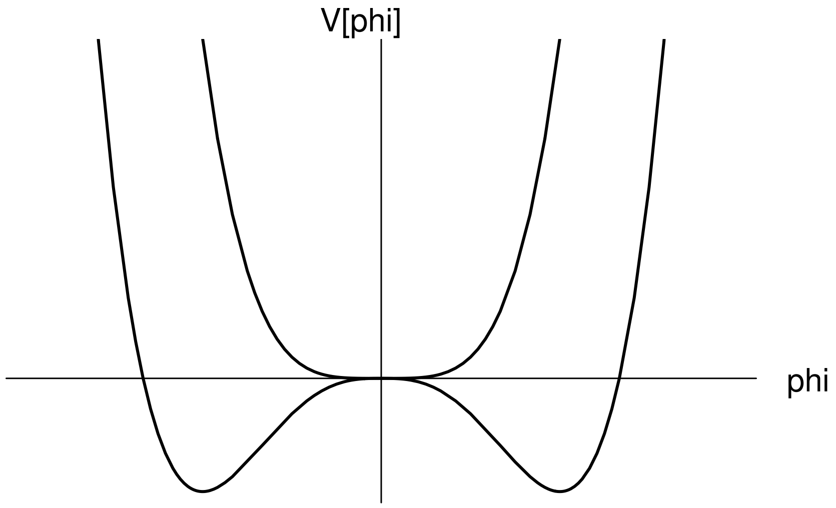
Aside: the Jacobian determinant is field independent.

One-loop Effective Potential for $d = 1$

The effective potential for $d = 1$ is given by

$$\mathcal{V}_{v_0}[v; d = 1] = \frac{\lambda^2}{8} v^4 - \mathcal{A} \frac{\lambda^3}{6\sqrt{2}\pi\nu^2} |v|^3 + O(\mathcal{A}^2) \; .$$

The qualitative form of the potential is given by



For large $|v|$ the classical potential dominates and for small $|v|$ the one-loop contribution drives the minimum of the potential away from the "classical" minimum at $v = 0$. The system undergoes dynamical symmetry breaking. One can estimate the location of the minimum of the one-loop effective potential

$$v^{(1)} = \pm \mathcal{A} \, \frac{\lambda}{2\sqrt{2}\pi\nu^2} + O(\mathcal{A}^2) \; .$$

Note that $v^{(1)} \to 0$ as $\mathcal{A} \to 0$, as it should, to recover the tree-level minimum $v^{(0)} = 0$. The presence of unknown $O(\mathcal{A}^2)$ terms makes it difficult to give a good estimate for the value of $v^{(1)}$. Detecting the occurrence of DSB is easier than finding the precise location of the minimum.

One-loop Effective Potential for $d = 2$

The effective potential for $d = 2$ is given by

$$\mathcal{V}_{v_0}[v; d = 2] = \frac{\lambda^2}{8} v^4 + \mathcal{A} \frac{\lambda^4}{32\pi^2 \nu^3} v^4 \log\left(\frac{v^2}{\mu^2}\right) + O(\mathcal{A}^2) \ .$$

This potential is zero at $v = 0$, then becomes negative, and for large enough fields ($|v| > \mu$) the potential becomes positive. This potential exhibits dynamical symmetry breaking. The minimum of the one-loop effective potential occurs for $v^{(1)} \neq 0$, but the presence of unknown $O(\mathcal{A}^2)$ terms makes it difficult to give a good estimate for the value of $v^{(1)}$. The location of the minimum is given by

$$v^{(1)} = \pm \mu \ \exp\left[-\frac{2\pi^2 \nu^3}{\lambda^2 \mathcal{A}} - \frac{1}{4} + O(\mathcal{A}) \right] \ .$$

Note that $v^{(1)} \to 0$ as $\mathcal{A} \to 0$, as it should, to recover the tree-level minimum $v^{(0)} = 0$. The one-loop beta function is derived from the fact that the bare effective potential does not depend on the renormalization scale

$$\frac{\mu \ \mathrm{d}}{\mathrm{d}\mu} \ \mathcal{V}_{v_0}[v; d = 2] = 0 \ ,$$

which yields

$$\beta_\lambda \stackrel{\mathrm{def}}{=} \frac{\mu \ \mathrm{d}}{\mathrm{d}\mu} \ \lambda = \frac{\mathcal{A}}{4\pi^2} \frac{\lambda^3}{\nu^3} + O(\mathcal{A})^2 \ .$$

We cannot extract the beta function for the wave-function renormalization of v from this analysis. This requires the calculation of the effective action for an inhomogeneous field. But to one-loop there is no wave-function renormalization for the KPZ field in this background.

Example II: RDD systems

1. Geometrical patterns are ubiquitous: from galaxies to living systems, examples abound where a particular spatial distribution of some material is preferred versus others out of a seemingly unlimited variety.
2. In many cases, these patterns are successfully described by systems of coupled parabolic non-linear partial differential equations (SPDEs).
3. In chemical kinetics such equations summarize the space-time evolution of chemical species diffusing and reacting in some confined geometrical region, which makes its presence felt in the boundary conditions for the problem.
4. In these phenomena the values of "reaction constants" play a role which reminds one of the role played by coupling constants in determining the vacuum (or ground) state in a quantum field theory undergoing spontaneous symmetry breaking, or in the description of phase transitions in condensed matter systems.
5. What is the effect on an existing pattern of the elimination of fast degrees of freedom?
6. How do fluctuations affect the stability of an established pattern?
7. The basic equation for the reaction-diffusion-decay system is

$$\left(\delta_i{}^j \, \frac{\partial}{\partial t} - \nu_i{}^j \; \vec{\nabla}^2 \right) \phi_j \quad = \quad P_i(\phi_j) + \eta_i \ ,$$

   with $P(\phi)$ a polynomial that describes nucleation, decay, two-particle reaction, etc.

Aside: the Jacobian determinant is field dependent.

Noise and Stability: Application

1. Consider the system with reaction polynomial $P(\phi) = a\phi^2 + b\phi + c$.
2. Consider the zero-noise system, and homogeneous and static concentrations. The solutions are given by $P(\phi_0) = 0$ such that
$$\phi_0^{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \,.$$
3. Carry out the zero-noise linear stability analysis by defining $\psi^{\pm}(\vec{x}, t) = \phi - \phi_0^{\pm}$. Fourier transform and keep up to linear order in the perturbation $\psi_{\vec{k}}^{\pm}(t)$, to obtain
$$\frac{\partial \psi_{\vec{k}}^{\pm}}{\partial t} \stackrel{\text{def}}{=} \lambda_k \psi_{\vec{k}}^{\pm} = \left(-\nu k^2 \pm \sqrt{b^2 - 4ac}\right) \psi_{\vec{k}}^{\pm} \,.$$
4. Hopf bifurcations occur when $\mathbf{R}e(\lambda_0) = 0$. If $b_{\mathrm{H}}^2 = 4ac$, for $b > b_{\mathrm{H}}$ linearized perturbations about ϕ_0^- decay and those about ϕ_0^+ grow.
5. Turing instabilities occur when $\mathbf{R}e(\lambda_k) > 0$ for $k \neq 0$. Consider the case ϕ_0^+ and $b > b_{\mathrm{H}}$. For the band of momenta $0 \leq q^2 < q_T^2 = \sqrt{b^2 - 4ac}/\nu$ one expects the onset of spatial structures to form with length scales corresponding to this momentum band.
6. The one-loop equation that determines the new (one-loop) homogeneous and static solutions is $P(\varphi_0) + \mathcal{A}h(\varphi_0) + O(\mathcal{A}^2) = 0$. This means that one obtains new conditions on the parameters that govern the onset of Hopf and Turing instabilities. In general $\phi_0 \neq \varphi_0$.
7. By expanding to linear order in the perturbation $\Psi(\vec{x}, t) = \phi - \varphi_0$, one obtains the corresponding one-loop eigenvalues
$$\Lambda_k = -\nu k^2 + P'(\varphi_0) + \mathcal{A}h'(\varphi_0) + O(\mathcal{A}^2) \,.$$

The effect of the noise is to shift the symmetric states of the system, as well as to change the nature of the linear instabilities that may be induced by perturbations around these new states.

Conclusions and Work in Progress

1. SPDEs model relevant systems: biology, chemistry, physics, geology, etc. In general these equations do not arise from a Lagrangian formalism (variational principle), due to the presence of non-hermitian operators.
2. The experience from QFT tells us that concepts such as the effective action and the effective potential have all the information regarding the dynamics and "ground states" of the system, respectively.
3. Start from an arbitrary SPDE and build up a functional integral formalism: characteristic functional, effective action, and effective potential.
4. This "direct" approach is complementary to the Martin-Siggia-Rose (MSR) formalism (without the presence of the adjoint fields). [MSR, Physical Review A **8**, 423 (1973)].
5. The effective action gives rise naturally to the concept of an effective potential, which serves to identify "ground states" and allows one to investigate the symmetry properties of the system and patterns of symmetry breaking.
6. Example I: KPZ equation. In one and two space dimensions linear field configurations ($\phi = -\vec{v} \cdot \vec{x}$) experience dynamical symmetry breaking due to noise effects.
7. Example II: reaction-diffusion systems. The presence of noise alters the value of the parameters for which there exist Hopf and/or Turing bifurcations.
8. Limitations: weak noise expansion versus strong noise (summation of the perturbation expansion?) [See G. Eyink in Physical Review E, **54**, 3419-3435 (1996) for strong noise amplitude analysis.]; is the system such that its dynamical evolution is driven to the homogeneous and static field configurations that minimize the effective potential?
9. Carry out a detailed study of the effective action to be able to analyze inhomogeneous field configurations (dynamical behaviour).
10. Carry out numerical computations to see how relevant those "ground field configurations" are.