Acoustic black holes

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Short Abstract:

There is a powerful analogy that relates acoustic wave propagation in moving fluids to classical scalar field theory in an "effective" curved spacetime.

This analogy lets you develop a number of useful connections between supersonic fluid flow and black holes in general relativity.

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Long Abstract:

It is a deceptively simple question to ask how acoustic disturbances propagate in a non-homogeneous flowing fluid. Subject to suitable restrictions, this question can be answered by invoking the language of Lorentzian differential geometry: If the fluid is barotropic and inviscid, and the flow is irrotational (though possibly time dependent), then the equation of motion for the velocity potential describing a sound wave is identical to that for a minimally coupled massless scalar field propagating in a (3+1)-dimensional Lorentzian geometry. The acoustic metric governing the propagation of sound depends algebraically on the density, flow velocity, and local speed of sound. Even though the underlying fluid dynamics is Newtonian, non-relativistic, and takes place in flat space plus time, the fluctuations (sound waves) are governed by an "effective" (3+1)-dimensional Lorentzian space-time geometry. This simple physical system is the basis underlying a deep and fruitful analogy between the black holes of Einstein gravity and supersonic fluid flows. Many results and definitions can be carried over directly from one system to another. For example, I will show how to define the ergo-sphere, trapped regions, acoustic apparent horizon, and acoustic event horizon for a supersonic fluid flow, and will exhibit the close relationship between the acoustic metric for the fluid flow surrounding a point sink and a particular form of the Schwarzschild metric for a black hole. This analysis can be used either to provide a concrete non-relativistic analogy for black hole physics, or to provide a framework for attacking acoustics problems with the full power of Lorentzian differential geometry.

Basic Idea:

Consider sound waves in a flowing fluid.

If the fluid is moving faster than sound, then the sound waves are swept along with the flow, and cannot escape from that region.

This sounds awfully similar to a black hole in general relativity — is there any connection?

— # # # YES! # # # —

Key points:

Acoustic propagation in fluids can be described in terms of Lorentzian geometry.

The acoustic metric depends algebraically on the fluid flow.

Acoustic geometry shares kinematic aspects of general relativity, but not the dynamics.

Einstein equations versus Euler equation.

In particular:

Acoustic black holes divorce kinematic aspects of black hole physics from the dynamics due to the Einstein equations.

Geometrical acoustics:

In a flowing fluid, if sound moves a distance $\mathrm{d}\vec{x}$ in time $\mathrm{d}t$ then

$$||\mathrm{d}\vec{x} - \vec{v}\,\mathrm{d}t|| = c_s\,\mathrm{d}t.$$

Write this as

$$(\mathrm{d}\vec{x} - \vec{v}\,\mathrm{d}t)\cdot(\mathrm{d}\vec{x} - \vec{v}\,\mathrm{d}t) = c_s^2\mathrm{d}t^2.$$

Now rearrange a little:

$$-(c_s^2 - v^2)\,\mathrm{d}t^2 - 2\,\vec{v}\cdot\mathrm{d}\vec{x}\;\;\mathrm{d}t + \mathrm{d}\vec{x}\cdot\mathrm{d}\vec{x} = 0.$$

Notation — four-dimensional coordinates:

$$x^\mu = (x^0; x^i) = (t; \vec{x}).$$

Then you can write this as

$$g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = 0.$$

With an effective acoustic metric

$$g_{\mu\nu}(t,\vec{x}) \propto \left[\begin{array}{ccc} -(c_s^2 - v^2) & \vdots & -\vec{v} \\ \cdots\cdots\cdots\cdots & \cdot & \cdots\cdots \\ -\vec{v} & \vdots & I \end{array}\right].$$

Eikonals:

The sound paths of geometrical acoustics are the null geodesics of this effective metric.

Geometrical acoustics, by itself, does not give you enough information to fix an overall multiplicative factor (conformal factor).

Note: This also works for geometrical optics in a flowing fluid, with $c_s \to c/n$; replace the speed of sound by the speed of light in the medium (speed of light divided by refractive index).

This is already enough to give you some very powerful results:

Fermat's principle is now a special case of geodesic propagation.

Sound focussing can be described by the Riemann tensor of this effective metric.

But there is a lot more hiding in the woodwork.

The Acoustic metric:

Suppose you have a non-relativistic flowing fluid, governed by the Euler equation plus the continuity equation.

Suppose the fluid flow is barotropic, irrotational, and inviscid.

Suppose we look at linearized fluctuations.

Then the linearized fluctuations (*aka* sound waves, *aka* phonons) are described by a massless minimally coupled scalar field propagating in a (3+1)-dimensional acoustic metric

$$g_{\mu\nu}(t,\vec{x}) \equiv \frac{\rho}{c} \left[\begin{array}{ccc} -(c^2 - v^2) & \vdots & -\vec{v} \\ \cdots\cdots\cdots\cdots & \cdot & \cdots\cdots \\ -\vec{v} & \vdots & I \end{array} \right].$$

Proof: Unruh81, Visser93, Unruh94, Visser97.

Other representations:

$$g^{\mu\nu}(t,\vec{x}) \equiv \frac{1}{\rho c} \left[\begin{array}{ccc} -1 & \vdots & -v^j \\ \cdots\cdots & \cdot & \cdots\cdots\cdots\cdots \\ -v^i & \vdots & (c^2\delta^{ij} - v^i v^j) \end{array} \right].$$

$$ds^2 \equiv g_{\mu\nu}\, dx^\mu\, dx^\nu.$$

$$ds^2 = \frac{\rho}{c}\left[-c^2 dt^2 + ||d\vec{x} - \vec{v}\, dt||^2\right].$$

If you move with the fluid, null cones spread out at the speed of sound.

The conformal factor is required to get a nice minimally coupled d'Alembertian equation of motion for the velocity potential.

Key points:

— The signature is $(-,+,+,+)$.

— There are two distinct metrics:
the *physical spacetime metric*, and
the *acoustic metric* .

— A completely general $(3+1)$–dimensional Lorentzian geometry has 6 degrees of freedom per point in spacetime. (4×4 symmetric matrix $\Rightarrow$ 10 independent components; then subtract 4 coordinate conditions).

—The acoustic metric is specified completely by the three scalars $\psi(t,\vec{x})$ $[\vec{v} = -\vec{\nabla}\psi]$, $\rho(t,\vec{x})$, and $c(t,\vec{x})$. It has at most 3 degrees of freedom per point in spacetime. Continuity reduces this to 2 degrees of freedom which can be taken to be $\psi(t,\vec{x})$ and $c(t,\vec{x})$.

Apparent horizons:

Trapped surface: Take a closed two-surface. If the fluid velocity is everywhere inward, and the normal component of the fluid velocity is everywhere greater than the local speed of sound, then no matter what direction a sound wave propagates, it will be swept inward by the fluid flow and be trapped inside the surface. The surface is then said to be outer-trapped.

Trapped region: The region containing outer trapped surfaces.

Apparent horizon: The boundary of the trapped region.

Event horizons:

Event horizon: (absolute horizon) the boundary of the region from which null geodesics (phonons; sound rays) cannot escape.

This is the future event horizon.

A past event horizon can be defined in terms of the boundary of the region that cannot be reached by incoming phonons

(Strictly speaking this requires us to define notions of past and future null infinities, but I will simply take all relevant incantations as understood.)

In particular the event horizon is a null surface, the generators of which are null geodesics.

(Meaning, there's a special class of sound rays that just skims along the surface of the horizon, neither escaping nor being sucked in.)

Example: draining bathtub

A $(2+1)$ dimensional flow with a sink.

Continuity implies

$$\rho \, v^{\hat{r}} \propto \frac{1}{r}.$$

Vorticity-free implies

$$v^{\hat{t}} \propto \frac{1}{r}.$$

Conservation of angular momentum implies

$$\rho \, v^{\hat{t}} \propto \frac{1}{r}.$$

Combine: the density ρ must be constant throughout the flow (which automatically implies that the pressure p and speed of sound c are also constant throughout the fluid flow).

The velocity of the fluid flow is

$$\vec{v} = \frac{(A \, \hat{r} + B \, \hat{\theta})}{r}.$$

Example: draining bathtub

The acoustic metric is

$$ds^2 = -c^2 dt^2 + \left(dr - \frac{A}{r}dt\right)^2 + \left(r\, d\theta - \frac{B}{r}dt\right)^2.$$

The acoustic event horizon forms once the radial component of the fluid velocity exceeds the speed of sound, that is at

$$r_{horizon} = \frac{|A|}{c}.$$

Supersonic flow sets in outside the event horizon, when the magnitude of the velocity equals the speed of sound.

In general relativity this is called an ergo-region and is important for spinning black holes.

Example: oscillating bubble

Given a spherically symmetric flow of constant density fluid, what is the acoustic metric?

Continuity implies $v \propto 1/r^2$.

If ρ is position independent then (because of the barotropic assumption) so is the pressure, and hence the speed of sound as well.

Define a normalization constant $r_0(t)$ and set

$$v = \pm c \, \frac{r_0(t)^2}{r^2}.$$

The acoustic metric is

$$ds^2 = -c^2 dt^2 + \left(dr \pm c \, \frac{r_0(t)^2}{r^2} \, dt \right)^2$$

$$+ r^2 (d\theta^2 + \sin^2\theta \, d\phi^2).$$

This acoustic metric is very easy to set up experimentally.

Example: oscillating bubble

For a spherically-symmetric oscillating bubble of radius $R(t)$:

$$r_0(t) = R(t)\,\sqrt{\frac{|\dot{R}(t)|}{c}}.$$

We should only use this metric for the fluid region outside the bubble, and only in the approximation that the ambient fluid is constant density (e.g. water).

For the compressible medium inside the bubble (e.g. air) we should use a separate acoustic metric.

It is experimentally easy to generate acoustic apparent horizons in this manner: In cavitating bubbles (typically air bubbles in water) it is easy to get the bubble wall moving at supersonic speeds (up to Mach 10 in extreme cases).

Example: oscillating bubble

As soon as the bubble wall is moving supersonically an acoustic apparent horizon forms.

The apparent horizon first forms at the bubble wall itself but then will typically detach itself from the bubble wall (since the apparent horizon will continue to be the surface at which the fluid achieves Mach 1) as the bubble wall goes supersonic.

Since the bubble must eventually stop its collapse and re-expand, there is no acoustic event horizon (no absolute horizon) in this experimental situation, merely a temporary apparent horizon.

(The apparent horizon must by construction last less than one sound-crossing-time for the collapsing bubble.)

Example: Schwarzschild

The Schwarzschild geometry in ingoing (+) and outgoing (−) Painlevé–Gullstrand coordinates is:

$$ds^2 = -dt^2 + \left(dr \pm \sqrt{\frac{2GM}{r}} dt \right)^2$$
$$+ r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right).$$

Equivalently

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 \pm \sqrt{\frac{2GM}{r}} \, dr \, dt$$
$$+ dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right).$$

This representation of the Schwarzschild geometry is not particularly well-known and has been rediscovered several times this century.

Surface gravity: Static acoustic geometries

If we restrict attention to a static geometry, we can apply all of the standard tricks for calculating the "surface gravity" developed in general relativity.

The surface gravity is a useful characterization of general properties of the event horizon and is given in terms of a normal derivative by

$$g_H = \frac{1}{2}\frac{\partial(c^2 - v^2)}{\partial n} = c\,\frac{\partial(c - v)}{\partial n}.$$

This is not quite Unruh's result since he implicitly took the speed of sound to be a position-independent constant.

The fact that ρ drops out of the final result for the surface gravity can be justified by appeal to the known conformal invariance of the surface gravity.

Surface gravity: Static acoustic geometries

Though derived in a totally different manner, this result is also compatible with the expression for "surface-gravity" obtained in the solid-state black holes of Reznik, where a position dependent (and singular) refractive index plays a role analogous to the acoustic metric.

Because we are now considering a static geometry, the relationship between the Hawking temperature and surface gravity may be verified in the usual fast-track manner — using the Wick rotation trick to go to "imaginary time" and analytically continue to "Euclidean signature".

If you don't like Euclidean signature techniques (which are in any case only applicable to equilibrium situations) you should go back to the original Hawking derivation.

Hawking radiation:

As discussed by Unruh81, (and subsequent papers — Jacobson91, Jacobson93, Unruh94, Brout, Jacobson95, Jacobson96, Corley-Jacobson96, Corley-Jacobson97, Corley97a, Corley97b, Hochberg97, Reznik96, Reznik97) an acoustic event horizon will emit Hawking radiation in the form of a thermal bath of phonons at a temperature

$$k\, T_H = \frac{\hbar\, g_H}{2\pi\, c_s}.$$

(Yes, this really is the speed of sound, and g_H is really normalized to have the dimensions of a physical acceleration.)

$$T_H = (1.2 \times 10^{-6} K\, mm) \left[\frac{c}{km\, s^{-1}}\right] \left[\frac{1}{c}\frac{\partial(c - v_\perp)}{\partial n}\right].$$

Experimental verification of this acoustic Hawking effect will be rather difficult.

Laws of black hole mechanics:

The dynamical origin of the laws of black hole mechanics is evident from the fact that the various proofs explicitly use either the Einstein equations plus the energy conditions, or at an absolute minimum, the existence of a diffeomorphisim invariant Lagrangian (built up out of the metric and its derivatives) governing the evolution of the Lorentzian geometry.

In the acoustic model, with no Einstein equations, no energy conditions, and not even the guarantee of a diffeomorphisim invariant Lagrangian, the usual laws of black hole mechanics are moot.

Laws of black hole mechanics:

Going from fluid dynamics to the acoustic metric is relatively easy; working backwards to try to re-derive all of hydrodynamics from the acoustic metric, is not so easy and looks downright impossible.

The notion of black hole entropy may not even be meaningful, never mind the generalized second law and such-like.

There is an important lesson here for string theory:
(1) Finding Hawking radiation in your theory does not imply that you have discovered quantum gravity.
(2) If you find Hawking radiation, and you have a theory that approximates classical Einstein gravity, then you must get black hole entropy approximately proportional to area.

Conclusions:

Acoustic geometries are very good toy models that guide us in logically separating the kinematics of gravity from the dynamics.

Hawking radiation is kinematics — it occurs for any test field on any Lorentzian geometry with event horizon independent of whether or not the Lorentzian geometry is dynamical.

Black hole entropy is dynamics — to even define black hole entropy requires a diffeomorphisim invariant dynamics for the Lorentzian geometry.

The acoustic analog model for black holes can teach us a lot.