# From Lagrangians to pseudo-Riemannian geometry: Modelling Einstein's gravity

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30 October 2001

## Abstract:

The idea of modelling (aspects of) Einstein's theory of gravity is attracting a lot of attention.

Idea: Use radically different physical systems that nevertheless share much of the mathematical structure.

Method: In particular, for any hyperbolic system of PDEs you can use the characteristic curves to define a precursor to the "light-cones" of general relativity, and under suitable algebraic restrictions can then deduce the existence of a pseudo-Riemannian metric.

I will introduce these ideas, present simple examples, and discuss the extent to which these analog models capture the essence of general relativity.

## Overview:

Einstein gravity (general relativity) is based on two things:

- pseudo-Riemannian geometry (Lorentzian geometry).
- field equations for the Ricci tensor.

Q: Are there *other* physical systems that *naturally* lead to the notion of pseudo-Riemannian geometry?

A: Yes, *lots* of them...

Q: Is there something deeper going on?

A: Yes, hyperbolicity plus field-theory normal modes...

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## Example 1:

Lagrangian:

$$\mathcal{L}(\partial_\mu \phi, \phi).$$

Convention:

$$\partial_\mu \phi = (\partial_t \phi \ ; \ \partial_i \phi) = (\partial_t \phi \ ; \ \nabla \phi).$$

Action:

$$S[\phi] = \int \mathrm{d}^{d+1}x \, \mathcal{L}(\partial_\mu \phi, \phi).$$

Euler–Lagrange equations:

$$\partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$

Linearize the field around a solution:

$$\phi(t,\vec{x}) = \phi_0(t,\vec{x}) + \epsilon \phi_1(t,\vec{x}) + \frac{\epsilon^2}{2} \phi_2(t,\vec{x}) + O(\epsilon^3).$$

## Example 1:

Linearized action

$$
\begin{aligned}
S[\phi] &= S[\phi_0] \\
&+ \frac{\epsilon^2}{2} \int \mathrm{d}^{d+1}x \left[ \left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\, \partial(\partial_\nu \phi)} \right\} \partial_\mu \phi_1 \, \partial_\nu \phi_1 \right. \\
&\qquad \left. + \left( \frac{\partial^2 \mathcal{L}}{\partial \phi \, \partial \phi} - \partial_\mu \left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\, \partial \phi} \right\} \right) \phi_1 \, \phi_1 \right] \\
&+ O(\epsilon^3).
\end{aligned}
$$

Linear pieces $[O(\epsilon)]$ vanish by equations of motion.

Quadratic in $\phi_1 \Rightarrow$ field-theory normal modes.

Linearized equations of motion:

$$
\begin{aligned}
&\partial_\mu \left( \left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\, \partial(\partial_\nu \phi)} \right\} \partial_\nu \phi_1 \right) \\
&\qquad - \left( \frac{\partial^2 \mathcal{L}}{\partial \phi \, \partial \phi} - \partial_\mu \left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\, \partial \phi} \right\} \right) \phi_1 = 0.
\end{aligned}
$$

Formally self-adjoint.

## Example 1:

Geometrical interpretation:

$$[\Delta(g(\phi_0)) - V(\phi_0)]\,\phi_1 = 0.$$

Metric:

$$\sqrt{-g}\; g^{\mu\nu} \equiv f^{\mu\nu} \equiv \left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\; \partial(\partial_\nu \phi)} \right\}\Bigg|_{\phi_0}.$$

Potential:

$$V(\phi_0) \;=\; \frac{1}{\sqrt{-g}} \left( \frac{\partial^2 \mathcal{L}}{\partial\phi\; \partial\phi} - \partial_\mu \left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\; \partial\phi} \right\} \right).$$

And

linearization $\Rightarrow$ metric;

hyperbolic $\Rightarrow$ pseudo-Riemannian;

parabolic $\Rightarrow$ degenerate;

elliptic $\Rightarrow$ Riemannian.

## Example 2:

### Barotropic irrotational inviscid fluid dynamics

Lagrangian (two fields):

$$\mathcal{L} = -\rho\, \partial_t \theta - \frac{1}{2}\rho(\nabla\theta)^2 - \int_0^\rho d\rho'\, h(\rho').$$

$$h(\rho) = h[p(\rho)] = \int_0^{p(\rho)} \frac{dp'}{\rho(p')}.$$

Vary $\rho \Rightarrow$ Bernoulli equation (Euler equation).

$$\partial_t \theta + \frac{1}{2}(\nabla\theta)^2 + h(\rho) = 0.$$

Vary $\theta \Rightarrow$ continuity equation.

$$\partial_t \rho + \nabla(\rho\, \nabla\theta) = 0.$$

Use the Bernoulli equation to algebraically eliminate $\rho$:

$$\rho = h^{-1}(z) = h^{-1}\left(-\partial_t\theta - \frac{1}{2}(\nabla\theta)^2\right).$$

## Example 2:

Reduced Lagrangian:

$$\mathcal{L}(z) = z\,\rho(z) - \int_0^{\rho(z)} d\rho'\, h(\rho').$$

But:

$$p(\rho) = \rho\, h(\rho) - \int_0^{\rho} d\rho'\, h(\rho').$$

Proof: Differentiate

$$\frac{\mathrm{d}[\mathrm{RHS}]}{\mathrm{d}\rho} = \rho\,\frac{\mathrm{d}h}{\mathrm{d}\rho} + h(\rho) - h(\rho) = \rho(z)\frac{\mathrm{d}h}{\mathrm{d}\rho} = \frac{\mathrm{d}p}{\mathrm{d}\rho}.$$

Finally:

$$\mathcal{L} = p(\rho(z)) = p\left(h^{-1}\left(-\partial_t\theta - \frac{1}{2}(\nabla\theta)^2\right)\right).$$

This reduces the Lagrangian to the form of Example 1.

There is a metric hiding here waiting to be found...

## Example 2:

Apply the result of Example 1:

$$\sqrt{-g}\; g^{\mu\nu} \equiv f^{\mu\nu} \equiv \left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\; \partial(\partial_\nu \phi)} \right\}\Bigg|_{\phi_0}.$$

Use:

$$\frac{\partial z}{\partial(\partial_\mu \phi)} = -(1; \nabla\theta)^\mu = -(1; \nabla^i \theta).$$

And:

$$\frac{\partial^2 z}{\partial(\partial_\mu \phi)\; \partial(\partial_\nu \phi)} = -\delta^{ij}.$$

Therefore:

$$f^{\mu\nu} = \frac{\mathrm{d}^2 p}{\mathrm{d}z^2}\,(1; \nabla\theta)^\mu\,(1; \nabla\theta)^\nu - \frac{\mathrm{d}p}{\mathrm{d}z}\,\delta^{ij}.$$

## Example 2:

But

$$\frac{\mathrm{d}p}{\mathrm{d}z} = \rho;$$

while

$$\frac{\mathrm{d}^2 p}{\mathrm{d}z^2} = \frac{\mathrm{d}\rho}{\mathrm{d}z} = \frac{\mathrm{d}\rho}{\mathrm{d}p}\,\frac{\mathrm{d}p}{\mathrm{d}z} = \rho\, c_s^{-2}.$$

Collecting terms:

$$f^{\mu\nu} = -\rho\, c_s^{-2} \left[ \begin{array}{c|c} -1 & -\nabla^i\theta \\ \hline -\nabla^j\theta & c_s^2\,\delta^{ij} - \nabla^i\theta\,\nabla^j\theta \end{array} \right].$$

This is equivalent to the standard (d+1) dimensional "acoustic metric". Use

$$g^{\mu\nu} = |\mathrm{det} f|^{-1/(d-1)}\; f^{\mu\nu}.$$

And note an overall minus sign is irrelevant.

## Example 2:

When the dust settles

$$g^{\mu\nu} \propto \left[ \begin{array}{ccc} -1 & \vdots & -\nabla^i\theta \\ \dots\dots & \cdot & \dots\dots\dots\dots\dots\dots \\ -\nabla^j\theta & \vdots & c_s^2\,\delta^{ij} - \nabla^i\theta\,\nabla^j\theta \end{array} \right].$$

Inverting

$$g_{\mu\nu} \propto \left[ \begin{array}{ccc} -(c_s^2 - [\nabla\theta]^2) & \vdots & -\nabla_i\theta \\ \dots\dots\dots\dots\dots & \cdot & \dots\dots \\ -\nabla_j\theta & \vdots & \delta_{ij} \end{array} \right].$$

Equivalently

$$\mathrm{d}s^2 \propto -c_s^2\,\mathrm{d}t^2 + (\mathrm{d}x - \nabla\theta\,\mathrm{d}t)^2.$$

Natural way of assigning a pseudo-Riemannian (Lorentzian) metric to this physical system.

This metric governs the propagation of linearized fluctuations — in this context, sound waves.

## Example 3:

### Multiple fields

The reason the previous examples are interesting is because they are part of a much more general pattern.

Suppose we have many interacting fields $\phi^A(t,\vec{x})$.

Lagrangian:

$$\mathcal{L}(\partial_\mu \phi^A, \phi^A).$$

Action:

$$S[\phi^A] = \int \mathrm{d}^{d+1}x\; \mathcal{L}(\partial_\mu \phi^A, \phi^A).$$

Linearize the fields:

$$\phi^A(t,\vec{x}) = \phi^A_0(t,\vec{x}) + \epsilon \phi^A_1(t,\vec{x}) + \frac{\epsilon^2}{2}\phi^A_2(t,\vec{x}) + O(\epsilon^3).$$

## Example 3:

Linearize the action:

$$
\begin{aligned}
&S[\phi^A] = S[\phi_0^A] \\
&+\frac{\epsilon^2}{2}\int \mathrm{d}^{d+1}x \left[ \left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^A)\, \partial(\partial_\nu \phi^B)} \right\} \partial_\mu \phi_1^A \, \partial_\nu \phi_1^B \right. \\
&\qquad +2 \left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^A)\, \partial \phi^B} \right\} \partial_\mu \phi_1^A \, \phi_1^B \\
&\qquad \left. + \left\{ \frac{\partial^2 \mathcal{L}}{\partial \phi^A \, \partial \phi^B} \right\} \phi_1^A \, \phi_1^B \right] \\
&+O(\epsilon^3).
\end{aligned}
$$

NB: The fields now carry indices $(AB)$. The linear term still vanishes by Euler–Lagrange.

Still quadratic $\Rightarrow$ field-theory normal modes.

The equation of motion for the linearized fluctuations is now more complicated.

## Example 3:

Equation of motion:

$$\partial_\mu \left( \left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^A)\, \partial(\partial_\nu \phi^B)} \right\} \partial_\nu \phi_1^B \right)$$
$$+\partial_\mu \left( \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^A)\, \partial \phi^B}\, \phi_1^B \right)$$
$$-\partial_\mu \phi_1^B \, \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^B)\, \partial \phi^A}$$
$$-\left( \frac{\partial^2 \mathcal{L}}{\partial \phi^A\, \partial \phi^B} \right) \phi_1^B = 0.$$

To simplify it we need several definitions.

First, generalize $f^{\mu\nu}$:

$$f^{\mu\nu}{}_{AB} \equiv \frac{1}{2} \left( \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^A)\, \partial(\partial_\nu \phi^B)} + \frac{\partial^2 \mathcal{L}}{\partial(\partial_\nu \phi^A)\, \partial(\partial_\mu \phi^B)} \right).$$

Symmetric in $(\mu\nu)$ and $(AB)$.

## Example 3:

Second, define:

$$\Gamma^{\mu}{}_{AB} \equiv +\frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^A)\,\partial\phi^B} - \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^B)\,\partial\phi^A}$$
$$+\frac{1}{2}\partial_\nu \left( \frac{\partial^2 \mathcal{L}}{\partial(\partial_\nu \phi^A)\,\partial(\partial_\mu \phi^B)} - \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^A)\,\partial(\partial_\nu \phi^B)} \right).$$

This "connexion" is anti-symmetric in $[AB]$.

Third:

$$K_{AB} = -\frac{\partial^2 \mathcal{L}}{\partial\phi^A\,\partial\phi^B} + \frac{1}{2}\partial_\mu \left( \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^A)\,\partial\phi^B} \right)$$
$$+\frac{1}{2}\partial_\mu \left( \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi^B)\,\partial\phi^A} \right).$$

This "potential" or "mass matrix" is, by construction, symmetric in $(AB)$.

## Example 3:

Linearized equations of motion:

$$
\begin{aligned}
&\partial_\mu \left( f^{\mu\nu}{}_{AB}\, \partial_\nu \phi_1^B \right) \\
&+\frac{1}{2}\left[ \Gamma^\mu_{AB}\, \partial_\mu \phi_1^B + \partial_\mu (\Gamma^\mu_{AB}\, \phi_1^B) \right] \\
&+K_{AB}\, \phi_1^B = 0.
\end{aligned}
$$

Now transparent that this is a formally self-adjoint second-order linear *system* of PDEs.

Analyze causal structure using theory of characteristics.

Leading *symbol* of the PDE system.

Courant and Hilbert,
with suitable generalizations.

Causal structure is a surrogate for the pseudo-Riemannian metric.

## Example 3:

Normal cone:

$$\mathcal{N}(q) \equiv \{p_\mu \,|\, \det\left(f^{\mu\nu}{}_{AB}\;\; p_\mu\, p_\mu\right) = 0\}\,.$$

(locus of normals to the characteristic surfaces)

With $N$ fields this "normal cone" will generically consist of $N$ nested sheets each with the topology (not necessarily the geometry) of a cone.

Often several of these cones will coincide.

Common for some of these cones to be degenerate, which is more problematic.

> *It may be remarked that the present state of the theory of algebraic surfaces does not permit entirely satisfactory applications to the questions of reality of geometric structures which confront us here.*

## Example 3:

Define $Q(q,p)$ on the co-tangent bundle

$$Q(q,p) \equiv \det\left(f^{\mu\nu}{}_{AB}(q)\, p_\mu\, p_\mu\right).$$

Monge cone: (*aka* "ray cone", *aka* "characteristic cone", *aka* "null cone")

$$\mathcal{M}(q) = \left\{ t^\mu = \frac{\partial Q(q,p)}{\partial p_\mu} \,\middle|\, p_\mu \in \mathcal{N}(q) \right\}.$$

Envelope of the set of characteristic surfaces through the point $q$.

The "Monge cone" is dual to the "normal cone".

> *Even if [the normal cone] is a relatively simple algebraic cone of degree [2N], the ray cone [Monge cone/null cone] may have singularities, or isolated rays, and need not consist of separate smooth conical shells.*

## Example 3:

## Field redefinitions:

$$\phi^A \to \bar{\phi}^A = h^A(\phi^B).$$

$$\phi_1^A \to \bar{\phi}_1^A = \left.\frac{\partial h^A}{\partial \phi^B}\right|_{\phi_0^C} \phi_1^B = [L^{-1}(\phi_0^C)]^A{}_B \;\; \phi_1^B.$$

Matrix notation:

$$\boldsymbol{\phi}_1 \to \bar{\boldsymbol{\phi}}_1 = \boldsymbol{L}^{-1} \, \boldsymbol{\phi}_1.$$

$$\partial_\mu \left(\boldsymbol{f}^{\mu\nu} \, \partial_\nu \boldsymbol{\phi}_1\right) + \boldsymbol{\Gamma}^\mu \, \partial_\mu \boldsymbol{\phi}_1 + \frac{1}{2} \partial_\mu (\boldsymbol{\Gamma}^\mu) \, \boldsymbol{\phi}_1 + \boldsymbol{K} \, \boldsymbol{\phi}_1 = 0.$$

$$\boldsymbol{f}^{\mu\nu} \to \bar{\boldsymbol{f}}^{\mu\nu} = \boldsymbol{L}^T \, \boldsymbol{f}^{\mu\nu} \, \boldsymbol{L}.$$

## Example 3:

## Case A: Einstein equivalence principle.

Uniqueness of free fall for "normal matter" implies that to high accuracy ($10^{-14}$) there is a field redefinition such that:

$$\bar{f}^{\mu\nu}{}_{AB} = \delta_{AB}\, f^{\mu\nu} = \delta_{AB}\, \sqrt{-g}\; g^{\mu\nu}.$$

In arbitrary field variables:

$$f^{\mu\nu}{}_{AB} = h_{AB}\, f^{\mu\nu} = h_{AB}\, \sqrt{-g}\; g^{\mu\nu}.$$

This "factorization" condition on $f^{\mu\nu}{}_{AB}$ is a necessary and sufficient condition for strict adherence to the Einstein Equivalence Principle.

## Example 3:

## Case B: Multiple metrics.

For a multi-metric theory, there must be some choice of field variables so that all the linearized fields $\bar{\phi}_1^A$ "decouple" and see independent metrics:

$$\begin{aligned} \bar{f}^{\mu\nu}{}_{AB} &= \text{diag}\{f_1^{\mu\nu}, f_2^{\mu\nu}, f_3^{\mu\nu}, \cdots, f_N^{\mu\nu}\} \\ &= \text{diag}\{\sqrt{-g_1}\, g_1^{\mu\nu}, \cdots, \sqrt{-g_N}\, g_N^{\mu\nu}\}. \end{aligned}$$

The necessary and sufficient condition for the $f^{\mu\nu}{}_{AB}$ to be simultaneously diagonalizable in field space is that $\forall\, \mu, \nu, \alpha, \beta$:

$$f^{\mu\nu}{}_{AB}\, f^{\alpha\beta}{}_{BC} = f^{\alpha\beta}{}_{AB}\, f^{\mu\nu}{}_{BC}.$$

That is

$$[\, \boldsymbol{f}^{\mu\nu}, \boldsymbol{f}^{\alpha\beta}\, ] = 0.$$

## Example 3:

## Case C: pseudo-Finsler geometry.

Riemann $\to$ pseudo-Riemann (Lorentzian).

Finsler $\to$ pseudo-Finsler.

Lorentzian geometry $(norm)^2$:

$$Q_2(q,p) \equiv g^{\mu\nu}(q)\, p_\mu\, p_\nu.$$

pseudo-Finsler geometry $(norm)^{2N}$:

$$Q_{2N}(q,p) = \det\left(f^{\mu\nu}{}_{AB}(q)\, p_\mu\, p_\mu\right).$$

Homogeneous order $N$:

$$Q_{2N}(q,\lambda p) = \lambda^{2N}\, Q_{2N}(q,p). \tag{1}$$

## Example 3:

## Case C: pseudo-Finsler geometry.

(Complex) Lorentzian norm

$$||p|| = [Q_2(q,p)]^{1/2}.$$

(Complex) Finsler metric

$$d_F(q,p) = [Q(q,p)]^{1/2N}.$$

The standard notions of Finsler geometry must be modified...

(Note: Elliptic systems $\Rightarrow$ Finsler geometry;

Hyperbolic $\Rightarrow$ pseudo-Finsler geometry.)

Much more general than we really need for GR.

## Basic message:

Hyperbolic PDE $\Rightarrow$ pseudo-Finsler geometry.

Finsler geometries and pseudo-Finsler geometries now have physics-motivated uses.

But in the most interesting cases:

Hyperbolic PDE $\Rightarrow$ pseudo-Riemannian geometry.

There are “effective metrics” hiding in the woodwork.

Used in physics to generate “analog models” of Einstein gravity.

## Physics example:

Sound in a Bose–Einstein condensate.

Mathematical description close to Example 2.

$$c_s \approx 6\ cm/sec.$$

Relatively easy to generate "acoustic horizons" from supersonic flow; black hole analogues (dumb holes).

Should exhibit Hawking radiation...

Now *phonons* not *photons*.

$$T_{Hawking} \approx 70\ nK.$$

$$T_{condensate} \approx 90\ nK.$$

Experiments in 5 to 10 years?

## Conclusions:

- Analogue models connect fundamental issues of PDE systems with abstract differential geometry.

- Lagrangians $\to$ pseudo-Riemannian geometry.

- Rich mathematical structure;
brings together ideas from condensed matter, fluid dynamics, general relativity, etc.

- Analogue models are fun.

## Publications:

Acoustic propagation in fluids:
An Unexpected example of Lorentzian geometry.
e-Print Archive: gr-qc/9311028

Acoustic black holes: Horizons, ergospheres,
and Hawking radiation.
Class.Quant.Grav.15:1767-1791,1998
e-Print Archive: gr-qc/9712010

Acoustic black holes.
e-Print Archive: gr-qc/9901047

Unexpectedly large surface gravities for acoustic horizons?
(with Stefano Liberati and Sebastiano Sonego).
Class.Quant.Grav.17:2903,2000
e-Print Archive: gr-qc/0003105

Analog gravity from Bose-Einstein condensates.
(with Carlos Barcelo and Stefano Liberati)
Class.Quant.Grav.18:1137,2001
e-Print Archive: gr-qc/0011026

Analog gravity from field theory normal modes?
(with Carlos Barcelo and Stefano Liberati)
Class.Quant.Grav.18:3595-3610,2001
e-Print Archive: gr-qc/0104001

Einstein gravity as an emergent phenomenon?
(with Carlos Barcelo and Stefano Liberati)
IJMPD (in press)
e-Print Archive: gr-qc/0106002

## Publications:

Wave equation for sound in fluids with vorticity.
(with Santiago Bergliaffa, Katrina Hibberd, and Michael Stone)
e-Print Archive: cond-mat/0106255

Acoustics in Bose-Einstein condensates as an example of broken Lorentz symmetry.
(with Carlos Barcelo and Stefano Liberati)
e-Print Archive: hep-th/0109033

Riemannian geometry of irrotational vortex acoustics.
(with Uwe R. Fischer)
e-Print Archive: cond-mat/0110211

Towards the observation of Hawking radiation in Bose-Einstein condensates.
(with Carlos Barcelo and Stefano Liberati)
e-Print Archive: gr-qc/0110036

Refringence, field theory, and normal modes?
(with Carlos Barcelo and Stefano Liberati)
in preparation.