Acoustics in BECs as an example of broken Lorentz symmetry

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Overview:

What?

Acoustic propagation in Bose–Einstein condensates has the interesting property that the low-energy spectrum exhibits an effective Lorentz invariance (in terms of the speed of sound) while at high energies the dispersion relation turns over and becomes Newtonian.

Why?

This very concrete physical model provides an explicit example of a "broken" Lorentz invariance.

Acoustics in Bose-Einstein condensates:

BECs are described by the nonlinear Schrodinger equation (Gross–Pitaevskii equation).

$$-i\hbar\, \partial_t \psi(t,\vec{x}) = -\frac{\hbar^2}{2m}\nabla^2\psi(t,\vec{x}) + \lambda\, ||\psi||^2\, \psi(t,\vec{x}).$$

Use the Madelung representation to put the Schrodinger equation in "hydrodynamic" form:

$$\psi = \sqrt{\rho}\ \exp(-i\theta\, m/\hbar).$$

Take real and imaginary parts: You get a continuity equation and something that looks like the Euler equation.

Continuity:

$$\partial_t \rho + \nabla\cdot(\rho\, \nabla\theta) = 0.$$

Velocity field:

$$v \equiv \nabla\theta.$$

Madelung form:

Hamilton–Jacobi equation:

$$\frac{\partial}{\partial t}\theta + \frac{1}{2}(\nabla\theta)^2 + \frac{\lambda\,\rho}{m} - \frac{\hbar^2}{2m^2}\,\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} = 0.$$

This is completely equivalent to irrotational inviscid hydrodynamics with an enthalpy

$$h = \int \frac{\mathrm{d}p}{\rho} = \frac{\lambda\,\rho}{m},$$

plus a peculiar self-interaction:

$$V_Q = -\frac{\hbar^2}{2m^2}\,\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} = 0.$$

EOS:

$$p = \frac{\lambda\,\rho^2}{2m}; \qquad \frac{\mathrm{d}p}{\mathrm{d}\rho} = \frac{\lambda\,\rho}{m}.$$

The quantum metric:

Linearize around some background.

In the low-momentum limit neglect V_Q.

Phonon: massless minimally-coupled scalar.

d'Alembertian equation/effective geometry:

$$g^{\mu\nu}(t, \vec{x}) \equiv \frac{\rho_0}{c_s} \left[\begin{array}{ccc} -1 & \vdots & -v_0 \\ \cdots\cdots & \cdot & \cdots\cdots\cdots\cdots \\ -v_0 & \vdots & (c_s^2\, \mathbf{I} - v_0 \otimes v_0) \end{array} \right].$$

Here

$$c_s^2 \equiv \frac{\lambda\ \rho_0}{m}; \qquad\qquad v_0 = \nabla\theta_0.$$

Low-momentum phonon physics is QFT in curved spacetime.

(Garay, Cirac, Anglin, Zoller;
PRL gr-qc/0002015, PRA gr-qc/0005131.
Barceló, Liberati, Visser; CQG gr-qc/0011026.)

ADM formalism:

$$g_{\mu\nu}(t,\vec{x}) \equiv \frac{\rho_0}{c_s} \left[\begin{array}{ccc} -(c_s^2 - v_0^2) & \vdots & -\vec{v}_0 \\ \cdots\cdots\cdots\cdots & \cdot & \cdots\cdots \\ -\vec{v}_0 & \vdots & \mathbf{I} \end{array} \right].$$

$$\mathrm{d}s^2 = \frac{\rho_0}{c_s} \left[-c_s^2 \, \mathrm{d}t^2 + ||\mathrm{d}\vec{x} - \vec{v}_0 \, \mathrm{d}t||^2 \right].$$

Shift vector:

$$\vec{\beta} = -\vec{v}_0 = -\nabla\theta_0.$$

Lapse function:

$$N = c_s.$$

The low-momentum physics looks completely Lorentz invariant...

(Acoustic Lorentz invariance mind you...)

Question:

But wait...

We started with the nonlinear Schrodinger equation...

That equation is parabolic...

Characteristics move at infinite speed.

How did we get a hyperbolic d'Alembertian equation?

The subtlety resides in neglecting the higher-derivative term V_Q.

Breaking the Lorentz invariance:

Keep V_Q. Go to the eikonal approximation:

$$\left(\omega - \vec{v}_0 \cdot \vec{k}\right)^2 = c_s^2 \, k^2 + \left(\frac{\hbar}{2m} k^2\right)^2.$$

This is the curved-space generalization of the Bogolubov dispersion relation. Equivalently

$$\omega = \vec{v}_0 \cdot \vec{k} \pm \sqrt{c_s^2 \, k^2 + \left(\frac{\hbar}{2m} k^2\right)^2}.$$

Group velocity:

$$\vec{v}_g = \frac{\partial \omega}{\partial \vec{k}} = \vec{v}_0 + \frac{\left(c_s^2 + \frac{\hbar^2}{2m^2} k^2\right)}{\sqrt{c_s^2 k^2 + \left(\frac{\hbar}{2m} \, k^2\right)^2}} \, \vec{k}.$$

Phase velocity:

$$\vec{v}_p = \frac{\omega \, \hat{k}}{||k||} = (v_0 \cdot \hat{k}) \, \hat{k} + \sqrt{c_s^2 + \frac{\hbar^2 \, k^2}{4m^2}} \; \hat{k}.$$

Generalized Bogolubov relation:

Consider:

$$\omega(k) = \sqrt{m_0^2 + k^2 + \left(\frac{k^2}{2m_\infty}\right)^2}.$$

(BEC condensates $m_0 = 0$; $c = \hbar = 1$.)

Low momenta ($k \ll m_0$):

$$\omega(k) = m_0 + \frac{k^2}{2m_0} + O(k^4).$$

Intermediate momenta ($m_0 \ll k \ll m_\infty$):
Approximately relativistic.

Large momenta ($k \gg m_\infty$):

$$\omega(k) = \frac{k^2}{2m_\infty} + m_\infty + O(k^{-2}).$$

Surprise!

Comments:

BECs are examples of physically realizable systems with "broken" Lorentz invariance.

They are not the only ones —
condensed matter physics is littered with "analog models" for low-energy Lorentz invariance.

Other examples:

1) Acoustics plus viscosity.
(Visser, CQG, gr-qc/9712010)

2) Lattice phonons.

3) It's really just a matter of doing a low-momentum field-theory normal-modes analysis.
(Barceló, Liberati, Visser, CQG, gr-qc/0104001)

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