

## BUCHDAHL-LIKE TRANSFORMATIONS FOR PERFECT FLUID SPHERES

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In two previous articles [*Phys. Rev. D* **71** (2005) 124307 (gr-qc/0503007) and *Phys. Rev. D* **76** (2006) 0440241 (gr-qc/0607001)] we have discussed several “algorithmic” techniques that permit one (in a purely mechanical way) to generate large classes of general-relativistic static perfect fluid spheres. Working in Schwarzschild curvature coordinates, we used these algorithmic ideas to prove several “solution-generating theorems” of varying levels of complexity. In the present article we consider the situation in other coordinate systems. In particular, in general diagonal coordinates we shall generalize our previous theorems, in isotropic coordinates we shall encounter a variant of the so-called “Buchdahl transformation,” and in other coordinate systems (such as Gaussian polar coordinates, Sygne isothermal coordinates, and Buchdahl coordinates) we shall find a number of more complex “Buchdahl-like transformations” and “solution-generating theorems” that may be used to investigate and classify the general-relativistic static perfect fluid sphere. Finally, by returning to general diagonal coordinates and making a suitable *ansatz* for the functional form of the metric components, we place the Buchdahl transformation in its most general possible setting.

*Keyword:* Fluid spheres.

### 1. Introduction

Understanding perfect fluid spheres in general relativity is an important topic, recently deemed worthy of a brief chapter in an updated edition of the premier book summarizing and surveying “exact solutions” in general relativity.<sup>1</sup> For considerably more details, the extensive review in Ref. 2 is an essential reference. Physically, perfect fluid spheres are a first approximation to realistic models for a general-relativistic star. The study of perfect fluid spheres is a long-standing topic with a venerable history,<sup>1–9</sup> and continuing interest.<sup>10–12</sup> Nevertheless the topic continues to provide surprising and novel results.<sup>14–18</sup> In particular, as derived in Refs. 14 and 15, and as further described in Refs. 16–18, we have developed several “algorithmic” techniques that permit one to generate large classes of perfect fluid

spheres from first principles in a purely mechanical way. We now generalize these algorithmic ideas, originally derived for Schwarzschild curvature coordinates (and to a lesser extent isotropic coordinates), to a number of other coordinate systems. This sometimes leads to much simpler results, and sometimes more general results. In the current article we shall aim to:

- (1) Derive a number of new and significant “transformation theorems” and “solution-generating theorems” that allow us to map perfect fluid spheres into perfect fluid spheres;
- (2) Investigate regularity conditions at the center of the fluid sphere from the geometrical, physical, and purely mathematical points of view;
- (3) Develop a coherent picture of how these various theorems and various coordinate systems interrelate with one another.

## 2. Strategy

We start the discussion by considering static, spherically symmetric distributions of matter, which implies (purely by symmetry) that in orthonormal components the stress energy tensor takes the specific form

$$T_{\hat{a}\hat{b}} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p_r & 0 & 0 \\ 0 & 0 & p_t & 0 \\ 0 & 0 & 0 & p_t \end{bmatrix}. \quad (1)$$

If the matter is a perfect fluid, then in addition we must have

$$p_r = p_t. \quad (2)$$

Invoking the Einstein equations, this then implies a differential relation on the space–time geometry arising from equating the appropriate orthonormal components of the Einstein tensor

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{r}\hat{r}} = G_{\hat{\phi}\hat{\phi}}. \quad (3)$$

Equivalently, one could work with the appropriate orthonormal components of the Ricci tensor

$$R_{\hat{\theta}\hat{\theta}} = R_{\hat{r}\hat{r}} = R_{\hat{\phi}\hat{\phi}}. \quad (4)$$

In terms of the metric components, this now leads to an ordinary differential equation (ODE), which constrains the space–time geometry for *any* general-relativistic static perfect fluid sphere.

This equation constraining the space–time geometry of static perfect fluid spheres is now analyzed in several different coordinate systems: general diagonal coordinates, Schwarzschild curvature coordinates, isotropic coordinates, and lesser-known coordinate systems such as Gaussian polar coordinates, Synge isothermal

coordinates, and Buchdahl coordinates. In the various coordinate systems we discuss below, we exhibit the metric (in the form of the line element), the orthonormal components ( $G_{\hat{r}\hat{r}}$ ,  $G_{\hat{\theta}\hat{\theta}}$ , and  $G_{\hat{t}\hat{t}}$ ) of the Einstein tensor, and the ODE resulting from the pressure isotropy condition  $G_{\hat{r}\hat{r}} = G_{\hat{\theta}\hat{\theta}}$ . We then classify this ODE as to its type, and whenever possible, derive suitable transformation theorems and solution-generating theorems.

To place the use of “unusual” coordinate systems in perspective, one might observe that Finch–Skea<sup>3</sup> estimate that about 55% of all work on fluid spheres is carried out in Schwarzschild curvature coordinates, that isotropic coordinates account for about 35% of related research, and that the remaining 10% is spread over multiple specialized coordinate systems. We take the viewpoint that the “unusual” coordinate systems are useful *only* insofar as they enable us to obtain particularly simple analytic results, and the central point of this article is to see just how much we can do in this regard. We also consider what conditions are required on the space–time metric and transformation theorems to guarantee regularity and physicality of both the geometry and the matter distribution near the center of the configuration.

### 3. General Diagonal Coordinates

We begin by setting the notation. Choose coordinates to put the metric into the form

$$ds^2 = -\zeta(r)^2 dt^2 + \frac{dr^2}{B(r)} + R(r)^2 d\Omega^2. \quad (5)$$

A brief calculation yields

$$G_{\hat{r}\hat{r}} = -\frac{1 - B(R')^2}{R^2} + \frac{2B\zeta'R'}{R\zeta}, \quad (6)$$

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = \frac{1}{2} \frac{2B\zeta'R' + 2\zeta R''B + \zeta B'R' + 2R\zeta''B + R\zeta'B'}{R\zeta}, \quad (7)$$

$$G_{\hat{t}\hat{t}} = \frac{1 - B(R')^2}{R^2} + \frac{2BR'' - B'R'}{R}. \quad (8)$$

#### 3.1. ODEs

If this space–time geometry is to be a perfect fluid sphere, then we must have  $G_{\hat{r}\hat{r}} = G_{\hat{\theta}\hat{\theta}}$ . This isotropy constraint supplies us with an ODE, which we can write in the form

$$[R(R\zeta)']B' + 2[RR''\zeta + R^2\zeta'' - RR'\zeta' - (R')^2\zeta]B + 2\zeta = 0. \quad (9)$$

In this form the ODE is a first-order, linear, nonhomogeneous ODE in  $B(r)$ , and hence *explicitly* solvable. (Though it must be admitted that the explicit solution is sometimes rather messy.) The physical impact of this ODE (9) is that it reduces the freedom to choose the three *a priori* arbitrary functions in the general diagonal

space–time metric (5) to two arbitrary functions (and we still have some remaining coordinate freedom in the  $r$ – $t$  plane, which then allows us to specialize the ODE even further).

The *same* ODE (9) can also be rearranged as

$$[2R^2B]\zeta'' + [R^2B' - 2BRR']\zeta' + [2B(RR'' - [R']^2) + RB'R' + 2]\zeta = 0. \quad (10)$$

In this form the ODE is a second-order, linear, homogeneous ODE in  $\zeta(r)$ . While this ODE is not explicitly solvable in closed form, it is certainly true that a lot is known about its generic behavior.

We can also view the two equivalent ODEs (9) and (10) as an ODE for  $R(r)$ :

$$[2B\zeta]RR'' + [B'\zeta - 2B\zeta']RR' - [2B\zeta](R')^2 + [2B\zeta'' + B'\zeta']R^2 + 2\zeta = 0. \quad (11)$$

Viewed in this manner it is a second-order nonlinear ODE of no discernible special form — and this approach does not seem to lead to any useful insights.

### 3.2. *Solution-generating theorems*

Two rather general solution-generating theorems can be derived for the general diagonal line element.

**Theorem 1 (general diagonal 1).** *Suppose that we adopt general diagonal coordinates and that  $\{\zeta(r), B(r), R(r)\}$  represents a perfect fluid sphere. Define*

$$\Delta(r) = \lambda \left( \frac{R(r)\zeta(r)}{(R(r)\zeta(r))'} \right)^2 \exp \left( 2 \int \frac{\zeta'(r)}{\zeta(r)} \cdot \frac{(R'(r)\zeta(r) - R(r)\zeta'(r))}{(R'(r)\zeta(r) + R(r)\zeta'(r))} dr \right). \quad (12)$$

*Then, for all  $\lambda$ , the geometry defined by holding  $\zeta(r)$  fixed and setting*

$$ds^2 = -\zeta(r)^2 dt^2 + \frac{dr^2}{B(r) + \Delta(r)} + R(r)^2 d\Omega^2 \quad (13)$$

*is also a perfect fluid sphere. That is, the mapping*

$$\mathsf{T}_{\text{gen1}}(\lambda) : \{\zeta, B, R\} \mapsto \{\zeta, B + \Delta, R\} \quad (14)$$

*takes perfect fluid spheres into perfect fluid spheres.*

**Proof.** The proof is based on the techniques used in Refs. 14 and 16. No new principles are involved and we quickly sketch the argument. Assume that  $\{\zeta(r), B(r), R(r)\}$  is a solution for Eq. (9). Under what conditions does  $\{\zeta(r), \tilde{B}(r), R(r)\}$  also satisfy Eq. (9)? Without loss of generality, we write

$$\tilde{B}(r) = B(r) + \Delta(r). \quad (15)$$

Substitute  $\tilde{B}(r)$  in Eq. (9):

$$[R(R\zeta)'](B + \Delta)' + 2[RR''\zeta + R^2\zeta'' - RR'\zeta' - (R')^2\zeta](B + \Delta) + 2\zeta = 0. \quad (16)$$

Rearranging, we see that

$$[R(R\zeta)']B' + 2[RR''\zeta + R^2\zeta'' - RR'\zeta' - (R')^2\zeta]B + 2\zeta + [R(R\zeta)']\Delta' + 2[RR''\zeta + R^2\zeta'' - RR'\zeta' - (R')^2\zeta]\Delta = 0. \tag{17}$$

But we know that the first line in Eq. (17) is zero, because  $\{R, \zeta, B\}$  corresponds by hypothesis to a perfect fluid sphere. Therefore

$$[R(R\zeta)']\Delta' + 2[RR''\zeta + R^2\zeta'' - RR'\zeta' - (R')^2\zeta]\Delta = 0, \tag{18}$$

which is an ordinary homogeneous, first-order differential equation in  $\Delta$ . A straightforward calculation, including an integration by parts, now leads to

$$\Delta = \lambda \left( \frac{R\zeta}{(R\zeta)'} \right)^2 \exp \left( 2 \int \frac{\zeta'}{\zeta} \cdot \frac{(R'\zeta - R\zeta')}{(R'\zeta + R\zeta')} dr \right), \tag{19}$$

as required. □

**Theorem 2 (general diagonal 2).** *Suppose that we adopt general diagonal that coordinates, and  $\{\zeta(r), B(r), R(r)\}$  represents a perfect fluid sphere. Define*

$$Z(r) = \sigma + \epsilon \int \frac{R(r)}{\sqrt{B(r)} \zeta(r)^2} dr. \tag{20}$$

*Then, for all  $\sigma$  and  $\epsilon$ , the geometry defined by holding  $B(r)$  and  $R(r)$  fixed and setting*

$$ds^2 = -\zeta(r)^2 Z(r)^2 dt^2 + \frac{dr^2}{B(r)} + R(r)^2 d\Omega^2 \tag{21}$$

*is also a perfect fluid sphere. That is, the mapping*

$$\mathsf{T}_{\text{gen2}}(\sigma, \epsilon) : \{\zeta, B, R\} \mapsto \{\zeta Z(\zeta, B, R), B, R\} \tag{22}$$

*takes perfect fluid spheres into perfect fluid spheres.*

**Proof.** The proof is based on the technique of “reduction in order” as used in Refs. 14 and 16. No new principles are involved and we quickly sketch the argument. Assuming that  $\{\zeta(r), B(r), R(r)\}$  solves Eq. (10), write

$$\zeta(r) \rightarrow \zeta(r)Z(r) \tag{23}$$

and demand that  $\{\zeta(r)Z(r), B(r), R(r)\}$  also solve Eq. (10). Then

$$[2R^2B](\zeta Z)'' + [R^2B' - 2BRR'](\zeta Z)' + [2B(RR'' - [R']^2) + RB'R' + 2](\zeta Z) = 0. \tag{24}$$

We expand the above equation to

$$[2R^2B](\zeta''Z + 2\zeta'Z' + \zeta Z'') + [R^2B' - 2BRR'](\zeta'Z + \zeta Z') + [2B(RR'' - [R']^2) + RB'R' + 2](\zeta Z) = 0, \tag{25}$$

and then regroup to obtain

$$\begin{aligned} & \{[2R^2B]\zeta'' + [R^2B' - 2BRR']\zeta' + [2B(RR'' - [R']^2) + RB'R' + 2]\zeta\} Z \\ & \times [2R^2B](2\zeta'Z' + \zeta Z'') + [R^2B' - 2BRR']\zeta Z' = 0. \end{aligned} \quad (26)$$

This is a linear, homogeneous, second order ODE for  $Z$ . But under the current hypotheses the entire first line simplifies to zero — so the ODE now simplifies to

$$[R^2B'\zeta + 4R^2B\zeta' - 2BRR'\zeta]Z' + (2R^2B\zeta)Z'' = 0. \quad (27)$$

This is a first-order, linear ODE in the dependent quantity  $Z'(r)$ . Rearrange the above equation into

$$\frac{Z''}{Z'} = -\frac{R^2B'\zeta + 4R^2B\zeta' - 2BRR'\zeta}{2R^2B\zeta}. \quad (28)$$

Simplifying,

$$\frac{Z''}{Z'} = -\frac{1}{2}\frac{B'}{B} - 2\frac{\zeta'}{\zeta} + \frac{R'}{R}. \quad (29)$$

Rewrite  $Z''/Z' = d\ln(Z')/dr$ , and integrate twice over both sides of Eq. (29), to obtain

$$Z(r) = \sigma + \epsilon \int \frac{R}{\sqrt{B}\zeta^2} dr, \quad (30)$$

depending on the old solution  $\{\zeta, B, R\}$ , and two arbitrary integration constants,  $\sigma$  and  $\epsilon$ .  $\square$

### 3.3. Regularity conditions

Now consider the result of placing various regularity conditions on the geometry of the perfect fluid sphere. (See particularly the discussion by Delgaty and Lake regarding the importance of regularity conditions.<sup>2</sup>) At a minimum we want the space–time geometry to be well defined and smooth everywhere. This requires:

- $\zeta(r) > 0$ , so that a clock held at any fixed but arbitrary “place”  $(r, \theta, \phi)$  is well behaved, and at least continues to “tick” (even if it is fast or slow).
- Now the “center” of the space–time is at some *a priori* arbitrary coordinate  $r_0$  such that the very specific condition  $R(r_0) = 0$  is satisfied; by a simple coordinate change we can without loss of generality set  $r_0 \rightarrow 0$  so that at the center of the space–time  $R(0) = 0$ .
- So purely for geometrical reasons we should demand that

- $R(0) = 0$ ,
- $\zeta(0) > 0$ .

- Now consider the spatial three-geometry

$$\frac{dr^2}{B(r)} + R(r)^2 d\Omega^2, \tag{31}$$

and consider a small “ball” centered on the origin with coordinate radius  $r$ . The surface area of such a ball is

$$S = 4\pi R(r)^2, \tag{32}$$

while its proper radius is

$$\ell = \int_0^r \frac{dr}{\sqrt{B(r)}}. \tag{33}$$

If the geometry at the center is to be smooth (i.e. locally Euclidean), then these two measures of the radius must asymptotically agree (for small balls):

$$\ell = R + \mathcal{O}(R^2). \tag{34}$$

Thus, in terms of the original coordinate  $r$

$$\left. \frac{\partial \ell}{\partial r} \right|_{r=0} = \frac{1}{\sqrt{B(0)}} = \left. \frac{\partial R}{\partial r} \right|_{r=0}, \tag{35}$$

implying that

$$B(0) [R'(0)]^2 = 1, \quad B(0) \neq 0, \quad \text{i.e.} \quad B(0) = \frac{1}{[R'(0)]^2}.$$

- Now, if the space-time is regular at the center we must avoid “kinks” in the metric components. Specifically, in terms of proper distance from the center we should have  $\zeta(\ell) = \zeta(0) + \mathcal{O}(\ell^2)$ . But in view of the already-established relation between  $\ell$  and  $r$  this implies that  $\zeta(r) = \zeta(0) + \mathcal{O}(r^2)$  and we must have

$$\zeta'(0) = 0. \tag{36}$$

- This does not complete the list of regularity conditions. We can also look at the orthonormal components of the Einstein tensor. Consider

$$G_{\hat{r}\hat{r}} = -\frac{1 - B(R')^2}{R^2} + \frac{2B\zeta'R'/\zeta}{R}. \tag{37}$$

From the conditions already established, applying the *l'Hospital* rule to the last term leads to a finite result,  $2B(0)\zeta''(0)/\zeta(0)$ . For the remaining term, noting that the denominator ( $R^2$ ) has a double zero at the center, we deduce that the *derivative* of the numerator must likewise be zero, in order that there be any chance for the *l'Hospital* rule to deliver a finite answer. This now yields the additional constraint that at the center

$$[B(R')^2]' \rightarrow 0, \tag{38}$$

i.e.

$$B'(0)R'(0) + 2B(0)R''(0) = 0. \tag{39}$$

In view of our earlier result for  $B(0)$  this can be written as

$$B'(0) = -\frac{2R''(0)}{[R'(0)]^3} = \left( \frac{1}{[R'(r)]^2} \right)' \Big|_{r=0}. \quad (40)$$

- To check consistency, consider

$$G_{\hat{r}\hat{r}} + G_{\hat{t}\hat{t}} = \frac{2B\zeta'R'/\zeta - B'R' - 2BR''}{R}, \quad (41)$$

and note that this is finite at the center provided that  $\zeta' \rightarrow 0$  and  $[B(R')^2]' \rightarrow 0$ .

- As a final consistency check consider

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = \frac{B\zeta'R'}{R\zeta} + \frac{2\zeta''B + \zeta'B'}{2\zeta} + \frac{2R''B + B'R'}{2R}, \quad (42)$$

and again note that this is finite at the center provided that  $\zeta' \rightarrow 0$  and  $[B(R')^2]' \rightarrow 0$ .

- So now let us collect all our regularity conditions:

$$R(0) = 0, \quad \zeta(0) > 0, \quad B(0) = \frac{1}{R'(0)^2} > 0, \quad (43)$$

$$\zeta'(0) = 0, \quad B'(0) = -\frac{2R''(0)}{[R'(0)]^3}. \quad (44)$$

- Finally, inserting all these conditions into the  $r \rightarrow 0$  limit we quickly find that at the center of the fluid sphere the Einstein tensor has the limit

$$G_{\hat{r}\hat{r}}|_0 = G_{\hat{\theta}\hat{\theta}}|_0 = G_{\hat{\phi}\hat{\phi}}|_0 = \frac{B''(0)}{2} + \frac{2\zeta''(0)}{\zeta(0)[R'(0)]^2} + \frac{R'''(0)}{[R'(0)]^3} - \frac{3[R''(0)]^2}{[R'(0)]^4}, \quad (45)$$

$$G_{\hat{t}\hat{t}}|_0 = -\frac{3}{2} \left\{ B''(0) + \frac{2R'''(0)}{[R'(0)]^3} - \frac{6[R''(0)]^2}{[R'(0)]^4} \right\}, \quad (46)$$

and therefore

$$G_{\hat{t}\hat{t}}|_0 + 3G_{\hat{r}\hat{r}}|_0 = \frac{6\zeta''(0)}{\zeta(0)[R'(0)]^2}. \quad (47)$$

- There is another, largely independent way of getting these regularity conditions, either by hand or with the aid of a symbolic manipulation program such as Maple. Let us start by assuming that near the origin all metric functions have Taylor series expansions

$$\zeta(r) = \zeta_0 + \zeta_1 r + \frac{\zeta_2 r^2}{2!} + \frac{\zeta_3 r^3}{3!} + \mathcal{O}(r^4), \quad (48)$$

$$B(r) = B_0 + B_1 r + \frac{B_2 r^2}{2!} + \frac{B_3 r^3}{3!} + \mathcal{O}(r^4), \quad (49)$$

$$R(r) = R_1 r + \frac{R_2 r^2}{2!} + \frac{R_3 r^3}{3!} + \mathcal{O}(r^4). \quad (50)$$

Now use this *ansatz* to calculate  $G_{\hat{t}\hat{t}}$ ,  $G_{\hat{r}\hat{r}}$ , and  $G_{\hat{\theta}\hat{\theta}}$  as Laurent series in  $r$ . A symbolic manipulation program such as Maple will quickly tell you that

$$G_{\hat{a}\hat{b}} = \frac{A_{\hat{a}\hat{b}}}{r^2} + \frac{B_{\hat{a}\hat{b}}}{r} + C_{\hat{a}\hat{b}} + \mathcal{O}(r). \tag{51}$$

Regularity now has the obvious meaning that you want the coefficients  $A_{\hat{a}\hat{b}}$  and  $B_{\hat{a}\hat{b}}$  of the  $1/r^2$  and  $1/r$  pieces to vanish — which implies constraints on the Taylor series coefficients  $\{\zeta_i, B_i, R_i\}$ . A brief computation should convince one that the pole residues  $A_{\hat{a}\hat{b}}$  and  $B_{\hat{a}\hat{b}}$  vanish provided that

$$R_0 = 0, \quad \zeta_0 > 0, \quad B_0 = \frac{1}{R_1^2} > 0, \quad \zeta_1 = 0, \tag{52}$$

$$B_1 = -\frac{2R_2}{(R_1)^3}, \tag{53}$$

i.e.

$$R(0) = 0, \quad \zeta(0) > 0, \quad B(0) = \frac{1}{R'(0)^2} > 0, \quad \zeta'(0) = 0, \tag{54}$$

$$B'(0) = -\frac{2R''(0)}{[R'(0)]^3}. \tag{55}$$

These are the same regularity conditions as we previously found by hand. Imposing these conditions, symbolic manipulation software such as Maple can then easily evaluate the limits as you move to the center of the star.

The situation for general diagonal coordinates is the most complicated case considered in this article; other cases, discussed below, are simpler.

### 3.4. Implications

Note that the regularity conditions do feed back into the theorem General diagonal 2. In that theorem we were interested in the quantity

$$Z(r) = \sigma + \epsilon \int \frac{R(r)}{\sqrt{B(r)} \zeta(r)^2} dr. \tag{56}$$

In view of the regularity conditions we now know that for a physically reasonable fluid sphere the integrand is finite all the way to the origin so that we can without loss of generality set

$$Z(r) = \sigma + \epsilon \int_0^r \frac{R(r)}{\sqrt{B(r)} \zeta(r)^2} dr. \tag{57}$$

But to ensure regularity we want both  $\zeta(0)$  and the new  $\zeta(0) \rightarrow \zeta(0)Z(0)$  to be finite and positive, hence  $Z(0) > 0$ , and therefore  $\sigma > 0$ . This reproduces in a purely mathematical way one of the results of Ref. 15, where we used the TOV and boundedness of the central pressure and density to impose a similar constraint.

### 3.5. Summary

Let us now assess what we have done so far: for arbitrary spherically symmetric static space–times in general diagonal coordinates we have investigated the perfect fluid constraint, derived two “solution-generating theorems,” and applied regularity conditions to the space–time. The analysis has so far been rather general, and we still have the freedom to make one further specific coordinate choice to simplify the line element, Einstein tensor, solution-generating theorem, and regularity conditions. Making the extra coordinate choice will sometimes simplify the mathematical structure sufficiently to allow more information to be extracted.

## 4. Schwarzschild Curvature Coordinates

The Schwarzschild curvature coordinate system is by far the most popular coordinate system used in the study of perfect fluid spheres, and this coordinate choice [corresponding to  $R(r) \rightarrow r$ ] accounts for approximately 55% of the research reported on perfect fluid spheres.<sup>3</sup> The Schwarzschild coordinate system is the basis of the various theorems derived and discussed in Refs. 14–18.

The line element of a spherically symmetric space–time in Schwarzschild curvature coordinates is

$$ds^2 = -\zeta(r)^2 dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega^2. \quad (58)$$

To begin, we calculate

$$G_{\hat{r}\hat{r}} = \frac{2B\zeta'r - \zeta + \zeta B}{r^2\zeta}, \quad (59)$$

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = \frac{1}{2} \frac{B'\zeta + 2B\zeta' + 2r\zeta''B + r\zeta'B'}{r\zeta}, \quad (60)$$

$$G_{\hat{t}\hat{t}} = -\frac{B'r - 1 + B}{r^2}. \quad (61)$$

### 4.1. ODEs

Pressure isotropy leads to the ODE

$$[r(r\zeta)']B' + [2r^2\zeta'' - 2(r\zeta)']B + 2\zeta = 0. \quad (62)$$

This is first-order, linear, nonhomogeneous in  $B(r)$ . Solving for  $B(r)$  in terms of  $\zeta(r)$  is the basis of the analysis in Ref. 13, and is also the basis for Theorem 1 in Ref. 14. (See also Ref. 16 and the reports in Refs. 17 and 18. After rephrasing in terms of the TOV equation this is also related to Theorem P2 in Ref. 15.) If we regroup in terms of  $\zeta(r)$  we find that

$$2r^2\zeta'' + (r^2B' - 2rB)\zeta' + (rB' - 2B + 2)\zeta = 0, \quad (63)$$

which is a linear, homogeneous, second-order ODE. This is the basis of Theorem 2 in Ref. 14, and after being recast as a Riccati equation is also the basis of Theorem P1 in Ref. 15.

**4.2. Regularity conditions**

Now consider the regularity conditions which we demand to keep the space–time regular at the center. First we have:

- The “center” of the space–time is by construction at  $r = 0$ .
- We must again demand that  $\zeta(r) > 0$ , so that a clock held at fixed  $(r, \theta, \phi)$  is well behaved and at least continues to “tick” (even if it is fast or slow).
- In particular,  $\zeta(0) > 0$ .
- We now consider the spatial geometry

$$\frac{dr^2}{B(r)} + r^2 d\Omega^2, \tag{64}$$

and slightly modify the discussion (previously developed in general diagonal coordinates) for a small “ball” centered on the origin with coordinate radius  $r$ . The surface area of such a ball is

$$S = 4\pi r^2, \tag{65}$$

while its proper radius is

$$\ell = \int_0^r \frac{dr}{\sqrt{B(r)}}. \tag{66}$$

If the geometry at the center is to be smooth (locally Euclidean), then these two measures of the radius must asymptotically agree (for small balls):

$$\ell = r + \mathcal{O}(r^2). \tag{67}$$

But in terms of the original coordinate  $r$  this implies that

$$B(0) \neq 0, \quad \sqrt{B(0)} = 1, \quad \text{i.e.} \quad B(0) = 1. \tag{68}$$

Note that we now have the much tighter constraint  $B(0) = 1$ .

- The same argument used for general diagonal coordinates still implies that

$$\zeta'(0) = 0. \tag{69}$$

- By considering

$$G_{\hat{r}\hat{r}} = \frac{2B\zeta'}{r\zeta} + \frac{B-1}{r^2} \tag{70}$$

we deduce that at the center

$$B' \rightarrow 0, \tag{71}$$

i.e.

$$B'(0) = 0. \tag{72}$$

- To check consistency note that

$$G_{\hat{r}\hat{r}} + G_{\hat{t}\hat{t}} = \frac{-B' + 2B\zeta'/\zeta}{r}, \quad (73)$$

which is finite at the center provided that  $B' \rightarrow 0$  and  $\zeta' \rightarrow 0$ .

- For the final consistency check note that

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = \frac{B'}{2r} + \frac{2B\zeta'' + B'\zeta'}{2\zeta} + \frac{B\zeta'}{r\zeta}, \quad (74)$$

which is finite at the center provided that  $B' \rightarrow 0$  and  $\zeta' \rightarrow 0$ .

- So now let us collect all our regularity conditions:

$$\zeta(0) > 0, \quad B(0) = 1, \quad \zeta'(0) = 0, \quad B'(0) = 0. \quad (75)$$

- Inserting all these conditions into the limit, we quickly find that

$$G_{\hat{r}\hat{r}}|_0 = G_{\hat{\theta}\hat{\theta}}|_0 = G_{\hat{\phi}\hat{\phi}}|_0 = \frac{2\zeta''(0)}{\zeta(0)} + \frac{B''(0)}{2}, \quad (76)$$

$$G_{\hat{t}\hat{t}}|_0 = -\frac{3B''(0)}{2}, \quad (77)$$

$$G_{\hat{t}\hat{t}}|_0 + 3G_{\hat{r}\hat{r}}|_0 = \frac{6\zeta''(0)}{\zeta(0)}. \quad (78)$$

- The regularity conditions derived above can again be extracted in another manner, by using a Taylor series expansion for the metric components to derive a Laurent series for the Einstein tensor — using (for example) *Maple*, this will still give the same regularity conditions as those we derived by hand. This is a specialization of the method developed in the previous general diagonal coordinates. A little work with *Maple* should convince you that the pole pieces vanish provided that

$$\zeta(0) > 0, \quad B(0) = 1, \quad \zeta'(0) = 0, \quad B'(0) = 0. \quad (79)$$

The ultimate consistency check on this formalism now results from considering general diagonal coordinates and simply setting  $R(r) \rightarrow r$ . That the results agree under this substitution gives us confidence that the formalism is internally consistent.

## 5. Isotropic Coordinates

Isotropic coordinates are again commonly used coordinates in the investigation of perfect fluid spheres. This is the second-most-popular coordinate system, accounting for about 35% of published research in this field.<sup>3</sup> The space-time metric for an arbitrary static, spherically symmetric space-time in isotropic coordinates is conveniently given by

$$ds^2 = -\zeta(r)^2 dt^2 + \frac{1}{\zeta(r)^2 B(r)^2} \{dr^2 + r^2 d\Omega^2\}. \quad (80)$$

We calculate

$$G_{\hat{r}\hat{r}} = -\frac{2BB'\zeta^2}{r} + (B')^2\zeta^2 - \zeta'^2B^2, \tag{81}$$

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = -\frac{BB'\zeta^2}{r} + (B')^2\zeta^2 - BB''\zeta^2 + B^2\zeta'^2, \tag{82}$$

$$G_{\hat{t}\hat{t}} = 2B^2\zeta\zeta'' + \frac{4B^2\zeta\zeta'}{r} - 3B^2\zeta'^2 - 2BB'\zeta\zeta' + 2BB''\zeta^2 - 3B'^2\zeta^2 + \frac{4BB'\zeta^2}{r}. \tag{83}$$

**5.1. ODEs**

The pressure isotropy condition leads to the very-simple-looking ODE<sup>11</sup>

$$\left(\frac{\zeta'}{\zeta}\right)^2 = \frac{B'' - B'/r}{2B}. \tag{84}$$

There are several ways of improving this. For instance, if we write  $\zeta(r) = \exp(\int g(r)dr)$ , then we have an algebraic equation for  $g(r)$ <sup>11</sup>:

$$g(r) = \pm\sqrt{\frac{B'' - B'/r}{2B}}. \tag{85}$$

Conversely, the isotropy condition can be written in terms of  $B(r)$  as

$$B'' - \frac{B'}{r} - 2g^2B = 0. \tag{86}$$

There is also an improvement obtained by writing  $B(r) = \exp(2\int h(r)dr)$  so that

$$g(r)^2 = 2h(r)^2 + h'(r) - \frac{h(r)}{r}, \tag{87}$$

which is the basis of the analysis in Ref. 11.

**5.2. Solution-generating theorems**

Let us now develop two transformation theorems appropriate to isotropic coordinates.

**Theorem 3 (Isotropic 1 — Buchdahl transformation).** *In Isotropic coordinates, if  $\{\zeta(r), B(r)\}$  describes a perfect fluid, then so does  $\{\zeta(r)^{-1}, B(r)\}$ . This is the well-known Buchdahl transformation in disguise. That is, if*

$$ds^2 = -\zeta(r)^2 dt^2 + \frac{1}{\zeta(r)^2 B(r)^2} \{dr^2 + r^2 d\Omega^2\} \tag{88}$$

represents a perfect fluid sphere, then the geometry defined by

$$ds^2 \rightarrow -\frac{1}{\zeta(r)^2} dt^2 + \frac{\zeta(r)^2}{B(r)^2} \{dr^2 + r^2 d\Omega^2\} \tag{89}$$

is also a perfect fluid sphere. Alternatively, the mapping

$$\mathbb{T}_{\text{Iso1}} : \{\zeta, B\} \mapsto \{\zeta^{-1}, B\} \tag{90}$$

takes perfect fluid spheres into perfect fluid spheres, and furthermore is a “square root of unity” in the sense that

$$\mathbb{T}_{\text{Iso1}} \circ \mathbb{T}_{\text{Iso1}} = I. \tag{91}$$

**Proof.** By inspection. *Vide* Eq. (84). □

**Theorem 4 (Isotropic 2).** Let  $\{\zeta(r), B(r)\}$  describe a perfect fluid sphere. Define

$$Z = \left\{ \sigma + \epsilon \int \frac{rdr}{B(r)^2} \right\}. \tag{92}$$

Then, for all  $\sigma$  and  $\epsilon$ , the geometry defined by holding  $\zeta(r)$  fixed and setting

$$ds^2 = -\zeta(r)^2 dt^2 + \frac{1}{\zeta(r)^2 B(r)^2 Z(r)^2} \{dr^2 + r^2 d\Omega^2\} \tag{93}$$

is also a perfect fluid sphere. That is, the mapping

$$\mathbb{T}_{\text{Iso2}}(\sigma, \epsilon) : \{\zeta, B\} \mapsto \{\zeta, BZ(B)\} \tag{94}$$

takes perfect fluid spheres into perfect fluid spheres.

**Proof.** The proof is based on the technique of “reduction in order,” and is a simple variant on the discussion in Ref. 14. Assuming that  $\{\zeta(r), B(r)\}$  solves Eq. (86), write

$$B(r) \rightarrow B(r)Z(r) \tag{95}$$

and demand that  $\{\zeta(r), B(r)Z(r)\}$  also solve Eqs. (84) and (86). We find that

$$(BZ)'' - \frac{(BZ)'}{r} - 2g^2(BZ) = 0. \tag{96}$$

Regrouping,

$$\left\{ B'' - \frac{B'}{r} - 2g^2 B \right\} Z + \left( 2B' - \frac{B}{r} \right) Z' + BZ'' = 0. \tag{97}$$

This is a linear, homogeneous, second-order ODE for  $Z$  which now simplifies [in view of (86)] to

$$\left( 2B' - \frac{B}{r} \right) Z' + BZ'' = 0, \tag{98}$$

which is an ordinary homogeneous, second-order differential equation, depending only on  $Z'$  and  $Z''$ . (So it can be viewed as a first-order, homogeneous differential equation in  $Z'$ , which is solvable.) Separating the unknown variable to one side,

$$\frac{Z''}{Z'} = -2\frac{B'}{B} + \frac{1}{r}. \tag{99}$$

Rewrite  $Z''/Z' = d\ln(Z')/dr$ , and integrate twice over both sides of Eq. (99), to obtain

$$Z(r) = \left\{ \sigma + \epsilon \int \frac{rdr}{B(r)^2} \right\}, \tag{100}$$

depending on the old solution  $\{\zeta(r), B(r)\}$ , and two arbitrary integration constants,  $\sigma$  and  $\epsilon$ . □

**Theorem 5 (Isotropic 3).** *The transformations  $T_{\text{Iso1}}$  and  $T_{\text{Iso2}}$  commute.*

**Proof.** By inspection. □

Note that the fact that these two transformation theorems commute is specific to isotropic coordinates. Such behavior certainly does not occur in Schwarzschild coordinates or in general diagonal coordinates.

### 5.3. Regularity conditions

We now investigate the regularity conditions for isotropic coordinates. Some details are a little tricky, and have led to confusion in some of the previous published literature.

- The “center” of the space–time is at some coordinate  $r_0$ ; by a simple coordinate change we can set  $r_0 \rightarrow 0$  so that at the center of the space–time is at  $r = 0$ .
- We still require  $\zeta(r) > 0$ , so that a clock held at fixed  $(r, \theta, \phi)$  is well behaved and at least continues to “tick” (even if it is fast or slow). In particular,

$$\zeta(0) > 0. \tag{101}$$

- Now consider the spatial geometry

$$\frac{dr^2}{\zeta(r)^2 B(r)^2} + \frac{r^2}{\zeta(r)^2 B(r)^2} d\Omega^2 = \frac{\{dr^2 + r^2 d\Omega^2\}}{\zeta(r)^2 B(r)^2} \tag{102}$$

and consider a small “ball” centered on the origin with coordinate radius  $r$ . The surface area of such a ball is

$$S = 4\pi \frac{r^2}{\zeta(r)^2 B(r)^2}, \tag{103}$$

while its proper radius is

$$\ell = \int_0^r \frac{dr}{\zeta(r)B(r)}. \tag{104}$$

If the geometry at the center is to be smooth, then these two measures of the radius must asymptotically agree (for small balls):

$$\int_0^r \frac{dr}{\zeta(r)B(r)} = \frac{r}{\zeta(r)B(r)} + \mathcal{O}(r^2), \tag{105}$$

implying that

$$\frac{1}{\zeta(r)B(r)} = \left( \frac{r}{\zeta(r)B(r)} \right)' + \mathcal{O}(r), \quad (106)$$

i.e.

$$\frac{1}{\zeta(r)B(r)} = \frac{1}{\zeta(r)B(r)} + \mathcal{O}(r). \quad (107)$$

This implies *no constraint* on  $\zeta(r)$  or  $B(r)$ , beyond the obvious constraint that these quantities remain finite and nonzero. Note in particular that in isotropic coordinates it is *not true* that we need to enforce  $B(0) = 1$ . [If  $B(0) \neq 1$  we could think of arbitrarily rescaling it by a constant factor to force it to be unity; this would at worst multiply all curvature invariants by a constant, and so a nonsingular geometry remains nonsingular under such rescaling. This point is commonly misunderstood in the literature.]

- On the other hand, we could also find all our regularity conditions for isotropic coordinates by deducing them from the analysis developed for general diagonal coordinates. Identifying coefficients in the two metrics we see that

$$\zeta_{\text{iso}} = \zeta_{\text{gen}}; \quad (\zeta B)_{\text{iso}}^2 = B_{\text{gen}}, \quad (108)$$

$$\frac{r}{(\zeta B)_{\text{iso}}} = R_{\text{gen}}. \quad (109)$$

Now consider

$$[B(R')^2]_{\text{gen}} = (\zeta B)_{\text{iso}}^2 \left[ \left( \frac{r}{(\zeta B)_{\text{iso}}} \right)' \right]^2 = \left[ 1 - \frac{r\zeta'}{\zeta} - \frac{rB'}{B} \right]_{\text{iso}}^2. \quad (110)$$

So at the center  $[B(R')^2]_{\text{gen}} \rightarrow 1$  provides no extra information about  $B_{\text{iso}}$  or  $\zeta_{\text{iso}}$  (beyond the fact that these quantities are finite and nonzero).

- The same argument used for general diagonal coordinates and Schwarzschild coordinates still implies that

$$\zeta'(0) = 0. \quad (111)$$

- This does not complete the list of regularity conditions. Consider

$$G_{\hat{r}\hat{r}} = -\frac{2BB'\zeta^2}{r} + B'^2\zeta^2 - \zeta'^2B^2. \quad (112)$$

Finiteness of this quantity now implies that

$$B'(0) = 0. \quad (113)$$

- On the other hand, finiteness of

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = \frac{-BB'\zeta^2}{r} + B'^2\zeta^2 - BB''\zeta^2 + B^2\zeta'^2 \quad (114)$$

implies no additional constraint.

- Similarly, finiteness of

$$G_{\hat{t}\hat{t}} = 2B^2\zeta\zeta'' + \frac{4B^2\zeta\zeta'}{r} - 3B^2\zeta'^2 - 2BB'\zeta\zeta' + 2BB''\zeta^2 - 3B'^2\zeta^2 + \frac{4BB'\zeta^2}{r} \quad (115)$$

implies no additional constraint.

- Summarizing, our regularity conditions are

$$B(0) > 0, \quad B'(0) = 0, \quad \zeta(0) > 0, \quad \zeta'(0) = 0. \quad (116)$$

- Now, inserting the above conditions into the central limit we quickly find that

$$G_{\hat{r}\hat{r}}|_0 = G_{\hat{\theta}\hat{\theta}}|_0 = G_{\hat{\phi}\hat{\phi}}|_0 = -2B(0)B''(0)\zeta(0)^2, \quad (117)$$

$$G_{\hat{t}\hat{t}}|_0 = 6B(0)B''(0)\zeta(0)^2 + 6B(0)^2\zeta(0)\zeta''(0), \quad (118)$$

$$G_{\hat{t}\hat{t}}|_0 + 3G_{\hat{r}\hat{r}}|_0 = 6B(0)^2\zeta(0)\zeta''(0).$$

- On the other hand, the regularity conditions can also be easily obtained by using Taylor series for the metric components to compute Laurent series for the orthonormal components of the Einstein tensor. A brief computation with Maple should convince you that the pole pieces vanish provided that

$$\zeta(0) > 0, \quad B(0) > 0, \quad \zeta'(0) = 0, \quad B'(0) = 0.$$

#### 5.4. Implications

Note that the regularity conditions do feed back into the theorem Isotropic 2. In that theorem we were interested in the quantity

$$Z(r) = \sigma + \epsilon \int \frac{rdr}{B(r)^2}. \quad (119)$$

In view of the regularity conditions we now know that for a physically reasonable fluid sphere the integrand is finite all the way to the origin so that we can without loss of generality set

$$Z(r) = \sigma + \epsilon \int_0^r \frac{rdr}{B(r)^2}. \quad (120)$$

But to ensure regularity we want both  $B(0)$  and the new  $B(0) \rightarrow B(0)Z(0)$  to be finite and positive, hence  $Z(0) > 0$ , and therefore  $\sigma > 0$ .

### 6. Gaussian Polar Coordinates

Having now dealt with the most common coordinate systems, viz. Schwarzschild curvature coordinates and isotropic coordinates, we will spend some time studying other, more unusual coordinate systems. Although Gaussian polar coordinates are typically less convenient than Schwarzschild or isotropic coordinates, we have found them to be occasionally useful in the theory of symmetric, static perfect

fluid spheres. Consider, for instance, the metric in Gaussian polar (proper radius) coordinates:

$$ds^2 = -\zeta(r)^2 dt^2 + dr^2 + R(r)^2 d\Omega^2. \quad (121)$$

The following expressions can easily be calculated:

$$G_{\hat{r}\hat{r}} = \frac{\zeta R'^2 - \zeta + 2\zeta' RR'}{R^2 \zeta}, \quad (122)$$

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = \frac{\zeta R'' + \zeta' R' + \zeta'' R}{R\zeta}, \quad (123)$$

$$G_{\hat{t}\hat{t}} = \frac{-R'^2 - 2RR'' + 1}{R^2}. \quad (124)$$

The isotropy condition supplies a second-order, linear, homogeneous ODE for  $\zeta(r)$ :

$$\zeta'' - \zeta' \frac{R'}{R} + \zeta \left\{ \frac{1 - (R')^2 + R'' R}{R^2} \right\} = 0. \quad (125)$$

In contrast, when rearranged and viewed as an ODE for  $R(r)$ ,

$$\zeta RR'' - \zeta' RR' - \zeta (R')^2 + \zeta'' R^2 + \zeta = 0, \quad (126)$$

this has no particularly desirable properties.

### 6.1. *Solution-generating theorem*

**Theorem 6 (Gaussian polar).** *Suppose that we are in Gaussian polar coordinates, and that  $\{\zeta(r), R(r)\}$  represents a perfect fluid sphere. Define*

$$\Lambda(r) = \left\{ \sigma + \epsilon \int \frac{R(r) dr}{\zeta(r)^2} \right\}. \quad (127)$$

*Then, for all  $\sigma$  and  $\epsilon$ , the geometry defined by holding  $R(r)$  fixed and setting*

$$ds^2 = -\zeta(r)^2 \Lambda(r)^2 dt^2 + dr^2 + R(r)^2 d\Omega^2 \quad (128)$$

*is also a perfect fluid sphere. That is, the mapping*

$$\mathbb{T}_{\text{Gaussian}}(\sigma, \epsilon) : \{\zeta, R\} \mapsto \{\zeta \Lambda(\zeta, R), R\} \quad (129)$$

*takes perfect fluid spheres into perfect fluid spheres.*

**Proof.** The proof is again based on the technique of “reduction in order.” No new principles are involved and the argument is quickly sketched. Assuming that  $\{\zeta(r), R(r)\}$  solves Eq. (125), write

$$\zeta(r) \rightarrow \zeta(r)\Lambda(r), \quad (130)$$

and demand that  $\{\zeta(r)\Lambda(r), R(r)\}$  also solve Eq. (125). Then

$$(\zeta\Lambda)'' - (\zeta\Lambda)' \frac{R'}{R} + (\zeta\Lambda) \left\{ \frac{1 - (R')^2 + R''R}{R^2} \right\} = 0, \tag{131}$$

which simplifies to

$$\left( 2\zeta' - \zeta \frac{R'}{R} \right) \Lambda' + \zeta\Lambda'' = 0. \tag{132}$$

This is a first-order, homogeneous differential equation in  $\Lambda'$ , which is solvable. Separating the unknown variable and integrating twice, one obtains

$$\Lambda(r) = \left\{ \sigma + \epsilon \int \frac{R(r)dr}{\zeta(r)^2} \right\}, \tag{133}$$

depending on the old solution  $\{\zeta(r), R(r)\}$ , and two arbitrary integration constants,  $\sigma$  and  $\epsilon$ . □

**6.2. Regularity conditions**

In the following, we briefly describe the regularity conditions that must hold at the center of the fluid sphere.

- Adapting the arguments that by now should be completely standard, we have

$$R(0) = 0, \quad \zeta(0) > 0, \quad \zeta'(0) = 0. \tag{134}$$

- Now consider the spatial geometry

$$dr^2 + R(r)^2 d\Omega^2, \tag{135}$$

and consider a small “ball” centered on the origin with coordinate radius  $r$ . The surface area of such a ball is

$$S = 4\pi R(r)^2, \tag{136}$$

while its proper radius is

$$\ell = \int_0^r 1 dr = r. \tag{137}$$

If the geometry at the center is to be smooth, these two measures of the radius must asymptotically agree (for small balls):

$$r = R + \mathcal{O}(R^2). \tag{138}$$

But in terms of the original coordinate  $r$  this implies that

$$R'(0) = 1. \tag{139}$$

- Now consider

$$G_{\hat{r}\hat{r}} = \frac{R'^2 - 1}{R^2} + \frac{2\zeta'R'/\zeta}{R}. \tag{140}$$

The last term is finite by the *l'Hospital* rule and the previously derived condition  $\zeta' \rightarrow 0$ . Applying the *l'Hospital* rule twice, the first term will be finite at the center iff

$$[(R')^2]' \rightarrow 0, \tag{141}$$

i.e. iff

$$R''(0) = 0. \tag{142}$$

- For a consistency check, note that

$$G_{\hat{r}\hat{r}} + G_{\hat{t}\hat{t}} = \frac{2\zeta'R'/\zeta - 2R''}{R}, \tag{143}$$

which implies no new constraint.

- For a consistency check, note that

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = \frac{R''}{R} + \frac{R'\zeta'}{R\zeta} + \frac{\zeta''}{\zeta}, \tag{144}$$

which implies no new constraint.

- Now let us collect all our regularity conditions:

$$R(0) = 0, \quad R'(0) = 1, \quad R''(0) = 0, \quad \zeta(0) > 0, \quad \zeta'(0) = 0. \tag{145}$$

- Inserting all these conditions into the limit we quickly deduce that

$$G_{\hat{r}\hat{r}}|_0 = G_{\hat{\theta}\hat{\theta}}|_0 = G_{\hat{\phi}\hat{\phi}}|_0 = R'''(0) + \frac{2\zeta''(0)}{\zeta(0)}, \tag{146}$$

$$G_{\hat{t}\hat{t}}|_0 = -3R'''(0), \tag{147}$$

$$G_{\hat{t}\hat{t}}|_0 + 3G_{\hat{r}\hat{r}}|_0 = \frac{6\zeta''(0)}{\zeta(0)}. \tag{148}$$

- These regularity conditions can again be verified by using Taylor series in the metric components to calculate Laurent series for the orthonormal components of the Einstein tensor. Using Maple, the pole pieces vanish provided that

$$\zeta(0) > 0, \quad \zeta'(0) = 0, \quad R(0) = 0, \quad R'(0) = 1, \quad R''(0) = 0. \tag{149}$$

### 6.3. *Implications*

Note that the regularity conditions do feed back into the theorem Gaussian polar. In that theorem we were interested in the quantity

$$Z(r) = \sigma + \epsilon \int \frac{R(r)dr}{\zeta(r)^2}. \tag{150}$$

In view of the regularity conditions we now know that for a physically reasonable fluid sphere the integrand is finite all the way to the origin so that we can without

loss of generality set

$$Z(r) = \sigma + \epsilon \int_0^r \frac{R(r) dr}{\zeta(r)^2}. \quad (151)$$

But to ensure regularity we want both  $\zeta(0)$  and the new  $\zeta(0) \rightarrow \zeta(0)Z(0)$  to be finite and positive, hence  $Z(0) > 0$ , and therefore  $\sigma > 0$ .

## 7. Sygne Isothermal Coordinates

Consider the metric in Sygne isothermal (tortoise) coordinates:

$$ds^2 = -\zeta(r)^{-2} \{dt^2 - dr^2\} + \{\zeta(r)^{-2} R(r)^2 d\Omega^2\}. \quad (152)$$

A brief computation yields

$$G_{\hat{r}\hat{r}} = \frac{3R^2(\zeta')^2 - 4R\zeta R'\zeta' + (R')^2\zeta^2 - \zeta^2}{R^2}, \quad (153)$$

$$G_{\hat{\theta}\hat{\theta}} = \frac{3R(\zeta')^2 - 2\zeta R'\zeta' - 2R\zeta\zeta'' + \zeta^2 R''}{R}, \quad (154)$$

$$G_{\hat{t}\hat{t}} = \frac{-3R^2\zeta'^2 + 2R^2\zeta\zeta'' + 4RR'\zeta\zeta' + \zeta^2 - R'^2\zeta^2 - 2RR''\zeta^2}{R^2}. \quad (155)$$

Pressure isotropy supplies us with a second-order, homogeneous ODE for  $\zeta(r)$ :

$$\zeta'' - \zeta' \frac{R'}{R} - \zeta \left\{ \frac{1 - (R')^2 + RR''}{2R^2} \right\} = 0. \quad (156)$$

This is very similar to the isotropy equation in Gaussian polar coordinates. On the other hand, rearranging to obtain an ODE for  $R(r)$  yields a second-order, nonlinear differential equation of no particularly useful form.

### 7.1. Solution-generating theorem

Our first key result can be phrased as a simple theorem:

**Theorem 7 (Synge).** *Suppose that  $\{\zeta(r), R(r)\}$  represents a perfect fluid sphere. Define*

$$A(r) = \left\{ \sigma + \epsilon \int \frac{R(r) dr}{\zeta(r)^2} \right\}. \quad (157)$$

*Then, for all  $\sigma$  and  $\epsilon$ , the geometry defined by holding  $R(r)$  fixed and setting*

$$ds^2 = -\frac{1}{\zeta(r)^2 A(r)^2} \{dt^2 - dr^2\} + \frac{R(r)^2}{\zeta(r)^2 A(r)^2} d\Omega^2 \quad (158)$$

*is also a perfect fluid sphere. That is, the mapping*

$$\mathbb{T}_{\text{Synge}}(\sigma, \epsilon) : \{\zeta, R\} \mapsto \{\zeta A(\zeta, R), R\} \quad (159)$$

*takes perfect fluid spheres into perfect fluid spheres.*

**Proof.** The proof is again based on the technique of “reduction in order.” No new principles are involved and the argument is very quickly sketched. Assuming that  $\{\zeta(r), R(r)\}$  solves Eq. (156), write

$$\zeta(r) \rightarrow \zeta(r)A(r) \tag{160}$$

and demand that  $\{\zeta(r)A(r), R(r)\}$  also solve Eq. (156). We eventually find that

$$\left(2\zeta' - \zeta \frac{R'}{R}\right) A' + \zeta A'' = 0, \tag{161}$$

which is solvable. Separating the unknown variable to one side and integrating twice, we obtain

$$A(r) = \left\{ \sigma + \epsilon \int \frac{R(r)dr}{\zeta(r)^2} \right\}, \tag{162}$$

depending on the old solution  $\{\zeta(r), R(r)\}$ , and two arbitrary integration constants,  $\sigma$  and  $\epsilon$ .  $\square$

## 7.2. Regularity conditions

- As usual,

$$R(0) = 0, \quad \zeta(0) > 0, \quad \zeta'(0) = 0. \tag{163}$$

- Now consider the spatial geometry

$$\frac{dr^2}{\zeta(r)^2} + \frac{R(r)^2}{\zeta(r)^2} d\Omega^2 = \frac{dr^2 + R(r)^2 d\Omega^2}{\zeta(r)^2} \tag{164}$$

and consider a small “ball” centered on the origin with coordinate radius  $r$ . The surface area of such a ball is

$$S = 4\pi \frac{R(r)^2}{\zeta(r)^2}, \tag{165}$$

while its proper radius is

$$\ell = \int_0^r \frac{1}{\zeta(r)} dr. \tag{166}$$

If the geometry at the center is to be smooth, then these two measures of the radius must asymptotically agree (for small balls):

$$\ell = \frac{R}{\zeta} + \mathcal{O} \left[ \left( \frac{R}{\zeta} \right)^2 \right]. \tag{167}$$

But in terms of the original coordinate  $r$  this implies that

$$R'(0) = 1. \tag{168}$$

Note that there is no additional constraint on  $\zeta(0)$  apart from the fact that it is nonzero and finite.

- Consider

$$G_{\hat{r}\hat{r}} = 3R'^2 - \frac{4\zeta\zeta'R'}{R} + \frac{R'^2\zeta^2 - \zeta^2}{R^2}. \quad (169)$$

Regularity at the center requires that

$$[(R')^2]' \rightarrow 0, \quad (170)$$

which implies that

$$R''(0) = 0. \quad (171)$$

- The combination

$$G_{\hat{r}\hat{r}} + G_{\hat{t}\hat{t}} = 2\zeta\zeta'' - \frac{2R''\zeta^2}{R} \quad (172)$$

does not lead to any new constraint.

- The component

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = 3(\zeta')^2 - 2\zeta\zeta'' + \frac{2R''\zeta^2 - 2\zeta R'\zeta'}{R} \quad (173)$$

does not lead to any new constraint.

- Collect all our regularity conditions:

$$\zeta(0) > 0, \quad \zeta'(0) = 0, \quad R(0) = 0, \quad R'(0) = 1, \quad R''(0) = 0. \quad (174)$$

- Finally, inserting all these conditions into the limit we quickly find that

$$G_{\hat{r}\hat{r}}|_0 = G_{\hat{\theta}\hat{\theta}}|_0 = G_{\hat{\phi}\hat{\phi}}|_0 = -4\zeta(0)\zeta''(0) + \zeta(0)^2 R'''(0), \quad (175)$$

$$G_{\hat{t}\hat{t}}|_0 = 6\zeta(0)\zeta''(0) - 3\zeta(0)^2 R'''(0), \quad (176)$$

$$G_{\hat{t}\hat{t}}|_0 + 3G_{\hat{r}\hat{r}}|_0 = -6\zeta(0)\zeta''(0). \quad (177)$$

- Again working with Taylor series to derive Laurent series, the pole pieces vanish and the center of the sphere is regular provided that

$$\zeta(0) > 0, \quad \zeta'(0) = 0, \quad R(0) = 0, \quad R'(0) = 1, \quad R''(0) = 0. \quad (178)$$

### 7.3. Implications

Note that the regularity conditions do feed back into theorem Synge. In that theorem we were interested in the quantity

$$A(r) = \sigma + \epsilon \int \frac{R(r)dr}{\zeta(r)^2}. \quad (179)$$

In view of the regularity conditions we now know that for a physically reasonable fluid sphere the integrand is finite all the way to the origin so that we can without loss of generality set

$$A(r) = \sigma + \epsilon \int_0^r \frac{R(r)dr}{\zeta(r)^2}. \quad (180)$$

But to ensure regularity we want both  $\zeta(0)$  and the new  $\zeta(0) \rightarrow \zeta(0)Z(0)$  to be finite and positive, hence  $Z(0) > 0$ , and therefore  $\sigma > 0$ .

## 8. Buchdahl Coordinates

Without loss of generality we can put the metric in “Buchdahl coordinates” and choose the coefficients to be

$$ds^2 = -\zeta(r)^2 dt^2 + \zeta(r)^{-2} \{dr^2 + R(r)^2 d\Omega^2\}. \quad (181)$$

This coordinate system is a sort of cross between Synge isothermal (tortoise) coordinates and Gaussian polar (proper radius) coordinates. We calculate

$$G_{\hat{r}\hat{r}} = -(\zeta')^2 - \frac{\zeta^2[1 - (R')^2]}{R^2}, \quad (182)$$

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = +(\zeta')^2 + \frac{\zeta^2 R''}{R}, \quad (183)$$

$$G_{\hat{t}\hat{t}} = -3(\zeta')^2 + 2\zeta\zeta'' - \frac{2\zeta(\zeta R'' - 2R'\zeta')}{R} + \frac{\zeta^2[1 - (R')^2]}{R^2}. \quad (184)$$

Imposing pressure isotropy supplies us with a particularly simple first-order, homogeneous ODE for  $\zeta(r)$ :

$$\left(\frac{\zeta'}{\zeta}\right)^2 = -\frac{1 - (R')^2 + RR''}{2R^2}. \quad (185)$$

This is very similar to the equation we obtained in isotropic coordinates. [Rearranging this into an ODE for  $R(r)$  yields a second-order, nonlinear differential equation which does not seem to be particularly useful.]

**Theorem 8 (Buchdahl).** *If  $\{\zeta(r), R(r)\}$  describes a perfect fluid, then so does  $\{\zeta(r)^{-1}, R(r)\}$ . This is the Buchdahl transformation in yet another disguise. The geometry defined by holding  $R(r)$  fixed and setting*

$$ds^2 = -\zeta(r)^{-2} dt^2 + \zeta(r)^2 \{dr^2 + R(r)^2 d\Omega^2\} \quad (186)$$

*is also a perfect fluid sphere. That is, the mapping*

$$\mathbb{T}_{\text{Buchdahl}} : \{\zeta, R\} \mapsto \{\zeta^{-1}, R\} \quad (187)$$

*takes perfect fluid spheres into perfect fluid spheres.*

**Proof.** By inspection. (Note the strong similarities to the discussion for isotropic coordinates.) □

### 8.1. Regularity conditions

- As usual,

$$R(0) = 0, \quad \zeta(0) > 0, \quad \zeta'(0) = 0. \quad (188)$$

- Now consider the spatial geometry

$$\zeta(r)^{-2} \{dr^2 + R(r)^2 d\Omega^2\} \quad (189)$$

and consider a small “ball” centered on the origin with coordinate radius  $r$ . The surface area of such a ball is

$$S = 4\pi\zeta(r)^{-2}R(r)^2, \tag{190}$$

while its proper radius is

$$\ell = \int_0^r \zeta(r)^{-1} dr. \tag{191}$$

If the geometry at the center is to be smooth, then these two measures of the radius must asymptotically agree (for small balls):

$$\ell = \frac{R}{\zeta} + \mathcal{O}\left[\left(\frac{R}{\zeta}\right)^2\right]. \tag{192}$$

But in terms of the original coordinate  $r$  this implies that

$$R'(0) = 1. \tag{193}$$

- Consider

$$G_{\hat{r}\hat{r}} = -(\zeta')^2 - \frac{\zeta^2[1 - (R')^2]}{R^2}. \tag{194}$$

This now yields the additional constraint

$$[(R')^2]' \rightarrow 0, \tag{195}$$

which implies that

$$R''(0) = 0. \tag{196}$$

- The combination

$$G_{\hat{r}\hat{r}} + G_{\hat{t}\hat{t}} = -4(\zeta')^2 + 2\zeta\zeta'' - \frac{2\zeta(\zeta R'' - 2\zeta' R')}{R} \tag{197}$$

does not lead to any new constraint.

- Similarly,

$$G_{\hat{\theta}\hat{\theta}} = (\zeta')^2 + \frac{\zeta^2 R''}{R} \tag{198}$$

does not lead to any new constraint.

- So now let us collect all our regularity conditions:

$$R(0) = 0, \quad R'(0) = 1, \quad R''(0) = 0, \quad \zeta(0) > 0, \tag{199}$$

$$\zeta'(0) = 0. \tag{200}$$

- Finally, inserting all these conditions into the limit we quickly find that

$$G_{\hat{r}\hat{r}}|_0 = G_{\hat{\theta}\hat{\theta}}|_0 = G_{\hat{\phi}\hat{\phi}}|_0 = 2\zeta(0)^2 R'''(0), \tag{201}$$

$$G_{\hat{t}\hat{t}}|_0 = 6\zeta(0)\zeta''(0) - 6\zeta(0)^2 R'''(0), \tag{202}$$

$$G_{\hat{t}\hat{t}}|_0 + 3G_{\hat{r}\hat{r}}|_0 = 6\zeta(0)\zeta''(0). \tag{203}$$

- An analysis based on Taylor series and Laurent series again quickly leads to

$$R(0) = 0, \quad R'(0) = 1, \quad R''(0) = 0, \quad \zeta(0) > 0, \quad \zeta'(0) = 0. \quad (204)$$

We have now seen the Buchdahl transformation  $\zeta(r) \rightarrow 1/\zeta(r)$  arise in two separate settings — both in isotropic coordinates and in the Buchdahl coordinates we have just investigated. This raises the question as to whether we can expect to encounter Buchdahl-like transformations in more general settings, perhaps by suitably choosing the functional form of the metric. This will be our last topic to be investigated in this article.

### 9. Generalized Buchdahl Ansatz

We now return to general diagonal coordinates and, based on our insight from dealing with isotropic coordinates and Buchdahl coordinates, make a specific *ansatz* for the functional form of the metric components. Without loss of generality we choose the metric coefficients to be

$$ds^2 = -\zeta(r)^2 dt^2 + \zeta(r)^{-2} \left\{ \frac{dr^2}{E(r)} + R(r)^2 d\Omega^2 \right\}. \quad (205)$$

We calculate

$$G_{\hat{r}\hat{r}} = -E(\zeta')^2 - \frac{\zeta^2[1 - E(R')^2]}{R^2}, \quad (206)$$

$$G_{\hat{\theta}\hat{\theta}} = G_{\hat{\phi}\hat{\phi}} = +E(\zeta')^2 + \frac{\zeta^2[2ER'' + E'R']}{2R}, \quad (207)$$

$$G_{\hat{t}\hat{t}} = -3E(\zeta')^2 + 2E\zeta\zeta'' + \zeta E'\zeta' - \frac{2E\zeta(\zeta R'' - 2R'\zeta')}{R} + \frac{\zeta^2[1 - E(R')^2]}{R^2} - \frac{\zeta^2 E'R'}{R}. \quad (208)$$

Imposing pressure isotropy supplies us with a first-order, homogeneous ODE for  $\zeta(r)$ :

$$\left( \frac{\zeta'}{\zeta} \right)^2 = - \frac{2 - 2E(R')^2 + 2ERR'' + RE'R'}{4ER^2}. \quad (209)$$

This is very similar to the equation we obtained in isotropic coordinates and Buchdahl coordinates, but now in general diagonal coordinates — the key point is that we have carefully chosen the functional form of the metric components. Rearranging this into an ODE for  $E(r)$  yields a first-order, linear differential equation:

$$[\zeta^2 RR']E' + [4R^2(\zeta')^2 + 2R\zeta^2 R'' - 2\zeta^2(R')^2]E + 2\zeta^2 = 0. \quad (210)$$

[In contrast, rearranging this into an ODE for  $R(r)$  yields a second-order, nonlinear differential equation which does not seem to be particularly useful.]

**Theorem 9 (Generalized Buchdahl).** *If  $\{\zeta(r), E(r), R(r)\}$  describes a perfect fluid, then so does  $\{\zeta(r)^{-1}, E(r), R(r)\}$ . This is the Buchdahl transformation in yet*

another disguise (now in its most general setting). The geometry defined by holding  $E(r)$  and  $R(r)$  fixed and transforming the given perfect fluid sphere

$$ds^2 = -\zeta(r)^2 dt^2 + \zeta(r)^{-2} \left\{ \frac{dr^2}{E(r)} + R(r)^2 d\Omega^2 \right\} \quad (211)$$

into the new geometry

$$ds^2 = -\zeta(r)^{-2} dt^2 + \zeta(r)^2 \left\{ \frac{dr^2}{E(r)} + R(r)^2 d\Omega^2 \right\} \quad (212)$$

yields a space-time that is also a perfect fluid sphere. That is, the mapping

$$\mathbb{T}_{\text{Generalized Buchdahl}} : \{\zeta, E, R\} \mapsto \{\zeta^{-1}, E, R\} \quad (213)$$

takes perfect fluid spheres into perfect fluid spheres.

**Proof.** By inspection. (Note the very strong similarities to the discussion for isotropic coordinates and Buchdahl coordinates. Note also that because we are now working in general diagonal coordinates, where the metric has three independent components, this is the most general result we can hope for with regard to a static, spherically symmetric space-time.)  $\square$

The regularity conditions in this case can be extracted from a minor variant of the previous analysis, and reduce to

$$R(0) = 0, \quad E(0)[R'(0)]^2 = 1, \quad E'(0) = -\frac{2R''(0)}{[R'(0)]^3} = 0, \quad (214)$$

$$\zeta(0) > 0, \quad \zeta'(0) = 0. \quad (215)$$

Finally, inserting all these conditions into the central limit we quickly find that

$$G_{\hat{r}\hat{r}}|_0 = G_{\hat{\theta}\hat{\theta}}|_0 = G_{\hat{\phi}\hat{\phi}}|_0 = \frac{\zeta(0)^2 R'''(0)}{R'(0)^3} - \frac{3\zeta(0)^2 R''(0)^2}{R'(0)^4} + \frac{\zeta(0)^2 E''(0)}{2}, \quad (216)$$

$$G_{\hat{t}\hat{t}}|_0 = \frac{-3\zeta(0)^2 R'''(0)}{R'(0)^3} + \frac{9\zeta(0)^2 R''(0)^2}{R'(0)^4} - \frac{3\zeta(0)^2 E''(0)}{2} + \frac{6\zeta(0)\zeta''(0)}{R'(0)^2}, \quad (217)$$

$$G_{\hat{t}\hat{t}}|_0 + 3G_{\hat{r}\hat{r}}|_0 = \frac{6\zeta(0)\zeta''(0)}{R'(0)^2}. \quad (218)$$

In addition, we could use the fact that the isotropy condition, when viewed as a differential equation in  $E(r)$ , is first-order and linear to develop yet another variant on the theorem General diagonal 1. As no new significant insight is gained, we suppress the details.

## 10. Discussion

In this article, we have developed several new and significant transformation theorems that map perfect fluid spheres to perfect fluid spheres using both “usual” and “unusual” coordinate systems — such as Schwarzschild (curvature), isotropic,

Gaussian polar (proper radius), Synge isothermal (tortoise), and Buchdahl coordinates. In each case we developed at least one such transformation theorem, while in several cases we have been able to develop multiple transformation theorems.

We have also investigated regularity conditions at the center of the fluid sphere in all these coordinate systems. The meaning of regularity is that (at a minimum) we want the space–time geometry to be well defined and smooth everywhere. We can calculate the regularity conditions by hand, but note that there is another way to more mechanically derive these regularity conditions by using Taylor and Laurent series, with calculations aided by symbolic manipulation programs such as Maple. We have found it convenient to use this observation to provide a crosscheck on our explicit computations.

In summary, we have now extended the algorithmic technique introduced in Ref. 11 and extended in Refs. 12 and 13–18 to many other coordinate systems and many other functional forms for the metric.

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