Riemannian Geometry of Irrotational Vortex Acoustics

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We consider acoustic propagation in an irrotational vortex, using the technical machinery of differential geometry to investigate the "acoustic geometry" that is probed by the sound waves. The acoustic space-time curvature of a constant circulation hydrodynamical vortex leads to deflection of phonons at appreciable distances from the vortex core. The scattering angle for phonon rays is shown to be quadratic in the small quantity $\Gamma/2\pi cb$, where Γ is the vortex circulation, c the speed of sound, and b the impact parameter.

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Introduction.—The last few years have seen considerable interest in the development of analog models of and for general relativity [1–3]. These analog models provide a two-way street: sometimes they illuminate aspects of general relativity and sometimes the machinery of differential geometry can be used to illuminate aspects of the analog model. In this article, we use mathematical methods developed in the framework of differential geometry to study acoustic propagation in an irrotational vortex. We analyze the "acoustic" space-time geometry created by the vortex in terms of its metric tensor, Killing vectors, and geodesics. The medium is assumed to be "almost incompressible," by which we mean that we take the background density and the speed of sound relative to the medium to be constant, and focus on the effects due to motion of the medium. If the flow and its perturbations are irrotational there is a rigorous theorem to the effect that sound can be described by a curved-space scalar wave equation using an effective acoustic geometry, this theorem being derived by using the Euler and continuity equations of classical hydrodynamics [1,2]. (Even if distributed vorticity is present, which is not the case considered here, then in the eikonal approximation there are rigorous theorems to the effect that sound ray propagation can be described by curved-space geodesics in the same "effective acoustic geometry" [4].)

Using the acoustic space-time approach, we shall show that the quasiclassical scattering process of phonons by the vortex leads to a scattering angle quadratic in the small quantity $\Gamma/2\pi cb$, where Γ is the vortex circulation, c the speed of sound, and b the impact parameter of the phonon trajectory relative to the vortex. Because of the fact that the lowest order scattering angle is quadratic in $\Gamma/2\pi cb$ and not linear, there is (up to third order) no net modification of the transverse force exerted by a phonon beam directed towards the vortex. The effect of the acoustic space-time curvature is that the vortex acts as a converging lens on quasiclassically moving phonons.

Riemannian geometry of vortex acoustics.—For an irrotational vortex, the velocity profile is given in terms of the circulation $\Gamma \equiv \oint \vec{v} \cdot d\vec{x} \equiv 2\pi\gamma$ by

$$\vec{v} = \frac{\Gamma}{2\pi r}\,\hat{\theta} = \frac{\gamma}{r}\,\hat{\theta}\,. \tag{1}$$

If we adopt cylindrical coordinates r, θ, and z, so that the spatial metric is

$$g_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{2}$$

the covariant and contravariant components of the velocity are $v_\theta = \gamma$; $v^\theta = \gamma/r^2$. Now go to a four-dimensional description in terms of the coordinates t, r, θ, and z. Then, in terms of the speed of sound c, the $(3 + 1)$-dimensional acoustic geometry of the vortex is described by the space-time metric [1,2]

$$g_{\mu\nu}^{\text{vortex}} = \begin{bmatrix} -c^2 + (\gamma/r)^2 & 0 & -\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma & 0 & r^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3}$$

The scalar velocity potential of phonons lives in the curved space-time world given by $g_{\mu\nu}^{\text{vortex}}$. This form of the metric is often referred to as the Painlevé-Gullstrand form [2,5]. It is important to note that this vortex metric is *not* identical to the metric of a massless spinning cosmic string, a solution of the Einstein equations with cylindrical symmetry. That metric corresponds to

$$g_{\mu\nu}^{\text{string}} = \begin{bmatrix} -c^2 & 0 & -\gamma & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma & 0 & r^2 - \gamma^2/c^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4}$$

The two metrics (3) and (4), vortex and cosmic string, agree asymptotically at large r,

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$$g_{\mu\nu}^{\text{vortex}} = g_{\mu\nu}^{\text{string}}[1 + O(r_c^2/r^2)]. \quad (5)$$

Here we have defined the "core radius"

$$r_c = \frac{|\gamma|}{c}. \quad (6)$$

This denotes the radius at which the flow goes supersonic. The assumption of incompressibility as well as the profile (1) certainly break down at this radius, but the actual dimensions of the vortex core (for a superfluid vortex being set by the coherence length ξ_0) are often larger than r_c. The two metrics are markedly different at intermediate values of r. In particular, we shall see below that they differ significantly well outside the vortex core radius r_c. The metric of the spinning cosmic string is everywhere flat (the space-time curvature is identically zero). In contrast, even at intermediate distances, the acoustic geometry of the vortex is not flat (the space-time Riemann tensor is not zero), and there are significant effects on the propagation of null geodesics (sound rays) at values of r well outside the vortex core. That there is a region of vanishing classical deflection outside a vortex or a magnetic flux tube, as conventionally assumed in the standard formulation of the Aharonov-Bohm problem (see, e.g., [6,7]), is thus only *asymptotically* true for a hydrodynamical vortex, even if the generalized vorticity (hydrodynamical vorticity and/or magnetic flux) vanishes everywhere outside the vortex core. It is hence not *a priori* clear that computations of the Iordanskiĭ force based on the spinning string metric (the analog-gravitational Aharonov-Bohm effect [8,9]), give the full force exerted by a phonon beam on an actual hydrodynamical vortex.

It is straightforward to compute the Ricci curvature scalar corresponding to the metric (3),

$$R^{\text{vortex}} = \frac{2\gamma^2}{c^2 r^4} = \frac{2r_c^2}{r^4}. \quad (7)$$

The curvature of the space-time experienced by the quasiparticles is thus significant at distances which can be well outside the core domain. That the flow be considered well outside the core domain is required by the relation (6), which states that at a distance r_c from the rotation axis the velocity around the core equals the speed of sound, so that compressibility is no longer negligible.

A simple criterion for the curvature of the effective space-time to be significant for the motion of the phonon around the vortex at a given distance r is how the phonon wave vector magnitude k compares with the inverse space-time curvature radius at that distance. The wave vector magnitude is then to be compared with

$$k_c(r) = 2\pi\sqrt{R} = \frac{2\pi\sqrt{2}}{r_c}\frac{1}{(r/r_c)^2}. \quad (8)$$

Well outside the core, $k_c r_c \ll 1$. The phonon "sees" the curvature of the space-time if k exceeds $k_c(r)$, and behaves as a particle moving on a geodesic in that space-time. Vice versa, the topological structure of the acoustic space-time related to the Iordanskiĭ force (the Aharonov-Bohm effect in the space-time of the spinning string) dominates the behavior of the phonons if $k \ll k_c(r)$.

For a specific physical example of an irrotational vortex geometry, in the dilute limit of a Bose-condensed atomic vapor (BEC) [10], we can relate the circulation Γ, the coherence length ξ_0, and the speed of sound by using the relations $\xi_0^2 = \hbar^2/(2mgn)$ and $c = \sqrt{gn/m}$. They combine to give the relation

$$2\pi r_c c = 2\pi\sqrt{2}\, c\xi_0 = |\Gamma| = 2\pi|\gamma| \qquad \text{(BEC)} \quad (9)$$

for a singly quantized vortex with $\Gamma = h/m$, where g is the strength of interaction related to the s-wave scattering length a by $g = 4\pi\hbar^2 a/m$. The "acoustic" core size is thus in this example of the same order as the actual (quantum-mechanical) core size of varying density. The hydrodynamic circulation, calculated with the speed of sound along a core circumference with radius $r_c = \sqrt{2}\,\xi_0$, equals the quantum of circulation in the dilute gas limit. This relation qualitatively also holds in the dense, strongly correlated superfluid helium II (superfluid ^{4}He), where ξ_0 is of order the atomic size. The relation (9) yields for the curvature scalar in (7)

$$R^{\text{vortex}} = \frac{4\xi_0^2}{r^4}. \qquad \text{(BEC)} \quad (10)$$

Curvature of space-time implies that particle worldlines on geodesics in that space-time deviate from being initially parallel after some proper distance of travel along the geodesic. From the general space-time interval of the Painlevé-Gullstrand metric in the form [2]

$$ds^2 = -dt^2 + \delta_{ij}(dx^i - v^i dt)(dx^j - v^j dt), \quad (11)$$

it is apparent that a constant time slice of the effective space-time of quasiparticles is just ordinary Euclidean space. Hence spatial distances measured on constant time slices in the effective space-time are identical to distances in the Newtonian lab world, and a real force is acting upon the phonon. It is, however, to be stressed that the actual motion of the phonon in the lab world is described correctly by the motion in the full effective space-time, that is, the phonon is *not* just a (Lagrangian) particle dragged along by the flow, if that flow is inhomogeneous.

Phonon motion in acoustic geometry.—To explicitly see how the vortex flow affects acoustic propagation, and thereby get a handle on how acoustic influences can affect the vortex, we use the eikonal approximation and consider *phonons* instead of sound waves. Phonons then follow null geodesics in the acoustic geometry [2]. Associated with space-time symmetries are so-called Killing vectors [11], along which the metric is invariant. For both geometries (vortex and spinning string) there are three such Killing vectors, corresponding to translations in the t, θ, and z

directions:

$$(K_1)^\mu = (1,0,0,0); \qquad (K_2)^\mu = (0,0,1,0); \qquad \text{and} \qquad (K_3)^\mu = (0,0,0,1). \quad (12)$$

This leads to three conserved quantities along each geodesic [11]

$$(K_A)^\mu g_{\mu\nu} \frac{dx^\nu}{d\lambda} = -\tilde{k}_A . \quad (13)$$

Here λ is some arbitrary affine parameter for the geodesic [11]. We are interested in null geodesics which represent the paths of sound rays in the acoustic geometry. From these three conservation laws we see that for the vortex geometry

$$\left[-c^2 + \left(\frac{\gamma}{r}\right)^2\right]\frac{dt}{d\lambda} - \gamma\,\frac{d\theta}{d\lambda} = -\tilde{k}_1 ; \quad (14)$$

$$-\gamma\,\frac{dt}{d\lambda} + r^2\,\frac{d\theta}{d\lambda} = -\tilde{k}_2 ; \quad (15)$$

and

$$\frac{dz}{d\lambda} = -\tilde{k}_3 . \quad (16)$$

By elimination between the first two equations

$$\frac{dt}{d\lambda} = \left(\tilde{k}_1 + \frac{\tilde{k}_2\gamma}{r^2}\right)c^{-2}. \quad (17)$$

We are furthermore particularly interested in sound rays that come in from infinity, so without loss of generality it is possible to rescale λ to choose $\tilde{k}_1 = c^2$, and to then define $\tilde{k}_2 = c^2 k_2$ and $\tilde{k}_3 = -k_3$. Using the new affine parameter, a brief computation yields

$$\frac{dz}{dt} = \frac{k_3}{1 + k_2\gamma/r^2} \quad (18)$$

and

$$\frac{d\theta}{dt} = \frac{\gamma}{r^2} - \frac{k_2 c^2/r^2}{1 + k_2\gamma/r^2} . \quad (19)$$

To now calculate dr/dt we use the fact that the sound rays are null curves so that

$$g_{\mu\nu}\,\frac{dx^\mu}{dt}\,\frac{dx^\nu}{dt} = 0 . \quad (20)$$

Therefore

$$\left[-c^2 + \left(\frac{\gamma}{r}\right)^2\right] - 2\gamma\,\frac{d\theta}{dt} + \left(\frac{dr}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = 0 , \quad (21)$$

leading to, by substituting (18) and (19),

$$\frac{dr}{dt} = \sqrt{c^2 - \left(\frac{k_2 c^2/r}{1 + k_2\gamma/r^2}\right)^2 - \left(\frac{k_3}{1 + k_2\gamma/r^2}\right)^2} . \quad (22)$$

The three equations (18), (19), and (22) completely specify the path of the sound ray in terms of the time parameter t and the two nontrivial constants of the motion k_2 and k_3. These have the physical interpretation that $k_3 = v_\infty^z = \beta c < c$, while in terms of the impact parameter b

$$k_2 = \frac{\gamma - cb\sqrt{1-\beta^2}}{c^2} . \quad (23)$$

The radial motion is more usefully recast as an "energy equation"

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 + V(r) = \frac{1}{2}\,c^2, \quad (24)$$

with the "potential"

$$V(r) = \frac{1}{2}\,c^2\,\frac{(k_2 c/r)^2 + \beta^2}{(1 + k_2\gamma/r^2)^2} . \quad (25)$$

The form of this potential is sketched in Fig. 1 (assuming $k_2\gamma > 0$ and setting $\beta = 0$). Note that at large distances it has the standard centrifugal barrier proportional to $1/r^2$, while at short distances (however, already inside the "acoustic" core) it falls quadratically to zero.

The general deflection angle of the ray as a function of k_2 and k_3 is obtained by integrating

$$\Delta\theta = 2\int_{r_{\rm turn}}^{\infty} \frac{d\theta}{dr}\,dr , \quad (26)$$

where the turning point $r_{\rm turn}$ is determined by solving $dr/d\theta = 0$. For simplicity, we set $\beta = k_3 = 0$ in the following. To lowest order in the small parameter $|\gamma|/cb = r_c/b$

$$r_{\rm turn} = |b|\left[1 - \left(\frac{r_c}{b}\right)^2 + O\left(\frac{r_c^3}{b^3}\right)\right]. \quad (27)$$

For $b \to \infty$, at fixed γ and c (fixed core radius r_c), the deflection then evaluates from integrating

$$\frac{d\theta}{dr} = \frac{b(1 - r_c^2/r^2)}{\sqrt{(r^2 - r_{\rm turn}^2)\,(r^2 - r_c^2)}} + O(r_c^3/b^3) \quad (28)$$

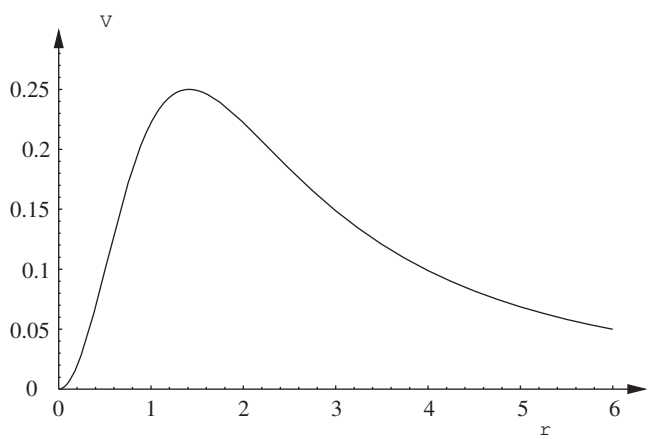

FIG. 1. The effective potential of phonon motion in the vortex space-time, with $b = -r_c$ and $k_3 = v_z^\infty = 0$, $r_c = c = 1$.

along the ray, according to Eq. (26). The result is to quadratic order in r_c/b

$$\Delta\theta = \pi\,\mathrm{sgn}(b)\left[1 + \frac{3}{4}\frac{r_c^2}{b^2} + O(r_c^3/b^3)\right]. \qquad (29)$$

The limit of zero classical force corresponds to the limit of vanishing core size r_c($c \to \infty$ and/or $\gamma \to 0$), as may be inferred from the vanishing of the curvature scalar (7). This is directly seen in the vanishing, to quadratic order in r_c, of the classical scattering of the ray away from a straight line caused by the acoustic space-time curvature. The action of the vortex on the phonons is schematically depicted in Fig. 2.

From the deflection angle (29), one can easily obtain the focal length as a function of impact parameter. Indeed, two initially parallel phonon beams that pass by opposite sides of the vortex with impact parameter b will intersect at a distance $\ell = 2f$ beyond the vortex, where

$$f = \frac{2}{3\pi}\frac{b^3}{r_c^2}\left[1 + O\left(\frac{r_c}{b}\right)\right]. \qquad (30)$$

The fact that the focal length depends on impact parameter shows that the vortex exhibits what is known in optics as "spherical abberation," which in the present context is more correctly termed "cylindrical abberation."

In an experiment, one should take into account that the transverse size of a phonon beam is limited by the wavelength λ, which has to be less than or equal to the impact parameter b for the quasiclassical phonon scattering picture to make sense. Thus the minimum focal length is given by

$$f_{\min} = \frac{2}{3\pi}\frac{\lambda^3}{r_c^2}\left[1 + O\left(\frac{r_c}{\lambda}\right)\right]. \qquad (31)$$

In superfluid ^{4}He, where $r_c \simeq 0.6$ Å, the roton minimum occurs at $\lambda_{\mathrm{roton}} \simeq 3$ Å. The phonon wavelength has to be about three times the roton wavelength, for an approximately linear phonon dispersion to apply, so that $\lambda \gtrsim 15 r_c$, which results in $f_{\min} \simeq 700 r_c$. Accordingly, the focal length is bounded by $f \gtrsim 40$ nm.

In dilute Bose-Einstein condensates, due to the Bogoliubov-type spectrum of these systems, which does not display a roton minimum, the phonon dispersion is to a good approximation linear up to $\lambda \simeq 2\pi\xi_0 = \sqrt{2}\,\pi r_c$, hence $f_{\min} \simeq 20 r_c$. With $r_c \simeq 0.3$ μm, this results in $f \gtrsim 6$ μm for Bose-Einstein condensates. The latter estimate indicates that vortical focusing effects are potentially within the realm of experimental feasibility.

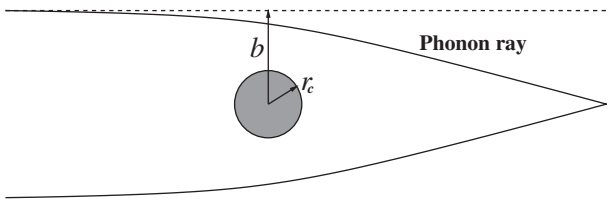

FIG. 2. Deflection of phonons by the vortex, which acts as a converging lens.

Discussion.—An incoming phonon of finite momentum $\boldsymbol{k}$ passing a singular vortex is deflected by a classical force acting upon it. This force is equivalent to an acoustic space-time curvature induced in the vicinity of the vortex by that flow. From (29), it follows that the vortex acts as a converging lens. The fact that around a hydrodynamical vortex there is a classical force field at appreciable distances outside the core, entails that calculations based on the assumption that there is no force in the vorticity free region outside the core are potentially misleading. In particular, there are finite distance effects associated with a hydrodynamic vortex over and above the familiar Aharonov-Bohm effect.

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