Lorentzian geometry from Bose-Einstein condensates.

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Propagation of a massless scalar wave ϕ
on a background spacetime

$$\Downarrow$$

$$\Delta\phi \equiv \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\; g^{\mu\nu}\; \partial_\nu\phi\right) = 0. \qquad (1)$$

Depending on $g_{\mu\nu}(x)$ one can find very
different behaviours

$$g_{tt}(\text{Surface}) = 0 \qquad \Longrightarrow \qquad \begin{cases} \text{ergosphere} \\ \text{event horizons} \end{cases}$$

Equation (1) can be reproduced in
different condense matter systems !

Bose-Einstein condensate

An ensemble of indistinguishable bosons occupying at the same time the same single-particle state

No interaction:

$$\widehat{H}\Psi = i\hbar\frac{\partial}{\partial t}\Psi$$

$$\widehat{H} = \sum_{k=1}^{N}\left(\frac{\widehat{p}_i^2}{2m} + V(x_i)\right) \qquad (2)$$

$$\Downarrow$$

$$\Psi_{\text{BEC}} = \psi_0(x_1)\cdots\psi_0(x_N) \qquad (3)$$

Weak interaction (dilute BE gas):

$$\Psi_{\text{BEC}} = \psi(x_1)\cdots\psi(x_N) \qquad (4)$$

$$\boxed{-\frac{\hbar^2}{2m}\nabla^2\psi + V\,\psi + \lambda_I\,(\psi^*\psi)\,\psi = i\hbar\,\frac{\partial}{\partial t}\psi} \qquad (5)$$

Gross-Pitaevskii equation

Hydrodynamic (Madelung) representation

$$\psi(t,\vec{x}) = \sqrt{\rho(t,\vec{x})}\,\exp[-i\phi(t,\vec{x})/\hbar]. \qquad (6)$$

$$\partial_t \phi = -\frac{1}{2m}\left(\vec{\nabla}\phi\cdot\vec{\nabla}\phi\right) - V - \lambda_I \rho \qquad (7)$$

$$+\boxed{\frac{\hbar^2}{2m}\frac{1}{\sqrt{\rho}}\left(\nabla^2\sqrt{\rho}\right)}\ \Big\}\ \text{“quantum potential”}$$

$$\partial_t \rho = -\frac{1}{m}\left[\vec{\nabla}\left(\rho\vec{\nabla}\phi\right)\right]. \qquad (8)$$

Linearize now around a solution

$$\rho = \rho_0 + \rho_1, \qquad \phi = \phi_0 + \phi_1, \qquad (9)$$

$$\Downarrow$$

$$\partial_t \phi_1 = -\frac{1}{m}\left(\vec{\nabla}\phi_0\cdot\vec{\nabla}\phi_1\right) - \lambda_I \rho_1 \qquad (10)$$

$$+\boxed{\frac{\hbar^2}{2m}\left[\frac{1}{2\sqrt{\rho_0}}\nabla^2\left(\frac{\rho_1}{\sqrt{\rho_0}}\right) - \rho_1\frac{\nabla^2\sqrt{\rho_0}}{2(\sqrt{\rho_0})^3}\right]}$$

$$\partial_t \rho_1 = -\frac{1}{m}\left[\vec{\nabla}(\rho_0\vec{\nabla}\phi_1) + \vec{\nabla}(\rho_1\vec{\nabla}\phi_0)\right]. \qquad (11)$$

Let us define

$$c^2 \equiv \frac{\lambda_I}{m}\rho_0, \qquad \vec{v}_0 \equiv \frac{1}{m}\vec{\nabla}\phi_0, \tag{12}$$

Taking ρ_1 from (10) and substituting in (11) one obtains and differential equation for ϕ_1:

$$\partial_\mu \left(\sqrt{-g}\; g^{\mu\nu}\; \partial_\nu \phi_1\right) = 0. \tag{13}$$

with

$$g_{\mu\nu} = \frac{c}{\lambda_I}\left[\begin{array}{ccc} -(c^2 - \vec{v}_0^2) & \vdots & -\vec{v}_0 \\ \cdots\cdots\cdots\cdots & \cdot & \cdots\cdots \\ -\vec{v}_0 & \vdots & I \end{array}\right]. \tag{14}$$

This is the effective metric!

(Garay et al, gr-qc/0002015 and gr-qc/0005131)

Spacetime metric	Effective metric
$\Downarrow$	$\Downarrow$
6 arbitrary functions	2 arbitrary functions

We cannot reproduce all possible spacetime metrics

Regime of validity

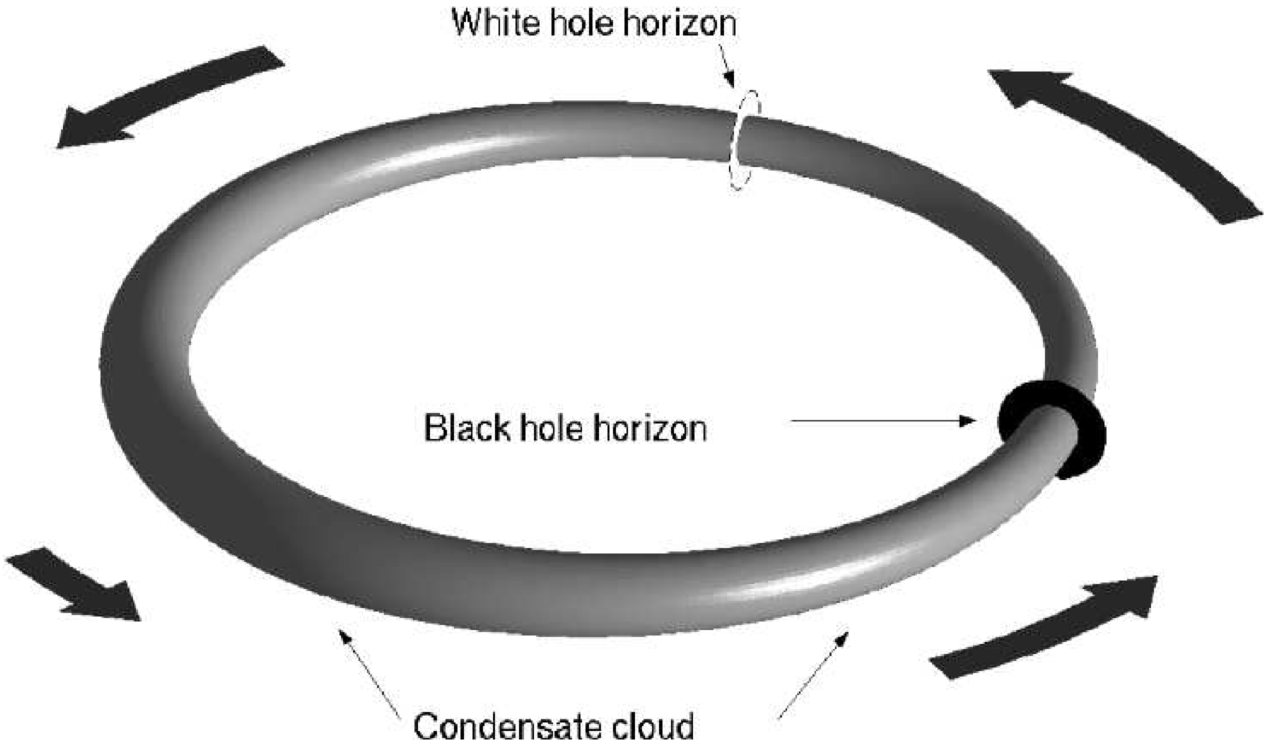
- Interatomic spacing $>>$ scattering length
- $N >> 1$
- $\lambda >> \frac{\lambda_c}{2}$ with $\lambda_c = \frac{h}{mc}$

Possible constructions

- uni-dimensional ring black hole
- uni-dimensional sink black hole
- others

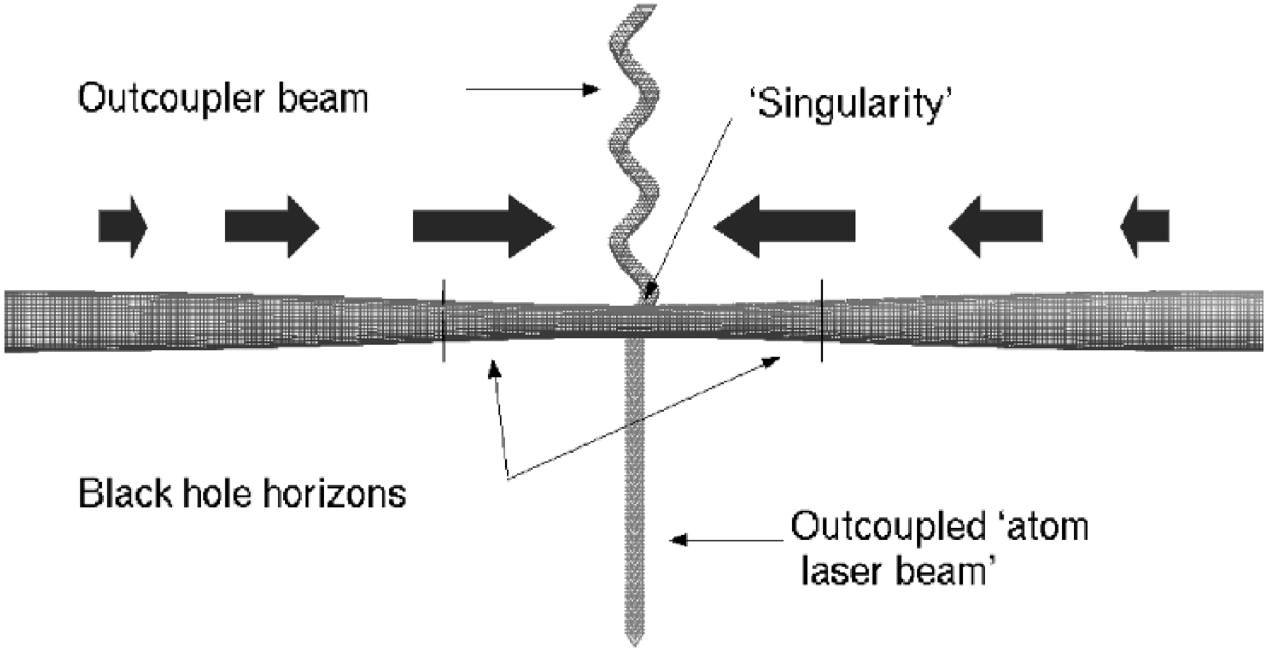
Lab limitations

- present velocities: $c \sim$ 1mm/s
- present sizes: $R \sim$ 100μ with $N = 10^3 - 10^6$ atoms



Ring black hole, Garay et al

$$c(\theta) = A(1 + b\cos\theta), \qquad v(\theta) = \frac{B}{(1 + b\cos\theta)^2}$$



Sink black hole, Garay et al

$$c(x) = \begin{cases} c_0 & |x| < L \\ c_0[1 + (\sigma - 1)x/\epsilon] & L < |x| < L + \epsilon \\ \sigma c_0 & L + \epsilon < |x| \end{cases}$$

$$v(x) = -\frac{A}{c^2(x)} \frac{x}{|x|}$$

First corrections to the Garay et al analysis:

Within the validity of the Gross-Pitaevskii equation:

- The quantum potential takes into account departures from homogeneity of the background.

  Defining $\lambda_b^{-2} = \frac{1}{8\pi^2}\left[\frac{\nabla^2 c}{c} - \frac{(\nabla c)^2}{c^2}\right]$ and remembering $\lambda_c \equiv \frac{h}{mc}$

  $$\Downarrow$$

  Same acoustic propagation for phase perturbations but with
  $$\tilde{c} = c\sqrt{(1 + \lambda_c^2/\lambda_b^2)}$$

  again $\lambda >> \frac{\lambda_c}{2}$

- For $\lambda \sim \frac{\lambda_c}{2}$ the effective metric is replaced by an operator-valued metric.

$$\partial_\mu (f^{\mu\nu}\, \partial_\nu \phi_1) = 0. \tag{15}$$

Here

$$f^{00} = -\left[\Pi'(\rho_0) - \frac{\hbar^2}{2\mu}\, D_2\right]^{-1} \tag{16}$$

$$f^{0j} = -\left[\Pi'(\rho_0) - \frac{\hbar^2}{2\mu}\, D_2\right]^{-1} \frac{h^{jk}\, \nabla_k \phi_0}{\mu} \tag{17}$$

$$f^{i0} = -\frac{h^{ik}\, \nabla_k \phi_0}{\mu} \left[\Pi'(\rho_0) - \frac{\hbar^2}{2\mu}\, D_2\right]^{-1} \tag{18}$$

$$f^{ij} = \frac{\rho_0\ {}^{(3)}h^{ij}}{\mu} \tag{19}$$

$$-\frac{h^{ik}\, \nabla_k \phi_0}{\mu} \left[\Pi'(\rho_0) - \frac{\hbar^2}{2\mu}\, D_2\right]^{-1} \frac{h^{jl}\, \nabla_l \phi_0}{\mu}. \tag{20}$$

$$D_2 \equiv \rho_0^{-2} (\vec{\nabla} \rho_0^{1/2})^2 - \rho_0^{-3/2} [\Delta_h (\rho_0^{1/2})] \\ -\rho_0^{-3/2}\, \vec{\nabla} \rho_0^{1/2} \cdot \vec{\nabla} + \frac{1}{2} \rho_0^{-1} \Delta_h \tag{21}$$

Bogoliubov dispersion relation

Over homogeneous background regions the phonon propagation on the "moving fluid" is modified by a classical kinetic energy term

$$\omega = v_0^i \, k_i \pm \sqrt{c^2 k^2 + \left(\frac{\hbar}{2m} \, k^2 \right)^2} \qquad (22)$$

- The low energy (long wavelengths) regime is the effectively Lorentzian.
- There is no longer a maximum velocity for the propagation of perturbations.

Beyond the Garay et al construction

- Collective phenomena in dimension $d \leq 2$:

  $\lambda_I \, (\psi^* \psi) \, \psi \;\; \rightarrow \;\; \Pi(x, (\psi^* \psi)) \, \psi$ in GP.

- BEC for bosons with anisotropic effective masses

$$m \;\; \rightarrow \;\; m_{ij}(x) = \mu \, h_{ij}(x)$$

  (e.g. paraexcitons in semiconductors)

An analogous analysis can be carry over:

$$\Downarrow$$

$$g_{\mu\nu} \equiv \frac{c}{\Pi'} \left[\begin{array}{ccc} -(c^2 - v_0^2) & \vdots & -v_0^k \, h_{kj} \\ \cdots\cdots\cdots\cdots & \cdot & \cdots\cdots \\ -v_0^k \, h_{ki} & \vdots & h_{ij} \end{array} \right]. \qquad (23)$$

with $c \equiv \frac{\Pi'}{\mu} \rho_0$ and $v_0^i \equiv [m^{-1}]^{ij} \nabla_j \phi$

Now we can mimic generic gravitational fields