ROLE OF SONIC METRIC IN RELATIVISTIC SUPERFLUID

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Abstract. *After recapitulating the basic principles of ordinary relativistic superfluid mechanics, and in particular the irrotational barotropic case that is relevant to superfluidity at zero temperature, this article provides a self contained introduction to the natural relativsitic generalisation of the Landau two constituent model for a superfluid at finite temperature, with special emphasis on the role of the Unruh type sonic metric in the low temperature limit.*

1 Introduction.

The present discussion will be limited to scalar (spin 0) superfluid models such as are appropriate for the familiar case of Helium 4 (though not Helium 3) and for the interpenetrating neutron superfluid fluid in the crust of a neutron star, which is the part most relevant to the analysis of pulsar frequency variations. The first three section consist of a recapitulation from modern standpoint of the long well known essentials of the relativistic version of the single constituent kind of superfluid model that is appropriate for the description of Helium 4 at zero temperature, with special attention to the theory of small perturbations, in which an Unruh type sonic metric plays a role that is more directly important than that of he background spacetime metric. The fourth section provides a presentation of the more recently developed generalisation [1, 2, 3] to a two constituent model (of the kind whose non-relativistic analogue was originally developped by Landau) in which the second constituent represents a gas of phonon excitations.

Quite appart from the fact that it provides a clearer picture of the dynamics of superfluid Helium - 4 in an ordinary laboratory context (in which many aspects are complicated, not simplified, by the introduction of an artificially preferred rest frame such is necessary for the traditional kind of non-relativistic treament) the development of relativistic superfluid models is motivated by the need to treat neutron superfluidity in the cores and particularly the lower layers of the crusts of neutrons stars, in order to account for pulsar frequency data, in which the most salient feature is the common occurence of glitch discontinuities. The generalised Landau model described here is not by itself sufficiently elaborate to account for such effects: its construction is just an intermediate step in a long and still far from complete program of development of the more sophisticated models that will be needed.

The way to combined a the strictly fluid kind of model decrobed here with the kind of elastic solid model that is appropriate for the outer crust in order to obtain the rather elaborate kind of hybrid model needed for the mesoscopic description of the inner neutron star crust layers (at densities exceeding the neutron drip threshold) has been outlined elsewhere [4] and will not be discussed here, and anyway it is to be noted that the validity of such a mesoscopic solid - superfluid hybrid model is limited to scales small compared with the intervortex spacing which is much less than a millemetre in typical pulsars. For a macroscopic description of the inner crust of a neutron star, on scales larger than the intervortex spacing an elegant conservative model providing a fully relativistic description allowing for the anisotropic stress resulting from the averaged effect of the quantised vortices (that must be present wherever there is rotation) has recently been developped [5] and is described [6] elsewhere. Such anisotropy has been neglected in a cruder non conservative model that been developped even more recently [8] for the purpose of providing a global description of the long term evolution.

2 Single constituent perfect fluid models

Our purpose here is to offer a self contained presentation of the basic principles of the relatively simple models that suffice for the familiar laboratory case of superfluid Helium 4 (though not for Helium 3, which is so complicated [10] that a relativistic description is not yet available), starting with the kind of single constituent model (for which I have provided a more detailed discussion in the lecture notes of a recent Les Houches school [6]) that is applicable to the zero temperature limit.

This zero temperature limit will be describable by an ordinary barotropic fluid model subject to the restriction that the vorticity, meaning the exterior derivative of the relevant momentum covector (whose definition in a relativistic context will be recapitulated below) should vanish, so that the momentum covector itself has the form of the gradient of a scalar. The meaning of the term barotropic is simply that the perfect fluid pressure scalar P should be determined by an equation of state as a function only of the fluid mass density ρ. The barotropic equation of state determines a corresponding speed $c_{\rm I}$ say, of ordinary "first" sound, that will be given by the familiar formula

$$c_{\rm I}^2 = dP/d\rho\,, \tag{1}$$

and that must be subluminal, $c_{\rm I}^2 \leq c^2$ (where c is the speed of light) in order for the usual causality requirement to be respected.

Before proceeding it is desirable to recapitulate the essential elements of the relativistic kinematics and dynamics that will be required. This is particularly necessary in view of the regretable tradition in non-relativistic fluid theory – and particularly in non relativistic superfluid theory – of obscuring the essential distinction between velocity (which formally belongs in a tangent bundle) and momentum (which formally belongs in a cotangent bundle) despite the fact that the distinction is generally respected in other branches of non-relativistic condensed matter theory, such as solid state physics, where the possibility of non-alignment between the 3-velocity v^a, and the effective 3-momentum p_a of an electron travelling in a metallic lattice is well known. Failure to distinguish between contravariant entities such as the velocity v^a and covariant entities such as the momentum p_a is something that one can get away with in a non-relativistic treatment only at a price that includes restriction to strictly Cartesian (rather than for example cylindrical or comoving) coordinates.

In a relativistic treatment, even using coordinates $x^\mu \leftrightarrow \{t, x^a\}$ of Minkowski type, with a flat spacetime metric $g_{\mu\nu}$ whose components are of the fixed standard form diag$\{-c^2, 1, 1, 1\}$, the necessity of distinguishing between raised and lowered indices is inescapable. Thus for a trajectory parametrised by proper time τ, the correponding unit tangent vector

$$u^\mu = \frac{dx^\mu}{d\tau} \tag{2}$$

is automatically, by construction, a contravariant vector: its space components, $u^a = \gamma v^a$ with $\gamma = (1 - v^2/c^2)^{-1/2}$ will be unaffected by the index lowering operation $u^\mu \mapsto u_\mu = g_{\mu\nu}u^\nu$, but its time component $u^0 = dt/d\tau = \gamma$ will differ in sign from the corresponding component $u_0 = -\gamma c^2$ of the associated covector u_μ. On the other hand the 3-momentum p_a and energy E determine a 4-momentum covector μ_ν with components $\pi_a = p_a$, $\mu_0 = -E$ that are intrinsically covariant. The covariant nature of the momentum can be seen from the way it is introduced by the defining equation,

$$\mu_\nu = \frac{\partial L}{\partial u^\nu}\,, \tag{3}$$

in terms of the relevant position and velocity dependent Lagrangian function L, from which the corresponding equation of motion is obtained in the well known form

$$\frac{d\mu_\nu}{d\tau} = \frac{\partial L}{\partial x^\nu}\,. \tag{4}$$

In the case of a free particle trajectory, and more generally for fluid flow trajectories in a model of the simple barotropic kind that is relevant in the zero temperature limit, the Lagrangian function will have the familiar standard form

$$L = \tfrac{1}{2}\mu g_{\mu\nu}u^\mu u^\nu - \tfrac{1}{2}\mu c^2\,, \tag{5}$$

in which (unlike what is needed for more complicated chemically inhomogeneous models[11, 4]) it is the same scalar spacetime field μ that plays the role of mass in the first term and that provides the potential energy contribution in the second term. The momentum will thus be given by the simple proportionality relation

$$\mu_\nu = \mu u_\nu\,, \tag{6}$$

so that one obtains the expressions $E = \gamma\mu c^2$, $p_a = \mu\gamma v_a$, in which the field μ is interpretable as the relevant effective mass.

In the case of a free particle model, the effective mass μ will of course just be a constant, $\mu = m$. This means that if, as we have been supposing so far, the metric $g_{\mu\nu}$ is that of flat Minkowski type, the resulting free particle trajectories will be obtainable trivially as straight lines. However the covarant form of the equations (2) to (6) means that they will still be valid for less trivial cases for which, instead of being flat, the metric $g_{\mu\nu}$ is postulated to have a variable form in order to represent the effect of a gravitational field, such as that of a Kerr black hole (for which, as I showed in detail in a much earlier Les Houches school [12], the resulting non trivial geodesic equations still turn out to be exactly integrable).

In the case of the simple "barotropic" perfect fluid models with which we shall be concerned here, the effective mass field μ will be generically non-uniform. In these models the equation of state giving the pressure P as a function of the mass density ρ can most conveniently be specified by first giving ρ in terms of the corresponding conserved number density n by an expression that will be decomposible in the form

$$\rho = mn + \frac{\epsilon}{c^2}\,, \tag{7}$$

in which m is a fixed "rest mass" characterising the kind of particle (e.g. a Cooper type neutron pair) under consideration, while ϵ represents an extra compression energy contribution. The pressure will then be obtainable using the well known formula

$$P = (n\mu - \rho)c^2\,, \tag{8}$$

in which the effective dynamical mass is defined by

$$\mu = \frac{d\rho}{dn} = m + \frac{1}{c^2}\frac{d\epsilon}{dn}\,. \tag{9}$$

It is this same quantity μ (sometimes known as the "specific enthalpy") that is to be taken as the effective mass function appearing in the specification (5) of the relevant Lagrangian function (on what is formally identifiable as the tangent bundle of the spacetime manifold).

When one is dealing not just with a single particle trajectory but a spacefilling fluid flow, it is possible and for many purposes desirable to convert the Lagrangian dynamical equation (4) from particle evolution equation to equivalent field evolution equations [11, 4]. Since the momentum covector μ_ν will be obtained as a field over spacetime, it will have a well defined gradient tensor $\nabla_\rho \mu_\nu$ that can be used to rewrite the right hand side of (4) in the form $d\mu_\nu/d\tau = u^\rho \nabla_\rho \mu_\nu$. Since the value of the Lagrangian will also be obtained as a scalar spacetime field L, it will also have a well defined gradient which will evidently be given by an expression of the form

$$\nabla_\nu L = \frac{\partial L}{\partial x^\nu} + \frac{\partial L}{\partial \mu_\rho}\nabla_\nu \mu_\rho\,.$$

We can thereby rewrite the Lagrangian dynamical equation (4) as a field equation of the form

$$u^\rho \nabla_\rho \mu_\nu + \mu_\rho \nabla_\nu u^\rho = \nabla_\nu L\,. \tag{10}$$

An alternative approach is of course to start from the corresponding Hamiltonian function, as obtained in terms of the position and momentum variables (so that formally it should be considered as a function on the spacetime cotangent bundle) via the Legendre transformation

$$H = \mu_\nu u^\nu - L\,. \tag{11}$$

In this approach the velocity vector is recovered using the formula

$$\frac{dx^\mu}{d\tau} = \frac{\partial H}{\partial \mu_\nu}\,, \tag{12}$$

and the associated dynamical equation takes the form

$$\frac{d\mu_\nu}{d\tau} = -\frac{\partial H}{\partial x^\nu}\,. \tag{13}$$

The consideration that we are concerned not just with a single trajectory but with a space-filling fluid means that, as in the case of the preceeding Lagrangian equations, so in a similar way this familiar Hamiltonian dynamical equation can also be converted to a field equation which takes the form

$$2u^\rho \nabla_{[\rho}\mu_{\nu]} = -\nabla_\nu H\,, \tag{14}$$

with the usual convention that square brackets are used to indicate index antisymmetrisation. On contraction with u^ν the left hand side will evidently go out, leaving the condition

$$u^\nu \nabla_\nu H = 0\,, \tag{15}$$

expressing the conservation of the value of the Hamiltonian allong the flow lines.

The actual form of the Hamiltonian function that is obtained from the simple barotropic kind of Lagrangian function (5) with which we are concerned will evidently be given by

$$H = \frac{1}{2\mu} g^{\nu\rho}\mu_\nu\mu_\rho + \frac{\mu c^2}{2}\,, \tag{16}$$

in which it is again the same scalar spacetime field μ that plays the role of mass in the first term and that provides potential energy contribution in the second term.

In order to ensure the proper time normalisation for the parameter τ the equations of motion (in whichever of the four equivalent forms (4), (10), (13), (14) may be preferred) are to be solved subject to the constraint that the numerical value of the Hamiltonian should vanish,

$$H = 0\,, \tag{17}$$

initially , and hence also by (15) at all other times. This is evidently equivalent to imposing the standard normalisation condition

$$u^\mu u_\mu = -c^2\,, \tag{18}$$

on the velocity four vector. In more general "non-barotropic" systems, such as are needed for some purposes, the Hamiltonian may be constrained in a non uniform manner [11, 4] so that the term on the right of (14) will be non zero, but in the simpler systems that suffice for our present purpose the restraint (17) ensures that this final term will drop out, leaving a Hamiltonian equation that takes the very elegant and convenient form

$$u^\nu w_{\nu\rho} = 0\,. \tag{19}$$

in terms of the relativistic vorticity tensor that is defined as the antisymmetrised ("exterior") derivative of the momentum covector, i.e.

$$w_{\nu\rho} = 2\nabla_{[\nu}\mu_{\rho]}\,. \tag{20}$$

It is an evident consequence (and, as discussed in greated detail in the above cited Les Houches notes [6], would still be true even if (17) were not satisfied) that if $w_{\mu\nu}$ is zero initially it will remain zero throughout the flow, which in this case will be describable as "irrotational".

In cases for which the vorticity is non-zero, the "barotropic" dynamical equation (19) is interpretable as requiring the flow vector u^μ to be a zero eigenvalue eigenvector of the vorticity tensor $w_{\mu\nu}$, which is evidently possible only if its determinant vanishes, a requirement that is expressible as the degeneracy condition

$$w_{\mu[\nu}w_{\rho\sigma]} = 0\,. \tag{21}$$

Since the the possibility of it having matrix rank 4 is thus excluded, it follows that unless it actually vanishes the vorticity tensor must have rank 2 (since an antisymmetric matrix can never have odd integer rank). This means that the flow vector u^μ is just a particular case within a whole 2-dimensional tangent subspace of zero eigenvalue vorticity eigenvectors, which (by a well known theorem of differential form theory) will mesh together to form well defined vorticity 2-surfaces as a consequence of the Poincaré closure property,

$$\nabla_{[\mu}w_{\nu\rho]} = 0\,, \tag{22}$$

that follows from the definition (20).

Although it has long been well known to specialists [16], the simple form (19) of what is interpretable just as the relativistic version of the classical Euler equation is still not as widely familiar as it ought to be, perhaps because its Hamiltonian interpretation was not recognised until relatively recently [11, 4]. It does not constitute by itself the complete set of dynamical equations of motion for the perfect fluid, but must be supplemented by a particle conservation equation of the usual form for the particle number current

$$n^\nu = nu^\nu\,, \tag{23}$$

which must of course satisfy the condition

$$\nabla_\nu n^\nu = 0\,. \tag{24}$$

A much more widely known, but for computational purposes (particularly in curved spacetime) less useful form of the perfect fluid dynamical equations is to express them in terms of the stress momentum energy density tensor, which is given in terms of the mass density ρ and the pressure P by

$$T^{\mu\nu} = (\rho + \frac{P}{c^2})u^\mu u^\nu + Pg^{\mu\nu}\,, \tag{25}$$

and which must satisfy a so called conservation law of the standard form

$$\nabla_\nu T^{\mu\nu} = 0\,. \tag{26}$$

Although it is conveniently succinct, a disadvantage of this traditional formulation is that it is directly interpretable as a law of conservation of momentum and energy in the strict sense only in the case of a flat (Minkowski type) spacetime, but not in a curved background such as that of a neutron star. The possibility in the barotropic case (i.e. when P is a function only of ρ) of decomposing the combined set of dynamical equations (25) as the combination of the convergence condition (24) (obtained by contracting (26) with u_μ) and the relativistic Euler equation (19), which can be written out more explicitly as

$$n^\nu \nabla_{[\nu} \mu_{\rho]} = 0\,, \tag{27}$$

has the advantage that these are interpretable as genuine conservation laws – for particle number flux and vorticity respectively – even in an arbitrarily curved spacetime background.

3 Single constituent superfluid models

The simplest superfluid models, namely those pertaining to the zero temperature limit, are just ordinary perfect fluid models subject to the restraint of irrotationality, with a momentum covector given as the gradient

$$\mu_\nu = \nabla_\nu S\,, \tag{28}$$

of a periodic scalar field S that is to interpreted as being proportional to the angle of the mesoscopic phase factor, $\mathrm{e}^{i\phi}$ say, of an underlying bosonic condensate that might consist of Helium-4 atoms or Cooper type neutron pairs) in which the phase angle ϕ is given according to the usual correspondence principle by

$$\phi = S/\hbar\,. \tag{29}$$

In a multiconnected configuration of a classical irrotational fluid the Jacobi action field S obtained from (28) might have an arbitrary periodicity, but in a superfluid there will be a U(1) periodicity quantisation requirement that the periodicity of the phase angle ϕ should be a multiple of 2π, and thus that the periodicity of the Jacobi action S should be a multiple of $2\pi\hbar$. The simplest configuration for any such superfluid is a uniform stationary state in a flat Minkowski background, for which the phase will have the standard plane wave form

$$S/\hbar = k_a x^a - \omega t\,, \tag{30}$$

from which one obtains the correspondence $\mu_\nu \leftrightarrow \{-\hbar\omega, \hbar k_a\}$, which means that the effective energy per particle will be given by $E = \gamma\mu c^2 = \hbar\omega$ and that the 3-momentum will be given by $p_a = \mu\gamma v_a = \hbar k_a$.

It is to be remarked that for ordinary timelike superfluid particle trajectories the corresponding phase speed ω/k of the wave characterised by (30) will always be superluminal, a fact of which most laboratory handlers of Helium-4 are blissfully unaware, and can usually safely ignore, since what matters for most practical purposes is not the phase speed but the group velocity of perturbation wave packets.

In the irrotational case characterised by (28) the Euler equation (19) is satisfied automatically, so the only dynamical equation that remains is (24). When the phase scalar is subject to a small perturbation, $\delta\phi = \varphi$ say, it can be seen that the corresponding perturbation of the conservation law (24) provides a wave equation of the form

$$\widetilde{\Box}\varphi = 0 \tag{31}$$

in which $\widetilde{\Box}$ is a modified Dalembertian type operator that is constructed from an appropriately modified space-time metric tensor $\tilde{g}^{\mu\nu}$ in the same way that the ordinary Dalembertian operator $\Box \equiv \nabla^\mu\nabla_\mu$ is constructed from the ordinary spacetime metric tensor $g^{\mu\nu}$. The appropriately modified spacetime metric, namely the relativistic version of what is known in the context of Newtonian fluid[13, 14] and superfluid [15] mechanics as the Unruh metric, can be read out in terms of the light speed c and the (first) sound speed $c_{\rm I}$ given by (1) as

$$\tilde g^{\mu\nu} = \frac{\mu}{n}(c_{\rm I}\gamma^{\mu\nu} - c_{\rm I}^{\,-1}u^\mu u^\nu)\,, \tag{32}$$

where $\gamma^{\mu\nu}$ is the spacially projected (positive indefinite) part of the ordinary space time metric, as defined by

$$\gamma^{\mu\nu} = g^{\mu\nu} + c^{-2}u^\mu u^\nu\,. \tag{33}$$

The quantum excitations of the linearised perturbation field φ governed by (31) are what are known as phonons. For such excitations the phase speed and the group velocity are the same, both being given with respect to the unperturbed background by the ordinary ("first") soundspeed, $c_{\rm I}$, as given by (1), which will of course be subluminal. Phonons do nevertheless have a tachyonic aspect of their own, because the fact that their phase speed is subluminal automatically implies that they have a 4-momentum covector that is spacelike, in contrast with that of a ordinary fluid or superfluid particle which is timelike. This means that whereas the effective energy E of an ordinary fluid or superfluid particle is always positive, the effective energy E of a phonon may be positive or negative, depending on whether the frame of reference with respect to which it is measured is moving subsonically or supersonically. The well known implication is that if the superfluid is in contact with a supersonically moving boundary there will inevitably be an instability giving rise to dissipative phonon creation.

Given a dynamical system, one of the first things any physicist is inclined to ask is whether it is derivable from a Lagrangian type variation principle. We have already seen in the previous sections that (19) by itself is obtainable from Lagrangian equations of motion for the individual trajectories, which are of course obtainable from a one dimensional action integral of the form $\int L\, d\tau$ with L as given by (5). The question to be adressed now is how to obtain the complete set of dynamical equations (26), including (19) as well as (24), from an action integral over the 4-dimensional background manifold $\mathcal{S}^{(4)}$ of the form

$$\mathcal{I} = \int \mathcal{L}\, d\mathcal{S}^{(4)}\,, \qquad d\mathcal{S}^{(4)} = \frac{\Vert g\Vert^{1/2}}{c} d^4x\,, \tag{34}$$

for some suitable scalar Lagrangian functional $\mathcal{L}$.

There are several available procedures for doing this for a generic perfect fluid with rotation, involving radically different choices of the independent variables to be varied: although they are all ultimately equivalent "on shell" the "off shell" bundles over which the variations are taken differ not only in structure but even in dimension. These methods notably include the worldline variation procedure (the most economical from a dimensional point of view) developed by Taub[17], and the Clebsch type variation proceedure developed by Schutz[18], as well as the more recently developed Kalb-Ramond type method[19, 6] that has been specifically designed for generalisation [5] to allow for the anisotropy arising from the averaged effect of vortex tension in the treatment of superfluidity at a macroscopic level. None of these various methods is sufficiently simple to have become widely popular.

The problem is much easier to deal with if, to start off with, one restricts oneself to the purely irrotational case characterised by (28), which is all that is needed for the description of zero temperature superfluidity at a mesoscopic level. In this case a very simple and well known procedure is available. In this procedure, the independent variable is taken to be just the Jacobi action S, or equivalently in a superfluid context, the phase φ as given by (29), and the action is simply taken to be the pressure P expressed as a function of the effective mass μ, with the latter constructed as proportional to the amplitude of the 4-momentum, according to the prescription

$$\mu^2 c^2 = -\mu_\nu \mu^\nu \tag{35}$$

with the 4-momentum itself given by the relation (28) that applies in the irrotational case, i.e.

$$\mu_\nu = \hbar \nabla_\nu \phi \,. \tag{36}$$

Thus setting

$$\mathcal{L} = P \,, \tag{37}$$

and using the standard pressure variation formula

$$\delta P = c^2 n \delta \mu \,, \tag{38}$$

one sees that the required variation of the Lagrangian will be given by

$$\delta \mathcal{L} = -n^\nu \delta \mu_\nu = -\hbar n^\nu \nabla_\nu (\delta \phi) \,. \tag{39}$$

Demanding that the action integral (34) be invariant with respect to infinitesimal variations of $\varphi = \delta\phi$ then evidently leads to the required conservation law (24).

It is to be noted that this variational principle can be reformulated in terms of an independently variable auxiliary field amplitude Φ and an appropriately constructed potential function $V\{\Phi\}$ as a function of which the action takes the desirably fashionable form

$$\mathcal{L} = -\frac{\hbar^2}{2} \Phi^2 (\nabla_\nu \phi) \nabla^\nu \phi - V\{\Phi\} \tag{40}$$

which is interpretable as the classical limit of a generalised Landau Ginzburg type model. In this formulation, as discussed in greater detail elsewhere [19, 6], the auxiliary amplitude is to be identified as being given by the formula

$$\Phi = \frac{n}{\sqrt{\rho + P/c^2}} = \Big(\frac{n}{\mu}\Big)^{1/2} \,, \tag{41}$$

while the prescription for the corresponding potential energy density function is that it should be given by

$$V = \frac{\rho c^2 - P}{2} \,. \tag{42}$$

Having evaluated V as a function of Φ one can recover the effective mass μ, number density n, mass density ρ and pressure P of the fluid using the formulae

$$\mu^2 = \frac{1}{c^2 \Phi} \frac{dV}{d\Phi} \,, \qquad n = \Phi^2 \mu \,, \tag{43}$$

and

$$\rho = \tfrac{1}{2}\, \Phi^2 \mu^2 + \frac{V}{c^2} \,, \qquad P = \tfrac{1}{2}\, \Phi^2 \mu^2 c^2 - V \,, \tag{44}$$

which are derivable from (8) and (9). It is to be remarked that the covariant inverse of the generalised Unruh tensor (32) is expressible in this notation as

$$\tilde{g}^{-1}_{\ \mu\nu} = \Phi^2 ({c_{_\mathrm{I}}}^{-1}\gamma_{\mu\nu} - c_{_\mathrm{I}} c^{-2} u_\mu u_\nu) \tag{45}$$

A particularly noteworthy example is the conformally invariant special case [6] characterised by a potential function that is homogeniously quartic, $V \propto \Phi^4$, which is what is obtained for a radiation gas type equation of state of the familiar form $P = \rho c^2/3$, and for which the (first) sound speed is given by $c_{_\mathrm{I}}^2 = c^2/3$.

4 Landau type 2-constituent superfluid models.

As an intermediate step between the very simple single constituent superfluid models described in the previous section and the more elaborate models needed in the context of neutron stars, the purpose of this section is to describe the relativistic version of the category of non dissipative 2-constituent superfluid that was originally developed by Landau for the description of ordinary superfluid Helium-4 at non-zero temperature. As well as the relevant conserved particle number current n^μ (representing the flux of Helium atoms in that particular application) such a model involves another independently conserved current vector, s^μ say, representing the flux of entropy. In the single constituent case characterised by the variation rule (39) we saw how the current vector n^ν was associated with a dynamically conjugate covector μ_ν that is interpretable as representing the effective mean 4-momentum per particle. In a similar way in a 2-constituent model the second current vector s^ν will be analogously associated with its own dynamically conjugate 4-momentum covector Θ_ν.

The earliest presentations of the generic category of non-dissipative 2-constituent superfluid were on one hand a generalisation [1] of the relativistic Clebsch formulation [18] based on the variation of a generalised pressure function Ψ depending on the 4-momentum covectors μ_ν and Θ_ν according to the partial differentiation rule

$$d\Psi = -n^\nu \, d\mu_\nu - s^\nu \, d\Theta_\nu \, , \tag{46}$$

and on the other hand a generalisation [20, 4] of the world line variational formulation due to Taub [17] based on the variation of a master function Λ depending on the currents n^μ and s^ν according to the partial differentiation rule

$$d\Lambda = \mu_\nu \, dn^\nu + \Theta_\nu \, ds^\nu \, . \tag{47}$$

Although they were originally developed independently these alternative formulations were subsequently shown to be equivalent to each other and to an intermediate mongrel version [2] based on a Lagrangian density

$$\mathcal{L} = \Psi + s^\nu \Theta_\nu = \Lambda - n^\mu \mu_\nu \, , \tag{48}$$

depending on the particle 4-momentum covector μ_ν and the entropy current s^ν according to the partial differentiation rule

$$d\mathcal{L} = \Theta_\nu \, ds^\nu - n^\nu \, d\mu_\nu \, . \tag{49}$$

All of these variational formulations are subject to the complication that the allowable field variations are not free but must be suitably constrained to avoid giving overdetermined field equations. Although it violates the symmetry between the two kinds of conserved current n^ν and s^ν that are involved, the mongrel formulation characterised by (49) is the one that allows the simplest specification of the constraints required to get the appropriate dynamical equations for the superfluid case. In this formulation [2] the constraint on the particle 4-momentum covector is simply that it should have the same phase gradient form (28) as in the zero temperature limit in which the entropy constituent is absent, namely

$$\mu_\nu = \hbar \nabla_\nu \phi \,. \tag{50}$$

The corresponding constraint on the current vector s^ν of the "normal" constituent is the not quite so simple Taub type requirement that its variation should be determined by the displacement of the flow lines generated by an arbitrary vector field ζ^ν say, which means [4] that it must have the form

$$ds^\nu = \zeta^\rho \nabla_\rho s^\nu - s^\nu \nabla_\rho \zeta^\nu + s^\nu \nabla_\rho \zeta^\nu \,, \tag{51}$$

whose derivation is obtainable by a procedure that will be explained more explicitly in the next Section. Demanding invariance of the volume integral of $\mathcal{L}$ with respect to infinitesimal local variations of the phase variable ϕ then gives the usual particle conservation law in the same form (24) as for the single constituent limit, while demanding invariance for an arbitrary local displacement field ζ gives not only the analogous entropy conservation law

$$\nabla_\nu s^\nu = 0 \,, \tag{52}$$

but also the dynamical equation

$$s^\nu \nabla_{[\nu} \Theta_{\rho]} = 0 \tag{53}$$

that governs the evolution of the thermal 4-momentum covector in a manner analogous to that whereby the relativistic Euler equation (27) governs the evolution of the momentum covector in an ordinary perfect fluid. These dynamical equations entail (but unlike the single constituent case are not entirely contained in) an energy momentum pseudo-conservation law of the usual form (26) for a stress-momentum-energy density tensor that can be written in the form

$$T^\nu_\rho = n^\nu \mu_\rho + s^\nu \Theta_\rho + \Psi g^\nu_\rho \,, \tag{54}$$

which will in fact (although it is not obvious in this particular expression) be automatically symmetric, $T^{[\nu\rho]} = 0$.

The category of models characterised by the preceeding specifications for various conceivable forms of the equation of state specifying $\mathcal{L}$ as a scalar function of μ_ν and s^ν is very large. The use of what is interpretable [21] as a special subcategory therein, on the basis of a particular kind of separation ansatz, was proposed in early work of Israel [22] and Dixon [23] and has been advocated more recently by Olsen [24]. Unfortunately however, the simplification provided by the Israel Dixon ansatz (effectively the relativistic generalisation of the obsolete Tisza-London theory that was superceded by that of Landau) is incompatible with the kind of equation of state that is needed for even a minimally realistic treatment of a real superfluid.

A satisfactory treatment of what goes on at temperatures high enough for non-linear "roton" type excitations to be important is not yet available, but in the low temperature "cool" regime, in which only linar "phonon" type excitations are important, it is not difficult to provide a straightforward analytic derivation of the kind of equation of state that is appropriate. Following the lines developed in a non-relativistic context by Landau himself [25] the relativistic version of the approriate "cool" equation of state has recently been derived [3] by considering perturbations of the single constituent model – with equation of state specified as a pressure function, $P\{\mu\}$ – that describes the relevant zero temperature limit. The result is obtained in an analytically explicit form that (despite the fact that it is not of the separable Israel Dixon kind) can be given a very simple expression in terms of what we referred to as the "sonic" metric, which is specifiable by the conformal relation

$$\mathcal{G}^{\rho\sigma} = \Phi^2 {c_{_\mathrm{I}}}^{-1} \tilde{g}^{\rho\sigma} \tag{55}$$

in terms the Unruh phonon metric (32) that is associated with the relevant zero temperature limit state as specified by the relevant momentum covector μ_ν, which by (35) determines the relevant value of the scalar μ and hence (via the zero temperature equation of state, using the formalism of Section 2) also of the relevant phonon speed $c_{_\mathrm{I}}$ and field amplitude Φ. While the Unruh metric is more convenient for many purposes, the advantage of the conformal modification we have used, namely

$$\mathcal{G}^{\rho\sigma} = g^{\rho\sigma} + (c^{-2} - {c_{_\mathrm{I}}}^{-2}) u^\rho u^\sigma \,, \tag{56}$$

is that its spacially projected part agrees with that of the ordinary space metric, from which it differs only in the measurement of time.

The result that is obtained [3] is given by a Lagrangian of the form

$$\mathcal{L} = P - 3\psi \tag{57}$$

in which the deviation from the zero pressure limit value $P\{\mu\}$ is given as a function not just of the particle 4-momentum covector μ_ν but also of the entropy flux s^ν (postulated to be sufficiently weak to be constituted only of phonons) by the formula

$$\psi = \frac{\tilde{\hbar}}{3} {c_{_\mathrm{I}}}^{-1/3} |\mathcal{G}^{-1}_{\rho\sigma} s^\rho s^\sigma|^{2/3} \,, \tag{58}$$

where $\tilde{\hbar}$ is identifiable to a very good approximation with the usual Dirac-Planck constant $\hbar$, its exact value being given by

$$\tilde{\hbar} = \frac{9}{4\pi}\Big(\frac{5\pi}{6}\Big)^{1/3} \simeq 0.99\,\hbar \,. \tag{59}$$

This is equivalent to taking the generalised pressure function to be

$$\Psi = P + \psi \,, \tag{60}$$

with

$$\psi = \frac{c_{\mathrm{I}}}{4}\Big(\frac{3}{4\tilde{\hbar}}\Big)^3 (\mathcal{G}^{\rho\sigma}\Theta_\rho\Theta_\sigma)^2 \,, \tag{61}$$

in which the effective thermal 4-momentum per unit of entropy is given (according to the partial differentiation formula (49) by

$$\Theta_\rho = \frac{4\tilde{\hbar}}{3} |c_{\mathrm{I}} \mathcal{G}^{-1}_{\mu\nu} s^\mu s^\nu|^{-1/3} \mathcal{G}^{-1}_{\rho\sigma} s^\sigma \,, \tag{62}$$

with

$$\mathcal{G}^{-1}_{\rho\sigma} = g_{\rho\sigma} + \Big(1 - \frac{c_{_\mathrm{I}}{}^2}{c^2}\Big)|\mu^\nu\mu_\nu|^{-1}\mu_\rho\mu_\sigma \,. \tag{63}$$

An concrete illustration, allowing the explicit evaluation of the relevant quantities, is provided by the polytropic case, as characterised by a (single consitituent) equation of state giving the mass density ρ as a function of the number density n in terms of a fixed ("rest") mas per particle m, a scale constant κ and a fixed dimensionless index γ in the form

$$\rho = mn + \kappa n^\gamma \Leftrightarrow \mu = m + \kappa\gamma n^{\gamma-1} \,, \tag{64}$$

which corresponds to taking the pressure to be given by

$$P = \kappa c^2(\gamma-1)n^\gamma = \kappa c^2(\gamma-1)\Big(\frac{\mu - m}{\kappa\gamma}\Big)^{\gamma/(\gamma-1)} \,, \tag{65}$$

while the corresponding sound speed will be given (independently of κ) by

$$c_{_\mathrm{I}}{}^2 = (\gamma-1)\Big(1 - \frac{m}{\mu}\Big)c^2 \,. \tag{66}$$

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