

Dimensional Analysis in Operations Research

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Summary

Dimensional Analysis is a general method of determining the form of solutions to physical problems. It is illustrated by an example from physics and descriptions of applications to some OR problems. The Ehrhardt power approximation for computing (s, S) inventory policies is examined from this point of view and found to be flawed.

1 Introduction

In a paper published in 1966, Naddor [9] encouraged those in Operations Research to use dimensional methods and gave examples in queueing, inventory, and linear programming. Following Huntley [8], he listed several ways the technique could aid the OR worker but noted some restrictions. This paper sets out to reinforce Naddor's views, explaining what Dimensional Analysis is and giving illustrations of its use in some applications. We also test Ehrhardt's power approximation [3, 4, 5] for inventory policies and find it dimensionally incorrect.

Dimensional Analysis (DA) is a technique that has been used by physicists and engineers for many years to obtain preliminary solutions to physical problems. Assuming that the phenomenon can be described by a dimensionally correct equation among a set of variables, DA quickly determines a general form of the solution form constraints put on it by their dimensions. DA is most useful where the derivation of an analytical solution is difficult but the variables that take part in the problem are understood or can be postulated. DA will not provide a complete solution, nor does it substitute for a knowledge of the working of the phenomenon involved.

Newton, in proposing the principle of similitude in his *Principia*, recognised three primary distinct attributes, length, inertia (mass), and time from which other measures such as speed, force, and acceleration are derived. Fourier, in his work in the theory of heat, postulated them as "fundamental units", and suggested that every physical quantity has "dimensions" derived from powers of these units. He introduced the idea of a "dimensional formula" and showed that equations should have "dimensional homogeneity". From this requirement follows the constraints that DA uses to obtain the general form of a solution. Using the same method Lord Rayleigh later developed a range of solutions to physical problems such as the oscillation of liquid drops under surface tension. In the late 19th Century many of the great classical physicists used the

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method as a normal and powerful tool of their work and it remains a standard tool of the applied mathematicians and mathematical physicists.

Buckingham's Pi theorem is a more formal description of the requirement for dimensional homogeneity. It asserts that the formula (or physical law) connecting the set of variables can be expressed as a function of dimensionless arguments. Each argument is a dimensionless product of a combination of the original variables raised to integer powers. Depending on the number of fundamental dimensions, fewer of these products will exist than of the original variables. Thus to determine the general form of the solution, we only need to find the complete collection of such groupings that can be formed from the physical parameters and express the formula as some function of these Dimensionless Numbers.

Engineers, faced with complicated physical problems, have been enthusiastic users of the method and have named many Dimensionless Numbers to describe particular situations. The Reynolds number, Re , for example, is associated with the velocity and size of a wide range of conditions Re describes the situation succinctly and can be used as a precise indicator of system state for this type of problem. The Froude number, Fr , is related to the ratio of inertial and gravitational forces and is associated with free surface movement in a liquid. Fr is used to determine the wave resistance of ships of similar shapes running at different speeds. By using such dimensionless numbers, engineers can carry out experiments on models and extrapolate the results to full-scale systems. Their utility is perhaps one reason why engineers tend to report measurements on systems as power functions of the parameters involved.

Traditionally, physicists introduce the methods of dimensions by demonstrating how to derive the formula for the period of oscillation of a simple pendulum. We will write the dimensions of a measured quantity in square brackets next to its symbol. The dimensions are established from the quantity's definition and are usually stated directly in the units of measurement. The physical parameters of the pendulum problem, in addition to the period of oscillation, t [T], are the mass of the pendulum bob, m [M], the length of the string, L [L], the acceleration of gravity, g [LT^{-2}] and (perhaps) the maximum angle of oscillation measured in radians, A . A is a ratio of two lengths and can therefore be considered to be dimensionless (but see the discussion of directed lengths in Huntley [8]). The acceleration of gravity, g , has the dimensions of rate of change of velocity, i.e. [LT^{-2}]. There are two dimensionless numbers in the pendulum problem. The first, A , is given to us immediately. It is a special type called a scale factor, or, in physical problems, a shape factor, formed from the ratio of two variables with identical dimensions. The second number can be discovered by considering how a product of powers of t , m , L , and g can be made dimensionless in all three of the dimensions, M , L , and T . The variable, m , cannot be combined with any others in the problem to form a dimensionless product since none of them have an [M] dimension to cancel out that term. Thus m cannot form part of the solution with the problem as specified. The only way the remaining three variables can be grouped is as follows:

$$\left(\frac{L}{gt^2}\right) \quad (1)$$

The solution of the problem is then given to us as a function of these two dimensionless products:

$$f\left\{\left(\frac{L}{gt^2}\right), (A)\right\} = \text{const} \quad (2)$$

The simplest possible function of the first dimensionless parameter is a linear form. Assuming

this and expressing it in terms of the other variables, we obtain:

$$t = k \left(\frac{L}{g} \right)^{0.5} f(A) \quad (3)$$

Here $f(\)$ is some yet unknown function of A and k is an arbitrary but dimensionless constant. Because it is dimensionless, k does not change and the relationship still holds even if the measurement units for mass, length, and time are changed. Comparing this form with the known solution to this problem obtained from the complete mathematical analysis, for small angles of oscillation:

$$t = 2\pi \left(\frac{L}{g} \right)^{0.5} \left(1 + \frac{A^2}{16} + \dots \right) \quad (4)$$

This demonstrates that, though it can neither provide the precise functional nor the value of the constant in the formula, DA can quickly determine the general structure of the solution.

The method of dimensions has had its greatest use in physics and applied mathematics and there appears to have been some resistance to its introduction to the non-physical sciences. Coyle and Ballico-Lay [1] argue strongly for its recognition in management science and describe using a computer package to detect several common dimensional errors in management science models.

The idea of using dimensionless numbers as succinct ways of expressing effects in models is well known even in the non-physical sciences. Sivazlian [13] uses it to reduce the number of variables he had to plot to present the results of determining (s, S) inventory policies with gamma distributed demand in each period. Silver [12] uses some dimensionless numbers to plot indifference curves in a continuous review stochastic (s, S) system and Ewing [6] proposed the use of dimensionless expressions rather than the original measurements in analysing data in the social sciences.

In the earliest classical textbooks in OR, Sasieni, Yaspan and Friedman [11] and Flagle, Huggins and Roy [7] use dimensional methods to check inventory formulae derived using more complicated analyses. Naddor [10] also uses the concept of dimensions extensively in his book on inventory theory. The ideas are particularly powerful in this topic because several types of costs, differing only in their dimensional structure, can be defined. For example, Naddor uses three types of stockout cost with dimensions $[\$/Q/T]$, $[\$/Q]$, and $[\$]$, all of which lead to different solutions for the inventory problem. Introducing the technique to economists, De Jong [2] uses the same fundamental dimensions as we do in OR and discusses at some length the dimensional problems associated with interest rates and discounting.

2 Fundamental Dimensions in Operations Research

The fundamental dimensions involved in Operations Research are not as familiar as the three $[MLT]$ dimensions of mechanics and physics. OR problems will often have a component of time $[T]$ in common with mechanics but will often also have some criterion of optimisation, perhaps cost or profit, for which we will use the symbol $[\$]$. Reorder cost will therefore have dimension of $[\$]$ and average cost per unit time dimensions of $[\$/T]$. Often there are components of material $[Q]$, such as lot sizes or production quantities. Demand, for example, has dimensions $[Q/T]$ and the cost of an item, $[\$/Q]$. It is sometimes necessary to distinguish between different types

of such materials and each can be given a different symbol. Linear programming models, as Naddor points out, typically contain very many different types of quantities and constraints. Some queue models will include people, $[P]$ or customers, $[C]$. Probability is dimensionless in nature but random variables usually have a dimension of their own, depending on the particular model.

3 Queueing examples

As an illustration of the application of Dimensional Analysis to OR problems, consider a simple queue system. The average number of customers in a queue system, L , can be considered either as an integer and hence dimensionless [1] or measuring the number of customers $[C]$. In contrast to Naddor, I will take the second approach and set the dimensions of the arrival rate, A , to be $[C/T]$. The mean time in the system, W , has dimensions $[T]$. the only dimensionless expression that can associate these three variables is (AW/L) and we are led to a solution form:

$$f\left(\frac{AW}{L}\right) = \text{const} \quad (5)$$

with both the function and the constant unspecified. The simplest function of this form is given

$$L = kAW \quad (6)$$

where the dimensionless constant, k , is unknown until a more complicated analysis gives $k = 1.0$ and hence Little's law. We can add further factors to our simple model: s , the number of servers $[S]$, and the service rate for each server, R $[C/T/S]$. We then obtain the additional dimensionless group (A/sR) , the traffic intensity, and a general functional form of:

$$f\left(\frac{AW}{L}, \frac{A}{sR}\right) = \text{const} \quad (7)$$

Thus the simplest form of function for average waiting time is

$$W = \frac{L}{A} f\left(\frac{A}{sR}\right) \quad (8)$$

Adding more variables to the model, let $E(t)$ $[T]$ be the mean service time and $\text{var}(t)$ its variance $[T^2]$. This introduces two new dimensionless groupings, a scale factor $W/E(t)$ and $\text{var}(t)/E^2(t)$. By dropping the group involving W and including the new groups, we obtain a facsimile of the Pollaczek-Khintchine formula:

$$L = A E(t) f\left(\frac{A}{sR}, \frac{\text{var}(t)}{E^2(t)}\right)$$

We can go on from here to consider different types of costs – of service, of customer waiting time, and of the system.

4 Inventory Models

A dimensional analysis of the simple continuous stock control problem, excluding leadtime effects, gives the following function of a set of dimensionless groupings:

$$f\left(\underbrace{\frac{MK}{hY^2}}_{(1)}, \underbrace{\frac{Yh}{MC_5}}_{(2)}, \underbrace{\frac{M}{A}}_{(3)}, \underbrace{\frac{h}{p}}_{(4)}, \underbrace{\frac{K}{C_4}}_{(5)}\right) = \text{const}$$

where Y is the order quantity (lot size) [Q]

M is the demand rate [Q/T]

h is the holding cost [$\$/Q/T$]

p is the penalty cost proportional to average stockout [$\$/Q/T$]

K is the order cost [$\$$]

A is the production rate [Q/T]

and the C 's are penalty costs of different dimensional forms:

C_4 a fixed penalty cost per stockout [$\$$]

C_5 proportional only to average size of a stockout [$\$/Q$]

Describing the dimensionless numbers in order, we can see that:

- (1) balances reorder cost K [$\$$] against holding cost, h [$\$/Q/T$] for given demand, M , and reorder quantity, Y . This group gives rise to the Wilson Lot-size formula, if the influences of the other factors are ignored but it appears in many other models in this form. (I suggest it be christened the Wilson Number.)
- (2) balances the size-of-backlog penalty cost, C_5 , and holding cost, h . This group appears in the continuous review stochastic models as the probability of a stockout per leadtime.
- (3) is a scale factor which becomes active for cases of non-infinite production rate, A .
- (4) is a scale factor giving the ratio of holding to penalty costs and is active where the penalty cost is not infinite and hence stockouts are allowed.
- (5) is a scale factor balancing reorder costs and fixed penalty costs.

Ignoring one of these groups corresponds to making constraining assumptions about the model. For example, ignoring group (3) is equivalent to assuming an infinite replenishment rate. Ignoring all but (1) gives the simplest solution to the model, expressing in terms of Y :

$$Y^2 = k \frac{KM}{h} \tag{9}$$

This is the standard lot-size formula except for the particular value of the dimensionless constant, k . This is found to be 2 in more detailed analysis.

Allowing group (4) also to be active gives us a formula for the lot size, Y , with a finite stockout cost, p .

$$Y^2 = l \frac{KM}{h} f\left(\frac{h}{p}\right) \tag{10}$$

We can get a closer approximation to the solution by considering reasonable forms for the function of $\frac{h}{p}$. We know that when p is infinite, the function must tend to a constant, since Y still has a value even when stockouts are forbidden. Without loss of generality we can assume that this is 1.0 for the limiting case. It is reasonably easy to convince ourselves that a simple first model for this component is $\left(1 + \frac{ah}{p}\right)$, where a is a dimensionless constant. The textbook solution for this problem gives $a = 1$ and the functional form as $\left(1 + \frac{h}{p}\right)$. Similar logic gives us a first attempt at the form of the production rate function using group (3) as $\left(1 - \frac{M}{A}\right)$ which is a good first approximation to $\left(1 + \frac{M}{A}\right)^{-1}$. The latter expression is still dimensionally correct.

In the next section we will extend this model to the case of stochastic periodic-review.

5 Ehrhardt's Power Approximation

An illustration of the utility of the dimensional analysis approach comes from examining the results of experiments conducted by Ehrhardt [3] and Ehrhardt and Mosier [4]. They attempt to find an efficient power approximation for computing (s, S) policies in a periodic-review, single-item, backlogged inventory system with a fixed leadtime and independent stochastic demand in each period.

We will study only part of this investigation, the determination of a power approximation to the order size, $D = S - s$, which corresponds in dimensional terms to our previous Y [Q]. Ehrhardt starts from Robert's model, which here is the same as the Wilson Lot size formula. He generalises it to include four additional factors. First the mean, M_r , and standard deviation, s_r [Q] of demand each review period, r . We distinguish here M_r from the demand rate, M [$\frac{Q}{T}$], with $M_r = M/r$. In his analysis, r [T] is taken as 1 throughout so although they are of different dimensions there is no numeric difference between M_r and M (until scale changes take place). Ehrhardt then adds the penalty costs, p [\$/Q/T] and the leadtime, L . We will assume that L is a number of review periods and is therefore dimensionless [1]. Ehrhardt uses $(1 + L)$ instead of L since full analysis shows that this is needed to allow for extra security because reviews occur only every period. Dimensionally there is, of course, no difference. He arrives at the following, dimensionally incorrect, general model:

$$D = k M_r^a \left(\frac{K}{h}\right)^b (1 + L)^c (s_r)^d \left(\frac{p}{h}\right)^e \quad (11)$$

Ehrhardt then generates 288 cases by varying each of these parameters (except h which was fixed at 1) and three demand distributions in a grid design. Using the optimal value of D determined from a full iterative method for each of the 288 cases, he uses a regression to determine the best fitting values of the exponents, arriving at the following expression:

$$D = 1.463 M_r^{0.364} \left(\frac{K}{h}\right)^{0.498} (s_L)^{0.138} \quad (12)$$

where $s_L = s_r(1 + L)^{0.5}$ [Q] is the standard deviation of demand in $(1 + L)$ periods. This expression is given external to the model and reflects the assumption of statistical independence of demand between periods. the factor p/h was found to have a negligible effect. This formula is

dimensionally incorrect as was pointed out by the author when it was revised. A change of units of M or M_r , for example, would require a change in the size of the constant 1.463. Ehrhardt also notes that the coefficient of M “is significantly lower than the Wilson value of 0.5”. Note that if we move the M into the MK/h form, the formula becomes

$$D = 1.463 \left(\frac{MK}{h} \right)^{0.498} \left(\frac{s_L}{M_r} \right)^{0.138} M_r^{0.004}$$

Not surprisingly, the model is now very close to that predicted by DA:

$$D = k \left(\frac{MK}{h} \right)^{0.5} f \left(\frac{s_r}{M_r}, (1 + L), \frac{p}{h} \right) \quad (13)$$

The exponent of 0.5 for the first term is derived from the Wilson Number connecting D , M , K , and h . The dimensional technique cannot predict the forms of the function or the value of the constant (though we would be surprised if it were far from 1.4). If we assume simple power functions of the different dimensionless products, we note that Ehrhardt finds the exponent of $\frac{s_r}{M_r}$ to be about 0.138 and absorbs at least part of the $(1 + L)$ term if s_L is to be used instead of s_r . Our analysis, of course, cannot say whether there is any further lead-time effect, though the formula allows it as a possibility.

In the 1984 paper revising the power approximation, Ehrhardt and Mosier [4] first forbid a zero variance forcing a zero value for D by changing the function of s_L to be fitted to $1 + \left(\frac{s_L}{M}\right)^2$. To find it they use similar arguments to those used for the lot-size model in the previous section to derive the form of the function of $\frac{h}{p}$. In this instance DA helps to constrain the parameter to be dimensionless but is of little help beyond this.

Secondly they adjust the model to be dimensionally correct for changes in units of demand. They do this by constraining the regression so that $a = 1 - b$. The regression is now limited to fitting one constant and two exponents.

The Revised Power Approximation for D can now be written as follows:

$$D = 1.30 \left(\frac{KM}{h} \right)^{0.506} \left(1 + \left(\frac{s_L}{M} \right)^2 \right)^{0.116} M^{-0.012} \quad (14)$$

Despite their efforts, this is still not dimensionally correct. One can see that the constant must change by a small amount if the time unit is changed. The regression is very close to the DA solution and any change in the constant is probably within the errors of estimation even for large changes in time unit. Nevertheless a preliminary dimensional analysis would show that the true exponent for the first term is 0.5 (and not 0.498 or 0.506) and the correct form is the same as that given in equation (13). The function involving $\frac{s_L}{M}$ is acceptably dimensionless if it is corrected to $\frac{s_L}{M_r}$.

Dimensional considerations would alternatively suggest a different experimental design to find a power law model. The design would include regressing a range of values of the dimensionless Wilson Number or its reciprocal, $D \left(\frac{MK}{h} \right)^{-0.5}$ against values of the other dimensionless groups $\frac{s_L}{M_r}$, $(1 + L)$ and $\frac{p}{h}$ or, if further information is available, as in this case, against functions of these such as $1 + \left(\frac{s_L}{M_r} \right)^2$. Apart from starting with at least part of the correct model for D , this would have the advantage of either reducing the number of points to be analysed, or, while

keeping the same number, increasing the range of situations to be fitted. It would also give a much clearer view of the target functions to be approximated, essentially breaking the main formula down into sensible components.

6 Conclusion

Dimensional Analysis, though it has had any occasional use in Operations Research and Management Science, is potentially useful in those applications such as inventory and queueing where problems involve combinations of time, quantities, and objective function, and where the factors and effects involved are understood but the form of the solutions is not known.

A preliminary dimensional analysis of the problem, while it will not provide a complete solution, will give a general form of the solution and may save much unnecessary work.

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