Duality for Some Categories of Coalgebras

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Abstract

A contravariant duality is constructed between the category of coalgebras of a given signature, and a category of Boolean algebras with operators, including modal operators corresponding to state transitions in coalgebras, and distinguished elements abstracting the sets of states defined by observable equations.

This duality is used to give a new proof that a class of coalgebras is definable by Boolean combinations of observable equations if it is closed under disjoint unions, domains and images of coalgebraic morphisms, and ultrafilter enlargements. The proof reduces the problem to a direct application of Birkhoff's variety theorem characterising equational classes of algebras.

Contents

1 Introduction and Overview

This paper continues a study begun in [Gol99] of equational logic for certain kinds of *coalgebra*. These structures include automata and other state-transition systems, and also encompass simple constructions of *classes* in object-oriented programming languages [Rei95, Jac96, Rut95, Rut00]. A typical coalgebra of the kind we consider has a non-empty set X of *states* (which may be thought of as the possible realisations of some notion of object), and a set of operations which are classified as *methods* or *attributes*. Methods are of the type $m : X \times I \to X$ and attributes of the type $a : X \times I \to O$, where I is a set of inputs and O a set of outputs. A method can be thought of as a system of state transitions $x \mapsto m(x, b)$ parameterised by inputs b, while an attribute assigns outputs or *observable values* $a(x, b)$ to states x relative to an input.

A calculus of *terms* for these coalgebras is developed in [Gol99]. A term is an expression whose value is either a state, or a member of a *data set*, i.e. an input or output set. Symbols for methods and attributes are used to construct terms out of variables ranging over data sets, constants naming members of data sets, and a special state symbol σ which may be thought of as a parameter denoting the "current" state. An *observable equation* is an expression of the form $t_1 \approx t_2$, where t_1 and t_2 are terms taking observable values. The equation is interpreted as asserting the identity of the values of t_1 and t_2. One of the principal conclusions of [Gol99] is that Boolean combinations of observable equations form a natural language of formulas for the specification of classes of coalgebras. Moreover this is implicitly a *modal* language. Corresponding to the state-transition $x \mapsto m(x, b)$ we may introduce a modality $[m.b]$, with a formula $[m.b]\varphi$ expressing the statement "after applying method m with parameter b, φ is true". It turns out that this assertion is expressed by the formula $\varphi(m(\sigma, b))$, obtained from φ by substituting the term $m(\sigma, b)$ for σ (see (2.v) below). Thus by taking Boolean combinations of equations we are already dealing with a modal language, one in which the modalities are definable from other constructs.

The *ultrafilter enlargement* of a coalgebra A is defined as a new coalgebra A^* whose states are certain "observationally rich" ultrafilters on the state set of A. These ultrafilters are required to contain special sets of states defined by observable equations. The A^* construction is central to the characterisation of classes of coalgebras defined by combinations of observable equations. The main result (Theorem 9.2) of [Gol99] states that

> *a class of coalgebras is definable by Boolean combinations of observable equations if and only if it is closed under disjoint unions, domains and images of coalgebraic morphisms, and ultrafilter enlargements.*

This result may be viewed as an analogue of Birkhoff's celebrated variety theorem, stating that a class of algebras is equationally definable iff it is closed under homomorphic images, subalgebras, and direct products. The purpose of the present paper is to make this analogy quite explicit, by developing a categorical duality between our coalgebras and a certain kind of *Boolean algebra with operators* (BAO), and then giving a new proof of the above result about coalgebras in which direct use is made of Birkhoff's theorem for these BAO's.

The kind of BAO that we use has operators $[m.b]^{\mathbf{R}}$ corresponding to the modalities $[m.b]$ described above, and distinguished elements $(t_1 \approx t_2)^{\mathbf{R}}$ corresponding to observable equations. An *$\mathcal{S}$-algebra* is defined as an algebra of this type satisfying certain equations that are set out in Definition 3.1. Associated with any coalgebra A is the $\mathcal{S}$-algebra $\mathcal{P}A$ whose underlying Boolean algebra is the powerset algebra of the state-set of A, with $[m.b]^{\mathcal{P}A}(Y) = \{x : m(x,b) \in Y\}$, and with $(t_1 \approx t_2)^{\mathcal{P}A}$ being the set of states in which the equation $(t_1 \approx t_2)$ is satisfied. Also associated with coalgebra A is the *minimal* subalgebra $\mathcal{D}A$ of $\mathcal{P}A$. $\mathcal{D}A$ is the $\mathcal{S}$-algebra generated by the distinguished elements $(t_1 \approx t_2)^{\mathcal{P}A}$, and plays a vital part in our theory: satisfaction of algebraic equations by $\mathcal{D}A$ is expressible in terms of satisfaction of observable coalgebraic equations by A (see Theorem 3.5(2)).

In the opposite direction, starting with an abstract $\mathcal{S}$-algebra $\mathbf{R}$ we construct a coalgebra $\mathsf{co}\mathbf{R}$ by an adaptation of the representation theory for BAO's [JT51, Gol89]. The states of $\mathsf{co}\mathbf{R}$ are certain ultrafilters F of $\mathbf{R}$ that satisfy the following "observational richness" condition:

> for each ground observable term t, taking values in output set O^t,
> there exists some data element $b \in O^t$ such that $(t \approx b)^{\mathbf{R}} \in F$.

This condition enables attribute operations to be defined in $\mathsf{co}\mathbf{R}$. But its imposition is a restriction on the standard Stone representation of a Boolean algebra, in which all ultrafilters are involved. In fact some $\mathcal{S}$-algebras fail to have any rich ultrafilters, in which case $\mathsf{co}\mathbf{R}$ does not exist. We give an example whose underlying Boolean algebra is the well-known atomless quotient of the powerset $\mathcal{P}(\omega)$ of the natural numbers by the ideal of finite sets. It is therefore necessary to restrict attention to those $\mathcal{S}$-algebras that have at least one rich ultrafilter. The class of such algebras, with $\mathcal{S}$-algebraic homormorphisms between them, forms the category $\mathcal{S}$-$\mathbf{Alg}$. The constructions $\mathbf{R} \mapsto \mathsf{co}\mathbf{R}$ and $A \mapsto \mathcal{P}A$ extend to contravariant functors that provide a dual adjunction between $\mathcal{S}$-$\mathbf{Alg}$ and the category of coalgebras with coalgebraic morphisms. The components of this adjunction are certain maps $\xi^A : A \to \mathsf{co}\mathcal{P}A$ and $\eta^{\mathbf{R}} : \mathbf{R} \to \mathcal{P}\mathsf{co}\mathbf{R}$. The coalgebra $\mathsf{co}\mathcal{P}A$ is precisely the ultrafilter enlargement A^* of A mentioned above, and $\xi^A(x)$ is the principal ultrafilter generated by state x on the state-set of A. ξ^A was shown in [Gol99, Theorem 8.2] to be an injective morphism. $\eta^{\mathbf{R}}(r)$ is the set of rich ultrafilters containing the element r of $\mathbf{R}$, and the proof that $\eta^{\mathbf{R}}$ is a homomorphism involves an extensive analysis (see results 4.3–4.5). But, unlike the purely Boolean case, $\eta^{\mathbf{R}}$ need not be injective. We show this by constructing an infinite $\mathcal{S}$-algebra that has *exactly one* observationally rich ultrafilter, and hence has by no means enough such ultrafilters to separate its members.

Under this duality between coalgebras and $\mathcal{S}$-algebras, subcoalgebras correspond to homomorphic images, images of coalgebraic morphisms correspond to subalgebras, and disjoint unions of coalgebras correspond to direct products of algebras. These facts are all used in applying Birkhoff's variety theorem for

$\mathcal{S}$-algebras to the proof of our characterisation of observably definable classes of coalgebras (Theorem 6.1).

The structure of this paper is as follows. Section 2 summarizes the syntax and semantics of terms and formulas for coalgebras. Section 3 introduces $\mathcal{S}$-algebras and develops their basic theory. Section 4 constructs the coalgebra $\mathsf{co}\mathbf{R}$ of an $\mathcal{S}$-algebra, and gives the counter-examples concerning existence of rich ultrafilters. Section 5 establishes the functorial duality between $\mathcal{S}$-algebras and coalgebras. Section 6 contains the application of Birkhoff's theorem to characterisation of classes of coalgebras. Section 7 discusses connections with earlier work of the author in modal algebraic logic.

2 Syntax and Semantics of Coalgebras

A theory of terms and formulas for coalgebras, and their semantic interpretation, is set out in [Gol99]. We now survey the parts of that theory which will be needed here.

A *signature for coalgebras* is a sequence $\mathcal{S} = (In, Out, Meth, Att)$, with In being a collection of *input sets*, Out a collection of *output sets*, $Meth$ a collection of *method symbols*, and Att a collection of *attribute symbols*.

Each method symbol $m \in Meth$ is assigned an input set $I^m \in In$, called the *sort* of m. Each attribute symbol $a \in Att$ is assigned a pair (I^a, O^a) as its *sort*, with $I^a \in In$ and $O^a \in Out$.

The set $Data = \bigcup(In \cup Out)$ consists of all elements appearing in any input or output set. Members of $Data$ are *data elements*. Members of an output set are said to be *observable*.

A *coalgebra* for signature $\mathcal{S}$ is a structure

$$A = (X^A, \{m^A : m \in Meth\}, \{a^A : a \in Att\}),$$

consisting of a set X^A of *states*, a *method function* $m^A : X^A \times I^m \to X^A$ for each method symbol m of sort I^m, and an *attribute function* $a^A : X^A \times I^a \to O^a$ for each attribute symbol a of sort (I^a, O^a). If an input set has one element then it may be ignored, allowing us to consider methods of the form $X^A \to X^A$ and attributes of the form $X^A \to O^a$ without any associated input sets.

For an arbitrary set X, let

$$T^{\mathcal{S}}(X) = \prod_{m \in Meth} X^{I^m} \times \prod_{a \in Att} (O^a)^{I^a}.$$

Then an $\mathcal{S}$-coalgebra A as above can be identified with the function $\chi : X^A \to T^{\mathcal{S}}(X^A)$ having

$$\chi(x) = \big\langle \langle m_x^A : m \in Meth \rangle, \langle a_x^A : a \in Att \rangle \big\rangle,$$

where $m_x^A \in X^{I^m}$ is the function $b \mapsto m^A(x, b)$ and $a_x^A \in (O^a)^{I^a}$ is the function $c \mapsto a^A(x, c)$. The function $T^{\mathcal{S}}$ itself lifts naturally to an endofunctor $T^{\mathcal{S}}$:

Set $\to$ **Set** on the category of sets, and a coalgebra as a pair (X^A, χ) with $\chi : X^A \to T^{\mathcal{S}}(X^A)$ is precisely a $T^{\mathcal{S}}$-coalgebra as defined in category theory [BW85, p. 100]. This accounts for the name "coalgebra" for the structures being studied here.

We now define the syntactic category of *terms*, which are expressions taking either data elements or states as values. They are therefore classified as *data terms* or *state terms*, with data terms further classified as *input terms* or *observable* (i.e. *output*) terms. Let $\{V_P : P \in In \cup Out\}$ be a $In \cup Out$-indexed collection of sets of *variables*. A *term* is any expression formed by finitely many applications of the following rules. These rules also specify the *type* of a data term, which is the input or output set that provides the values of the term.

- For each $P \in In \cup Out$, each v in V_P is a data term, called a *variable of type* P.
- For each $P \in In \cup Out$, each b in P is a data term, called a *constant of type* P.
- The symbol σ is a state term.
- If m is a method symbol of sort I^m, t_1 is a state term, and t_2 is an input term of type I^m, then $m(t_1, t_2)$ is a state term.
- If a is an attribute symbol of sort (I^a, O^a), t_1 is a state term, and t_2 is an input term of type I^a, then $a(t_1, t_2)$ is an observable term of type O^a.

The symbol σ may be thought of as a parameter denoting the *current state*. If $\bar{v} = (v_1, \ldots, v_n)$ is a tuple of variables, we write $t(\bar{v})$ to indicate that all the variables occurring in term t are among $v_1, \ldots, v_n$. A tuple $\bar{d} = (d_1, \ldots, d_k)$ of data elements *matches* $\bar{v}$ if it has the same length as $\bar{v}$ (i.e. $k = n$) and each v_i has the same type as the corresponding d_i. In that case $\bar{d}$ serves as a valuation, assigning values to the v_i's. A term takes a *value* once values have been given to its variables and a state has been specified as the denotation of σ. The symbol

$$t^A[x, \bar{d}]$$

denotes the *value of term* $t(\bar{v})$ *in coalgebra* A *at state* $x \in X^A$ *under the assignment of* $\bar{d}$ *to* $\bar{v}$. This is defined by induction on the length (or formation) of t as follows

$$\begin{aligned}
v_i^A[x, \bar{d}] &= d_i \\
b^A[x, \bar{d}] &= b \\
\sigma^A[x, \bar{d}] &= x \\
m(t_1, t_2)^A[x, \bar{d}] &= m^A(t_1^A[x, \bar{d}], t_2^A[x, \bar{d}]) \\
a(t_1, t_2)^A[x, \bar{d}] &= a^A(t_1^A[x, \bar{d}], t_2^A[x, \bar{d}]).
\end{aligned}$$

A *ground* term is one that has no variables, and so is constructed from amongst constants and the symbol σ by method and attribute symbols. Ground terms take values $t^A[x]$ that depend only on assignment of a state x to σ. We write $t(\bar{d})$ for the *ground* term obtained by replacing each occurrence of v_i in t by the constant d_i. It is readily seen that the value $t^A[x, \bar{d}]$ is the same as the value of $t(\bar{d})$ at x, i.e. $t^A[x, \bar{d}] = t(\bar{d})^A[x]$. Thus discussion of the values of terms can in principle be reduced to discussion of the values of ground terms. The set GOT of *ground observable terms* will play a major role in what follows.

An *observable equation* is an expression of the form $t_1 \approx t_2$, where t_1 and t_2 are observable terms. An *observable formula* is an expression built from observable equations by the logical connectives $\neg$ (negation) and $\wedge$ (conjunction). These are the only kinds of equations and formulas for coalgebras that will be used here (in [Gol99] certain equations between state terms were also considered).

The other standard Boolean connectives $\rightarrow$, $\vee$, $\leftrightarrow$ may be defined from $\neg$ and $\wedge$ in the usual way. A *ground* formula is one without any variables. The set of ground observable formulas will be denoted GOF, while the set of ground observable equations is GOE.

The notation

$$A, x \models \varphi[\bar{d}]$$

means that *formula* φ *is satisfied in coalgebra* A *by state* x *under assignment* $\bar{d}$, and is defined inductively by

$$\begin{array}{lll} A, x \models t_1 \approx t_2[\bar{d}] & \text{iff} & t_1^A[x, \bar{d}] = t_2^A[x, \bar{d}] \\ A, x \models \neg\varphi[\bar{d}] & \text{iff} & \text{not } A, x \models \varphi[\bar{d}] \\ A, x \models \varphi_1 \wedge \varphi_2[\bar{d}] & \text{iff} & A, x \models \varphi_1[\bar{d}] \text{ and } A, x \models \varphi_2[\bar{d}]. \end{array}$$

We write $\varphi(\bar{v})$ to indicate that the variables occurring in φ are amongst those of $\bar{v}$, and use $\varphi(\bar{d})$ to denote the ground formula obtained from φ by substituting d_i for v_i. Then $\varphi(\bar{v})$ is *satisfied by* x, written $A, x \models \varphi$, if $A, x \models \varphi[\bar{d}]$ for all $\bar{d}$ matching $\bar{v}$. In general we have

$$A, x \models \varphi[\bar{d}] \quad \text{iff} \quad A, x \models \varphi(\bar{d}),$$

and so satisfaction of φ by x reduces to satisfaction of the *set* of ground formulas

$$\{\varphi(\bar{d}) : \bar{d} \text{ matches } \bar{v}\}.$$

This helps to account for the emphasis on ground formulas in our work.

Coalgebra A is a *model* of formula φ, written $A \models \varphi$, if $A, x \models \varphi$ for all states $x \in X^A$. We may also say that A *models* φ, or that φ is *valid in* A, when this occurs. We write $Mod\,\varphi$ for the class of all models of φ, and

$$Mod\,\Phi = \{A : A \models \varphi \text{ for all } \varphi \in \Phi\}$$

for the class of all models of a set Φ of formulas.

Application of a method m^A relative to an input parameter b causes a state transition $x \mapsto m^A(x,b)$ that may alter the values of terms and the satisfaction of formulas. These changes can be expressed logically. To explain this, we write $t(u)$ for the term obtained by replacing every occurrence of σ in term t by a *state* term u. It was shown in [Gol99, Corollary 2.2] that the value of t at state $m^A(x,b)$ is the same as the value at x of the term $t(m(\sigma,b))$ obtained by substituting $m(\sigma,b)$ for σ in t, i.e.

$$t^A[m^A(x,b),\bar{d}\,] = t(m(\sigma,b))^A[x,\bar{d}\,].$$

Similarly, for formulas the notation $\varphi(u)$ is used for the formula obtained by substituting state term u for σ in formula φ. Corollary 4.4 of [Gol99] showed that

$$A, m^A(x,b) \models \varphi \quad \text{iff} \quad A, x \models \varphi(m(\sigma,b)). \tag{2.i}$$

State transitions give rise to operators on subsets of X^A. For each $m \in \mathit{Meth}$ and each $b \in I^m$, define the function $[m.b]^A : \mathcal{P}(X^A) \to \mathcal{P}(X^A)$, where $\mathcal{P}$ denotes powerset, by putting, for each $Y \subseteq X^A$,

$$[m.b]^A(Y) = \{x \in X^A : m^A(x,b) \in Y\}. \tag{2.ii}$$

The operator $[m.b]^A$ preserves the Boolean set operations, i.e.

$$\begin{aligned}
[m.b]^A(-Y) &= -[m.b]^A(Y)\\
[m.b]^A(Y_1 \cap Y_2) &= [m.b]^A(Y_1) \cap [m.b]^A(Y_2)\\
[m.b]^A(Y_1 \cup Y_2) &= [m.b]^A(Y_1) \cup [m.b]^A(Y_2)\\
[m.b]^A(X^A) &= X^A\\
[m.b]^A(\emptyset) &= \emptyset.
\end{aligned}$$

Algebraic structures that are dual to coalgebras will be defined in the next section by abstracting these properties of the operators $[m.b]^A$.

In any coalgebra A, each formula φ defines the set φ^A of all states that satisfy φ:

$$\varphi^A = \{x \in X^A : A, x \models \varphi\}.$$

Relationships between sets defined by ground formulas can be expressed by the standard Boolean set operations:

$$(\neg\varphi)^A = X^A - \varphi^A \tag{2.iii}$$

$$(\varphi_1 \wedge \varphi_2)^A = \varphi_1^A \cap \varphi_2^A \tag{2.iv}$$

(in fact the second equation holds even if φ_1, φ_2 have variables). Result (2.i) asserts that

$$m^A(x,b) \in \varphi^A \text{ iff } x \in \varphi(m(\sigma,b))^A,$$

which implies that

$$[m.b]^A(\varphi^A) = \varphi(m(\sigma, b))^A. \tag{2.v}$$

Thus (2.iii)–(2.v) show that the class

$$\mathcal{D}(X^A) = \{\varphi^A : \varphi \in GOF\} \tag{2.vi}$$

of subsets of X^A definable by ground formulas is closed under the Boolean set operations and under the operators $[m.b]^A$.

We will need to use certain facts about the consequences of substituting one term for another. The following formulation meets our requirements, and is essentially a version of routine results from the semantics of predicate logic.

Theorem 2.1 *Let $A, x \models t_1 \approx t_2$, where $t_1, t_2 \in GOT$.*

(1) *If u is any ground term, and u' is obtained from u by replacing some occurrence(s) of t_1 by t_2, then $u^A[x] = u'^A[x]$.*

(2) *If $A, x \models \varepsilon$, where $\varepsilon \in GOE$, and ε' is obtained from ε by replacing some occurrence(s) of t_1 by t_2, then $A, x \models \varepsilon'$.*

Proof. We have $t_1^A[x] = t_2{}^A[x]$.

(1) This is a straightforward induction on the length of u. If u is a constant or the symbol σ, then either $u = t_1$, in which case u' is t_2, so $u^A[x] = u'^A[x]$ by hypothesis, or else $u \neq t_1$, in which case there is no occurrence of t_1 in u, so $u = u'$.

If u is $m(u_1, u_2)$, and the result is assumed to hold for u_1 and u_2, then u' has the form $m(u_1', u_2')$ with u_i' (for $i = 1, 2$) obtained from u_i by replacing some occurrence(s) of t_1 by t_2. The induction hypothesis gives $u_i^A[x] = u_i'^A[x]$, which implies

$$u^A[x] = m^A(u_1^A[x], u_2^A[x]) = m^A(u_1'^A[x], u_2'^A[x]) = u'^A[x].$$

The inductive case for $u = a(u_1, u_2)$ is similar.

(2) Let ε be $u_1 \approx u_2$. Then ε' has the form $u_1' \approx u_2'$ with u_i' obtained from u_i by replacing some occurrence(s) of t_1 by t_2. Hence by (1), $u_1^A[x] = u_1'^A[x]$ and $u_2^A[x] = u_2'^A[x]$. Thus if $A, x \models \varepsilon$, then $u_1^A[x] = u_2{}^A[x]$, and so $u_1'^A[x] = u_2'^A[x]$, i.e. $A, x \models \varepsilon'$.

□

3 $\mathcal{S}$-Algebras

This section introduces $\mathcal{S}$-algebras, a class of Boolean algebras with operators whose equational logic is closely connected with the logic of observable formulas for coalgebras. These $\mathcal{S}$-algebras enable results of general algebra, such as Birkhoff's variety theorem, to be directly applied to give results about coalgebra.

Definition 3.1 *For a given signature $\mathcal{S}$, an $\mathcal{S}$-algebra is a structure*

$$\mathbf{R} = (R, \cdot^{\mathbf{R}}, -^{\mathbf{R}}, 1^{\mathbf{R}}, \{[m.b]^{\mathbf{R}} : m \in Meth \ \& \ b \in I^m\}, \{\varepsilon^{\mathbf{R}} : \varepsilon \in GOE\})$$

meeting the following specification.

(i) $(R, \cdot^{\mathbf{R}}, -^{\mathbf{R}}, 1^{\mathbf{R}})$ *is a Boolean algebra, in which $r \cdot^{\mathbf{R}} s$ is the Boolean product of elements r and s; $-^{\mathbf{R}} r$ is the complement of r; and $1^{\mathbf{R}}$ is the greatest element of R in the partial ordering $\leq^{\mathbf{R}}$ given by $r \leq^{\mathbf{R}} s$ iff $r \cdot^{\mathbf{R}} s = r$.*

The superscript $\mathbf{R}$ will usually be suppressed in the notation for Boolean operations. $\mathbf{R}$ also has the least element $0 = -1$ and the Boolean sum operation $r + s = -(-r \cdot -s)$.

(ii) $[m.b]^{\mathbf{R}}$ *is an endomorphism of the Boolean algebra of* (i)*. In other words,* $[m.b]^{\mathbf{R}} : R \to R$ *satisfies the equations*

$$\begin{aligned}
[m.b]^{\mathbf{R}} - r &= -[m.b]^{\mathbf{R}} r \\
[m.b]^{\mathbf{R}}(r \cdot s) &= [m.b]^{\mathbf{R}} r \cdot [m.b]^{\mathbf{R}} s \\
[m.b]^{\mathbf{R}}(r + s) &= [m.b]^{\mathbf{R}} r + [m.b]^{\mathbf{R}} s \\
[m.b]^{\mathbf{R}} 1 &= 1 \\
[m.b]^{\mathbf{R}} 0 &= 0.
\end{aligned}$$

(The first two of these equations imply the others.)

(iii) *For each ground observable equation ε, $\varepsilon^{\mathbf{R}}$ is an element of R. These distinguished elements satisfy the following equational conditions.*

(a) $(t \approx t)^{\mathbf{R}} = 1$

(b) $(b \approx c)^{\mathbf{R}} = 0$ *if $b, c \in Data$ and $b \neq c$.*

(c) $[m.b]^{\mathbf{R}} \varepsilon^{\mathbf{R}} = \varepsilon(m(\sigma, b))^{\mathbf{R}}$

(d) $(t_1 \approx t_2)^{\mathbf{R}} \cdot \varepsilon^{\mathbf{R}} \leq \varepsilon'^{\mathbf{R}}$ *whenever ε' is obtained from ε by replacing some occurrence(s) of t_1 by t_2.*

□

A structure satisfying this definition is an "algebra" in the sense of Universal Algebra [BS81, Grä68], in that it comprises a non-empty set that carries a collection of finitary operations (distinguished elements are regarded as nullary operations). Moreover the specification is given entirely in terms of satisfaction of *equations*. Thus the class of $\mathcal{S}$-algebras is closed under subalgebras, homomorphic images, and direct products.

Theorem 3.2 *In any $\mathcal{S}$-algebra* $\mathbf{R}$*:*

(1) $(b \approx c)^{\mathbf{R}} = \begin{cases} 1 & \text{if } b = c \\ 0 & \text{if } b \neq c \end{cases}, \qquad$ *for all* $b, c \in Data$.

(2) $(t_1 \approx t_2)^{\mathbf{R}} = (t_2 \approx t_1)^{\mathbf{R}}$.

(3) $(t_1 \approx t_2)^{\mathbf{R}} \cdot (t_2 \approx t_3)^{\mathbf{R}} \leq (t_1 \approx t_3)^{\mathbf{R}}$.

Proof.

(1) If $b = c$, then $(b \approx c)^{\mathbf{R}} = 1$ by axiom (a) of Definition 3.1(iii). The case of $b \neq c$ is axiom (b).

(2) As a direct instance of axiom (d) of 3.1(iii),

$$(t_1 \approx t_2)^{\mathbf{R}} \cdot (t_1 \approx t_1)^{\mathbf{R}} \leq (t_2 \approx t_1)^{\mathbf{R}}.$$

But $(t_1 \approx t_1)^{\mathbf{R}} = 1$ by axiom (a), so this gives (2).

(3) Axiom 3.1(iii)(d), with the roles of t_1 and t_2 interchanged, gives

$$(t_2 \approx t_1)^{\mathbf{R}} \cdot (t_2 \approx t_3)^{\mathbf{R}} \leq (t_1 \approx t_3)^{\mathbf{R}}.$$

But $(t_2 \approx t_1)^{\mathbf{R}} = (t_1 \approx t_2)^{\mathbf{R}}$ by (2).

□

Associated naturally with any $\mathcal{S}$-coalgebra A are two $\mathcal{S}$-algebras. First there is the *powerset algebra* $\mathcal{P}A$ based on the Boolean algebra $\mathcal{P}(X^A)$ of all subsets of the state set of A, with the usual Boolean set operations. $[m.b]^{\mathcal{P}A}$ is defined to be the operator $[m.b]^A$ of (2.ii), which was noted there to be a Boolean endomorphism of $\mathcal{P}(X^A)$. For each observable equation $(t_1 \approx t_2) \in GOE$ we put

$$(t_1 \approx t_2)^{\mathcal{P}A} = (t_1 \approx t_2)^A = \{x \in X^A : t_1[x] = t_2[x]\}.$$

It is immediate that axioms (a) and (b) of Definition 3.1(iii) are satisfied by $\mathcal{P}A$. Axiom (c) is satisfied by result (2.v), and axiom (d) by Theorem 2.1(2).

The second $\mathcal{S}$-algebra will be denoted $\mathcal{D}A$, and is the *algebra of definable subsets of* A. Its carrier is the collection $\mathcal{D}(X^A) = \{\varphi^A : \varphi \in GOF\}$ introduced in (2.vi), where it was noted that $\mathcal{D}(X^A)$ is closed under the Boolean set operations and the operators $[m.b]^{\mathcal{P}A}$. But $\mathcal{D}(X^A)$ also contains the elements $\varepsilon^{\mathcal{P}A}$ for $\varepsilon \in GOE$, by definition. Thus $\mathcal{D}A$ can be defined as the subalgebra of $\mathcal{P}A$ based on $\mathcal{D}(X^A)$.

In fact $\mathcal{D}A$ is the *minimal* subalgebra of $\mathcal{P}A$, since any subalgebra of $\mathcal{P}A$ must include $\mathcal{D}A$. This is because any subalgebra of $\mathcal{P}A$ contains the distinguished elements ε^A for all $\varepsilon \in GOE$ and is closed under set complements and intersections, so contains the elements φ^A for all $\varphi \in GOF$ by (2.iii) and (2.iv). More generally, each $\mathcal{S}$-algebra $\mathbf{R}$ has a minimal subalgebra, which will be denoted $\mathcal{D}\mathbf{R}$, with $\mathcal{D}\mathcal{P}A$ just being $\mathcal{D}A$. Since $\mathbf{R}$ has elements $\varepsilon^{\mathbf{R}}$ for all $\varepsilon \in GOE$

and is closed under Boolean complements and products, we can inductively define an element $\varphi^{\mathbf{R}}$ of R for all $\varphi \in GOF$ by putting

$$(\neg\varphi)^{\mathbf{R}} = -\varphi^{\mathbf{R}}, \tag{3.i}$$

$$(\varphi_1 \wedge \varphi_2)^{\mathbf{R}} = \varphi_1^{\mathbf{R}} \cdot \varphi_2^{\mathbf{R}}. \tag{3.ii}$$

Then $\mathcal{D}\mathbf{R}$ is the subalgebra of $\mathbf{R}$ with carrier $\mathcal{D}R = \{\varphi^{\mathbf{R}} : \varphi \in GOF\}$. The equations just given show that $\mathcal{D}R$ is closed under the Boolean operations of $\mathbf{R}$. The fact that it is closed under the operators $[m.b]^{\mathbf{R}}$ follows from the next result.

Lemma 3.3 $[m.b]^{\mathbf{R}}\varphi^{\mathbf{R}} = \varphi(m(\sigma, b))^{\mathbf{R}}$ *for all* $\varphi \in GOF$*.*

Proof. By induction on the formation of φ. The result holds when $\varphi \in GOE$ by axiom (c) of Definition 3.1(iii). If it holds for a particular φ, then

$$\begin{aligned}
[m.b]^{\mathbf{R}}(\neg\varphi)^{\mathbf{R}} &= [m.b]^{\mathbf{R}} - \varphi^{\mathbf{R}} && \text{by (3.i)}\\
&= -[m.b]^{\mathbf{R}}\varphi^{\mathbf{R}} && \text{as } [m.b]^{\mathbf{R}} \text{ preserves } -\\
&= -\varphi(m(\sigma, b))^{\mathbf{R}} && \text{by hypothesis on } \varphi\\
&= \neg(\varphi(m(\sigma, b)))^{\mathbf{R}} && \text{by (3.i)}\\
&= (\neg\varphi)(m(\sigma, b))^{\mathbf{R}},
\end{aligned}$$

so it holds for $\neg\varphi$. Similarly, if it holds for φ_1 and φ_2 then it holds for $\varphi_1 \wedge \varphi_2$ by (3.ii) and the fact that $[m.b]^{\mathbf{R}}$ preserves Boolean products. □

We can now use $\mathcal{S}$-algebras to give another semantics for ground observable formulas, stipulating that $\mathbf{R}$ is a *model of* φ, and that φ is *valid in* $\mathbf{R}$, if $\varphi^{\mathbf{R}} = 1$. We write $\mathbf{R} \models \varphi$ when this holds. Since $\varphi^{\mathcal{D}\mathbf{R}} = \varphi^{\mathbf{R}}$, it follows that $\mathcal{D}\mathbf{R} \models \varphi$ iff $\mathbf{R} \models \varphi$.

In the case that $\mathbf{R}$ is the powerset algebra $\mathcal{P}A$ of coalgebra A, the algebraic and coalgebraic semantics coincide. Since $A \models \varphi$ iff $\varphi^A = X^A$, and X^A is the greatest element of $\mathcal{P}A$, we find that

$$A \models \varphi \quad \text{iff} \quad \mathcal{P}A \models \varphi \quad \text{iff} \quad \mathcal{D}A \models \varphi. \tag{3.iii}$$

In effect each formula $\varphi \in GOF$ acts as a *ground term* denoting a fixed element $\varphi^{\mathbf{R}}$ of any $\mathcal{S}$-algebra $\mathbf{R}$. In order to fully discuss terms and equations for $\mathbf{R}$ we require a new set Var of *algebraic variables* that can be assigned values in R. The set of *algebraic terms* can then be generated in the usual way [BS81, p. 62] from members of Var, members of GOE as constant algebraic terms, as well as the constant 1, using the operation symbols $-$, $\cdot$, and $[m.b]$ for all $m \in Meth$ and $b \in I^m$.

The notation $\alpha(p_1, \ldots, p_n)$ signifies that α is an algebraic term whose variables are all amongst the list $p_1, \ldots, p_n$ of members of Var. Such a term induces an operation $\alpha^{\mathbf{R}} : R^n \to R$ on $\mathbf{R}$ that assigns to any $\bar{r} = (r_1, \ldots, r_n) \in R^n$ the value $\alpha^{\mathbf{R}}[\bar{r}]$ of α in $\mathbf{R}$ under the assignment of r_i to p_i for all $i \leq n$ [BS81,

p. 63]. An *algebraic equation* $\alpha_1 \approx \alpha_2$ is satisfied by $\mathbf{R}$ iff the term operations $\alpha_1^{\mathbf{R}}$ and $\alpha_2^{\mathbf{R}}$ are identical, i.e.

$$\mathbf{R} \models \alpha_1 \approx \alpha_2 \quad \text{iff} \quad \text{for all } \bar{r} \in R^n,\ \alpha_1^{\mathbf{R}}[\bar{r}\,] = \alpha_2^{\mathbf{R}}[\bar{r}\,].$$

Our plan is to translate the equational logic of $\mathcal{S}$-algebras into the logic of observable formulas in coalgebras. The Boolean operations allow a helpful simplification here: any algebraic equation is equivalent to one of the form $\alpha \approx 1$. Given algebraic terms α_1, α_2, let α be the term $-(\alpha_1 \cdot -\alpha_2) \cdot -(\alpha_2 \cdot -\alpha_1)$. Then $\alpha_1^{\mathbf{R}}[\bar{r}\,] = \alpha_2^{\mathbf{R}}[\bar{r}\,]$ iff $\alpha^{\mathbf{R}}[\bar{r}\,] = 1$, so

$$\mathbf{R} \models \alpha_1 \approx \alpha_2 \quad \text{iff} \quad \mathbf{R} \models \alpha \approx 1. \tag{3.iv}$$

An algebraic term $\alpha(p_1, \dots, p_n)$ can be translated into a ground observable formula by replacing each variable p_i by some $\varphi_i \in GOF$, replacing $-$ and $\cdot$ by $\neg$ and $\wedge$, and eliminating $[m.b]$ by substituting $m(\sigma, b)$ for σ as suggested by Lemma 3.3. The result is a formula $\alpha(\bar{\varphi}) \in GOF$, where $\bar{\varphi} = (\varphi_1, \dots, \varphi_n)$. This translation is spelled out by induction on the formation of α in the following table.

$\alpha(p_1, \dots, p_n)$	$\alpha(\bar{\varphi}) \in GOF$
p_i	φ_i
$-\alpha_1$	$\neg(\alpha_1(\bar{\varphi}))$
$\alpha_1 \cdot \alpha_2$	$\alpha_1(\bar{\varphi}) \wedge \alpha_2(\bar{\varphi})$
$[m.b]\alpha_1$	$(\alpha_1(\bar{\varphi}))(m(\sigma, b))$

Theorem 3.4 *In any $\mathcal{S}$-algebra $\mathbf{R}$, $\alpha(\bar{\varphi})^{\mathbf{R}} = \alpha^{\mathbf{R}}[\varphi_1^{\mathbf{R}}, \dots, \varphi_n^{\mathbf{R}}]$.*

Proof. Since $p_i^{\mathbf{R}}[\varphi_1^{\mathbf{R}}, \dots, \varphi_n^{\mathbf{R}}] = \varphi_i^{\mathbf{R}} = p_i(\bar{\varphi})^{\mathbf{R}}$, the result holds whenever α is an algebraic variable p_i.

If α is the term $-\alpha_1$ and the result holds for α_1, then

$$\begin{aligned}
&(-\alpha_1)^{\mathbf{R}}[\varphi_1^{\mathbf{R}}, \dots, \varphi_n^{\mathbf{R}}] & \\
&= -^{\mathbf{R}}(\alpha_1^{\mathbf{R}}[\varphi_1^{\mathbf{R}}, \dots, \varphi_n^{\mathbf{R}}]) & &\text{by definition of } (-\alpha_1)^{\mathbf{R}} \\
&= -^{\mathbf{R}}(\alpha_1(\bar{\varphi})^{\mathbf{R}}) & &\text{by hypothesis on } \alpha_1 \\
&= \neg(\alpha_1(\bar{\varphi}))^{\mathbf{R}} & &\text{by (3.i)} \\
&= (-\alpha_1)(\bar{\varphi})^{\mathbf{R}} & &\text{from the above table,}
\end{aligned}$$

and so the result holds for α in this case. The inductive case of α being the term $\alpha_1 \cdot \alpha_2$ is similarly straightforward.

Now suppose α is the term $[m.b]\alpha_1$ and the result holds for α_1. Then

$$
\begin{aligned}
&([m.b]\alpha_1)^{\mathbf{R}}[\varphi_1^{\mathbf{R}}, \dots, \varphi_n^{\mathbf{R}}] \\
&= [m.b]^{\mathbf{R}}(\alpha_1^{\mathbf{R}}[\varphi_1^{\mathbf{R}}, \dots, \varphi_n^{\mathbf{R}}]) && \text{by definition of } ([m.b]\alpha_1)^{\mathbf{R}} \\
&= [m.b]^{\mathbf{R}}(\alpha_1(\bar{\varphi})^{\mathbf{R}}) && \text{by hypothesis on } \alpha_1 \\
&= (\alpha_1(\bar{\varphi}))(m(\sigma, b))^{\mathbf{R}} && \text{by Lemma 3.3} \\
&= ([m.b]\alpha_1)(\bar{\varphi})^{\mathbf{R}} && \text{from the above table,}
\end{aligned}
$$

and again the result holds for α. Thus it holds for all algebraic terms α, by induction. □

Theorem 3.4 has an important consequence for the description of satisfaction of algebraic equations in minimal algebras $\mathcal{D}\mathbf{R}$, where every element is of the form $\varphi^{\mathbf{R}}$. To explain this, assign to each algebraic term $\alpha(p_1, \dots, p_n)$, the set

$$\Phi_\alpha = \{\alpha(\bar{\varphi}) : \bar{\varphi} \in GOF^n\} \tag{3.v}$$

of all ground observable formulas of the form $\alpha(\bar{\varphi})$.

Theorem 3.5

(1) $\mathcal{D}\mathbf{R} \models \alpha \approx 1$ *if and only if* $\mathcal{D}\mathbf{R} \models \Phi_\alpha$.

(2) *For any coalgebra A,*

$$\mathcal{D}A \models \alpha \approx 1 \textit{ if and only if } A \models \Phi_\alpha.$$

Proof.

$$
\begin{aligned}
(1) \quad & \mathcal{D}\mathbf{R} \models \alpha \approx 1 \\
&\text{iff } \alpha^{\mathcal{D}\mathbf{R}}[r_1, \dots, r_n] = 1 && \text{for all } r_1, \dots, r_n \in \mathcal{D}R \\
&\text{iff } \alpha^{\mathcal{D}\mathbf{R}}[\varphi_1^{\mathcal{D}\mathbf{R}}, \dots, \varphi_n^{\mathcal{D}\mathbf{R}}] = 1 && \text{for all } \varphi_1, \dots, \varphi_n \in GOF \\
&\text{iff } \alpha(\bar{\varphi})^{\mathcal{D}\mathbf{R}} = 1 && \text{for all } \bar{\varphi} \in GOF^n \text{, by Theorem 3.4} \\
&\text{iff } \mathcal{D}\mathbf{R} \models \alpha(\bar{\varphi}) && \text{for all } \bar{\varphi} \in GOF^n \\
&\text{iff } \mathcal{D}\mathbf{R} \models \Phi_\alpha.
\end{aligned}
$$

(2) By (1), $\mathcal{D}A \models \alpha \approx 1$ iff $\mathcal{D}A \models \Phi_\alpha$. But by (3.iii), $\mathcal{D}A \models \Phi_\alpha$ iff $A \models \Phi_\alpha$. □

4 The Coalgebra of an Algebra

Each coalgebra A gives rise to the $\mathcal{S}$-algebra $\mathcal{P}A$ (and its subalgebra $\mathcal{D}A$). In the converse direction we seek to construct a coalgebra $\mathsf{co}\mathbf{R}$ out of a given $\mathcal{S}$-algebra $\mathbf{R}$ by taking certain ultrafilters of $\mathbf{R}$ as states for $\mathsf{co}\mathbf{R}$. The motivating idea is that each state x of coalgebra A can be identified with the collection $\{Y \subseteq X^A : x \in Y\}$, which is an ultrafilter of $\mathcal{P}A$.

Recall that a *filter* of $\mathbf{R}$ is a non-empty set $F \subseteq R$ that contains a product $r \cdot s$ iff it contains both r and s. A filter is increasing: if $r \in F$ and $r \leq s$ then $s \in F$. Every filter contains 1. F is called *proper* if $0 \notin F$.

An ultrafilter is a filter that is maximally proper, or equivalently that contains exactly one of r and $-r$ for all $r \in R$. If an ultrafilter contains a Boolean sum $r + s$, then it contains either r or s.

The following standard result from the theory of Boolean algebras will be needed here.

Lemma 4.1 *If $\theta : \mathbf{R} \to \mathbf{S}$ is a homomorphism of Boolean algebras, and F is an ultrafilter of $\mathbf{S}$, then $\theta^{-1}(F) = \{r \in R : \theta(r) \in F\}$ is an ultrafilter of $\mathbf{R}$.* □

An ultrafilter F of $\mathcal{S}$-algebra $\mathbf{R}$ will be called *observationally rich*, or more briefly just *rich*, if it satisfies the following condition:

> for any ground observable term t, of type O^t, there exists some $b \in O^t$ such that $(t \approx b)^{\mathbf{R}} \in F$.

The data element b corresponding to t in this condition is unique. Using parts (2) and (3) of Theorem 3.2 and properties of F we get that if $(t \approx b)^{\mathbf{R}}$ and $(t \approx c)^{\mathbf{R}}$ belong to F, then $(b \approx t)^{\mathbf{R}} \cdot (t \approx c)^{\mathbf{R}} \in F$ and so $(b \approx c)^{\mathbf{R}} \in F$. Hence $(b \approx c)^{\mathbf{R}} \neq 0$ as F is proper. Therefore $b = c$ by part (1) of 3.2.

The coalgebra $\mathsf{co}\mathbf{R} = (X^{\mathsf{co}\mathbf{R}}, \{m^{\mathsf{co}\mathbf{R}} : m \in Meth\}, \{a^{\mathsf{co}\mathbf{R}} : a \in Att\})$ is defined as follows.

(i) $X^{\mathsf{co}\mathbf{R}}$ is the set of all rich ultrafilters of $\mathbf{R}$.

(ii) For all $F \in X^{\mathbf{R}}$ and $c \in I^a$, $a^{\mathsf{co}\mathbf{R}}(F, c) = b$ iff $(a(\sigma, c) \approx b)^{\mathbf{R}} \in F$.

Since $a(\sigma, c)$ is a ground observable term, the required data element $b \in O^a$ in this condition exists by the definition of "rich", and is unique as explained above.

(iii) For all $F \in X^{\mathbf{R}}$ and $b \in I^m$,

$$\begin{aligned} m^{\mathsf{co}\mathbf{R}}(F, b) &= \{r \in R : [m.b]^{\mathbf{R}} r \in F\} \\ &= ([m.b]^{\mathbf{R}})^{-1}(F). \end{aligned}$$

Since $[m.b]^{\mathbf{R}} : \mathbf{R} \to \mathbf{R}$ is a Boolean homomorphism, Lemma 4.1 guarantees that $m^{\mathsf{co}\mathbf{R}}(F, b)$ is an ultrafilter. To show it is rich, take any ground observable term t of type O^t. Then $t(m(\sigma, b))$ is also a ground observable term of type O^t, so as F is rich it follows that there is some $c \in O^t$ such that $(t(m(\sigma, b)) \approx c)^{\mathbf{R}} \in F$. But

$$(t(m(\sigma, b)) \approx c)^{\mathbf{R}} = [m.b]^{\mathbf{R}}(t \approx c),$$

by axiom (c) of 3.1(iii), so this implies that $(t \approx c)^{\mathbf{R}} \in m^{\mathsf{co}\mathbf{R}}(F, b)$, establishing that $m^{\mathsf{co}\mathbf{R}}(F, b)$ belongs to $X^{\mathbf{R}}$.

This completes the construction of $\mathsf{co}\mathbf{R}$. In the case that $\mathbf{R} = \mathcal{P}A$ for a coalgebra A, $\mathsf{co}\mathcal{P}A$ is precisely the *ultrafilter enlargement of* A defined in [Gol99], where it was denoted A^*. It contains, for each $x \in X^A$, the principal ultrafilter $F_x = \{Y \subseteq X^A : x \in Y\}$ of $\mathcal{P}A$, which is readily seen to be rich: if $t \in GOT$ and $b = t^A[x]$, then $(t \approx b)^{\mathcal{P}A} \in F_x$.

There is one vital feature missing from this discussion: it has not been shown that in general $X^{\mathbf{R}} \neq \emptyset$. But an $\mathcal{S}$-algebra may not have any rich ultrafilters at all (see below), in which case $\mathsf{co}\mathbf{R}$ does not exist. We therefore define $\mathbf{R}$ itself to be *rich* if it has at least one rich ultrafilter. It has just been noted that $\mathcal{P}A$ is always rich, and the same applies to $\mathcal{D}A$: if x is any state of A then $F_x^{\mathcal{D}A} = \{\varphi^A : A, x \models \varphi\}$ is a rich ultrafilter of $\mathcal{D}A$. Lemma 4.2 gives a further handle on the existence of rich $\mathcal{S}$-algebras, since it implies that if $\mathbf{S}$ is rich and there exists an $\mathcal{S}$-algebraic homomorphism from $\mathbf{R}$ to $\mathbf{S}$, then $\mathbf{R}$ is also rich.

Lemma 4.2 *If $\theta : \mathbf{R} \to \mathbf{S}$ is a Boolean homomorphism between two $\mathcal{S}$-algebras that preserves all distinguished elements $\varepsilon^{\mathbf{R}}, \varepsilon^{\mathbf{S}}$, and F is a rich ultrafilter of $\mathbf{S}$, then $\theta^{-1}(F)$ is a rich ultrafilter of $\mathbf{R}$.*

Proof. By Lemma 4.1, $\theta^{-1}(F)$ is an ultrafilter of $\mathbf{R}$. To show it is rich, take any term $t \in GOT$ of type O^t. Then as F is rich there is some $b \in O^t$ such that $(t \approx b)^{\mathbf{S}} \in F$. But $(t \approx b)^{\mathbf{S}} = \theta((t \approx b)^{\mathbf{R}})$ by hypothesis, so $(t \approx b)^{\mathbf{R}} \in \theta^{-1}(F)$. □

The class of all rich $\mathcal{S}$-algebras with $\mathcal{S}$-algebraic homomorphisms between them forms the category $\mathcal{S}$-**Alg**. In the next section a duality will be constructed between $\mathcal{S}$-**Alg** and a category of coalgebras.

Here now is an example of an $\mathcal{S}$-algebra that is not rich, based on a well known quotient of the Boolean algebra $\mathcal{P}(\omega)$, where $\omega = \{0, 1, 2, \dots\}$ is the set of natural numbers. The signature $\mathcal{S}$ involved has a single method symbol m and a single attribute symbol a, both of which have no input sort. The output sort of a is ω. Thus in any coalgebra, m^A is a function of the type $X^A \to X^A$, while a^A is of type $X^A \to \omega$.

Now let A be a coalgebra for this signature having state set $X^A = \omega$, with m^A and a^A being any two *injective* functions from ω to ω. Define an equivalence relation Q between subsets Y, Z of ω by putting YQZ iff the symmetric difference $Y \oplus Z = (Y - Z) \cup (Z - Y)$ is finite. It is a standard fact that Q is a congruence relation on the powerset Boolean algebra $\mathcal{P}(\omega)$, and that the quotient Boolean algebra $\mathcal{P}(\omega)/Q$ has no atoms. But if YQZ then $[m]^A(Y)Q[m]^A(Z)$, so Q is also a congruence for the operator $[m]^A : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$, and hence is a congruence of the $\mathcal{S}$-algebra $\mathcal{P}A$. This follows because $[m]^A$ is a Boolean endomorphism of $\mathcal{P}(\omega)$, and so

$$[m]^A(Y) \oplus [m]^A(Z) = [m]^A(Y \oplus Z) = \{x \in \omega : m^A(x) \in Y \oplus Z\},$$

hence if $Y \oplus Z$ is finite then $[m]^A(Y) \oplus [m]^A(Z)$ is finite too by the pigeonhole principle, because m^A is injective.

Let $\mathbf{R} = \mathcal{P}A/Q$ be the quotient algebra of $\mathcal{P}A$ determined by the congruence Q and based on the Boolean algebra $\mathcal{P}(\omega)/Q$. Then $\mathbf{R}$ is a homomorphic image of $\mathcal{P}A$, and so is an $\mathcal{S}$-algebra because the notion of $\mathcal{S}$-algebra is equationally specified. $\mathbf{R}$ is not rich, as may be seen by taking t to be the ground observable term $a(\sigma)$. For any $b \in O^{a(\sigma)} = \omega$, the set

$$(a(\sigma) \approx b)^{\mathcal{P}A} = \{x \in \omega : a^A(x) = b\}$$

has at most one element, since a^A is injective, and therefore is Q-equivalent to $\emptyset$. Thus

$$(a(\sigma) \approx b)^{\mathbf{R}} = (a(\sigma) \approx b)^{\mathcal{P}A}/Q = \emptyset/Q = 0^{\mathbf{R}}.$$

Hence if F is any ultrafilter of $\mathbf{R}$ there can be no $b \in O^t$ with $(a(\sigma) \approx b)^{\mathbf{R}} \in F$, and so F is not rich.

The fact that the output set O^a is infinite in this example is no accident. For a signature $\mathcal{S}$ that has only finite output sets it would be appropriate to add to the definition (3.1) of an $\mathcal{S}$-algebra the equation

$$(t \approx b_1)^{\mathbf{R}} + \cdots + (t \approx b_n)^{\mathbf{R}} = 1$$

for each $t \in GOT$, where t is of type $\{b_1, \ldots, b_n\}$. In that case any ultrafilter F of $\mathbf{R}$ will contain $(t \approx b_i)^{\mathbf{R}}$ for some i, and so will be rich.

The representation theory for Boolean algebras with operators [JT51, Gol89] suggests that there should be a homomorphism $\eta^{\mathbf{R}}$ from $\mathbf{R}$ to $\mathcal{P}\mathsf{co}\mathbf{R}$ defined by

$$\eta^{\mathbf{R}}(r) = \{F \in X^{\mathsf{co}\mathbf{R}} : r \in F\}. \tag{4.i}$$

The defining properties of ultrafilters ensure that this map preserves Boolean products and complements. It also preserves the operators $[m.b]^{\mathbf{R}}$ and distinguished elements $\varepsilon^{\mathbf{R}}$, but to prove that will require a substantial further analysis of the nature of $\mathsf{co}\mathbf{R}$ (see below).

However, in contrast to the purely Boolean case, $\eta^{\mathbf{R}}$ need not be injective, since $\mathbf{R}$ may not have enough *rich* ultrafilters to separate elements of R. We can verify this by adapting the non-rich example $\mathbf{R} = \mathcal{P}A/Q$ just discussed to construct an infinite $\mathcal{S}$-algebra $\mathbf{S}$ that has *exactly one* rich ultrafilter. Let B be any one-element coalgebra for the signature of the above example (thus the single state x of B has $m^B(x) = x$, while $a^B(x)$ can be any natural number). We employ the powerset algebra $\mathcal{P}B$, or more conveniently its isomorphic copy $\mathbf{2}$, based on the two element Boolean algebra $\{0, 1\}$, that has $[m.b]^{\mathbf{2}}1 = 1$, $[m.b]^{\mathbf{2}}0 = 0$, and $\varepsilon^{\mathbf{2}} = 1$ iff $B \models \varepsilon$ for all $\varepsilon \in GOE$. The unique ultrafilter $\{1\}$ of $\mathbf{2}$ is rich, since for each $t \in GOT$ we have $(t \approx b)^{\mathbf{2}} = 1$, where $b = t^B[x]$.

Now let $\mathbf{S}$ be the direct product $\mathbf{R} \times \mathbf{2}$, where $\mathbf{R} = \mathcal{P}A/Q$ as above. If $\theta : \mathbf{S} \to \mathbf{2}$ is the projection homomorphism $\mathbf{R} \times \mathbf{2} \twoheadrightarrow \mathbf{2}$, then $F = \theta^{-1}(\{1\})$ is a rich ultrafilter of $\mathbf{S}$, by Lemma 4.2. Here $F = R \times \{1\} = \{s \in S : (0, 1) \leq s\}$, showing that F is the principal ultrafilter of $\mathbf{S}$ generated by the element $(0, 1)$, which is an atom. In fact $(0, 1)$ is the only atom of $\mathbf{S}$, since any other element of the form $(x,1)$ is greater than $(0,1)$ in $\mathbf{S}$, so cannot be an atom, while $\mathbf{R}$ has no atoms at all, so $\mathbf{S}$ has no atoms of the form $(x,0)$.

To see that F is the only rich ultrafilter of $\mathbf{S}$, let G be any member of $X^{\mathrm{co}\mathbf{S}}$. Then there is some $b \in O^{a(\sigma)}$ such that $(a(\sigma) \approx b)^{\mathbf{S}} \in G$. But

$$(a(\sigma) \approx b)^{\mathbf{S}} = ((a(\sigma) \approx b)^{\mathbf{R}}, (a(\sigma) \approx b)^{\mathbf{2}}),$$

and $(a(\sigma) \approx b)^{\mathbf{R}} = 0$ as above, so $(a(\sigma) \approx b)^{\mathbf{2}} = 1$ or else $(a(\sigma) \approx b)^{\mathbf{S}} = (0,0) = 0^{\mathbf{S}} \notin G$. Thus the element $(a(\sigma) \approx b)^{\mathbf{S}}$ of G is the atom $(0,1)$ that generates F, which implies that $G = F$.

It follows that $X^{\mathrm{co}\mathbf{S}} = \{F\}$, hence

$$\eta^{\mathbf{S}}(s) = \begin{cases} \emptyset & \text{if } s \in R \times \{0\}, \\ \{F\} & \text{if } s \in R \times \{1\}, \end{cases}$$

and so $\eta^{\mathbf{S}}$ far from injective.

The $\mathcal{P}\mathsf{co}\mathbf{R}$ construction cannot be used to give a concrete representation of $\mathcal{S}$-algebras, since there can be no embedding $\mathbf{S} \rightarrowtail \mathcal{P}A$ of the algebra $\mathbf{S}$ just discussed into the powerset algebra of *any* coalgebra whatsoever. This is because the image $\mathbf{S}'$ of such an embedding would be a subalgebra of $\mathcal{P}A$ isomorphic to $\mathbf{S}$. But such a subalgebra of $\mathcal{P}A$ would have more than one rich ultrafilter, contradicting the isomorphism with $\mathbf{S}$. To see this, take any element r of $\mathbf{S}'$ that is a non-empty proper subset of X^A: this must exist as $\mathbf{S}$ has non-zero elements other than $1^{\mathbf{S}}$. Then there exist A-states $x_1 \in r$ and $x_2 \notin r$. Put $G_i = \{s \in \mathbf{S}' : x_i \in s\}$ for $i = 1,2$. Then G_1 and G_2 are rich ultrafilters of $\mathbf{S}'$ that are distinct, since $r \in G_1 - G_2$.

We now take up the demonstration that $\eta^{\mathbf{R}}$ is always a homomorphism.

Lemma 4.3 *Let t be any ground term and $F \in X^{\mathrm{co}\mathbf{R}}$.*

(1) *If t is observable of type O^t, then for all $b \in O^t$,*

$$t^{\mathrm{co}\mathbf{R}}[F] = b \quad \textit{iff} \quad (t \approx b)^{\mathbf{R}} \in F.$$

(2) *If t is a state term, then for all $\varepsilon \in GOE$,*

$$\varepsilon^{\mathbf{R}} \in t^{\mathrm{co}\mathbf{R}}[F] \quad \textit{iff} \quad \varepsilon(t)^{\mathbf{R}} \in F.$$

Proof. By induction on the length of t. Recall that in case (2), $\varepsilon(t)$ is the equation obtained by substituting t for σ in ε.

If t is an observable constant c, then $t^{\mathrm{co}\mathbf{R}}[F] = c$, and so by Theorem 3.2(1), if $c = b$ then $(c \approx b)^{\mathbf{R}} = 1 \in F$ and if $c \neq b$ then $(c \approx b)^{\mathbf{R}} = 0 \notin F$. Hence (1) holds for t in this case.

If t is the state term σ, then $t^{\mathrm{co}\mathbf{R}}[F] = F$ and $\varepsilon(t) = \varepsilon$, so (2) is just a tautology: $\varepsilon^{\mathbf{R}} \in F$ iff $\varepsilon^{\mathbf{R}} \in F$.

Now let t be the state term $m(t_1, t_2)$ and make the induction hypothesis that the result holds for the state term t_1 and the input term t_2. Let $b = t_2^{\mathsf{co}\mathbf{R}}[F]$, so that $(t_2 \approx b)^{\mathbf{R}} \in F$ by hypothesis (1) on t_2. Then

$$\begin{array}{ll} \varepsilon^{\mathbf{R}} \in m(t_1, t_2)^{\mathsf{co}\mathbf{R}}[F] & \\ \text{iff } \varepsilon^{\mathbf{R}} \in m^{\mathsf{co}\mathbf{R}}(t_1^{\mathsf{co}\mathbf{R}}[F], b) & \\ \text{iff } [m.b]^{\mathbf{R}}\varepsilon^r \in t_1^{\mathsf{co}\mathbf{R}}[F] & \text{by definition of } m^{\mathsf{co}\mathbf{R}} \\ \text{iff } \varepsilon(m(\sigma, b))^{\mathbf{R}} \in t_1^{\mathsf{co}\mathbf{R}}[F] & \text{by axiom 3.1(iii)(c)} \\ \text{iff } \varepsilon(m(\sigma, b))(t_1)^{\mathbf{R}} \in F & \text{by hypothesis (2) on } t_1 \\ \text{iff } \varepsilon(m(t_1, b))^{\mathbf{R}} \in F. & \end{array}$$

But as instances of axiom 3.1(iii)(d) we have

$$\begin{array}{l} (b \approx t_2)^{\mathbf{R}} \cdot \varepsilon(m(t_1, b))^{\mathbf{R}} \leq \varepsilon(m(t_1, t_2))^{\mathbf{R}} \\ (t_2 \approx b)^{\mathbf{R}} \cdot \varepsilon(m(t_1, t_2))^{\mathbf{R}} \leq \varepsilon(m(t_1, b))^{\mathbf{R}}. \end{array}$$

Since $(b \approx t_2)^{\mathbf{R}} = (t_2 \approx b)^{\mathbf{R}} \in F$ and F is a filter, these allow us to conclude that $\varepsilon(m(t_1, b))^{\mathbf{R}} \in F$ iff $\varepsilon(m(t_1, t_2))^{\mathbf{R}} \in F$. Altogether then we have shown that

$$\varepsilon^{\mathbf{R}} \in m(t_1, t_2)^{\mathsf{co}\mathbf{R}}[F] \quad \text{iff} \quad \varepsilon(m(t_1, t_2))^{\mathbf{R}} \in F,$$

which is result (2) for this case of t.

The final case is where t is the observable term $a(t_1, t_2)$, with the induction hypothesis that the result holds for t_1 and t_2. This time let $c = t_2^{\mathsf{co}\mathbf{R}}[F]$, so that $(t_2 \approx c)^{\mathbf{R}} \in F$ by hypothesis (1) on t_2. Then for any $b \in O^t$,

$$\begin{array}{ll} a(t_1, t_2)^{\mathsf{co}\mathbf{R}}[F] = b & \\ \text{iff } a^{\mathsf{co}\mathbf{R}}(t_1^{\mathsf{co}\mathbf{R}}[F], c) = b & \\ \text{iff } (a(\sigma, c) \approx b)^{\mathbf{R}} \in t_1^{\mathsf{co}\mathbf{R}}[F] & \text{by definition of } a^{\mathsf{co}\mathbf{R}} \\ \text{iff } (a(\sigma, c) \approx b)(t_1)^{\mathbf{R}} \in F & \text{by hypothesis (2) on } t_1 \\ \text{iff } (a(t_1, c) \approx b)^{\mathbf{R}} \in F. & \end{array}$$

Since $(t_2 \approx c)^{\mathbf{R}} \in F$, we can then use instances of axiom 3.1(iii)(d) as in the previous case to prove that $(a(t_1, c) \approx b)^{\mathbf{R}} \in F$ iff $(a(t_1, t_2) \approx b)^{\mathbf{R}} \in F$, showing altogether that result (1) holds for this case of t. □

Theorem 4.4 *For any $\varphi \in GOF$ and $F \in X^{\mathsf{co}\mathbf{R}}$,*

$$\varphi^{\mathbf{R}} \in F \quad \textit{iff} \quad \mathsf{co}\mathbf{R}, F \models \varphi.$$

Proof. Suppose first that φ is an equation $(t_1 \approx t_2)$ between ground observable terms. Let $b_i = t_i^{\mathsf{co}\mathbf{R}}[F]$ for $i = 1, 2$. Then

$$\mathsf{co}\mathbf{R}, F \models (t_1 \approx t_2) \quad \text{iff} \quad b_1 = b_2.$$

But by Lemma 4.3(1),

$$(t_1 \approx b_1)^{\mathbf{R}},\ (t_2 \approx b_2)^{\mathbf{R}} \in F. \tag{4.ii}$$

Thus if $(t_1 \approx t_2)^{\mathbf{R}} \in F$, then applying Theorem 3.2(2) and 3.2(3) leads to $(b_1 \approx b_2)^{\mathbf{R}} \in F$, and hence by 3.2(1), $b_1 = b_2$. Conversely if $b_1 = b_2$, then $(b_1 \approx b_2)^{\mathbf{R}} = 1 \in F$, leading via (4.ii) to $(t_1 \approx t_2)^{\mathbf{R}} \in F$.

This shows that the Theorem holds whenever $\varphi \in GOE$. If $\varphi = \neg\varphi_1$ and the result holds for φ_1, then

$$\begin{array}{ll} \quad (\neg\varphi_1)^{\mathbf{R}} \in F & \\ \text{iff } -\varphi_1^{\mathbf{R}} \in F & \text{by (3.i)} \\ \text{iff } \varphi_1^{\mathbf{R}} \notin F & \text{as } F \text{ is an ultrafilter} \\ \text{iff not } \mathsf{co}\mathbf{R}, F \models \varphi_1 & \text{by hypothesis on } \varphi_1 \\ \text{iff } \mathsf{co}\mathbf{R}, F \models \neg\varphi_1 & \text{by (2.iii),} \end{array}$$

so the result holds for φ in this case. The inductive case of $\varphi = \varphi_1 \wedge \varphi_2$ follows similarly, using (3.ii), the fact that F is a filter, and (2.iv). □

We are now in a position to clarify the properties of the function $\eta^{\mathbf{R}}$ defined in (4.i).

Theorem 4.5 *For any $\mathcal{S}$-algebra $\mathbf{R}$, the function $\eta^{\mathbf{R}} : R \to \mathcal{P}(X^{\mathsf{co}\mathbf{R}})$ is an $\mathcal{S}$-algebraic homomorphism $\mathbf{R} \to \mathcal{P}\mathsf{co}\mathbf{R}$.*

Proof. The fact that $\eta^{\mathbf{R}}(r \cdot s) = \eta^{\mathbf{R}}(r) \cap \eta^{\mathbf{R}}(s)$ and $\eta^{\mathbf{R}}(-r) = X^{\mathsf{co}\mathbf{R}} - \eta^{\mathbf{R}}(r)$ has already been noted , and is a standard consequence of properties of ultrafilters.

To see that $\eta^{\mathbf{R}}$ preserves the distinguished elements $\varepsilon^{\mathbf{R}}$, observe that Theorem 4.4 states that $F \in \eta^{\mathbf{R}}(\varepsilon)$ iff $F \in \varepsilon^{\mathsf{co}\mathbf{R}}$ for all $F \in X^{\mathsf{co}\mathbf{R}}$. Thus $\eta^{\mathbf{R}}(\varepsilon) = \varepsilon^{\mathsf{co}\mathbf{R}}$.

Finally we have to show that $\eta^{\mathbf{R}}$ preserves the operators $[m.b]^{\mathbf{R}}$, i.e. that

$$\eta^{\mathbf{R}}([m.b]^{\mathbf{R}} r) = [m.b]^{\mathsf{co}\mathbf{R}}(\eta^{\mathbf{R}}(r))$$

for all $m \in Meth$, $b \in I^m$, and $r \in R$. But if $F \in X^{\mathsf{co}\mathbf{R}}$, then

$$\begin{array}{ll} \quad F \in \eta^{\mathbf{R}}([m.b]^{\mathbf{R}} r) & \\ \text{iff } [m.b]^{\mathbf{R}} r \in F & \\ \text{iff } r \in m^{\mathsf{co}\mathbf{R}}(F, b) & \text{by definition of } m^{\mathsf{co}\mathbf{R}} \\ \text{iff } m^{\mathsf{co}\mathbf{R}}(F, b) \in \eta^{\mathbf{R}}(r) & \\ \text{iff } F \in [m.b]^{\mathsf{co}\mathbf{R}}(\eta^{\mathbf{R}}(r)) & \text{by (2.ii).} \end{array}$$

□

5 Duality

A *morphism* from $\mathcal{S}$-coalgebra A to $\mathcal{S}$-coalgebra B is a function $f : X^A \to X^B$ between their state sets that preserves methods and attributes, meaning that for any state $x \in X^A$ the equation

$$f(m^A(x,b)) = m^B(f(x),b)$$

holds for all $m \in Meth$ and all $b \in I^m$; while

$$a^A(x,c) = a^B(f(x),c)$$

holds for all $a \in Att$ and $c \in I^a$. If f is surjective, then B is the *image* of A under this morphism. If f is bijective then it is an *isomorphism*, making A and B *isomorphic*, which is denoted $A \cong B$ as usual.

The standard symbols $\rightarrowtail$ and $\twoheadrightarrow$ will be used for injective and surjective functions. The notations $A \rightarrowtail B$ and $A \twoheadrightarrow B$ indicate that there exists a morphism from A to B that is injective or surjective respectively. Likewise $\mathbf{R} \rightarrowtail \mathbf{S}$ and $\mathbf{R} \twoheadrightarrow \mathbf{S}$ indicate the existence of injective and surjective homomorphisms between $\mathcal{S}$-algebras.

If X^A is a subset of X^B, then A is called a *subcoalgebra* of B if the inclusion function $X^A \hookrightarrow X^B$ is a morphism. This means that X^A is closed under all methods of B, i.e. $m^B(x,b) \in X^A$ whenever $x \in X^A$; m^A is the restriction of m^B to $X^A \times I^m$; and each attribute a^A is the restriction of a^B to $X^A \times I^a$. For any morphism $f : A \to B$, the image $f(X^A)$ is a subcoalgebra of B which is isomorphic to A when f is injective.

An important illustration of these concepts is given by the function

$$\xi^A : x \longmapsto F_x^A = \{Y \subseteq X^A : x \in Y\}, \tag{5.i}$$

which was shown in [Gol99, Theorem 8.2] to be an injective morphism $A \rightarrowtail \mathsf{co}\mathcal{P}A$ making A isomorphic to a subcoalgebra of $\mathsf{co}\mathcal{P}A$. Similarly the function

$$x \longmapsto F_x^{\mathcal{D}A} = \{\varphi^A : A, x \models \varphi\}$$

is a morphism $A \to \mathsf{co}\mathcal{D}A$ (see below, after Theorem 5.4). It need not however be injective. Indeed $F_x^{\mathcal{D}A} = F_y^{\mathcal{D}A}$ iff states x and y satisfy the same ground observable formulas in A, which is true precisely when they are are *bisimilar* [Gol99, Theorem 3.2 and Section 4].

Preservation of satisfaction by a coalgebraic morphism $f : A \to B$ was studied in [Gol99, Theorem 5.1], where it was shown that

$$A, x \models \varphi \quad \text{iff} \quad B, f(x) \models \varphi \tag{5.ii}$$

for all $x \in X^A$ and all formulas φ. It follows that $B \models \varphi$ implies $A \models \varphi$, while if $A \models \varphi$ and f is a surjection $A \twoheadrightarrow B$ then $B \models \varphi$. Thus the class $Mod\,\varphi$ of all models of φ is closed under domains and images of morphisms.

The identity function on X^A is a morphism, and the composition of morphisms is a morphism, so the class of all $\mathcal{S}$-coalgebras with coalgebraic morphisms between them forms a category $\mathcal{S}$-**CoAlg**. The powerset $\mathcal{S}$-algebra construction $A \mapsto \mathcal{P}A$ gives rise to a contravariant functor $\mathcal{P}$ from $\mathcal{S}$-**CoAlg** to the category $\mathcal{S}$-**Alg** of rich $\mathcal{S}$-algebras. This functor assigns to each morphism $f : X^A \to X^B$ the function $f^{\mathcal{P}} : \mathcal{P}B \to \mathcal{P}A$ defined by $f^{\mathcal{P}}(Y) = f^{-1}(Y)$ for all $Y \in \mathcal{P}(X^B)$.

Theorem 5.1 *$f^{\mathcal{P}}$ is a homomorphism of $\mathcal{S}$-algebras that is surjective if f is injective, and injective if f is surjective. Hence $A \rightarrowtail B$ implies $\mathcal{P}B \twoheadrightarrow \mathcal{P}A$, and $A \twoheadrightarrow B$ implies $\mathcal{P}B \rightarrowtail \mathcal{P}A$.*

Proof. That $f^{\mathcal{P}}$ is a Boolean homomorphism from $\mathcal{P}(X^B)$ to $\mathcal{P}(X^A)$ is standard set theory, as are the facts about injectivity and surjectivity. Our task is to show that $f^{\mathcal{P}}$ preserves operators $[m.b]^{\mathbf{R}}$ and distinguished elements $\varepsilon^{\mathbf{R}}$.

Result (5.ii) states that $x \in \varphi^A$ iff $x \in f^{-1}(\varphi^B)$, so $\varphi^A = f^{\mathcal{P}}(\varphi^B)$ for all $\varphi \in GOF$. In particular $f^{\mathcal{P}}(\varepsilon^{\mathcal{P}B}) = \varepsilon^{\mathcal{P}A}$ for all $\varepsilon \in GOE$.

It remains to show that $f^{\mathcal{P}}([m.b]^{\mathcal{P}B}(Y)) = [m.b]^{\mathcal{P}A}(f^{\mathcal{P}}(Y))$. But

$$
\begin{aligned}
&x \in f^{\mathcal{P}}([m.b]^{\mathcal{P}B}(Y)) \\
\text{iff } &f(x) \in [m.b]^{\mathcal{P}B}(Y) \\
\text{iff } &m^B(f(x), b) \in Y \\
\text{iff } &f(m^A(x, b)) \in Y \qquad \text{as } f \text{ is a morphism} \\
\text{iff } &m^A(x, b) \in f^{\mathcal{P}}(Y) \\
\text{iff } &x \in [m.b]^{\mathcal{P}A}(f^{\mathcal{P}}(Y)).
\end{aligned}
$$

□

The proof just given noted that $\varphi^A = f^{\mathcal{P}}(\varphi^B)$ for all $\varphi \in GOF$. Hence $\varphi^B = 1^{\mathcal{P}B}$ implies $\varphi^A = 1^{\mathcal{P}A}$, which gives another explanation of why $B \models \varphi$ implies $A \models \varphi$ (see (3.iii)). If $f : A \twoheadrightarrow B$, then $f^{\mathcal{P}}$ is injective, in which case $\varphi^A = 1^{\mathcal{P}A}$ implies $\varphi^B = 1^{\mathcal{P}B}$, confirming that $A \models \varphi$ implies $B \models \varphi$ in that case.

The fact that $f^{\mathcal{P}}(\varphi^B) = \varphi^A$ entails that $f^{\mathcal{P}}$ maps the subalgebra $\mathcal{D}B$ of $\mathcal{P}B$ homomorphically *onto* $\mathcal{D}A$. This is because the carrier of $\mathcal{D}B$ is $\{\varphi^B : \varphi \in GOF\}$, while that of $\mathcal{D}A$ is $\{\varphi^A : \varphi \in GOF\}$. Therefore if f is surjective, the injective $f^{\mathcal{P}}$ becomes a bijection between $\mathcal{D}B$ and $\mathcal{D}A$:

Corollary 5.2 *If $A \twoheadrightarrow B$, then $\mathcal{D}A \cong \mathcal{D}B$.* □

The dual correspondence between coalgebraic morphisms and algebraic homomorphisms of Theorem 5.1 can be applied to derive the dual correspondence between direct products of algebras and disjoint unions of coalgebras. If $\{A_j : j \in J\}$ is a collection of $\mathcal{S}$-coalgebras, then their *disjoint union* is the coalgebra

$$A = \coprod_J A_j = (\bigcup_J (X^{A_j} \times \{j\}), \{m^A : m \in Meth\}, \{a^A : a \in Att\}),$$

where $m^A((x,j),b) = (m^{A_j}(x,b),j)$ and $a^A((x,j),c) = a^{A_j}(x,c)$.

Thus $\coprod_J A_j$ is the union of a set of pairwise disjoint copies $A_j \times \{j\}$ of the structures A_j. For each $k \in J$, the map $f_k : x \mapsto (x,k)$ gives an injective morphism $A_k \rightarrowtail \coprod_J A_j$, whose image $A_k \times \{k\}$ is a subcoalgebra of $\coprod_J A_j$ isomorphic to A_k.

Theorem 5.3 *The powerset algebra $\mathcal{P}\coprod_J A_j$ of the disjoint union is isomorphic to the direct product $\prod_J \mathcal{P}A_j$ of the powerset algebras of the coalgebras A_j.*

Proof. This is familiar from the theory of Boolean algebras with operators [Gol89, Lemma 3.4.1], but we give the explanation here.

For each $k \in J$, the injective morphism $f_k : A_k \rightarrowtail \coprod_J A_j$ induces the surjective homomorphism $f_k^{\mathcal{P}} : \mathcal{P}\coprod_J A_j \twoheadrightarrow \mathcal{P}A_k$. These homomorphisms in turn give rise to a homomorphism

$$\theta : \mathcal{P}\coprod_J A_j \twoheadrightarrow \prod_J \mathcal{P}A_j,$$

where $\theta(Y)(k) = f_k^{\mathcal{P}}(Y)$, i.e. θ is the direct product of the $f_k^{\mathcal{P}}$'s.

This θ is surjective, because if $(Z_j : j \in J) \in \prod_J \mathcal{P}A_j$, with $Z_j \in \mathcal{P}A_j$ for all $j \in J$, then

$$(Z_j : j \in J) = \theta(\bigcup_J Z_j \times \{j\})$$

since $Z_k = f_k^{\mathcal{P}}(\bigcup_J Z_j \times \{j\})$ in general.

θ is also injective, for if $\theta(Y) = \theta(Z)$, then $(x,k) \in Y$ implies

$$x \in f_k^{\mathcal{P}}(Y) = \theta(Y)(k) = \theta(Z)(k) = f_k^{\mathcal{P}}(Z),$$

so $(x,k) \in Z$. Interchanging Y and Z shows $Y = Z$. Hence altogether

$$\theta : \mathcal{P}\coprod_J A_j \cong \prod_J \mathcal{P}A_j.$$

□

A contravariant functor $\mathsf{co} : \mathcal{S}\text{-}\mathbf{Alg} \to \mathcal{S}\text{-}\mathbf{CoAlg}$ is given by the construction $\mathbf{R} \mapsto \mathsf{co}\mathbf{R}$ of the coalgebra of a rich $\mathcal{S}$-algebra $\mathbf{R}$. This functor assigns to each algebraic homomorphism $\theta : \mathbf{R} \to \mathbf{S}$ the function $\theta^{\mathsf{co}} : \mathsf{co}\mathbf{S} \to \mathsf{co}\mathbf{R}$ defined by $\theta^{\mathsf{co}}(F) = \theta^{-1}(F)$ for all $F \in X^{\mathbf{S}}$. Lemma 4.2 guarantees that $\theta^{\mathsf{co}}(F)$ belongs to $X^{\mathbf{R}}$, so θ^{co} is a map of the right type.

Theorem 5.4 *θ^{co} is a morphism of coalgebras that is surjective if θ is injective, and injective if θ is surjective. Hence $\mathbf{R} \rightarrowtail \mathbf{S}$ implies $\mathsf{co}\mathbf{S} \twoheadrightarrow \mathsf{co}\mathbf{R}$, and $\mathbf{R} \twoheadrightarrow \mathbf{S}$ implies $\mathsf{co}\mathbf{S} \rightarrowtail \mathsf{co}\mathbf{R}$.*

Proof. We show that θ^{co} preserves methods and attributes, the rest being standard set theory. For methods we want

$$\theta^{\mathsf{co}}(m^{\mathbf{S}}(F,b)) = m^{\mathbf{R}}(\theta^{\mathsf{co}}(F),b)$$

for all $F \in X^{\mathbf{S}}$, $m \in Meth$, and $b \in I^m$. But

$$\begin{aligned}
&r \in \theta^{\mathsf{co}}(m^{\mathbf{S}}(F,b)) \\
\text{iff } &\theta(r) \in m^{\mathbf{S}}(F,b) \\
\text{iff } &[m.b]^{\mathbf{S}}\theta(r) \in F \\
\text{iff } &\theta([m.b]^{\mathbf{R}}r) \in F \qquad\qquad \text{as } \theta \text{ is a homomorphism} \\
\text{iff } &[m.b]^{\mathbf{R}}r \in \theta^{\mathsf{co}}(F) \\
\text{iff } &r \in m^{\mathbf{R}}(\theta^{\mathsf{co}}(F), b).
\end{aligned}$$

For attributes we want $a^{\mathbf{S}}(F,c) = a^{\mathbf{R}}(\theta^{\mathsf{co}}(F),c)$ in general. Let $a^{\mathbf{S}}(F,c) = b \in O^a$. Then by definition, $(a(\sigma,c) \approx b)^{\mathbf{S}} \in F$. But θ preserves distinguished elements, so

$$\theta((a(\sigma,c) \approx b)^{\mathbf{R}}) = (a(\sigma,c) \approx b)^{\mathbf{S}},$$

showing that $(a(\sigma,c) \approx b)^{\mathbf{R}} \in \theta^{\mathsf{co}}(F)$, which implies that $a^{\mathbf{R}}(\theta^{\mathsf{co}}(F),c) = b$ as desired. □

As a first application of this Theorem, consider the subalgebra $\mathcal{D}A$ of the powerset algebra $\mathcal{P}A$ of a coalgebra A. The inclusion $\mathcal{D}A \hookrightarrow \mathcal{P}A$ is an injective homomorphism, so induces a surjective coalgebraic morphism $\mathsf{co}\mathcal{P}A \twoheadrightarrow \mathsf{co}\mathcal{D}A$, mapping $F \in X^{\mathcal{P}A}$ to $\mathcal{D}(X^A) \cap F = \{\varphi^A \in F : \varphi \in GOF\}$. This morphism $\mathsf{co}\mathcal{P}A \twoheadrightarrow \mathsf{co}\mathcal{D}A$ composes with the injective morphism $A \rightarrowtail \mathsf{co}\mathcal{P}A$ discussed earlier, to yield a morphism $A \to \mathsf{co}\mathcal{D}A$. The latter is just the map $x \mapsto F_x^{\mathcal{D}A} = \{\varphi^A : A, x \models \varphi\}$.

Recall that $\mathsf{co}\mathcal{P}A$ is the *ultrafilter enlargement* of A. $\mathsf{co}\mathcal{D}A$ may be called the *definable enlargement* of A.

Corollary 5.5 *Let K be a class of coalgebras that is closed under domains and images of morphisms. Then*

(1) $\mathsf{co}\mathcal{D}A \in K$ *implies* $A \in K$.

(2) *K is closed under ultrafilter enlargements if and only if it is closed under definable enlargements.*

Proof.

(1) We have just seen that there is morphism $A \to \mathsf{co}\mathcal{D}A$, so if $\mathsf{co}\mathcal{D}A \in K$, closure under domains of morphisms immediately gives $A \in K$.

(2) As above there is a surjective morphism $\mathsf{co}\mathcal{P}A \twoheadrightarrow \mathsf{co}\mathcal{D}A$ for any coalgebra A. Thus if K is closed under domains and images of morphisms, then $\mathsf{co}\mathcal{P}A \in K$ iff $\mathsf{co}\mathcal{D}A \in K$.

□

Theorem 5.6 *For any coalgebraic morphism $f : A \to B$ and any $\mathcal{S}$-algebraic homomorphism $\theta : \mathbf{R} \to \mathbf{S}$, the following diagrams commute.*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \big\downarrow{\scriptstyle \xi^A} & & \big\downarrow{\scriptstyle \xi^B} \\ \mathsf{co}\mathcal{P}A & \xrightarrow{f^{\mathcal{P}\mathsf{co}}} & \mathsf{co}\mathcal{P}B \end{array} \qquad\qquad \begin{array}{ccc} \mathbf{R} & \xrightarrow{\theta} & \mathbf{S} \\ \big\downarrow{\scriptstyle \eta^{\mathbf{R}}} & & \big\downarrow{\scriptstyle \eta^{\mathbf{S}}} \\ \mathcal{P}\mathsf{co}\mathbf{R} & \xrightarrow{\theta^{\mathsf{co}\mathcal{P}}} & \mathcal{P}\mathsf{co}\mathbf{S} \end{array}$$

Proof. Again this is essentially a version of standard theory [Gol89, Theorem 2.3.3]. For the left diagram we have

$$\begin{aligned} f^{\mathcal{P}\mathsf{co}} \circ \xi^A(x) &= f^{\mathcal{P}\mathsf{co}}(\{Y \subseteq X^A : x \in Y\}) \\ &= \{Z \subseteq X^B : x \in f^{\mathcal{P}}(Z)\} \\ &= \{Z \subseteq X^B : f(x) \in Z\} \\ &= \xi^B \circ f(x), \end{aligned}$$

and for the other

$$\begin{aligned} \theta^{\mathsf{co}\mathcal{P}} \circ \eta^{\mathbf{R}}(r) &= \theta^{\mathsf{co}\mathcal{P}}(\{F \in X^{\mathsf{co}\mathbf{R}} : r \in F\}) \\ &= \{G \in X^{\mathsf{co}\mathbf{S}} : r \in \theta^{\mathsf{co}}(G)\} \\ &= \{G \in X^{\mathsf{co}\mathbf{S}} : \theta(r) \in G\} \\ &= \eta^{\mathbf{S}} \circ \theta(r). \end{aligned}$$

□

In the language of categories, this theorem states that the morphisms ξ^A for all coalgebras A are the components of a natural transformation from the identity functor on $\mathcal{S}$-**CoAlg** to the composite functor $\mathsf{co} \circ \mathcal{P}$, while the morphisms $\eta^{\mathbf{R}}$ for all rich $\mathcal{S}$-algebras $\mathbf{R}$ are the components of a natural transformation from the identity functor on $\mathcal{S}$-**Alg** to the functor $\mathcal{P} \circ \mathsf{co}$.

For any coalgebra A and rich $\mathcal{S}$-algebra $\mathbf{R}$, a standard calculation shows that the diagrams

$$\begin{array}{ccc} \mathcal{P}A & \xrightarrow{\eta^{\mathcal{P}A}} & \mathcal{P}(\mathsf{co}\mathcal{P}A) \\ & \searrow^{id} & \big\downarrow{\scriptstyle (\xi^A)^{\mathcal{P}}} \\ & & \mathcal{P}A \end{array} \qquad\qquad \begin{array}{ccc} \mathsf{co}\mathbf{R} & \xrightarrow{\xi^{\mathsf{co}\mathbf{R}}} & \mathsf{co}\mathcal{P}\mathsf{co}\mathbf{R} \\ & \searrow^{id} & \big\downarrow{\scriptstyle (\eta^{\mathbf{R}})^{\mathsf{co}}} \\ & & \mathsf{co}\mathbf{R} \end{array}$$

commute, where id is the identity function in each case. This means that the functors co and $\mathcal{P}$ form a *dual adjunction* between the categories $\mathcal{S}$-**CoAlg** and $\mathcal{S}$-**Alg**. The adjunction provides a bijective correspondence between the set of morphisms $f : A \to \mathsf{co}\mathbf{R}$ and the set of homomorphisms $\theta : \mathbf{R} \to \mathcal{P}A$, assigning to f the dual adjoint homomorphism $ad(f) = f^{\mathcal{P}} \circ \eta^{\mathbf{R}}$, and to θ the dual adjoint morphism $ad(\theta) = \theta^{\mathsf{co}} \circ \xi^A$:

$$
\begin{array}{ccc}
\mathbf{R} & \xrightarrow{\eta^{\mathbf{R}}} & \mathcal{P}\mathsf{co}\mathbf{R} \\
 & \searrow^{ad(f)} & \Big\downarrow f^{\mathcal{P}} \\
 & & \mathcal{P}A
\end{array}
\qquad\qquad
\begin{array}{ccc}
A & \xrightarrow{\xi^{A}} & \mathsf{co}\mathcal{P}A \\
 & \searrow^{ad(\theta)} & \Big\downarrow \theta^{\mathsf{co}} \\
 & & \mathsf{co}\mathbf{R}
\end{array}
$$

This is not however a *dual equivalence* of categories, since that would require the components $\eta^{\mathbf{R}}$ and ξ^A to be isomorphisms. But ξ^A, while injective, need not be surjective, and $\eta^{\mathbf{R}}$ need not be either.

6 Definable Classes of Coalgebras

Using the duality we have developed, facts about algebras can be applied to the study of coalgebras. One such fact is Birkhoff's variety theorem. A *variety*, or *equational class*, of algebras is one that is the class of all models of some set of equations. Birkhoff's theorem states that if a class of algebras is closed under direct products, subalgebras, and homomorphic images, then it is a variety. A refinement of this, due to Tarski, is that for any class of algebras $\mathcal{V}$, the variety *generated by* $\mathcal{V}$, i.e. the smallest variety including $\mathcal{V}$, is equal to $\mathbf{H}(\mathbf{S}(\mathbf{P}(\mathcal{V})))$. Here $\mathbf{H}$, $\mathbf{S}$, and $\mathbf{P}$ denote the operations of closure of a class of algebras under (isomorphic copies of) homomorphic images, subalgebras, and direct products, respectively.

In [Gol99, Theorem 9.2] it was shown that

> if a class of coalgebras is closed under disjoint unions, domains and images of morphisms, and ultrafilter enlargements, then it is the class of all models of some set of (ground) observable formulas.

The proof was model-theoretic, and made extensive use of bisimulation relations between coalgebras. The result itself could be said to be an analogue for coalgebras of Birkhoff's theorem. Here we provide support for that view by giving a different proof that makes a direct application, via our duality theory, of Birkhoff's theorem for $\mathcal{S}$-algebras.

Theorem 6.1 *Let K be a class of coalgebras that is closed under disjoint unions, domains and images of morphisms, and ultrafilter enlargements. Then $K = Mod\,\Phi$ for some set Φ of ground observable formulas.*

Proof. Define the class $\mathcal{P}K$ of $\mathcal{S}$-algebras by

$$\mathcal{P}K = \{\mathbf{R} : \mathbf{R} \cong \mathcal{P}A \text{ for some } A \in K\}.$$

Let $\mathcal{V} = \mathbf{H}(\mathbf{S}(\mathbf{P}(\mathcal{P}K)))$, the variety generated by $\mathcal{P}K$.

But $\mathbf{P}(\mathcal{P}K) = \mathcal{P}K$, and so $\mathcal{V} = \mathbf{H}(\mathbf{S}(\mathcal{P}K))$. For, if an $\mathcal{S}$-algebra $\mathbf{R}$ is isomorphic to a direct product $\prod_J \mathbf{R}_j$ of algebras $\mathbf{R}_j \in \mathcal{P}K$, then for each $j \in J$, $\mathbf{R}_j \cong \mathcal{P}A_j$ for some $A_j \in K$. Using Theorem 5.3 we then get

$$\mathbf{R} \cong \textstyle\prod_J \mathcal{P}A_j \cong \mathcal{P}(\coprod_J A_j) \in \mathcal{P}K,$$

since $\coprod_J A_j \in K$ by closure under disjoint unions. Hence $\mathbf{R} \in \mathcal{P}K$.

Since $\mathcal{V}$ is a variety, it is the class of all models of some set $\mathcal{E}$ of algebraic equations. We now make use of the fact (3.iv) that every algebraic equation for $\mathcal{S}$-algebras is equivalent to one of the form $\alpha \approx 1$ for some algebraic term α. Hence we assume all members of $\mathcal{E}$ have this form. Put

$$\Phi = \bigcup\{\Phi_\alpha : (\alpha \approx 1) \in \mathcal{E}\},$$

where Φ_α is the set of ground observable formulas assigned to α as in (3.v).

Now if $A \in K$, then for each $(\alpha \approx 1) \in \mathcal{E}$, $\mathcal{P}A \models \alpha \approx 1$ as $\mathcal{P}A \in \mathcal{P}K \subseteq \mathcal{V}$, so $\mathcal{D}A \models \alpha \approx 1$ as $\mathcal{D}A$ is a subalgebra of $\mathcal{P}A$, hence $A \models \Phi_\alpha$ by Theorem 3.5(2). Thus $A \models \Phi$. This shows that $K \subseteq Mod\,\Phi$.

Conversely, let $A \in Mod\,\Phi$ with the aim of showing $A \in K$. For each $(\alpha \approx 1) \in \mathcal{E}$, $A \models \Phi_\alpha$ and so by 3.5(2) again $\mathcal{D}A \models \alpha \approx 1$. Thus $\mathcal{D}A$ is a model of $\mathcal{E}$, and hence a member of the variety $\mathcal{V} = \mathbf{H}(\mathbf{S}(\mathcal{P}K))$. Therefore $\mathcal{D}A$ is a homomorphic image of some $\mathcal{S}$-algebra $\mathbf{R}$ that is isomorphic to a subalgebra of $\mathcal{P}B$ for some $B \in K$. This gives the situation

$$\mathcal{P}B \hookleftarrow \mathbf{R} \twoheadrightarrow \mathcal{D}A.$$

The duality described in Theorem 5.4 then yields

$$\mathsf{co}\mathcal{P}B \twoheadrightarrow \mathsf{co}\mathbf{R} \hookleftarrow \mathsf{co}\mathcal{D}A.$$

But $\mathsf{co}\mathcal{P}B \in K$, by closure of K under ultrafilter extensions, so then $\mathsf{co}\mathbf{R} \in K$ by closure under images of morphisms, and therefore $\mathsf{co}\mathcal{D}A \in K$ by closure under domains of morphisms. Hence $A \in K$ as there is a morphism $A \to \mathsf{co}\mathcal{D}A$ (see Corollary 5.5(1)). □

7 Relations with Modal Definability

The proof of Theorem 6.1 uses similar methodology to a result of the author [GT75, Theorem 8] characterising certain classes of Kripke frames that are definable by formulas from propositional modal logic. This is also expounded in [Gol93, Theorem 1.20.6], and generalised in [Gol89, Theorem 3.7.6] to arbitrary relational structures and BAO's.

Briefly, a Kripke frame is a pair $A = (X^A, R^A)$ with R^A a binary relation on set X^A. R^A may be identified with the function $x \mapsto \{y : xR^Ay\}$ from X^A to $\mathcal{P}(X^A)$, so a frame can be viewed as a coalgebra for the powerset endofunctor $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$. In this context a coalgebraic morphism $A \to B$ between frames is a *bounded morphism*, i.e. a function $f : X^A \to X^B$ satisfying

$$f(x)R^Bz \quad \text{iff} \quad \exists y \in X^A(xR^Ay \text{ and } f(y) = z).$$

A substructure A of B is a *subframe* of B if it is a subcoalgebra, which means that it is R^B-closed, i.e. if $x \in X^A$ then $\{y \in X^B : xR^By\} \subseteq X^A$.

The duality theory for modal logic associates with each frame A the modal algebra $\mathcal{P}A$, comprising the powerset Boolean algebra $\mathcal{P}(X^A)$ with the unary operator $[R]^A$ defined by $[R]^A(Y) = \{x \in X^A : \{y : xR^Ay\} \subseteq Y\}$. Each formula φ of propositional modal logic corresponds to a modal-algebraic term α_φ, and frame A *validates* φ iff algebra $\mathcal{P}A$ satisfies the equation $\alpha_\varphi \approx 1$ (compare this with Theorem 3.5). The *canonical extension* of frame A is the frame A^* whose points are the ultrafilters on X^A, with $FR^{A^*}G$ iff $\{Y : [R]^A(Y) \in F\} \subseteq G$. The modal characterisation result of [GT75, Theorem 8] can be stated as follows.

Theorem 7.1 *Let K be a class of frames that is closed under first-order elementary equivalence. Then K is the class of all frames validating some set of modal formulas iff it is closed under disjoint unions, subframes, and images of bounded morphisms, and the complement of K is closed under canonical extensions (i.e. $A^* \in K$ implies $A \in K$).* □

There are several obvious points of difference between this result and Theorem 6.1. First of all, formulas of propositional modal logic are very different to the observable formulas of this paper, in that modal formulas have propositional variables that take arbitrary subsets of X^A as values. But we saw that observable formulas have definable modalities $[m.b]$ (2.v), and so we could say that the class of observable formulas is the restricted modal language generated by taking observable equations in place of propositional variables.

Next, note that 7.1 does not apply to arbitrary classes of frames, but only to those closed under elementary equivalence. In fact closure under *ultrapowers* is sufficent: together with closure under images of bounded morphisms, this entails that $A \in K$ implies $A^* \in K$ [Gol89, Theorem 3.6.1]. That explains why the statement of 7.1 does not need to refer to closure of K under canonical extensions, whereas 6.1 does refer to closure under ultrafilter enlargements.

The assumption in the first sentence of 7.1 can be weakened to closure under canonical extensions, but it cannot be absorbed into the second sentence to give a single biconditional statement characterising all modally definable classes of Kripke frames. This is because there exist "non-canonical" propositional modal formulas whose class of validating frames is not closed under canonical extensions. That fact has been the source of much research in modal logic [Gol93, Chapters 10, 11], [Gol00, Section 5.5]. It marks another point of difference with observable formulas φ for coalgebras, since $Mod\,\varphi$ is always closed under ultrafilter enlargements [Gol99, Corollary 8.7].

The requirement in 7.1 that $A^* \in K$ implies $A \in K$ does not have a counterpart in 6.1. That is because a coalgebra A is the domain of the (injective) morphism $\xi^A : A \to \mathsf{co}\mathcal{P}A$, so the assumption that K is closed under domains of morphisms makes it automatic that if $\mathsf{co}\mathcal{P}A \in K$ then $A \in K$. This also points to another significant distinction: if φ is an observable formula, the class of coalgebras $Mod\,\varphi$ is always closed under domains of coalgebraic morphisms [Gol99, Theorem 5.1]. But the class of Kripke frames validating a propositional modal formula need not be closed under domains of bounded morphisms.

However, we cannot weaken the requirement in 6.1 of closure under domains of morphisms to closure under subcoalgebras (as 7.1 might suggest). In Section

10 of [Gol99] an example is given of a class of coalgebras that is closed under disjoint unions, images of morphisms, subcoalgebras, and ultrafilter enlargements, but not closed under domains of morphisms.

Overall, these observations suggest that the theory of definable classes of coalgebras is somewhat simpler than the corresponding theory for modal frames. In support of this is the point that methods in coalgebras give rise to parameterised state transitions $x \mapsto m(x, b)$ that are *deterministic*, or singled-valued. This corresponds to Kripke frames whose relation R is functional, and such frames have a more tractable theory. Indeed a functional Kripke frame is an $\mathcal{S}$-coalgebra for the basic signature $\mathcal{S}$ having just one method symbol, with no input sort, and no attribute symbols. Whereas a coalgebra is always (isomorphic to) a subcoalgebra of its ultrafilter enlargement, a frame having infinite sets of the form $\{y : xR^A y\}$ will not be R^{A^*}-closed in A^*, so will not be a subframe of its canonical extension.

On the other hand, the presence of attribute functions in coalgebras adds another dimension with its own complications, making the work of this paper a new chapter in the study of BAO's. Features of this new dimension include the restriction to rich ultrafilters, the fact that the homomorphisms $\eta^{\mathbf{R}} : \mathbf{R} \to \mathcal{P}\mathsf{co}\mathbf{R}$ need not be injective, and the existence of $\mathcal{S}$-algebras that cannot be represented as subalgebras of powerset algebras $\mathcal{P}A$.

It should also be noted that the coalgebras we have been studying lack certain operations, involving *disjoint unions* $Y + Z$ of sets, that are needed to model some object classes and abstract machines. For example, in modelling a class X of stacks of bounded size we may want a method of the form

$$\mathtt{pop} : X \to X + (O \times X),$$

where application of pop to x just returns $x \in X$ if the stack x is full, and otherwise returns the pair $(d, y) \in O \times X$, where d is the top element of x and y is the stack obtained by removing d from x. The extension of these studies to coalgebras involving disjoint unions appears to require a much more elaborate calculus of terms and types, and is a matter for further research.

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