Under consideration for publication in Math. Struct. in Comp. Science

A Comonadic Account of Behavioural Covarieties of Coalgebras

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Received 4 June 2002; revised 14 April 2004

A class K of coalgebras for an endofunctor $T : \mathbf{Set} \to \mathbf{Set}$ is a *behavioural covariety* if it is closed under disjoint unions and images of bisimulation relations (hence closed under images and domains of coalgebraic morphisms, including subcoalgebras). K may be thought of as the class of all coalgebras that satisfy some computationally significant property. In any logical system suitable for specifying properties of state-transition systems in the Hennessy-Milner style, each formula will define a class of models that is a behavioural covariety.
Assume that the forgetful functor on T-coalgebras has a right adjoint, providing for the construction of cofree coalgebras, and let $\mathbb{G}^T$ be the comonad arising from this adjunction. Then we show that behavioural covarieties K are (isomorphic to) the Eilenberg-Moore categories of coalgebras for certain comonads $\mathbb{G}^K$ naturally associated with $\mathbb{G}^T$. These are called *pure subcomonads* of $\mathbb{G}^T$, and a categorical characterization of them is given, involving a pullback condition on the naturality squares of a transformation from $\mathbb{G}^K$ to $\mathbb{G}^T$.
We show that there is a bijective correspondence between behavioural covarieties of T-coalgebras and isomorphism classes of pure subcomonads of $\mathbb{G}^T$.

Introduction

Coalgebras of endofunctors $T : \mathbf{Set} \to \mathbf{Set}$ on the category of sets and functions have been of recent interest in theoretical computer science, due to their role in representing various data structures, as well as structures used in operational semantics. These include automata, labelled transition systems and other types of "state-based" system; lists, streams and trees; and classes in object-oriented programming languages (Reichel, 1995; Jacobs, 1996; Jacobs and Rutten, 1997). In many of these cases the forgetful functor $U : T\text{-}\mathbf{Coalg} \to \mathbf{Set}$ on the category of T-coalgebras has a right adjoint $\mathcal{G}^T : \mathbf{Set} \to T\text{-}\mathbf{Coalg}$, providing a cofree coalgebra $\mathcal{G}^T X$ over each set X. This situation has spurred the development of a general theory of **Set**-based coalgebras (Rutten, 1995; Rutten, 2000;

[†] The author thanks the referees for some helpful suggestions for improvement.

Gumm, 1999; Gumm and Schröder, 2000), by analogy with, and with many concepts categorically dual to, the classical theory of universal algebras.

For example, a *covariety* is defined as a class of coalgebras that is closed under disjoint unions (coproducts), images of coalgebraic morphisms, and subcoalgebras; thereby dualizing the classical notion of a *variety* as a class of universal algebras closed under direct products, subalgebras, and homomorphic images. A celebrated result of Garrett Birkhoff (1935) states that such varieties are precisely the *equationally definable* classes of algebras. There has been a spate of papers and theses discussing coalgebraic analogues of Birkhoff's theorem (Rutten, 1996; Rutten, 2000; Roşu, 1998; Awodey and Hughes, 2000; Gumm, 2000; Kurz, 2000; Hughes, 2001a; Hughes, 2001b; Kurz, 2001a; Adámek and Porst, 2001; Goldblatt, 2001b; Adámek and Porst, 2003; Awodey and Hughes, 2003).

This paper is concerned with covarieties that are closed under images of bisimulations, aptly named *behavioural* covarieties in (Awodey and Hughes, 2000). The original idea of a bisimulation (Park, 1981; Milner, 1983) was that of a binary relation of *observational indistinguishability* between states of two transition systems. States are hidden: all we observe is the behaviour of the system in response to inputs or other actions. States that cause the same observable behaviour are interchangeable, and so a computationally significant property should be invariant under bisimulation. Thus if K is the class of all systems whose states satisfy a certain computational property, then K is behavioural; for if $\mathcal{A} \in K$ and there is a bisimulation from $\mathcal{A}$ onto $\mathcal{B}$, then all states of $\mathcal{B}$ satisfy this property.

Our aim here is to describe the way that behavioural covarieties arise as the Eilenberg-Moore categories of coalgebras of certain comonads. To this end we introduce the notion of one comonad $\mathbb{F}$ being a *pure subcomonad* of another one $\mathbb{G}$. This means that there is a natural transformation $\sigma : F \to G$ between their underlying functors that is a comonad morphism (Barr and Wells, 1985, Section 3.6), such that all components $\sigma_X : FX \to GX$ are mono and the diagram

$$\begin{array}{ccc} FY & \xrightarrow{Ff} & FX \\ \sigma_Y \downarrow & & \downarrow \sigma_X \\ GY & \xrightarrow{Gf} & GX \end{array}$$

is a *pullback* for any function $f : Y \to X$. Of course naturality just means that the diagram commutes, but we shall see that the requirement that it be a pullback captures the condition for a covariety to be behavioural.

The category T-**Coalg** is itself isomorphic to the category $\mathbb{G}^T$-**Coalg** of all $\mathbb{G}^T$-coalgebras, where $\mathbb{G}^T$ is the comonad on **Set** arising from the adjunction between U and $\mathcal{G}^T$. We will show that any behavioural subcovariety of T-**Coalg** can be represented as $\mathbb{F}$-**Coalg** for some pure subcomonad $\mathbb{F}$ of $\mathbb{G}^T$, and that this gives a precise correspondence between behavioural subcovarieties of T-**Coalg** and pure subcomonads of $\mathbb{G}^T$.

A number of computational examples of behavioural covarieties are given (Section 2.1), as well as examples of covarieties that are not behavioural (2.2) . The essence of the contribution of the paper resides in the notion of pure subcomonad, the results of Theorem 3.1, Corollary 5.2 and Theorem 5.3, and the analysis of Section 6.

1. Coalgebras

Let $T : \mathbf{Set} \to \mathbf{Set}$ be a functor on the category of sets. T will be required to have certain preservation properties, to be described below. For now we recall the useful fact that any endofunctor on **Set** preserves monos with non-zero domain (Rutten, 2000, Proposition A.1). For if $X \neq \emptyset$, any injective $f : X \to Y$ has a left inverse $g : Y \to X$, and then Tf has a left inverse, namely Tg, implying that it is injective.

A T*-coalgebra* $\mathcal{A} = (A, \alpha_{\mathcal{A}})$ consists of a set A and a function $\alpha_{\mathcal{A}} : A \to TA$. Here A may be thought of as the set of *states* of $\mathcal{A}$ and $\alpha_{\mathcal{A}}$ as the *transition structure*. A T*-morphism* $f : \mathcal{A} \to \mathcal{B}$ between coalgebras $\mathcal{A} = (A, \alpha_{\mathcal{A}})$ and $\mathcal{B} = (B, \alpha_{\mathcal{B}})$ is given by a function $f : A \to B$ such that the following diagram commutes.

$$\begin{array}{ccc} A & \xrightarrow{\;f\;} & B \\ {\scriptstyle \alpha_{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle \alpha_{\mathcal{B}}} \\ TA & \xrightarrow{\;Tf\;} & TB \end{array}$$

$\mathcal{B}$ will be called the *image* of the morphism if f is surjective.

The T-coalgebras and T-morphisms constitute the category T-**Coalg**, for which the forgetful functor $U : T\text{-}\mathbf{Coalg} \to \mathbf{Set}$ maps $\mathcal{A}$ to its state set A and f as a morphism to f as a function.† We use many facts from the theory of **Set**-based coalgebras as developed in (Rutten, 2000; Gumm, 1999) and other references. For instance we will need the fact that a morphism is epi in T-**Coalg** iff it is surjective (Rutten, 2000, Proposition 4.7), so images of morphisms are the codomains of epi's. Injective morphisms are mono in T-**Coalg**, but the converse need not hold.

Subcoalgebras

A subset X of A is *closed in* $\mathcal{A}$ if there exists a transition $\alpha : X \to TX$ for which the inclusion function $X \hookrightarrow A$ is a T-morphism from (X, α) to $\mathcal{A}$. If such an α exists it is unique, and in that case (X, α), or just X, is a *subcoalgebra* of $\mathcal{A}$. The union of any collection of closed subsets of A is itself closed in $\mathcal{A}$ (Rutten, 2000, Theorem 6.4).

The property of being closed is preserved by images of morphisms: if $f : \mathcal{A} \to \mathcal{B}$ is a T-morphism and X is closed in $\mathcal{A}$, then $f(X)$ is closed in $\mathcal{B}$, and the restriction of f to X gives a T-morphism with image $f(X)$ (Rutten, 2000, Theorem 6.3.1). In particular $f(A)$ is a subcoalgebra of $\mathcal{B}$ that is the image of f.

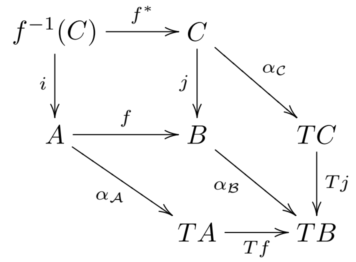
Preservation of closure by pre-images, i.e. pullbacks along morphisms, requires some restriction on T, so from now on we assume that

T weakly preserves pullbacks of monos.

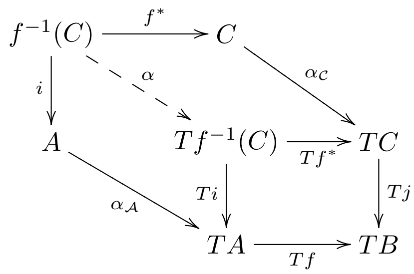
This condition means that the T-image of the diagram for a pullback of a mono is a diagram satisfying the existence but not necessarily the uniqueness part of the universal property of pullbacks. This implies that T preserves all monos, and in fact the condition

† The symbol "U" will be used indiscriminately for forgetful functors on various categories.

is equivalent to the ostensibly stronger requirement that T preserves pullbacks of monos . It is also equivalent to the requirement that if $f : \mathcal{A} \to \mathcal{B}$ is any T-morphism and $\mathcal{C}$ a subcoalgebra of $\mathcal{B}$, then $f^{-1}(C)$ is a subcoalgebra of $\mathcal{A}$ (see (Gumm, 1999, Theorem 8.19) or (Gumm and Schröder, 2000, Theorem 5.7)). Preservation of subcoalgebras by pre-images is central here, so we give an elementary proof. In the diagram



the upper square is a pullback in **Set**, where i and j are the inclusions and f^* is the restriction of f. The other two quadrangles commute because f and j are T-morphisms. Hence the outer perimeter of the diagram commutes, and this is the same as the perimeter of



But if T weakly preserves pullbacks of monos, then the lower right square of this last diagram is a weak pullback, so the transition α exists as indicated to make $f^{-1}(C)$ a subcoalgebra of $\mathcal{A}$ (by the left quadrangle) and f^* a T-morphism to $\mathcal{C}$ (by the upper one).

Disjoint Unions

The forgetful $U : T\textbf{-Coalg} \to \textbf{Set}$ creates colimits (Barr, 1993), and also preserves them as it is a left adjoint. In particular, any set $\{\mathcal{A}_i : i \in I\}$ of T-coalgebras has a coproduct $\Sigma_I \mathcal{A}_i$ in T-**Coalg**, which is realised as the disjoint union of the $\mathcal{A}_i$'s. Its underlying set is the disjoint union

$$\Sigma_I A_i = \bigcup_I (A_i \times \{i\})$$

of the underlying sets of the $\mathcal{A}_i$'s. Together with the evident injections $\iota_j : A_j \to \Sigma_I A_i$, this gives a coproduct of the A_i's in **Set**, and its co-universal property implies that there is a unique transition α_Σ such that

$$\begin{array}{ccc} A_j & \xrightarrow{\iota_j} & \Sigma_I A_i \\ {\scriptstyle \alpha_{\mathcal{A}_j}}\downarrow & & \downarrow{\scriptstyle \alpha_\Sigma} \\ TA_j & \xrightarrow{T\iota_j} & T\Sigma_I A_i \end{array} \tag{1.1}$$

commutes for every $j \in I$. Then $\Sigma_I \mathcal{A}_i$ is the T-coalgebra $(\Sigma_I A_i, \alpha_\Sigma)$, with each injection ι_j being a T-morphism $\mathcal{A}_j \to \Sigma_I \mathcal{A}_i$.

Bisimulations

If $\mathcal{A}$ and $\mathcal{B}$ are coalgebras for a functor T, then a relation $R \subseteq A \times B$ is a T*-bisimulation* from $\mathcal{A}$ to $\mathcal{B}$ if there exists a transition structure $\rho : R \to TR$ on R such that the projections from (R, ρ) to $\mathcal{A}$ and $\mathcal{B}$ are T-morphisms (Aczel and Mendler, 1989, p. 363):

$$\begin{array}{ccccc} A & \xleftarrow{\pi_1} & R & \xrightarrow{\pi_2} & B \\ {\scriptstyle \alpha_{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle \rho} & & \downarrow{\scriptstyle \alpha_{\mathcal{B}}} \\ TA & \xleftarrow[T\pi_1]{} & TR & \xrightarrow[T\pi_2]{} & TB \end{array} \tag{1.2}$$

Then R^{-1} is a T-bisimulation from $\mathcal{B}$ to $\mathcal{A}$ (Rutten, 2000, Theorem 5.2). If the right projection π_2 is surjective, $\mathcal{B}$ is the *image* of the bisimulation. In general the domain of R is a subcoalgebra of $\mathcal{A}$, being the image of the morphism π_1. If this domain is A, i.e. if π_1 is surjective, then R is called *total* on $\mathcal{A}$. Thus in general the image of a bisimulation from $\mathcal{A}$ is the image of a total bisimulation on a subcoalgebra of $\mathcal{A}$.

A function $f : A \to B$ is a T-morphism from $\mathcal{A}$ to $\mathcal{B}$ iff its graph $R_f = \{(a, f(a)) : a \in A\}$ is a bisimulation from $\mathcal{A}$ to $\mathcal{B}$ (Rutten, 2000, Theorem 2.5): a morphism is essentially a functional bisimulation. Moreover $\mathcal{B}$ is the image of the morphism f iff it is the image of the associated bisimulation R_f.

Lemma 1.1. A class K of T-coalgebras is closed under images of bisimulations if, and only if, it is closed under domains and images of T-morphisms.

Proof. Let K be closed under images of bisimulations and consider a morphism $f : \mathcal{A} \to \mathcal{B}$. If $\mathcal{A} \in K$ and $\mathcal{B}$ is the image of f, then $\mathcal{B}$ is the image of R_f, so $\mathcal{B}$ belongs to K by hypothesis. Also, since the domain of R_f is A, $\mathcal{A}$ is the image of $\mathcal{B}$ under the bisimulation R_f^{-1}. Hence if $\mathcal{B} \in K$ then $\mathcal{A} \in K$.

Conversely, let $\mathcal{B}$ be the image of $\mathcal{A}$ under bisimulation R. Then $\pi_1(R)$ is a subcoalgebra of $\mathcal{A}$ (see diagram (1.2)), and we have a sequence of T-morphisms

$$\mathcal{A} \longleftarrow \pi_1(R) \xleftarrow{\pi_1} (R, \rho) \xrightarrow{\pi_2} \mathcal{B},$$

where the first is the inclusion, and the last is surjective. Thus if $\mathcal{A} \in K$ and K is closed under domains and images of T-morphisms, then $\mathcal{B} \in K$.

□

2. Behavioural Covarieties

Let K be a class of T-coalgebras, viewed as a full subcategory of T-**Coalg**. K is a *quasi-covariety* if it is closed under disjoint unions and under images of morphisms, i.e. under codomains of epis. A *covariety* is a quasi-covariety that is also closed under subcoalgebras. K is *behavioural* if it is closed under images of bisimulations. By Lemma 1.1, this is equivalent to closure under domains and images of morphisms. Note that closure under domains of morphisms implies closure under subcoalgebras, since a subcoalgebra is the domain of the inclusion morphism. Thus each behavioural quasi-covariety is a covariety, while

> *a behavioural covariety is a class that is closed under disjoint unions and domains and images of morphisms.*

Rutten (1996; 2000) gave a characterisation of covarieties of T-coalgebras for the case that T has a "boundedness" property that implies the existence of cofree coalgebras over any set (see Section 3 below for the definition of cofreeness). The characterisation states that for any covariety K there is a set X and a subcoalgebra $\mathcal{A}_K$ of the cofree coalgebra $\mathcal{C}_X$ over X such that K is determined as the class of all coalgebras $\mathcal{A}$ with the property that each T-morphism $\mathcal{A} \to \mathcal{C}_X$ factors through $\mathcal{A}_K$. The fact that K is a quasi-covariety is enough to ensure that this $\mathcal{A}_K$ belongs to K, because it is constructed as the image of a morphism whose domain is a disjoint union of members of K (Rutten, 2000, Theorem 17.5). The characterisation may be viewed as a formulation in this setting of a dual version of Birkhoff's variety theorem. Formulations in more general settings involving coalgebras of functors on abstract categories were subsequently developed by Awodey and Hughes (2000; 2003; see also (Hughes, 2001b)), Kurz (2000; 2001a) and Adámek and Porst (2001; 2003).

Gumm and Schröder (1998; 2001) defined a *complete* covariety of coalgebras as one that is closed under images of *total* bisimulations. In the case that T-**Coalg** has a final (or terminal) object, they showed that a class K is a complete variety iff there is a subcoalgebra $\mathcal{A}$ of the final coalgebra such that K is the class of all coalgebras whose unique morphism to the final coalgebra factors through $\mathcal{A}$. This may be seen as a refinement and extension of Rutten's characterisation of covarieties, since a final T-coalgebra is one that cofree over a one-element set. A formulation of the result over abstract categories is given in (Awodey and Hughes, 2000; Hughes, 2001b). Complete covarieties are the same as behavioural ones, in view of the above observation that image of a bisimulation from $\mathcal{A}$ is the image of a total bisimulation on a subcoalgebra of $\mathcal{A}$.

Similar notions were independently developed by Roşu (1998; 2001) for an abstract category **C** having a final object and an *inclusion system*, which is a certain notion of factorization system first developed by J. Goguen. It comprises a pair $(\mathcal{I}, \mathcal{E})$ of subcategories of **C** whose properties include that each **C**-arrow factors uniquely as an $\mathcal{E}$-arrow followed by a $\mathcal{I}$-arrow, with $\mathcal{I}$ being a preorder category, i.e. it has at most one arrow between any two objects. An $\mathcal{I}$-arrow $\mathcal{A} \to \mathcal{B}$ is called an *inclusion*, with $\mathcal{A}$ being called a *subobject* of $\mathcal{B}$. If there exists an $\mathcal{E}$-arrow $\mathcal{A} \to \mathcal{B}$ then $\mathcal{B}$ is called a *quotient* of $\mathcal{A}$. Roşu showed that if a class of **C**-objects is closed under coproducts and quotients then it has a final object that is a subobject of the final **C**-object. He defined the *sink* (or *closure*)

$\mathbf{S}(\mathcal{A})$ determined by an object $\mathcal{A}$ to be the class of all domains of morphisms that have codomain $\mathcal{A}$, and proved that if a class K of objects is closed under coproducts, quotients and domains of morphisms then it is the sink $\mathbf{S}(\mathcal{A})$ determined by a unique subobject $\mathcal{A}$ of the final $\mathbf{C}$-object, with $\mathcal{A}$ being final in K. He also used this framework to give another kind of dualisation of Birkhoff's variety theorem that we will review in Section 7.

2.1. **Examples of Behavioural Covarieties**

A deterministic automaton with input set I has a set S of states and a state-transition function of the form $I \times S \to S$ which transposes to one of the form $S \to S^I$, revealing that it can be viewed as a T-coalgebra when $TS = S^I$. Pairing this with the transpose of an output function $I \times S \to O$, where O is a set of outputs, shows that an input-output automaton can be regarded as a coalgebra for the endofunctor $(-)^I \times (-)^O$ on **Set**.

A nondeterministic transition function assigns to each input-state pair (i, s) a *set* of possible next states. We may write $s \overset{i}{\mapsto} s'$ when s' belongs to this set, signifying that there is a possible transition from s to s' on input i. Putting $\alpha(s) = \{(i, s') : s \overset{i}{\mapsto} s'\}$ makes the transition function into a coalgebra $\alpha : S \to \mathcal{P}(I \times S)$ for the functor $\mathcal{P}(I \times -)$, where $\mathcal{P}$ is the covariant powerset functor on **Set**. We may also transpose it and view it as a $\mathcal{P}(-)^I$-coalgebra. In any case a morphism $f : (S, \alpha) \to (S', \alpha')$ is characterised as a function satisfying

$$f(s) \overset{i}{\mapsto} t \quad \text{iff} \quad \exists s' \in S(s \overset{i}{\mapsto} s' \text{ and } f(s') = t).$$

A subset $X \subseteq S$ is a subcoalgebra when it is closed under transitions in the sense that if $s \in X$ and $s \overset{i}{\mapsto} s'$ for some i then $s' \in X$.

In the world of process algebra, coalgebras of this kind are thought of as *labelled* transition systems, with I being a set of labels for possible transitions between various states/processes. For functors whose construction involves $\mathcal{P}$ there will typically be no final coalgebra. On the other hand a final coalgebra will exist for systems that model finitely-branching non-determinism. A system is called *image-finite* if the set $\{s' : s \overset{i}{\mapsto} s'\}$ is finite for all pairs (i, s). Such a system may be viewed as a $\mathcal{P}_\omega(-)^I$-coalgebra, where $\mathcal{P}_\omega(S)$ is the set of all finite subsets of S. The functor $\mathcal{P}_\omega(-)^I$ is *accessible* (Makkai and Paré, 1989), which implies that $\mathcal{P}_\omega(-)^I$-**Coalg** has a final object (Barr, 1993).

There are many properties that define a class of transition systems whose corresponding class of coalgebras is a behavioural covariety. Here are some examples:

— There are no deadlocked states, where s is deadlocked if there is no transition $s \overset{i}{\mapsto} s'$ starting from s.
— Every non-deadlocked state can reach a deadlocked one in finitely many transitions.
— Every transition path $s \overset{i}{\mapsto} s' \overset{i'}{\mapsto} s'' \mapsto \cdots$ is finite (i.e. eventually reaches a deadlocked state).
— Every state has an infinite transition path starting from it.

One way to specify properties that define behavioural covarieties is to use languages from modal logic. This approach was introduced by Hennessy and Milner (1980; 1985), associating a modality $\langle i \rangle$ with each label i. A formula $\langle i \rangle\varphi$ is satisfied by state s when there is an i-transition from s to a state satisfying φ. The dual form $[i]\varphi$ asserts that every i-transition leads to a state satisfying φ. Thus if $\top$ is a constant true formula, a terminal state is one satisfying $\neg\langle i \rangle\top$ for all i. The formula $[i]\langle i' \rangle\top$ expresses that every i-transition can be followed by an i'-transition, while $\langle i \rangle[i']\neg\top$ asserts that there is an i-transition that cannot be followed by any i'-transition, and so on. Numerous process calculi have been developed that have an associated modal language for expressing their behavioural properties: see for instance (Milner, 1989; Milner et al., 1993; Hennessy and Liu, 1995; Larsen, 1990) and several of the articles in the collections (Bergstra et al., 2001; Ponse et al., 1995).

(Moss, 1999) introduced modal languages with infinitary conjunctions for specifying properties of coalgebras for very general functors on sets. (Kurz, 2001b) and (Rößiger, 2001) developed finitary modal languages for coalgebras of polynomial functors. (Roşu, 1998; 2001) developed an equational approach to specification of properties of *hidden algebras*. These are certain many-sorted algebras which have “hidden” sorts of “internal values” and operations which take at most one hidden sort in their arguments. Many hidden algebras can be represented as coalgebras. A similar equational approach to polynomial coalgebras is taken up in (Goldblatt, 2001a; 2003b). In all these theories a semantic relation

$$\mathcal{A}, s \models \varphi$$

can be defined that expresses the notion that formula φ is satisfied at state s in coalgebra $\mathcal{A}$. The coalgebra is called a *model* of φ, written $\mathcal{A} \models \varphi$, if the formula is satisfied at every one of its states. If $f : \mathcal{A} \to \mathcal{B}$ is a coalgebraic morphism, then in general $\mathcal{A}, s \models \varphi$ iff $\mathcal{B}, f(s) \models \varphi$. Hence if $\mathcal{B}$ is a model of φ then so is $\mathcal{A}$, while conversely, if $\mathcal{A} \models \varphi$ and f is an epimorphism (surjective), then $\mathcal{B} \models \varphi$. So for any formula φ, the class

$$\mathit{Mod}\,\varphi = \{\mathcal{A} : \mathcal{A} \models \varphi\}$$

of all models of φ is closed under domains and images of morphisms. From the case that f is an insertion morphism $\mathcal{A}_j \rightarrowtail \Sigma\mathcal{A}_i$ it follows also that $\mathit{Mod}\,\varphi$ is closed under disjoint unions.

In fact a basic criterion of adequacy of a logic for a particular kind of transition system is that the property

$$\mathcal{A}, s \models \varphi \quad \text{iff} \quad \mathcal{B}, t \models \varphi$$

should hold whenever there is a bisimulation from $\mathcal{A}$ to $\mathcal{B}$ that relates s to t. This readily implies that $\mathit{Mod}\,\varphi$ is closed under disjoint unions and images of bisimulations, hence is a behavioural covariety. Moreover the class $\mathit{Mod}\,\Phi$ of models of a set Φ of formulas is also a behavioural covariety, being the intersection of the classes $\mathit{Mod}\,\varphi$ for all $\varphi \in \Phi$.

The moral of this story is that for any suitable logic for transition systems, the model classes defined by the formulas of the logic will be behavioural covarieties.

2.2. Counter–Examples

Here is a example of a covariety that is not behavioural. Let T be the identity functor on **Set**, so that a T-coalgebra $\mathcal{A}$ is just an endofunction $\alpha_{\mathcal{A}} : A \to A$. This may also be viewed as a one-label transition system having a transition $a \mapsto a'$ iff $a' = \alpha_{\mathcal{A}}(a)$.

Let K be the class of such coalgebras for which the transition function is an involution, i.e. $\alpha_{\mathcal{A}} \circ \alpha_{\mathcal{A}} = 1_A$. It is readily shown that K is closed under disjoint unions, images of T-morphisms and subcoalgebras. Now let $\mathcal{A}$ be the coalgebra having A as the set $\omega = \{0, 1, 2, \dots\}$ of natural numbers and $\alpha_{\mathcal{A}}$ the successor function $n \mapsto n + 1$. Let $\mathcal{B}$ be the unique T-coalgebra on the one-element set $\{0\}$. This $\mathcal{B}$ is a final T-coalgebra, and in particular the unique function $\omega \to \{0\}$ is a morphism from $\mathcal{A}$ to $\mathcal{B}$. But $\mathcal{B} \in K$ while $\mathcal{A} \notin K$, so K is not closed under domains of morphisms.

For a slightly more interesting example, let $\mathcal{B}'$ be the "mod 2" version of $\mathcal{A}$, with $B' = \{0, 1\}$, $\alpha_{\mathcal{B}'}(0) = 1$ and $\alpha_{\mathcal{B}'}(1) = 0$. The characteristic function of the set of even numbers is a T-epimorphism $\mathcal{A} \to \mathcal{B}'$, and again $\mathcal{B}' \in K$.

Here are some more properties of labelled transition systems that define covarieties. Each fails for $\mathcal{A}$ but is satisfied by at least one of $\mathcal{B}$ and $\mathcal{B}'$, so for the present T these covarieties are not behavioural.

— Every state s is recurrent, in the sense that there is a transition path $s \overset{i}{\mapsto} s' \overset{i'}{\mapsto} \cdots \mapsto s$ returning to s.
— Every transition $s \overset{i}{\mapsto} s'$ is reversible, i.e. $s' \overset{i'}{\mapsto} s$ for some i'.
— $s \overset{i}{\mapsto} s'$ implies $s' \overset{i}{\mapsto} s$.
— $s \overset{i}{\mapsto} s$ for some label i. Or for all labels i.

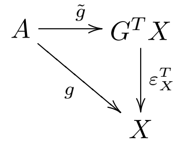
The class of all *image-finite* labelled transition systems is always a covariety of $\mathcal{P}(I \times -)$-coalgebras, but is not in general behavioural. To see this, take the one-label case again and consider $\mathcal{P}$-coalgebras. This time let $\mathcal{A} = (\omega, \alpha)$ with $\alpha(n) = \{m \in \omega : n < m\}$, so that $n \mapsto m$ iff $n < m$. Let $\mathcal{B} = (\{0\}, \beta)$ with $\beta(0) = \{0\}$, hence $0 \mapsto 0$. Again the unique function $\omega \to \{0\}$ is a morphism from $\mathcal{A}$ to $\mathcal{B}$, but now $\mathcal{B}$ is image finite while $\mathcal{A}$ is not. Note that this $\mathcal{B}$ is not a final $\mathcal{P}$-coalgebra (and indeed there is none). Any $\mathcal{P}$-coalgebra that has a deadlocked state can have no morphism to $\mathcal{B}$.

Existential assertions about transition systems provide a source of examples of quasi-covarieties that are not covarieties. For example, the class of systems that have at least one deadlocked state is closed under disjoint unions and images of morphisms, but not under subcoalgebras. Similarly the class of systems that have a recurrent state is a quasi-covariety that is not a covariety.

3. Cofreeness

Assume from now that the functor $T : \mathbf{Set} \to \mathbf{Set}$ is a *covarietor* (Adámek and Porst, 2001; 2003), which means that the forgetful functor $U : T\text{-}\mathbf{Coalg} \to \mathbf{Set}$ has a right adjoint $\mathcal{G}^T : \mathbf{Set} \to T\text{-}\mathbf{Coalg}$. We will usually write G^T for the endofunctor $U\mathcal{G}^T$: $\mathbf{Set} \to \mathbf{Set}$, so that for each set X, $G^T X$ is the underlying state set $U\mathcal{G}^T X$ of the coalgebra $\mathcal{G}^T X$. If $\varepsilon^T : U\mathcal{G}^T \to 1$ is the counit of the adjunction, then $\mathcal{G}^T X$ is *cofree over*

X in the sense that for any T-coalgebra $\mathcal{A}$, each function $g : A \to X$ has a unique lifting to a T-morphism $\tilde{g} : \mathcal{A} \to \mathcal{G}^T X$ such that the following diagram commutes:



If K is a class of T-coalgebras, then a T-coalgebra $\mathcal{G}^K X$ with a function $\varepsilon_X^K : U\mathcal{G}^K X \to X$ is called *cofree for* K *over the set* X if for any $\mathcal{A} \in K$, each function $f : A \to X$ has a unique lifting to a T-morphism $\tilde{f}^K : \mathcal{A} \to \mathcal{G}^K X$ such that $\varepsilon_X^K \circ \tilde{f}^K = f$.

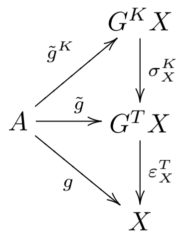
Suppose now that K is a quasi-covariety. Using the given functor $\mathcal{G}^T : \mathbf{Set} \to T\text{-}\mathbf{Coalg}$, cofree coalgebras for K that belong to K can be constructed by dualising the form of Birkhoff's classical construction in (Birkhoff, 1935) of a free algebra as a subalgebra of a direct product (see also (Burris and Sankappanavar, 1981, Theorem II.10.2)). Here we build an image of a coproduct of members of K. This construction appears to have originated in the proof of (Rutten, 2000, Theorem 17.5), as mentioned in Section 2, and was adapted in (Roşu, 1998, Lemmas 3.10, 4.7), (Gumm, 2000, Section 3.4) and (Kurz, 2001a, Proposition 2.2). Let

$$G^K X = \bigcup_{\mathcal{A} \in K} \{g(A) \,:\, \mathcal{A} \xrightarrow{g} \mathcal{G}^T X\}$$

be the union of all subcoalgebras of $\mathcal{G}^T X$ that are images of T-morphisms from members of K. $G^K X$ is itself the underlying set of a uniquely determined subcoalgebra $\mathcal{G}^K X$ of $G^T X$. To see this, let $\mathcal{S} = \{\mathcal{B}_i : i \in I\}$ be the set of all subcoalgebras of $\mathcal{G}^T X$ that are images of morphisms from members of K, let $f_i : \mathcal{A}_i \to \mathcal{B}_i$ be such a morphism for each i, with $\mathcal{A}_i \in K$, and let $f : \Sigma_I A_i \to G^T X$ be the coproduct of the f_i's. Then f is a T-morphism $\Sigma_I \mathcal{A}_i \to \mathcal{G}^T X$, and its image is a subcoalgebra of $\mathcal{G}^T X$ based on $G^K X$. This is the subcoalgebra we call $\mathcal{G}^K X$. By the closure of K under disjoint unions and images of morphisms, $\mathcal{G}^K X \in K$. Since each B_i is a subset of $G^K X$, it follows that $\mathcal{G}^K X$ is the largest member of $\mathcal{S}$. In particular

$\mathcal{G}^K X$ *is the largest subcoalgebra of* $\mathcal{G}^T X$ *that belongs to* K.

Let $\sigma_X^K : G^K X \to G^T X$ be the inclusion function, and put $\varepsilon_X^K = \varepsilon_X^T \circ \sigma_X^K$ to obtain the following situation.



(3.1)

If $\mathcal{A} \in K$, any function $g : A \to X$ lifts uniquely to a T-morphism $\tilde{g} : \mathcal{A} \to \mathcal{G}^T X$ over ε_X^T whose image is a subcoalgebra of $\mathcal{G}^T X$ included in $G^K X$ by definition of the latter, so $\tilde{g}$ factors (uniquely) through σ_X^K via a morphism $\tilde{g}^K : \mathcal{A} \to \mathcal{G}^K X$.

This construction extends to a functor $\mathcal{G}^K : \mathbf{Set} \to K$ that is right adjoint to the forgetful $U : K \to \mathbf{Set}$. As with $G^T = U\mathcal{G}^T$, we will usually write G^K for the endofunctor $U\mathcal{G}^K : \mathbf{Set} \to \mathbf{Set}$. For any function $f : Y \to X$, $\mathcal{G}^K f : \mathcal{G}^K Y \to \mathcal{G}^K X$ is defined as the restriction of $\mathcal{G}^T f$ to the subcoalgebra $\mathcal{G}^K Y$ of $\mathcal{G}^T Y$. Since $\mathcal{G}^K Y \in K$, $\mathcal{G}^T f$ does indeed map $G^K Y$ into $G^K X$, so the following diagram commutes.

$$\begin{array}{ccc} G^K Y & \xrightarrow{G^K f} & G^K X \\ {\scriptstyle \sigma^K_Y}\downarrow & & \downarrow{\scriptstyle \sigma^K_X} \\ G^T Y & \xrightarrow{G^T f} & G^T X \\ {\scriptstyle \varepsilon^T_Y}\downarrow & & \downarrow{\scriptstyle \varepsilon^T_X} \\ Y & \xrightarrow{f} & X \end{array} \tag{3.2}$$

The outer perimeter shows that $\mathcal{G}^K f$ is the unique lifting of $f \circ \varepsilon^K_Y$ to a morphism $\mathcal{G}^K Y \to \mathcal{G}^K X$ over ε^K_X. Note that the commuting of the top square of (3.2) shows that the inclusion functions σ^K_X are the components of a natural transformation $\sigma^K : G^K \to G^T$. The counit $\varepsilon^K : U\mathcal{G}^K \to 1$ of the adjunction between $\mathcal{G}^K$ and U is $\varepsilon^T \circ \sigma^K$ as above. The unit is $\eta^K : 1 \to \mathcal{G}^K U$, where $\eta^K_{\mathcal{A}}$ is the unique lifting of the identity map $1_A : A \to A$ to a T-morphism $\mathcal{A} \to \mathcal{G}^K A$ over ε^K_A. In other words, put $g = 1_A$ in diagram (3.1) to obtain $\eta^K_{\mathcal{A}}$ as $\tilde{g}^K$. Then $\varepsilon^K_A \circ \eta^K_{\mathcal{A}} = 1_A$, so as T is a functor,

$$T\varepsilon^K_A \circ T\eta^K_{\mathcal{A}} = 1_{TA} \tag{3.3}$$

for all $\mathcal{A} \in K$. Also when $g = 1_A$, $\tilde{g}$ itself in (3.1) is the component $\eta^T_{\mathcal{A}} : \mathcal{A} \to \mathcal{G}^T A$ of the unit $\eta^T : 1 \to \mathcal{G}^T U$ of the adjunction between $\mathcal{G}^T$ and $U : T\textbf{-Coalg} \to \mathbf{Set}$. Consequently

$$\eta^T_{\mathcal{A}} = \sigma^K_A \circ \eta^K_{\mathcal{A}} \tag{3.4}$$

whenever $\mathcal{A} \in K$. Equations (3.3) and (3.4) will be important later.

Theorem 3.1. If K is a behavioural covariety, then

$$\begin{array}{ccc} G^K Y & \xrightarrow{G^K f} & G^K X \\ {\scriptstyle \sigma^K_Y}\downarrow & & \downarrow{\scriptstyle \sigma^K_X} \\ G^T Y & \xrightarrow{G^T f} & G^T X \end{array}$$

is a pullback in $\mathbf{Set}$ for all functions $f : Y \to X$.

Proof. Let C be the pre-image $(G^T f)^{-1}(G^K X)$. Then $G^K Y \subseteq C$ because the diagram commutes. But the inclusion $C \hookrightarrow G^T Y$ is a pullback of σ^K_X along $G^T f$ in $\mathbf{Set}$, so to prove the Theorem it suffices to show that $C = G^K Y$. We use the fact that T-morphic pre-images of subcoalgebras are subcoalgebras, as explained in Section 1.

Now $\mathcal{G}^K X$ is a subcoalgebra of $\mathcal{G}^T X$, so its pre-image C is the underlying set of a subcoalgebra $\mathcal{C}$ of $\mathcal{G}^T Y$, and the restriction of $G^T f$ to C is a morphism from $\mathcal{C}$ to $\mathcal{G}^K X$. But $\mathcal{G}^K X \in K$ and K is closed under domains of morphisms, so $\mathcal{C} \in K$, which forces $C \subseteq G^K Y$ as $\mathcal{G}^K Y$ is the largest subcoalgebra of $\mathcal{G}^T Y$ belonging to K. □

4. Covarieties from Comonads

A *comonad* $\mathbb{G} = (G, \varepsilon, \delta)$ on **Set** consists of an endofunctor $G : \mathbf{Set} \to \mathbf{Set}$ and two natural transformations $\varepsilon : G \to 1$ and $\delta : G \to GG$ such that the following commute for each set X:

$$\begin{array}{ccc} GX & \xrightarrow{\delta_X} & G^2X \\ {\scriptstyle \delta_X}\downarrow & & \downarrow{\scriptstyle G\delta_X} \\ G^2X & \xrightarrow[\delta_{GX}]{} & G^3X \end{array} \qquad \begin{array}{ccccc} & & GX & & \\ & {\scriptstyle 1}\swarrow & \downarrow{\scriptstyle \delta_X} & \searrow{\scriptstyle 1} & \\ GX & \xleftarrow[G\varepsilon_X]{} & G^2X & \xrightarrow[\varepsilon_{GX}]{} & GX \end{array} \tag{4.1}$$

A $\mathbb{G}$-coalgebra is a G-coalgebra $\mathcal{A}$ for which the following commute:

$$\begin{array}{ccc} A & \xrightarrow{\alpha_{\mathcal{A}}} & GA \\ {\scriptstyle \alpha_{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle G\alpha_{\mathcal{A}}} \\ GA & \xrightarrow[\delta_A]{} & G^2A \end{array} \qquad \begin{array}{ccc} A & & \\ {\scriptstyle \alpha_{\mathcal{A}}}\downarrow & \searrow{\scriptstyle 1} & \\ GA & \xrightarrow[\varepsilon_A]{} & A \end{array} \tag{4.2}$$

For example, definition (4.1) itself states that (GX, δ_X) is a $\mathbb{G}$-coalgebra for any set X, a fact that will be used in Section 7. Note that the condition $\varepsilon_A \circ \alpha_{\mathcal{A}} = 1$ in (4.2) implies that $\alpha_{\mathcal{A}}$ is mono (injective) and ε_A is epi (surjective) in **Set**.

The full subcategory of G-**Coalg** consisting of the $\mathbb{G}$-coalgebras is denoted $\mathbb{G}$-**Coalg**. This is the Eilenberg-Moore category determined by the comonad $\mathbb{G}$ (Eilenberg and Moore, 1965). The assignment $X \mapsto (GX, \delta_X)$ is the object part of a functor $\mathbf{Set} \to \mathbb{G}\text{-}\mathbf{Coalg}$ that is right adjoint to the forgetful functor $U : \mathbb{G}\text{-}\mathbf{Coalg} \to \mathbf{Set}$. It follows that U preserves colimits. In particular it preserves epi's, and so a morphism is epi in $\mathbb{G}$-**Coalg** iff it is epi (surjective) in **Set**, because an epi as an arrow f for which

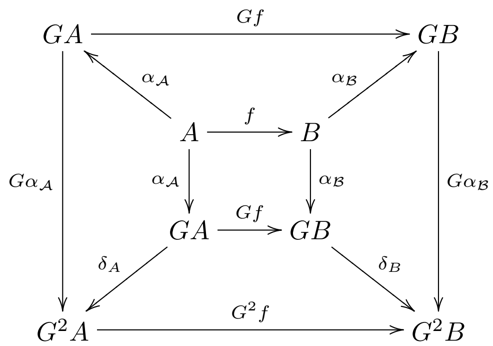
$$\begin{array}{ccc} \bullet & \xrightarrow{f} & \bullet \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle 1} \\ \bullet & \xrightarrow{1} & \bullet \end{array}$$

is a colimit diagram. Moreover, $U : \mathbb{G}\text{-}\mathbf{Coalg} \to \mathbf{Set}$ creates colimits (Borceux, 1994, dual of Proposition 4.3.1). This implies that every set of $\mathbb{G}$-coalgebras has a coproduct in $\mathbb{G}$-**Coalg** that is the same as its coproduct in G-**Coalg**, i.e. $\mathbb{G}$-**Coalg** is closed under disjoint unions.

Theorem 4.1. $\mathbb{G}$-**Coalg** is a covariety within G-**Coalg**.

Proof. We have just noted that $\mathbb{G}$-**Coalg** is closed under disjoint unions.

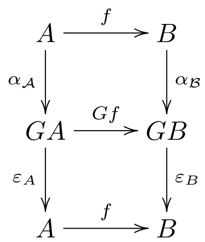
If $f : \mathcal{A} \to \mathcal{B}$ is a G-morphism, then the central square of



(4.3)

commutes (and is the same as the upper quadrangle). Hence as G is a functor, the outer perimeter commutes. Also the lower quadrangle commutes by naturality of δ.

Now suppose that $\mathcal{A}$ is a $\mathbb{G}$-coalgebra and f is epi in G-**Coalg**. Then the left quadrangle commutes by (4.2), allowing a chase around the diagram to show $G\alpha_{\mathcal{B}} \circ \alpha_{\mathcal{B}} \circ f = \delta_B \circ \alpha_{\mathcal{B}} \circ f$, and therefore $G\alpha_{\mathcal{B}} \circ \alpha_{\mathcal{B}} = \delta_B \circ \alpha_{\mathcal{B}}$ as f is epi in **Set**. Hence the right quadrangle commutes, which is the first requirement for $\mathcal{B}$ to be a $\mathbb{G}$-coalgebra. The second is that $\varepsilon_B \circ \alpha_{\mathcal{B}} = 1$. But this follows from the commuting diagram



because $\varepsilon_A \circ \alpha_{\mathcal{A}} = 1$ (4.2), so $\varepsilon_B \circ \alpha_{\mathcal{B}} \circ f = f$, and therefore the desired result follows as f is epi. This shows that $\mathbb{G}$-**Coalg** is closed under codomains of epis.

Now suppose instead that $\mathcal{A}$ is a subcoalgebra of $\mathcal{B}$ and $\mathcal{B}$ belongs to $\mathbb{G}$-**Coalg**, with $f : \mathcal{A} \to \mathcal{B}$ being the inclusion morphism. Then $\varepsilon_B \circ \alpha_{\mathcal{B}} = 1$, and so from the last diagram $f \circ \varepsilon_A \circ \alpha_{\mathcal{A}} = f$, hence we can cancel the **Set**-mono f to conclude $\varepsilon_A \circ \alpha_{\mathcal{A}} = 1$. Also the right quadrangle of the diagram (4.3) commutes, from which it follows that $G^2 f \circ G\alpha_{\mathcal{A}} \circ \alpha_{\mathcal{A}} = G^2 f \circ \delta_A \circ \alpha_{\mathcal{A}}$. Then if $A \neq \emptyset$, $G^2 f$ is mono because f is mono and G^2 preserves monos with non-empty domain, so $G\alpha_{\mathcal{A}} \circ \alpha_{\mathcal{A}} = \delta_A \circ \alpha_{\mathcal{A}}$, i.e. the left quadrangle commutes. But it commutes anyway if $A = \emptyset$, so altogether it is established that $\mathcal{A}$ is a $\mathbb{G}$-coalgebra. Hence $\mathbb{G}$-**Coalg** is closed under subcoalgebras. □

Now if K is a quasi-covariety of T-coalgebras, then arising from the adjunction $U \dashv \mathcal{G}^K$ is the comonad $\mathbb{G}^K = (G^K, \varepsilon^K, \delta^K)$ on **Set**, where $\varepsilon^K = \varepsilon^T \circ \sigma^K$ as defined earlier, and $\delta^K = U\eta^K \mathcal{G}^K$ in the sense that for each set A, δ^K_A is the function

$$\eta^K_{\mathcal{G}^K A} : G^K A \to G^K(G^K A)$$

that is the unique lifting of the identity on $G^K A$ to a T-morphism $\mathcal{G}^K A \to \mathcal{G}^K(G^K A)$ over

$\varepsilon^K_{G^K A}$ (Mac Lane, 1971, p. 134). There is a *comparison functor* $\chi^K : K \to \mathbb{G}^K$**-Coalg** such that for each T-coalgebra $\mathcal{A} \in K$, $\chi^K \mathcal{A}$ is the $\mathbb{G}^K$-coalgebra on A with transition structure $\eta^K_{\mathcal{A}} : A \to G^K A$. The closure of K under disjoint unions and images of morphisms is needed for $\mathbb{G}^K$ to be defined in the first place. If K is also closed under subcoalgebras, then χ^K is an isomorphism representing the covariety as the Eilenberg-Moore category of a comonad. This phenomenon is well-established in a number of formulations – see (Kurz, 2000, Corollary 2.5.11), (Adámek and Porst, 2003, Theorem 6.21) and (Awodey and Hughes, 2003, Corollary 4.8), as well as (Worrell, 2000, p. 101) for the case that $K = T$**-Coalg**. In each of these references the result is proved by application of a famous theorem of Beck (Mac Lane, 1971, Theorem VI.8.1) giving criteria for comonadicity in terms of creation of special equalisers. But for categories of coalgebras it is possible to give a direct and informative proof that χ^K is an isomorphism by explicitly defining a functor $\psi : \mathbb{G}^K$**-Coalg** $\to K$ that is inverse to χ^K, as we will now show (see also (Jacobs, 1995) for an approach of this kind for $K = T$**-Coalg** when T is a polynomial functor).

Theorem 4.2. If K is a covariety, then $\chi^K : K \to \mathbb{G}^K$**-Coalg** is an isomorphism of categories.

Proof. Any $\mathbb{G}^K$-morphism $f : \mathcal{A} \to \mathcal{B}$ gives rise to the commuting diagram

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha_{\mathcal{A}}} & G^K A & \xrightarrow{\alpha_{\mathcal{G}^K A}} & TG^K A & \xrightarrow{T\varepsilon^K_A} & TA \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle G^K f} & & \downarrow{\scriptstyle TG^K f} & & \downarrow{\scriptstyle Tf} \\ B & \xrightarrow[\alpha_{\mathcal{B}}]{} & G^K B & \xrightarrow[\alpha_{\mathcal{G}^K B}]{} & TG^K B & \xrightarrow[T\varepsilon^K_B]{} & TB \end{array} \tag{4.4}$$

The left square commutes as f is a G^K-morphism, the middle square as $G^K f$ is a T-morphism, and the right square by applying T to the boundary of diagram (3.2) for this f. Now let $\alpha_{\psi\mathcal{A}}$ be the composition

$$T\varepsilon^K_A \circ \alpha_{\mathcal{G}^K A} \circ \alpha_{\mathcal{A}} \tag{4.5}$$

of the three arrows forming the top edge of (4.4), and put $\psi\mathcal{A} = (A, \alpha_{\psi\mathcal{A}})$. Likewise $\psi\mathcal{B}$ is the T-algebra on B given by composing the bottom edge of (4.4). The commuting boundary of the diagram shows that f becomes a T-morphism $\psi\mathcal{A} \to \psi\mathcal{B}$, so we let ψf be f as a T-morphism. It is easily checked that ψ thus defined is a functor from $\mathbb{G}^K$**-Coalg** to T**-Coalg**.

First we show that ψ is left inverse to χ^K, i.e. $\psi \circ \chi^K = 1$. Let $\mathcal{C} = (A, \alpha_{\mathcal{C}})$ be any T-coalgebra on A that belongs to K. Then $\chi^K \mathcal{C}$ has transition $\eta^K_{\mathcal{C}} : A \to G^K A$, which is a morphism $\mathcal{C} \to \mathcal{G}^K A$, so the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{\eta^K_{\mathcal{C}}} & G^K A \\ {\scriptstyle \alpha_{\mathcal{C}}}\downarrow & & \downarrow{\scriptstyle \alpha_{\mathcal{G}^K A}} \\ TA & \xrightarrow{T\eta^K_{\mathcal{C}}} & TG^K A \end{array}$$

Taking $\mathcal{A}$ as $\chi^K\mathcal{C}$ in (4.5) shows that the transition structure of $\psi(\chi^K\mathcal{C})$ is $T\varepsilon^K_A \circ \alpha_{\mathcal{G}^K A} \circ$

$\eta^K_{\mathcal{C}}$, which is equal to $T\varepsilon^K_A \circ T\eta^K_{\mathcal{C}} \circ \alpha_{\mathcal{C}}$ by the last diagram. But $T\varepsilon^K_A \circ T\eta^K_{\mathcal{C}} = 1_{TA}$ by (3.3), so the transition of $\psi(\chi^K\mathcal{C})$ is just $\alpha_{\mathcal{C}}$, i.e. $\psi(\chi^K\mathcal{C}) = \mathcal{C}$ as desired.

Now we can use this result to show that ψ maps $\mathbb{G}^K$**-Coalg** into K. Given a $\mathbb{G}^K$-coalgebra $\mathcal{A}$ with underlying state set A, we have $\mathcal{G}^K A \in K$. Hence $\chi^K(\mathcal{G}^K A)$ is a $\mathbb{G}^K$-coalgebra with transition structure $\eta^K_{\mathcal{G}^K A}$, which is just $\delta^K_{\mathcal{A}}$ by definition of the later. So in this case the commuting square of (4.2) is

$$\begin{array}{ccc} A & \xrightarrow{\alpha_{\mathcal{A}}} & G^K A \\ {\scriptstyle \alpha_{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle G^K\alpha_{\mathcal{A}}} \\ G^K A & \xrightarrow[\eta^K_{\mathcal{G}^K A}]{} & G^K G^K A \end{array}$$

which shows that $\alpha_{\mathcal{A}}$ is itself a G^K-morphism from $\mathcal{A}$ to $\chi^K(\mathcal{G}^K A)$. Hence $\alpha_{\mathcal{A}}$ as $\psi\alpha_{\mathcal{A}}$ becomes a T-morphism from $\psi\mathcal{A}$ to $\psi\chi^K(\mathcal{G}^K A) = \mathcal{G}^K A$. But $\alpha_{\mathcal{A}}$ is injective, from the triangle in (4.2), so it makes $\psi\mathcal{A}$ isomorphic to a subcoalgebra of $\mathcal{G}^K A$. Since K is closed under subcoalgebras and isomorphism, this implies $\psi\mathcal{A} \in K$ as desired.

It now follows quickly that ψ is right inverse to χ^K. For if $\mathcal{A}$ is any $\mathbb{G}^K$-coalgebra, we have just seen that $\alpha_{\mathcal{A}}$ is a T-morphism of the form $\psi\mathcal{A} \to \mathcal{G}^K A$. But $\eta^K_{\psi\mathcal{A}}$ is, by definition, the unique such T-morphism lifting 1_A over ε^K_A. So $\mathcal{A} = (A, \eta^K_{\psi\mathcal{A}})$, and the latter is $\chi^K(\psi\mathcal{A})$. Hence $\chi^K \circ \psi = 1$.

□

In particular, taking K as the covariety of *all* T-coalgebras, we see that T**-Coalg** itself is represented as the Eilenberg-Moore category $\mathbb{G}^T$**-Coalg**, where $\mathbb{G}^T = (G^T, \varepsilon^T, \delta^T)$ is the comonad arising from the adjunction $U \dashv G^T$. The comparison functor $\chi^T : T\textbf{-Coalg} \to \mathbb{G}^T\textbf{-Coalg}$ is an isomorphism, and can be used to identify subcovarieties of T**-Coalg** with subcovarieties of $\mathbb{G}^T$**-Coalg**, and vice versa.

5. Pure Subcomonads

The results of the last two sections show that any behavioural covariety K in T**-Coalg** is isomorphic to $\mathbb{G}^K$**-Coalg**, where the comonad $\mathbb{G}^K$ is related to the comonad $\mathbb{G}^T$ by the fact that the naturality squares of the transformation $\sigma^K : G^K \to G^T$ are pullbacks, as in Theorem 3.1. Our aim now is to show how any comonad that relates to $\mathbb{G}^T$ in this way arises from a behavioural covariety.

A *morphism* from comonad $\mathbb{F} = (F, \varepsilon^F, \delta^F)$ to comonad $\mathbb{G} = (G, \varepsilon^G, \delta^G)$ on **Set** (Barr and Wells, 1985, Section 3.6) is a natural transformation $\sigma : F \to G$ making the diagrams

$$\begin{array}{ccc} FX & \xrightarrow{\sigma_X} & GX \\ & {\scriptstyle \varepsilon^F_X}\searrow & \downarrow{\scriptstyle \varepsilon^G_X} \\ & & X \end{array} \qquad \begin{array}{ccc} FX & \xrightarrow{\delta^F_X} & FFX \\ {\scriptstyle \sigma_X}\downarrow & & \downarrow{\scriptstyle (\sigma_X)^2} \\ GX & \xrightarrow{\delta^G_X} & GGX \end{array} \tag{5.1}$$

commute for all sets X, where $(\sigma_X)^2 = G\sigma_X \circ \sigma_{FX} = \sigma_{GX} \circ F\sigma_X$:

$$\begin{array}{ccc} FFX & \xrightarrow{\sigma_{FX}} & GFX \\ {\scriptstyle F\sigma_X}\downarrow & & \downarrow{\scriptstyle G\sigma_X} \\ FGX & \xrightarrow{\sigma_{GX}} & GGX \end{array} \tag{5.2}$$

The pair $(\mathbb{F}, \sigma)$ will be called a *subcomonad of* $\mathbb{G}$ if each component σ_X of σ is mono in **Set**. The subcomonad is *pure*[‡] if the diagram

$$\begin{array}{ccc} FY & \xrightarrow{Ff} & FX \\ {\scriptstyle \sigma_Y}\downarrow & & \downarrow{\scriptstyle \sigma_X} \\ GY & \xrightarrow{Gf} & GX \end{array}$$

is a pullback in **Set** for any function $f : Y \to X$.

Theorem 5.1. For any quasi-covariety K, $\sigma^K : G^K \to G^T$ is a comonad morphism from $\mathbb{G}^K$ to $\mathbb{G}^T$.

Proof. By definition $\varepsilon^K_X = \varepsilon^T_X \circ \sigma^K_X$, which is the first comonad morphism condition. The second is that $\delta^T_X \circ \sigma^K_X = (\sigma^K_X)^2 \circ \delta^K_X$. To show this we prove that each of $\delta^T_X \circ \sigma^K_X$ and $(\sigma^K_X)^2 \circ \delta^K_X$ is the underlying function of a T-morphism that composes with $\varepsilon^T_{G^T X}$ to give σ^K_X. In other words, each gives a lifting of σ^K_X to a T-morphism $\mathcal{G}^K X \to \mathcal{G}^T G^T X$ over $\varepsilon^T_{G^T X}$. But by the cofreeness of $\mathcal{G}^T G^T X$ over $\varepsilon^T_{G^T X}$ there is only one such morphism.

First, $\delta^T_X \circ \sigma^K_X = \eta^T_{\mathcal{G}^T X} \circ \sigma^K_X$ is the T-morphism restricting $\eta^T_{\mathcal{G}^T X} : \mathcal{G}^T X \to \mathcal{G}^T G^T X$ to the subcoalgebra $\mathcal{G}^K X$ of $\mathcal{G}^T X$, and

$$\varepsilon^T_{G^T X} \circ (\delta^T_X \circ \sigma^K_X) = (\varepsilon^T_{G^T X} \circ \eta^T_{\mathcal{G}^T X}) \circ \sigma^K_X = 1_{G^T X} \circ \sigma^K_X = \sigma^K_X.$$

Secondly, by the definition (5.2) of $(\sigma^K_X)^2$,

$$(\sigma^K_X)^2 \circ \delta^K_X = G^T \sigma^K_X \circ \sigma^K_{G^K X} \circ \eta^K_{\mathcal{G}^K X} = G^T \sigma^K_X \circ \eta^T_{\mathcal{G}^K X}$$

by (3.4), which is the underlying function of the T-morphism $\mathcal{G}^T \sigma^K_X \circ \eta^T_{\mathcal{G}^K X}$. Also, by the naturality of ε^T, $\varepsilon^T_{G^T X} \circ G^T \sigma^K_X = \sigma^K_X \circ \varepsilon^T_{G^K X}$, so

$$\varepsilon^T_{G^T X} \circ \big((\sigma^K_X)^2 \circ \delta^K_X\big) = \varepsilon^T_{G^T X} \circ G^T \sigma^K_X \circ \eta^T_{\mathcal{G}^K X} = \sigma^K_X \circ \varepsilon^T_{G^K X} \circ \eta^T_{\mathcal{G}^K X} = \sigma^K_X \circ 1_{G^K X} = \sigma^K_X.$$

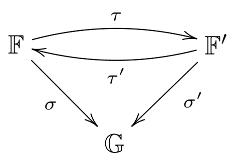
□

Corollary 5.2. If K is a behavioural covariety, then $(\mathbb{G}^K, \sigma^K)$ is a pure subcomonad of $\mathbb{G}^T$.

Proof. By Theorem 3.1. □

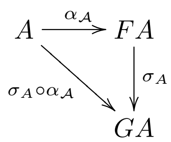
‡ The term "pure" subcomonad was suggested by Peter Johnstone, by association with the notion from module theory of a *pure submodule* as one with the property that everything which "ought to be" in the submodule is actually there.

The subcomonads of a comonad $\mathbb{G}$ can be pre-ordered by putting $(\mathbb{F}, \sigma) \leq (\mathbb{F}', \sigma')$ iff there exists a morphism $\tau : \mathbb{F} \to \mathbb{F}'$ that factors σ through σ', i.e. $\sigma = \sigma' \circ \tau$. In general there can be at most one such τ, as follows because the components of σ' are mono. The two subcomonads are *isomorphic*, $(\mathbb{F}, \sigma) \cong (\mathbb{F}', \sigma')$, if both $(\mathbb{F}, \sigma) \leq (\mathbb{F}', \sigma')$ and $(\mathbb{F}', \sigma') \leq (\mathbb{F}, \sigma)$. In that case there is a commuting diagram of morphisms

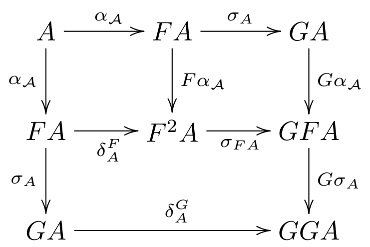


in which τ and τ' are mutually inverse and give a natural isomorphism between the underlying functors of $\mathbb{F}$ and $\mathbb{F}'$.

Now any morphism $\sigma : \mathbb{F} \to \mathbb{G}$ induces a functor $\varphi\sigma : \mathbb{F}$-**Coalg** $\to \mathbb{G}$-**Coalg**, taking each $\mathbb{F}$-coalgebra $\mathcal{A}$ to the G-coalgebra $\varphi\sigma\mathcal{A} = (A, \sigma_A \circ \alpha_{\mathcal{A}})$ on the same state set:



To see that $\varphi\sigma\mathcal{A}$ is in fact a $\mathbb{G}$-coalgebra, consider the diagram



(5.3)

The upper left square commutes because $\mathcal{A}$ is an $\mathbb{F}$-coalgebra (4.2), and the upper right one commutes by naturality of σ. Since $G\sigma_A \circ \sigma_{FA} = (\sigma_A)^2$ (5.2), the lower rectangle commutes because σ is a comonad morphism (5.1). Hence the outer perimeter of the diagram commutes, which is the first requirement for $\varphi\sigma\mathcal{A}$ to be a $\mathbb{G}$-coalgebra (4.2). The second is that $\varepsilon^G_A \circ (\sigma_A \circ \alpha_{\mathcal{A}}) = 1_A$. But that follows directly from $\varepsilon^G_A \circ \sigma_A = \varepsilon^F_A$ (5.1) and $\varepsilon^F_A \circ \alpha_{\mathcal{A}} = 1_A$ (as $\mathcal{A}$ is an $\mathbb{F}$-coalgebra).

If the components σ_X of σ are mono, then the transition structure of $\varphi\sigma\mathcal{A}$ comes via σ_A from a *unique* F-transition on A, and so $\varphi\sigma$ is injective on objects.

The action of $\varphi\sigma$ on an arrow $f : \mathcal{A} \to \mathcal{B}$ from $\mathbb{F}$-**Coalg** is indicated by the commuting diagram

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \alpha_{\mathcal{A}}}\downarrow & & \downarrow{\scriptstyle \alpha_{\mathcal{B}}} \\ FA & \xrightarrow{Ff} & FB \\ {\scriptstyle \sigma_A}\downarrow & & \downarrow{\scriptstyle \sigma_B} \\ GA & \xrightarrow{Gf} & GB \end{array}$$

whose outer perimeter shows that the function f also determines an arrow $\varphi\sigma f$ from $\varphi\sigma\mathcal{A}$ to $\varphi\sigma\mathcal{B}$, i.e. $U\varphi\sigma f = f$. This implies that $\varphi\sigma$ is faithful (injective on hom-sets). If $(\mathbb{F}, \sigma)$ is a subcomonad of $\mathbb{G}$, then $\varphi\sigma$ is also full (surjective on hom-sets). For, if $f : \varphi\sigma\mathcal{A} \to \varphi\sigma\mathcal{B}$ is any G-morphism, the outer perimeter of the last diagram commutes, so as the lower square commutes we get $\sigma_B \circ (\alpha_{\mathcal{B}} \circ f) = \sigma_B \circ (Ff \circ \alpha_{\mathcal{A}})$, hence when σ_B is mono it follows that the upper square commutes, and so f determines an F-morphism $\mathcal{A} \to \mathcal{B}$ whose $\varphi\sigma$-image is f.

We denote by Im$\varphi\sigma$ the full subcategory of $\mathbb{G}$-**Coalg** based on the collection of all $\mathbb{G}$-coalgebras that are of the form $\varphi\sigma\mathcal{A}$ for some $\mathbb{F}$-coalgebra $\mathcal{A}$. The above observations combine to show that if $(\mathbb{F}, \sigma)$ is a subcomonad of $\mathbb{G}$, then $\varphi\sigma$ is full, faithful and injective on objects, and hence makes $\mathbb{F}$-**Coalg** isomorphic to Im$\varphi\sigma$.

Note also that the assignment $\sigma \mapsto \varphi\sigma$ preserves composition: given another comonad morphism $\tau : \mathbb{G} \to \mathbb{H}$ we get $\varphi(\tau \circ \sigma) = \varphi\tau \circ \varphi\sigma$.

Theorem 5.3. Let $\sigma : \mathbb{F} \to \mathbb{G}$ be any comonad morphism.

(1) Im$\varphi\sigma$ is closed under disjoint unions.
(2) If $(\mathbb{F}, \sigma)$ is a subcomonad of $\mathbb{G}$, then Im$\varphi\sigma$ is closed under codomains of epis.
(3) If $(\mathbb{F}, \sigma)$ is a pure subcomonad of $\mathbb{G}$, then Im$\varphi\sigma$ is a behavioural covariety.

Proof.

(1) Both $\mathbb{F}$-**Coalg** and $\mathbb{G}$-**Coalg** are closed under disjoint unions (Theorem 4.1). Thus a set $\{\mathcal{A}_i : i \in I\}$ of $\mathbb{F}$-coalgebras has a disjoint union $\Sigma_I\mathcal{A}_i$ in $\mathbb{F}$-**Coalg**, and gives rise to the disjoint union $\Sigma_I\varphi\sigma\mathcal{A}_i$ in $\mathbb{G}$-**Coalg**. But it is readily seen that

$$\Sigma_I\varphi\sigma\mathcal{A}_i = \varphi\sigma\Sigma_I\mathcal{A}_i,$$

implying that Im$\varphi\sigma$ is closed under disjoint unions. The reason is conveyed by the commuting diagram

$$\begin{array}{ccc} A_j & \xrightarrow{\iota_j} & \Sigma_I A_i \\ {\scriptstyle \alpha_{\mathcal{A}_j}}\downarrow & & \downarrow{\scriptstyle \alpha_\Sigma} \\ FA_j & \xrightarrow{F\iota_j} & F\Sigma_I A_i \\ {\scriptstyle \sigma_{A_j}}\downarrow & & \downarrow{\scriptstyle \sigma_{\Sigma_I A_i}} \\ GA_j & \xrightarrow{G\iota_j} & G\Sigma_I A_i \end{array}$$

Here α_Σ is the transition structure of the F-coalgebra $\Sigma_I\mathcal{A}_i$ (see diagram (1.1)). The transition structure of $\varphi\sigma\Sigma_I\mathcal{A}_i$ is thus the right side $\sigma_{\Sigma_I A_i} \circ \alpha_\Sigma$ of the diagram, which, by the co-universal property of $\Sigma_I A_i$ in **Set**, is the unique function $\Sigma_I A_i \to G\Sigma_I A_i$ making the outer perimeter of the diagram commute for every $j \in I$. But that is just the transition structure of $\Sigma_I\varphi\sigma\mathcal{A}_i$, since the left side of the diagram is the transition of $\varphi\sigma\mathcal{A}_j$.

(2) Let $\mathcal{B}$ be an $\mathbb{F}$-coalgebra and $f : \varphi\sigma\mathcal{B} \to \mathcal{A}$ be an epi in $\mathbb{G}$-**Coalg**. We need to prove that $\mathcal{A}$ belongs to $\mathrm{Im}\varphi\sigma$ to show that $\mathrm{Im}\varphi\sigma$ is closed under codomains of epis.
Now $f : B \to A$ is epi in **Set**, i.e. surjective, as noted just prior to Theorem 4.1. Let $g : A \to B$ be a right inverse to f (i.e. $f \circ g = 1_A$) and put $\beta = Ff \circ \alpha_{\mathcal{B}} \circ g : A \to FA$:

$$\begin{array}{ccccc} B & \xrightarrow{\quad f \quad} & & & A \\ \downarrow{\scriptstyle \alpha_{\mathcal{B}}} & & & \swarrow{\scriptstyle\beta} & \\ FB & \xrightarrow{Ff} & FA & & \downarrow{\scriptstyle\alpha_{\mathcal{A}}} \\ \downarrow{\scriptstyle\sigma_B} & & & \searrow{\scriptstyle\sigma_A} & \\ GB & & \xrightarrow{\quad Gf \quad} & & GA \end{array} \tag{5.4}$$

The right triangle of this last diagram commutes. To prove this it suffices, since is f is epi, to show that $\alpha_{\mathcal{A}} \circ f = \sigma_A \circ \beta \circ f$. But from the definition of β, and the fact that the outer perimeter and lower quadrangle of this diagram commute, we get

$$\sigma_A \circ \beta \circ f = \sigma_A \circ Ff \circ \alpha_{\mathcal{B}} \circ g \circ f = \alpha_{\mathcal{A}} \circ f \circ g \circ f.$$

Since $f \circ g = 1_A$, this gives $\sigma_A \circ \beta \circ f = \alpha_{\mathcal{A}} \circ f$ as required, and we can cancel f on the right to deduce $\alpha_{\mathcal{A}} = \sigma_A \circ \beta$. It follows that if we can show (A, β) is an $\mathbb{F}$-coalgebra we will have $\mathcal{A} = \varphi\sigma(A, \beta) \in \mathrm{Im}\varphi\sigma$ as desired. To show this, consider the diagram

$$\begin{array}{ccccc} A & \xrightarrow{\beta} & FA & \xrightarrow{\sigma_A} & GA \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle F\beta} & & \downarrow{\scriptstyle G\beta} \\ FA & \xrightarrow[\delta^F_A]{} & F^2A & \xrightarrow[\sigma_{FA}]{} & GFA \\ \downarrow{\scriptstyle\sigma_A} & & & & \downarrow{\scriptstyle G\sigma_A} \\ GA & & \xrightarrow{\quad\delta^G_A\quad} & & GGA \end{array}$$

This is as for (5.3), but with β in place of $\alpha_{\mathcal{A}}$. We need to show that the upper left square commutes. But the upper right square and lower rectangle commute for the same reasons as given for (5.3), while the outer perimeter now commutes because $\sigma_A \circ \beta = \alpha_{\mathcal{A}}$ is the transition of a $\mathbb{G}$-coalgebra (see (4.2)). Hence

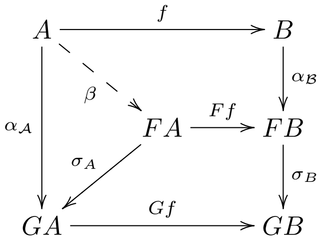
$$G\sigma_A \circ \sigma_{FA} \circ \delta^F_A \circ \beta = G\sigma_A \circ \sigma_{FA} \circ F\beta \circ \beta.$$

If $A \neq \emptyset$, then $G\sigma_A$ is mono, as σ_A is mono and G preserves monos with non-empty domain. Then $G\sigma_A \circ \sigma_{FA}$ is mono, implying $\delta^F_A \circ \beta = F\beta \circ \beta$ as desired. But if $UA = \emptyset$, the upper left square commutes anyway. Also

$$\varepsilon^F_A \circ \beta = \varepsilon^G_A \circ \sigma_A \circ \beta = \varepsilon^G_A \circ \alpha_{\mathcal{A}} = 1_A,$$

completing the proof that (A, β) is an $\mathbb{F}$-coalgebra.

(3) In view of (1) and (2), it remains in this case to show that $\mathrm{Im}\varphi\sigma$ is closed under domains of morphisms. Consider an arrow of the form $f : \mathcal{A} \to \varphi\sigma\mathcal{B}$ in $\mathbb{G}$-**Coalg**. This gives rise to a diagram analogous to (5.4):



This time the function β exists to make the whole diagram commute by the universal property of the lower quadrangle as a pullback. Since $\sigma_A \circ \beta = \alpha_{\mathcal{A}}$, the proof that (A, β) is an $\mathbb{F}$-coalgebra proceeds just as in part (2), so we again obtain $\mathcal{A} = \varphi\sigma(A, \beta)$, giving $\mathcal{A} \in \mathrm{Im}\varphi\sigma$.

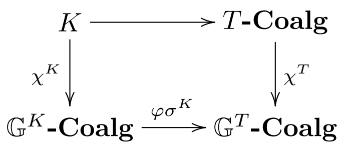
□

6. Correspondences

Given a pure subcomonad of the form $\sigma : \mathbb{F} \to \mathbb{G}^T$, we obtain a behavioural covariety K^σ of T-coalgebras by pulling $\mathrm{Im}\varphi\sigma$ back along the isomorphism of categories χ^T : T-**Coalg** $\to \mathbb{G}^T$-**Coalg**:

$$K^\sigma = (\chi^T)^{-1}\mathrm{Im}\varphi\sigma = \{\mathcal{A} \in T\textbf{-Coalg} : \chi^T\mathcal{A} \in \mathrm{Im}\varphi\sigma\}.$$

Any behavioural covariety K of T-coalgebras can presented in the form K^σ by taking σ to be the pure subcomonad $\sigma^K : \mathbb{G}^K \to \mathbb{G}^T$. This follows directly from the fact that $\varphi\sigma^K \circ \chi^K$ is just χ^T restricted to K:



(6.1)

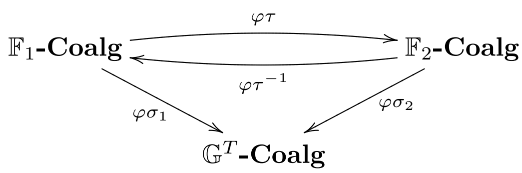
(the top arrow in this diagram is the inclusion functor). If $\mathcal{A} \in K$, then the coalgebras $\chi^K\mathcal{A}$, $(\varphi\sigma^K \circ \chi^K)\mathcal{A}$ and $\chi^T\mathcal{A}$ all have the same state set A. The transition structure of $\chi^K\mathcal{A}$ is $\eta^K_{\mathcal{A}}$, so that of $(\varphi\sigma^K \circ \chi^K)\mathcal{A}$ is $\sigma^K_A \circ \eta^K_{\mathcal{A}}$, while that of $\chi^T\mathcal{A}$ is $\eta^T_{\mathcal{A}}$. But $\eta^T_{\mathcal{A}} = \sigma^K_A \circ \eta^K_{\mathcal{A}}$ by (3.4), so $(\varphi\sigma^K \circ \chi^K)\mathcal{A} = \chi^T\mathcal{A}$ whenever $\mathcal{A} \in K$. This fact, together with the injectivity of χ^T, readily shows that

$$K = (\chi^T)^{-1}\mathrm{Im}\varphi\sigma^K = K^{\sigma^K}.$$

Isomorphic pure subcomonads of $\mathbb{G}^T$ will determine the same behavioural covariety, and indeed we have

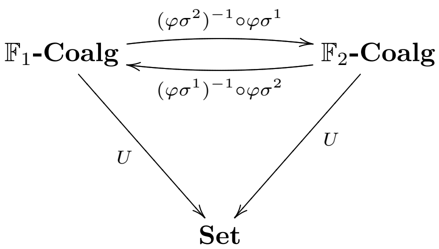
$$(\mathbb{F}_1, \sigma^1) \cong (\mathbb{F}_2, \sigma^2) \quad \text{iff} \quad K^{\sigma_1} = K^{\sigma_2}. \tag{6.2}$$

To see this, suppose first that $(\mathbb{F}_1, \sigma^1) \cong (\mathbb{F}_2, \sigma^2)$. Then there is an invertible morphism $\tau : \mathbb{F}_1 \to \mathbb{F}_2$ with $\sigma^1 = \sigma^2 \circ \tau$ and $\sigma^2 = \sigma^1 \circ \tau^{-1}$. Applying the $\varphi \mapsto \varphi\sigma$ construction of Section 5 gives a commuting functor diagram

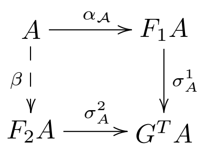


which implies that $\mathrm{Im}\varphi\sigma_1 = \mathrm{Im}\varphi\sigma_2$, and hence by definition that $K^{\sigma_1} = K^{\sigma_2}$.

The converse is more demanding. If $K^{\sigma_1} = K^{\sigma_2}$, then as χ^T is a surjection it follows that $\mathrm{Im}\varphi\sigma_1 = \mathrm{Im}\varphi\sigma_2$. But $\varphi\sigma^i : \mathbb{F}_i$-**Coalg** $\cong \mathrm{Im}\varphi\sigma^i$, so we get an isomorphism $\mathbb{F}_1$-**Coalg** $\cong \mathbb{F}_2$-**Coalg** given by $(\varphi\sigma^2)^{-1} \circ \varphi\sigma^1$, with inverse $(\varphi\sigma^1)^{-1} \circ \varphi\sigma^2$ and commutativity of



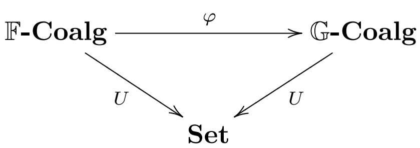
The action of $(\varphi\sigma^2)^{-1} \circ \varphi\sigma^1$ is conveyed by the diagram



(6.3)

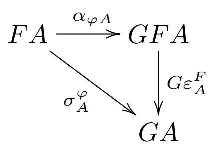
If $\mathcal{A}$ is an $\mathbb{F}_1$-coalgebra then $\varphi\sigma^1\mathcal{A} = (A, \sigma^1_A \circ \alpha_{\mathcal{A}})$ and $((\varphi\sigma^2)^{-1} \circ \varphi\sigma^1)\mathcal{A} = (A, \beta)$ for a unique function β making (6.3) commute, so that $\varphi\sigma^2(A, \beta) = \varphi\sigma^1\mathcal{A}$.

Now in general the assignment $\sigma \mapsto \varphi\sigma$ gives a bijection between comonad morphisms $\sigma : \mathbb{F} \to \mathbb{G}$ and functors $\varphi : \mathbb{F}$-**Coalg** $\to \mathbb{G}$-**Coalg** which are *U-invariant* in the sense that

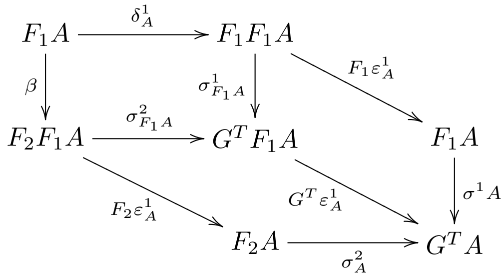


commutes. A proof of (the dual of) this is given in Theorem 3 of (Barr and Wells, 1985, Section 3.6). The inverse of $\sigma \mapsto \varphi\sigma$ assigns to each U-invariant φ the natural transformation $\sigma^\varphi : F \to G$ whose components σ^φ_A are defined as follows. For each set A the pair (FA, δ^F_A) is an $\mathbb{F}$-coalgebra (as noted just after diagrams (4.2)), so is sent by φ

to a $\mathbb{G}$-coalgebra $\alpha_{\varphi A} : FA \to GFA$. σ^{φ}_A is defined to be $G\varepsilon^F_A \circ \alpha_{\varphi A}$:



We now apply this analysis to the case that $\varphi = (\varphi\sigma^2)^{-1} \circ \varphi\sigma^1 : \mathbb{F}_1$-**Coalg** $\to \mathbb{F}_2$-**Coalg** in the discussion of (6.2). To simplify the superscript notation, let $\mathbb{F}_i = (F_i, \varepsilon^i, \delta^i)$. The situation is given by the following diagram.



The upper left square is diagram (6.3) with $\mathcal{A}$ replaced by (F_1A, δ^1_A), so that (F_1A, β) here is $\varphi(F_1A, \delta^1_A)$ in this case of $\varphi = (\varphi\sigma^2)^{-1} \circ \varphi\sigma^1$. The other two quadrangles commute by the naturality of σ^1 and σ^2. Hence the outer perimeter commutes. But in this perimeter we see that $F_2\varepsilon^1_A \circ \beta$ is just σ^{φ}_A in this case, while $F_1\varepsilon^1_A \circ \delta^1_A = 1$ by (4.1). So $\sigma^1_A = \sigma^2_A \circ \sigma^{\varphi}_A$.

This shows that the comonad morphism $\sigma^{\varphi} : \mathbb{F}_1 \to \mathbb{F}_2$ factors σ^1 through σ^2, proving that $(\mathbb{F}_1, \sigma^1) \leq (\mathbb{F}_2, \sigma^2)$. Interchanging the roles of the two pure subcomonads in this argument then leads to $(\mathbb{F}_1, \sigma^1) \cong (\mathbb{F}_2, \sigma^2)$ and completes the proof of (6.2). □

Now for any pure subcomonad $\sigma : \mathbb{F} \to \mathbb{G}^T$ we know that

$$\chi^T(K^\sigma) = \mathrm{Im}\varphi\sigma \cong \mathbb{F}\text{-}\mathbf{Coalg},$$

while putting $K = K^\sigma$ in diagram (6.1) gives

$$\chi^T(K^\sigma) = \mathrm{Im}\varphi\sigma^{K^\sigma} \cong \mathbb{G}^{K^\sigma}\text{-}\mathbf{Coalg}.$$

So $\mathrm{Im}\varphi\sigma = \mathrm{Im}\varphi\sigma^{K^\sigma}$, implying by the proof of (6.2) that there is a U-invariant isomorphism of categories $\mathbb{F}$-**Coalg** $\cong \mathbb{G}^{K^\sigma}$-**Coalg** which induces a subcomonad isomorphism $(\mathbb{F}, \sigma) \cong (\mathbb{G}^{K^\sigma}, \sigma^{K^\sigma})$, or more briefly $\sigma \cong \sigma^{K^\sigma}$. Together with (6.2) and the fact that $K = K^{\sigma^K}$, this establishes that

> *the assignments $K \mapsto (\mathbb{G}^K, \sigma^K)$ and $(\mathbb{F}, \sigma) \mapsto K^\sigma$ establish a bijection between behavioural covarieties of T-coalgebras and isomorphism classes of pure subcomonads of $\mathbb{G}^T$.*

7. Birkhoff's Theorem Revisited

The famous result of (Birkhoff, 1935) shows that a certain *logically* specified notion—an equationally definable class of algebras—has a *structural* characterisation: closure under homomorphic images (H), subalgebras (S) and direct products (P). Classical model theory subsequently developed many such "preservation theorems", characterising the class of models of arbitrary first-order sentences, or universal/ existential/ positive/ quasi-equational/ Horn ... sentences, in terms of their preservation by various constructions: ultraproducts, reduced products, direct products, substructures, extensions, homomorphic images, ... (see (Chang and Keisler, 1973, Section 3.2), (Burris and Sankappanavar, 1981, Section V.2)).

Now the term *variety* was originally used in abstract algebra to mean an equationally definable class of algebras,[§] adapting the use of that term from algebraic geometry where it meant the set of solutions of a system of equations. In recent decades Birkhoff's HSP-closure conditions have been taken as the *definition* of "variety"(e.g. (Burris and Sankappanavar, 1981, Section II.9)), and it is this structural definition that has been dualized in the definition of "covariety". A number of approaches to giving a categorical version of Birkhoff's result and its dual have sought to categorically re-formulate "equationally definable" and prove that this formulation is equivalent to the structural definition of "variety" or "covariety" (Herrlich and Ringel, 1972; Banaschewski and Herrlich, 1976; Rutten, 2000; Gumm, 2000; Awodey and Hughes, 2000; Kurz, 2000; Hughes, 2001b; Hughes, 2001a; Kurz, 2001a; Adámek and Porst, 2001; Adámek and Porst, 2003; Awodey and Hughes, 2003).

An alternative embodiment of the intention of Birkhoff's theorem would be to begin with the question of logical definability and develop an appropriate syntax and semantics that gives a notion of logical formula which naturally expresses properties of coalgebras. Then the objective would be to identify structural operations that lead to a characterisation of classes of models of such formulas as being those classes closed under the operations . An approach of this kind was given in (Roşu, 1998; Roşu, 2001) for hidden algebras, defining an algebra to *behaviourally satisfy* an equation $t = t'$ if it satisfies every equation $c[t] = c[t']$ where c is a term of visible sort having a single variable of the same hidden sort as t and t'. A class was shown (Roşu, 1998, Theorem 3.12) to be the class of all hidden algebras behaviourally satisfying some set of equations iff it is closed under coproducts, domains and images of morphisms, and "representative inclusions". Here an inclusion of algebras $\mathcal{A} \hookrightarrow \mathcal{B}$ is representative if $\mathcal{B}$ satisfies every equation that $\mathcal{A}$ does. Roşu then developed a categorical approach to this result by working in an abstract category $\mathbf{C}$ with a final object $\mathcal{F}$ and an inclusion system $(\mathcal{I}, \mathcal{E})$ as described earlier in Section 2. The existence was assumed of a map E assigning to each $\mathbf{C}$-object $\mathcal{A}$ a set $E\mathcal{A}$ to be thought of as "all equations satisfied by $\mathcal{A}$". In this situation a class K of objects can be called *definable* if there exists a set Γ such that K contains exactly those objects $\mathcal{A}$ for which $\Gamma \subseteq E\mathcal{A}$. E was assumed to have properties expressing the principle that satisfaction is preserved by domains of morphisms, quotients and coproducts, so a definable class is always an abstract behavioural covariety. It is also closed under *representative inclusions*, where an inclusion ($\mathcal{I}$-arrow) $\mathcal{A} \to \mathcal{B}$ is now defined to be representative when $E\mathcal{A} \subseteq E\mathcal{B}$. Roşu characterised the definable classes as being precisely the behavioural

[§] Philip Hall used the term in this way in lectures in the 1940's (information supplied to the author by Paul Cohn and Bernhard Neumann).

covarieties that are closed under representative inclusions, as well as being the sinks $\mathbf{S}(\mathcal{A})$ of those subobjects $\mathcal{A}$ of the final $\mathbf{C}$-object $\mathcal{F}$ that are *maximal* in the sense that any subobject $\mathcal{A}'$ of $\mathcal{F}$ with $E\mathcal{A} \subseteq E\mathcal{A}'$ is a subobject of $\mathcal{A}$.

Another approach to Birkhoff's theorem from the logical side was intitiated in (Goldblatt, 2001b) for coalgebras of certain "monomial" functors, constructible from constant-valued functors and the identity functor by forming binary products and exponentials $X \mapsto (TX)^D$ with constant exponent D. This involves a notion of *ultrafilter enlargement* of a coalgebra, an analogue of the Stone space of a Boolean algebra. The construction can be illustrated with the functor T on **Set** that has $TX = X \times \{0, 1\}$. The transition structure of a T-coalgebra $\mathcal{A}$ comprises in essence a pair of functions $\tau_{\mathcal{A}} : A \to A$ and $\omega_{\mathcal{A}} : A \to \{0, 1\}$, so $\mathcal{A}$ may be thought of as a simple kind of automaton that has an output function $\omega_{\mathcal{A}}$ and a state-transition function $\tau_{\mathcal{A}}$ that does not depend on an input parameter. Define a *term* to be an expression t of the form $\omega\tau\tau \cdots \tau x$, where x is a variable ranging over states. This induces an obvious function $t_{\mathcal{A}} : A \to \{0, 1\}$, viz. $a \mapsto \omega_{\mathcal{A}}\tau_{\mathcal{A}}\tau_{\mathcal{A}} \cdots \tau_{\mathcal{A}}(a)$. A state a *satisfies* an equation $t \approx t'$ between terms iff $t_{\mathcal{A}}(a) = t'_{\mathcal{A}}(a)$.

It turns out that states a and b from T-coalgebras $\mathcal{A}$ and $\mathcal{B}$ are observationally indistinguishable, i.e. related by some bisimulation, iff they satisfy the same Boolean combinations of equations between terms. Thus such combinations provide a natural notion of *formula* in this case.

Now the *ultrafilter enlargement* of a T-coalgebra $\mathcal{A}$ is the coalgebra $\mathcal{A}^*$ for which A^* is the set of all ultrafilters on the set A, with

$$\tau_{\mathcal{A}^*}(u) = \{X \subseteq A : \{a \in A : \tau_{\mathcal{A}}(a) \in X\} \in u\}$$

for each ultrafilter u, while

$$\omega_{\mathcal{A}^*}(u) = 1 \quad \text{iff} \quad \{a : \omega_{\mathcal{A}}(a) = 1\} \in u.$$

Then the following analogue of Birkhoff's theorem can be shown (Goldblatt, 2001b):

> *a class K of T-coalgebras is the class of all models of some set of Boolean combinations of equations iff K is a behavioural covariety that is closed under ultrafilter enlargements.*

The definition of A^* is particularly simple in the example just given because of the finiteness of $\{0, 1\}$. For many kinds of functor T, A^* has to be restricted to certain "observationally rich" ultrafilters u having a special closure property related to the construction of T. This requires that for each term τ taking values in some set D of "observable" elements, there is exactly one element $d \in D$ such that

$$\{a : \tau_{\mathcal{A}}(a) = d\} \in u.$$

A *polynomial* functor may be constructed using binary coproducts as well as the monomial operations mentioned above. Coproducts introduce considerable complexity associated with the partiality of certain "path functions" expressing the dynamics of the transition structure of a coalgebra. The construction of enlargements of T-coalgebras based on sets of observationally rich ultrafilters has been extended to polynomial T in

(Goldblatt, 2003a) and is itself derived from a construction of *observational ultrapowers* of polynomial coalgebras in (Goldblatt, 2003c). In this setting our analogue of Birkhoff's theorem holds for a suitably formulated notion of equation.

An example of a behavioural covariety that is not closed under ultrafilter enlargements is exhibited in (Goldblatt, 2001b, p. 881). It uses the functor $TX = X \times X \times \{0,1\}$, so that a T-coalgebra $\mathcal{A}$ may be viewed as having two transition functions $\tau_{\mathcal{A}}$, $\tau'_{\mathcal{A}} : A \to A$ and an output function $\omega_{\mathcal{A}} : A \to \{0,1\}$. The example consists of all T-coalgebras in which any application of $\tau'_{\mathcal{A}}$ is observationally indistinguishable from some finite number of iterations of $\tau_{\mathcal{A}}$: i.e.

> for each state $a \in A$ there is some natural number n such that $\tau'_{\mathcal{A}}(a)$ is related by a bisimulation to $(\tau_{\mathcal{A}})^n(a)$.

If a covariety K does have closure under ultrafilter enlargements, then the construction $\mathcal{A} \mapsto \mathcal{A}^*$ defines a *monad* on K (see (Goldblatt, 2003a, Section 6)), a refinement of the standard ultrafilter monad on **Set** (Manes, 1976, p. 30). This induces a monad on $\mathbb{G}^K$**-Coalg**. Thus a categorical account of behavioural covarieties closed under ultrafilter enlargements will need to analyse the relationship between a monad and a comonad, and is a natural topic for further study.

Finally, it might be asked whether, and how, this analysis of behavioural covarieties generalises to coalgebras for functors on underlying categories other than **Set**. We have made significant use of the fact that an endofunctor on **Set** preserves monos with non-empty domain because such monos have a left inverse. That is not a property enjoyed by other important categories: it fails in many topoi for example. An appropriate general setting would be that of the papers (Adámek and Porst, 2003; Awodey and Hughes, 2003) in which the underlying category has factorisation of each arrow as an epi followed by a *regular* mono (i.e. an equaliser), and a *subcoalgebra* is required to be a regular mono in T**-Coalg**. Then the components of a pure subcomonad should be required to be regular monos. These requirements hold over **Set**, where every mono is regular. This generalisation will be the subject of a separate report.

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