$\mathbb{R}^3$ IS THE DISJOINT UNION OF UNIT CIRCLES, IF AC IS TRUE

We prove that, assuming AC, there exists a partition of $\mathbb{R}^3$ into disjoint unit circles. We also show that such a decomposition cannot exist in $\mathbb{R}^2$.

Theorem 1 (AC)**.** *There exists a decomposition of $\mathbb{R}^3$ into unit circles.*

This theorem appears quite difficult to appreciate constructively—and like many theorems of ZF + AC, the fact that AC "gives" us functions and well-orderings without any need for intuition as to why they should exist in the first place, the constructions are mysterious in that sense, too.

We will need the following simple lemma.

Lemma 2 (AC)**.** *If $|x| = \kappa$ and $y \subset x$ with $|y| < \kappa$ then $|x \setminus y| = \kappa$.*

Proof. Otherwise $|x| = |y| + |x \setminus y| < \kappa$ by the Fundamental Theorem of Cardinal Arithmetic, a contradiction. □

Before we embark on the formal proof, a little bit of notation: for any set x, let

$$g_x : \mathfrak{P}(x) \setminus \{\emptyset\} \to x$$

denote a choice function: if $y \subset x$ is non-empty then $g_x(y) \in y$.

Proof of the theorem. All circles in this proof are unit circles.

Suppose $\{x_\alpha \mid \alpha < \mathfrak{c}\}$ is a well-ordering of $\mathbb{R}^3$. We argue in stages, by recursion. At each stage α we pick a circle C_α containing the respective point in the well-ordering of $\mathbb{R}^3$. In particular, we have two construction conditions: for all $\alpha < \mathfrak{c}$ we must have

(1) $C_\alpha \cap \left(\bigcup_{\beta<\alpha} C_\beta\right) = \emptyset$

(2) $x_\alpha \in \bigcup_{\beta\leq\alpha} C_\beta$

We now argue by recursion.

Suppose we have already chosen the circles $\{C_\beta \mid \beta < \alpha\}$ for $\alpha < \mathfrak{c}$. Consider x_α, the next real. Note that it might be the case that x_α is already covered by circles! In that case, we find another point not yet covered by circles, and cover that one instead.

If $x_\alpha \notin \bigcup_{\beta<\alpha} C_\beta$, then put $x = x_\alpha$. Otherwise, x_α is already covered by our chosen circles, so we find some other point that is not covered by circles yet, and deal with that one instead: we put

$$x = g_{\mathbb{R}^3}\left(\left\{y \in \mathbb{R}^3 \,\middle|\, y \notin \bigcup_{\beta<\alpha} C_\beta\right\}\right).$$

We now find a circle that contains x and is disjoint from all previously chosen circles C_β.

Let $\mathcal{P}$ denote the set of planes in $\mathbb{R}^3$ containing x.

Claim 1. $|\mathcal{P}| = \mathfrak{c}$

Proof of Claim 1. Each plane $P \in \mathcal{P}$ is determined by a normal vector (since we know $x \in P$); there are clearly $\mathfrak{c}$-many such normal vectors. ⊣

Of those planes, choose one plane that does not contain any previously enumerated circle. Observe that every circle in $\mathbb{R}^3$ lies in exactly one plain; in other words, there is a bijection between already enumerated circles and their planes. As we have only enumerated $|\alpha| < \kappa$ circles so far, lemma 2 tells us that $\mathfrak{c}$-many[1] planes without embedded circles remain; we choose one of them formally: define

$$P = g_{\mathcal{P}}(\{P \in \mathcal{P} \mid (\forall\beta < \alpha)(C_\beta \not\subset P)\})$$

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[1]Important here is only the fact that at least one plane remains.

Even though P does not properly contain any of the circles C_β, it might intersect some of them; hence we cannot pick *any* circle containing x that lies in P to satisfy the recursive construction. However, each C_β can intersect P in at most two points, of course; avoiding those points is easy by another counting argument. Define

$$\mathcal{D} = \{C \subset P \mid C \text{ is a circle} \wedge x \in C\}.$$

Claim 2. $|\mathcal{D}| = \mathfrak{c}$

Proof of Claim 2. The set of origins of all circles in $\mathcal{D}$ forms a circle itself. Clearly each origin determines a unique circle, and so there are $\mathfrak{c}$-many options. $\dashv$

Let $\mathcal{Q}$ be the set of points of intersections of P with any of the C_β; in particular:

$$\mathcal{Q} = \{y \in \mathbb{R}^3 \mid (\exists \beta < \alpha)(y \in P \cap C_\beta)\}.$$

It is now easily seen that $|\mathcal{Q}| \leq |\alpha| \times 2 = |\alpha|$.

We need to pick our circle from $\mathcal{D}$ so that it avoids $\mathcal{Q}$ (because then it also avoids all C_β). We count again: for each $y \in \mathcal{Q}$ there exist at most two circles in P that contain both x and y. Again, the set of origins of circles containing x form a circle S_x themself, and similarly for y. Hence any circle containing both x and y must have its origin in $S_x \cap S_y$. Since two circles intersect each other in at most two points, there can be at most two circles containing both x and y.

Hence we have an upper bound for the cardinality of the set of circles we need to avoid:

$$|\{C \in \mathcal{D} \mid x \in C \wedge C \cap \mathcal{Q} = \emptyset\}| \leq |\mathcal{Q}| \times 2 \leq |\alpha| \times 2 = |\alpha| < \mathfrak{c}.$$

Thus there are plenty of circles left to choose in $\mathcal{D}$ (by lemma 2 and claim 2). Let

$$C_\alpha = g_{\mathcal{D}}(\{C \in \mathcal{D} \mid x \in C \wedge C \cap \mathcal{Q} = \emptyset\})$$

to continue the construction. This completes the proof. $\square$

What if we do not insist on unit circles?

Theorem 3. *There exists a decomposition of $\mathbb{R}^3$ into circles.*

This result is due to Andrzey Szulkin from 1983, and can be found in his paper "$\mathbb{R}^3$ *is the Union of Disjoint Circles.*" in The American Mathematical Monthly, 90(9), pp. 640–641. Importantly, Szulkin's construction is constructive (no pun intended): it does not require AC.

Clearly there is no disjoint union of unit circles covering $\mathbb{R}^2$. What if we allow circles of all radii? Does such an extension of theorem 1 hold for $\mathbb{R}^2$? The answer is no. We need the following classical result from point-set topology first.

Lemma 4 (Cantor's Intersection theorem for $\mathbb{R}^2$)**.** *Let $(C_i)_{i<\omega}$ be a nested sequence of compact subsets of $\mathbb{R}^2$. Then the intersection $\bigcap C_i$ is non-empty.*

Theorem 5. *There is no decomposition of $\mathbb{R}^2$ circles.*

Proof. Here is a sketch. If there was one, then choose any point in $\mathbb{R}^2$ and find a nested sequence of circles whose radius tends to 0. By Cantor's Intersection theorem, the intersection of the discs bounded by the family of nested circles is a singleton; it cannot be covered by any circle without htting the boundary of some disc. $\square$

For more results of this kind take a look at Jonsson and Wästlund's "*Partitions of $\mathbb{R}^3$ into curves.*" in Math. Scand. 83 (1998), no. 2, 192–204. General constructions of strange subsets of $\mathbb{R}^n$ using AC can be found in Krzysztof Ciesielski's "*Set theory for the working mathematician.*" in London Mathematical Society Student Texts, 39. Cambridge University Press, Cambridge, 1997.