VUW LOGIC SEMINAR
CA-COUNTEREXAMPLES TO MARSTRAND'S THEOREM

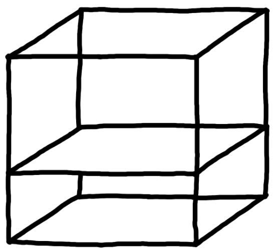
LINUS RICHTER

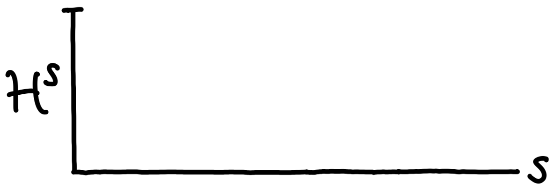
This talk is based on the paper [16]. By "counterexamples" I mean sets that fail the theorem.

History. Dimension is a measure of complexity.

For instance, take Lebesgue measure and the cube. Measuring tool are open coverings.

If I choose the wrong measure, I can't determine its volume! **Dimension is fundamental: it's the correct measure of the building blocks of a set!**





Was augmented by Hausdorff (1919) [4] to include non-integer dimension by introducing a type of *scaling factor* d: the contribution of each component of the covering *as it grows* is measured by d.

Note 1. One can also think of the dimension as the "scaling factor" of the set. If we insist all components of the covering have the same size, we recover the *box-counting*, or *Minkowski dimension.* If there are no restrictions, we get *Hausdorff dimension.*

A brief example for box-counting dimension: let $E \subset \mathbb{R}^2$ and $\epsilon > 0$. then denote by $N(\epsilon)$ the number of boxes of side-length ϵ needed to cover E. Then

$$\dim_{\mathsf{box}}(E) = \lim_{\epsilon \to 0} \frac{\log N(\epsilon)}{\log(1/\epsilon)}.$$

So if $\mathbb{I}^2$ is the unit square then it's easily seen that $N(1/2^n) = 2^{(2n)}$ for every $n < \omega$. Thus

$$\dim_{\mathsf{box}}(\mathbb{I}^2) = \lim_{n\to\infty} \frac{\log N(1/2^n)}{\log 2^n} = \lim_{n\to\infty} \frac{\log 2^{2n}}{\log 2^n} = \lim_{n\to\infty} \frac{2n \log 2}{n \log 2} = 2$$

as expected.

One can even define intermediate dimensions, where the restriction on covering sizes can be fixed. See [2] for examples.

Fact 2. Hausdorff dimension is invariant under isometries.

This follows since $\dim_H$ is also (implicitly) defined using open coverings.

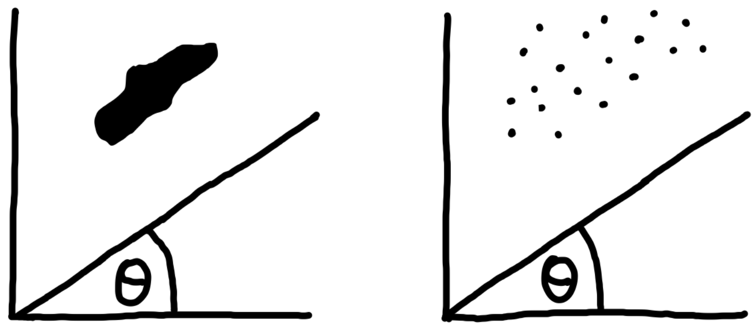
Date: 17 March 2023, Wellington, NZ.

Marstrand's Projection Theorem.

Theorem 3 (John Marstrand [13])**.** *Let $E \subset \mathbb{R}^2$ be analytic. For almost all θ*

$$\dim_H(p_\theta(E)) = \min\{1, \dim_H(E)\}.$$

Important: proof uses geometric measure theory, that's why the "for *almost* all". There is a newer proof due to Kaufman (1968) [7], and Mattila has extended[1] the result to orthogonal projections from $\mathbb{R}^m$ to $\mathbb{R}^n$ in [14].



What else do we know?

Theorem 4 ([12])**.** *If $\dim_H(E) = \dim_P(E)$ then Marstrand's Projection Theorem applies to E.*

Importantly, this is not a characterisation of analytic sets! And the proof uses effective methods!

Theorem 5 ((CH), Roy O. Davies [1])**.** *There exists a set E such that $\dim_H(E) = 1$ while $\dim_H(p_\theta(E)) = 0$ for all θ.*

What is least possible?

	$\boldsymbol{\Sigma}^1_1$		$\boldsymbol{\Sigma}^1_2$	
$\boldsymbol{\Delta}^1_1$		$\boldsymbol{\Delta}^1_2$		$\boldsymbol{\Delta}^1_3$
	$\boldsymbol{\Pi}^1_1$		$\boldsymbol{\Pi}^1_2$	

Connecting $\dim_H$ with K. Long history connecting classical notions of dimension with effective versions.

Began in earnest with J. Lutz' early 2000's work (2003) [8]: introduces *effective dimension* on elements of $2^{\mathbb{N}}$, via *term gales*, generalisations of martingales (were used to define *resource bounded measure*, which alludes to Ville's original definition of martingale [20] as well as Schnorr's characterisation of Kolmogorov-randomness in terms martingales [17]).

- proved that gales in fact characterise $\dim_H$ on Cantor space! [8, Theorem 3.6]
- introduced *constructive dimension* on subsets of $2^{\mathbb{N}}$, and used[2] that to define dim on individual strings in $2^{\mathbb{N}}$ (again using gales)! [9]

[1] in detail: if $0 < m \leq \dim_H(E) = s \leq n$ then for almost all hyperplanes L we have $\dim_H(p_L(E) = m$; and if $0 \leq s \leq m$ then $\dim_H(p_L(E)) = s$ for almost all L

[2] this is similar in nature to how Schnorr used constructive martingales to recover ML-randomness

This dim on $2^{\mathbb{N}}$ satisfies what we would naturally expect!

- if $A \in 2^{\mathbb{N}}$ is computable then $\dim(A) = 0$
- if $A \in 2^{\mathbb{N}}$ is (Kolmogorov/Chaitin) random then $\dim(A) = 1$

...and both relativise!

Also gives reals of all dimensions in between! Augments ML-randomness just like Hausdorff measure augments Lebesgue measure [15]

Note 6. Lutz already hoped for a deep connection: in [5] Hitchcock remarks that "Lutz conjectured that there should be a correspondence principle stating that the constructive dimension of every sufficiently simple set X coincides with its classical Hausdorff dimension." in lectures at Iowa State university in 2000. This would turn out to be true in a big way: see the *point-to-set principle* below.

It turns out: **we don't need gales!**

Theorem 7 ([15, Corollary 3.2])**.** For every $A \in 2^{\mathbb{N}}$

$$\dim(A) = \liminf_{n\to\infty} \frac{K(f \upharpoonright n)}{n}$$

where K denotes prefix-free Kolmogorov complexity.

Hence we also speak of *algorithmic information density*!

So we have:

- characterisation of dimension of reals in terms of prefix-free complexity K
- characterisation of Hausdorff dimension in terms of dimension of reals.

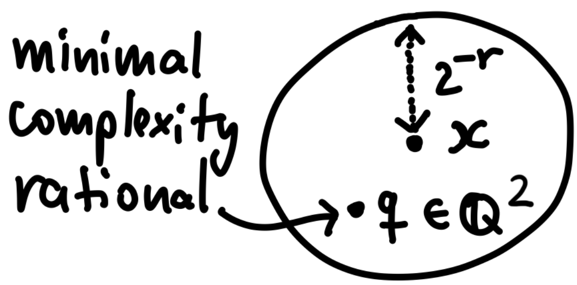
Can we get a characterisation of $\dim_H$ **in terms of** K? **YES!**

Theorem 8 ([5, Corollary 4.3])**.** If $X \subset 2^{\mathbb{N}}$ is a union (**arbitrary!**) of Π^0_1-sets (lightface) then

$$\dim_H(X) = \sup_{s\in X} \dim(s).$$

Can this be extended? YES!

• First: beyond Cantor space.



...but it can be done more easily:

Theorem 9 ([12])**.** If $\overline{x}$ is the binary expansion of $x \in \mathbb{R}$ then

$$|K_r(x) - K(\overline{x} \upharpoonright r)| \leq K(r) + c$$

This also works[3] in $\mathbb{R}^n$ **and in polar coordinates!** [16, Proposition 3.1] So now we can build reals by **determining the binary expansion!**

$$\dim(\overline{x}) = \dim(x)$$

Hence define

$$\begin{aligned}\dim(x) &= \liminf_{r\to\infty} \frac{K_r(x)}{r} \\ &= \liminf_{n\to\infty} \frac{K(\overline{x}\upharpoonright n)}{n}\end{aligned}$$

which also relativises. Also, observe this is the lim inf; so to get dim $= 0$ we only **need infinitely many drops!**

• Second: no restrictions.

Theorem 10 (Jack Lutz, Neil Lutz [10, Theorem 1])**.** For all $E \subset \mathbb{R}^2$ we have[4]

$$\dim_H(E) = \min_{A\in 2^{\mathbb{N}}} \sup_{x\in E} \dim^A(x).$$

(A similar theorem for Packing dimension is proven in the same paper.)

First counterexample. We assume $V = L$, and use the *point-to-set principle*!

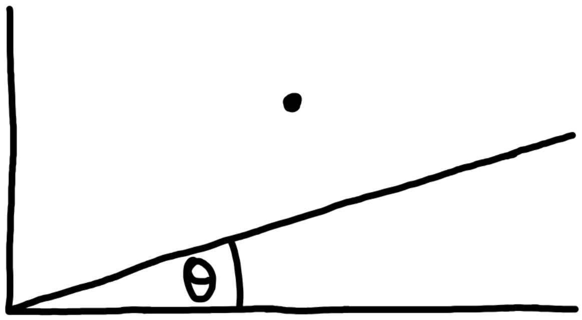
Theorem 11 (($V = L$), [16])**.** There exists a co-analytic set $E \subset \mathbb{R}^2$ such that $\dim_H(E) = 1$ while $\dim_H(p_\theta(E)) = 0$ for all θ.

Note 12. Observe this is *for all* θ!

Three questions:

(1) How do we control the dimension of each point in the projection?
(2) How do we control the dimension of E?
(3) How do we make sure the resulting set is co-analytic?

(1). This requires some work. **How do we compute the projection?**



In fact it's even easier: $\dim_H$ is **invariant under isometries!** So can only consider $r|\cos(\theta - \varphi_i)| = ra_i$! This introduces **projection factors**!

And we already know we can construct reals by **specifying their binary expansion**!

[3] provided reasonable coding
[4] in fact holds for $\mathbb{R}^n$

(2). We want to ensure E has $\dim_H = 1$. How do we force that? **Here is the power of the point-to-set principle** (usual proofs use geometric measure theory):

Theorem 13 (Folklore)**.** Suppose $E \subset \mathbb{R}^2$ intersects every straight line through the origin. Then $\dim_H(E) \geq 1$.

This proof is messy in geometric measure theory, but very straightforward here!

Proof. Work in polar coordinates. Fix some $A \in 2^{\mathbb{N}}$, take θ random relative to A. Then $\dim^A(\theta) = 1$. By assumption, there exists $(r, \theta) \in E$. Thus

$$\dim^A(r, \theta) \geq \dim^A(\theta) = 1.$$

Since A is arbitrary, the point-to-set principle yields the lower bound. □

So if we pick one point from each straight line through the origin then (2) is satisfied!

(3). Use Vidnyánszky's theorem [19] (just like last week), and argue by recursion.

The key idea: the point-to-set principle only needs the "best" oracle! By $V = L$, if we ensure at each step that the point we add projects "simply" onto all previous (countably many!) lines (i.e. all such projections have dimension 0), we are done!

Here is the structure of the recursion:

- conditions $P = [0, \pi/2]$
- suppose below $\alpha < \omega_1$ we are done, θ is the current condition.
- partial solution $\{(r_i, \varphi_i) \mid i < \omega\}$ ← since $V = L$
- let $a_i = |\cos(\theta - \varphi_i)|$ ← projection factors

Observe: every line appears as a condition in the recursion!

Goal: find r such that $\dim(ra_i) = 0$ for all $i < \omega$ ← this is the difficult part! Also must be cofinal in Turing degrees!

Idea: build r in stages, by specifying the binary expansion.

- start with r_0 the empty string;
- at stage k, decode as $\langle i, m \rangle$, so we deal[5] with condition a_i;
- find an extension of r_{n+1} such that $a[r_n]$ contains *enough* zeroes. ← recall that dim is given by lim inf!

What does *enough* mean? Recall that dim is given by the lim inf! So if at stage k we can force the complexity below $1/k$ then we're good!

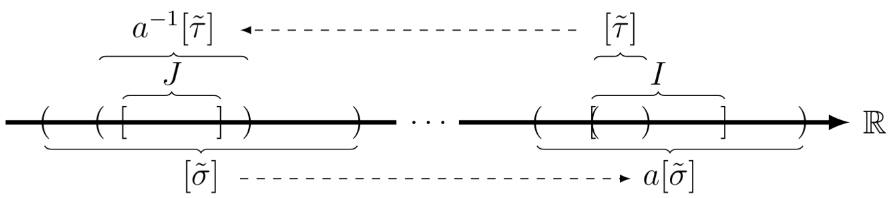


FIGURE 1. Taken from [16]. Important: dyadic intervals contain reals that are extensions of each other's binary expansions!

This costs bits, but not too many! In fact, if $\ell(r_{n+1}) = 2^{2^n}$ then we're done!

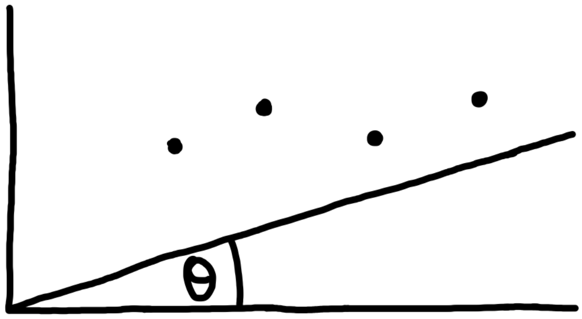
---

[5]we must ensure to deal with all conditions infinitely often, hence the double coding

Verifying it works. To show the projection of E onto θ has dimension 0: consider $(r, \varphi) \in E$:

EITHER: (r, φ) appeared after condition θ was satisfied. Then, by construction, $\dim(ra) = 0$ where a is the projection factor from L_φ onto L_θ;

OR: (r, φ) appeared before θ was satisfied. Take all (countably many!) points (r_i, φ_i) in E that appeared before θ, and take the join A of $r_i a_i$ (where a_i is the proj. factor of (r_i, φ_i) onto θ). Then A computes all projections onto θ.



By the point-to-set principle

$$\dim_H(p_\theta(E)) \leq \sup_{x \in E} \dim^A(p_\theta(x)) = 0.$$

Second counterexample.

Theorem 14 (($V = L$), [16])**.** For every $\epsilon \in [0, 1]$ there is a co-analytic set $E \subset \mathbb{R}^2$ such that $\dim_H(E) = 1 + \epsilon$ yet $\dim_H(p_\theta(E)) = \epsilon$ for all θ.

This is optimal!

Note 15. Note this is obvious for $\epsilon = 1$ (projections can at most lose 1 "unit of dimension"), and the case $\epsilon = 0$ is the first counterexample above.

This is more complicated.

The difficulty is: how do we force the dimensions to be exactly what we need?

The idea: for partial solution (a_i), code a string $T \in 2^{\mathbb{N}}$ (such that $\dim^Z(T) = \epsilon$ for some suitable Z) into each projection ra_i; later, long pieces of T can be recovered from r with knowledge of the partial solution, which gets us what we need.

Open questions.

- What about $\dim_H(E) < 1$?
- Packing dimension $\dim_P$? This is the dual to Hausdorff dimension, due to Tricot [18]. Algorithmic randomness characterisations exist!

Theorem 16 (Jack Lutz, Neil Lutz [10, Theorem 2])**.** If $E \subset \mathbb{R}^2$ then

$$\dim_P(E) = \min_{A \in 2^{\mathbb{N}}} \sup_{x \in E} \operatorname{Dim}^A(x)$$

where

$$\operatorname{Dim}(x) = \limsup_{r \to \infty} \frac{K_r(x)}{r}$$

...but this does not admit Marstrand-like result (Järvenpää (1994) [6]; Howroyd and Falconer (1996) [3])

- Extensions of point-to-set principle? Generalisations using gauge functions? [11]
- Other applications: Kakeya sets, Furstenberg sets (applications to harmonic analysis)...

References

[1] R. O. Davies. Two counterexamples concerning Hausdorff dimensions of projections. *Colloquium Mathematicum*, 42(1):53–58, 1979.

[2] K. J. Falconer, J. M. Fraser, and T. Kempton. Intermediate dimensions. *Math. Z.*, 296(1-2):813–830, 2020.

[3] K. J. Falconer and J. D. Howroyd. Projection theorems for box and packing dimensions. *Math. Proc. Cambridge Philos. Soc.*, 119(2):287–295, 1996.

[4] F. Hausdorff. Dimension und äußeres Maß. *Mathematische Annalen*, 79:157–179, 1919.

[5] J. M. Hitchcock. Correspondence principles for effective dimensions. *Theory Comput. Syst.*, 38(5):559–571, 2005.

[6] M. Järvenpää. On the upper Minkowski dimension, the packing dimension, and orthogonal projections. *Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes*, (99):34, 1994.

[7] R. Kaufman. On Hausdorff dimension of projections. *Mathematika*, 15(2):153–155, Dec. 1968. Publisher: London Mathematical Society.

[8] J. H. Lutz. Dimension in complexity classes. *SIAM J. Comput.*, 32(5):1236–1259, 2003.

[9] J. H. Lutz. The dimensions of individual strings and sequences. *Information and Computation*, 187(1):49–79, Nov. 2003.

[10] J. H. Lutz and N. Lutz. Algorithmic information, plane Kakeya sets, and conditional dimension. *ACM Trans. Comput. Theory*, 10(2):Art. 7, 22, 2018.

[11] J. H. Lutz, N. Lutz, and E. Mayordomo. Extending the reach of the point-to-set principle. In *39th International Symposium on Theoretical Aspects of Computer Science*, volume 219 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 48, 14. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2022.

[12] N. Lutz and D. M. Stull. Projection theorems using effective dimension. In *43rd International Symposium on Mathematical Foundations of Computer Science*, volume 117 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 71, 15. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2018.

[13] J. M. Marstrand. Some Fundamental Geometrical Properties of Plane Sets of Fractional Dimensions. *Proceedings of the London Mathematical Society*, s3-4(1):257–302, 1954.

[14] P. Mattila. Hausdorff dimension, orthogonal projections and intersections with planes. *Annales Academiae Scientiarum Fennicae Series A I Mathematica*, 1:227–244, 1975.

[15] E. Mayordomo. A Kolmogorov complexity characterization of constructive Hausdorff dimension. *Information Processing Letters*, 84(1):1–3, Oct. 2002.

[16] L. Richter. Co-analytic counterexamples to Marstrand's Projection Theorem. `https://arxiv.org/abs/2301.06684`, 2023.

[17] C.-P. Schnorr. Process complexity and effective random tests. *J. Comput. System Sci.*, 7:376–388, 1973.

[18] C. Tricot, Jr. Two definitions of fractional dimension. *Math. Proc. Cambridge Philos. Soc.*, 91(1):57–74, 1982.

[19] Z. Vidnyánszky. Transfinite inductions producing coanalytic sets. *Fund. Math.*, 224(2):155–174, 2014.

[20] J. Ville. *Étude critique de la notion de collectif.* NUMDAM, [place of publication not identified], 1939.