Suppose you have some object.
If it doesn't satisfy a property, by how much does it fail?

E.g.: If $f: \mathbb{R} \to \mathbb{R}$ (additive) is a Borel homomorphism, then it's continuous; it's of the form $f(x) = rx$ for some $r \in \mathbb{R}$.

$\mathcal{B}(X) \leftarrow$ Borel σ-algebra: algebra generated by open sets of X.

What if f satisfies

(f) $\quad f(x+y) - f(x) - f(y) \in G$ $\quad = 0$

for some countable group G?

Thm: Kanovei – Reeken (2000)

If f is Borel ← it's cts, and (*) holds then there is a Borel hom. g. s.t. $f(x) - g(x) \in G$.

General idea: can any almost homomorphism be approximated by a real homomorphism?

C^*-algebras, Boolean algebras, groups (Polish, because we can use descriptive set theory).

Finite groups:

simple (no non-trivial normal subgroups)

- Classification of finite groups.
- Every finite group is "constructed" from simple groups.

Very hard problem, no unified theory.

If G is finite group, we can write

$$G = E_n \trianglerighteq E_{n-1} \trianglerighteq \dots \trianglerighteq E_1 \trianglerighteq E_0 = 1$$

maximal normal

$\boxed{E_{n+1}/E_n \text{ is simple}}$

\ composition factors. $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$

- $\mathbb{Z}_4 \trianglerighteq \mathbb{Z}_2 \trianglerighteq 1$
- $S_4 \trianglerighteq A_4 \trianglerighteq \mathbb{Z}_2 \times \mathbb{Z}_2 \trianglerighteq \mathbb{Z}_2 \trianglerighteq 1$
- $S_5 \trianglerighteq A_5 \trianglerighteq 1$ ← not solvable (think Galois theory)

$$G = E_2 \trianglerighteq E_1 \trianglerighteq E_0 = 1$$

with factors $(\mathbb{Z}_3, \mathbb{Z}_2)$

$$E_1 \cong E_1/E_0 \cong \mathbb{Z}_3$$

$$E_2/E_1 \cong \mathbb{Z}_2 \Rightarrow E_2/\mathbb{Z}_3 \cong \mathbb{Z}_2$$

So $|E_2| = G$. $\leftarrow$ either S_3 or $\mathbb{Z}_6 = \mathbb{Z}_2 \times \mathbb{Z}_3$

<u>Dfn:</u> E_2 is an extension of E_2/E_1 by E_1.

$$G = E_2 \trianglerighteq E_1 \trianglerighteq E_0 = 1$$

$N \triangleleft G \Rightarrow G$ is an ext. of G/N by N.

<u>Equiv:</u>

$$1 \longrightarrow H \xrightarrow{i} E \xrightarrow{q} A \longrightarrow 1$$

injection

surjection

$\mathrm{im}(i) = \ker(q)$.

$$1 \rightarrow N \xrightarrow{i} G \xrightarrow{q} G/N \rightarrow 1$$

$$\boxed{\text{Assume } A, H \text{ are abelian.}}$$

$$\begin{tikzcd}
 & & E_1 \arrow[dd, dotted, "f"] \arrow[dr, "q"] & & \\
0 \arrow[r] & H \arrow[ur, "1 \mapsto 3"] \arrow[dr, "1 \mapsto 6"'] & & A \arrow[r] & 0 \\
 & & E_2 \arrow[ur, "q"'] & &
\end{tikzcd}$$

$H = \mathbb{Z}_3$, $E_1 = \mathbb{Z}_9$, $E_2 = \mathbb{Z}_9$, $A = \mathbb{Z}_3$

if this commutes then E_1 and E_2 are equivalent extensions. Then $E_1 \cong E_2$.

You can have distinct extensions with isom. middle groups.

If everything is abelian... we can use cocycles:

abelian

$$C : A^2 \to H$$

- $C(x,y) = C(y,x)$
- $C(x,y) + C(x+y,z) = C(x,z) + C(x+z,y).$

cocycle condition.

EACH SUCH C GIVES ME AN EXTENSION OF A BY H.
And vice versa.

A cocycle is a coboundary if

$$C(x,y) = \eta(x) + \eta(y) - \eta(x+y).$$

Look at $H^2(A, H)$ ← exactly give group ext of A by H.

$$C - C' = \eta(x) + \eta(y) - \eta(x+y)$$

$\Rightarrow C, C'$ are cohomologous.

Now look at Borel cocycles:

A, H are Polish spaces ← separable, comp. metrizable.

whose group op. is Borel.

Look at Borel cocycles:

$C : A^2 \to H$ is Borel as a map.

Thm: Kanovei - Reeken (2000)

$$H^2_{Bor}(\mathbb{R}, G) \neq 0 \text{ if } G \text{ is ctbl.}$$

$$\frac{Z^2_{Bor}(\mathbb{R}, G)}{B^2_{Bor}(\mathbb{R}, G)}$$

Thm: Lupini R.

$$H^2_{Bor}(\mathbb{R}^n, G) = 0 \text{ if } G \text{ is ctbl. for } n > 1.$$

Main argument of proof:

C is Borel so it's nice on some open interval I. (Baire category thm)