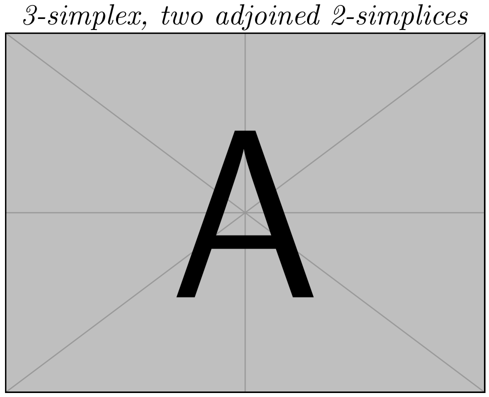
1 Introduction

Martino and I have been working on questions involving Topology, Algebra, and Logic. The connections are somewhat natural in the classical sense. We use algebra to answer topological questions. Fixing objects of a set theoretical nature that are subject to limitations courtesy of ZFC (Baire space, ordinals) allows another angle of approach. Set theoretical tools can now affect the topological properties.

$$\begin{tikzcd} \Big[\ \text{Topology} \arrow[r, bend left] & \text{Algebra}\ \Big] \arrow[l, bend left] & \text{Logic} \arrow[l] \end{tikzcd}$$

We outline an example: consider simplicial homology. This can be likened to graphs: A simplex is a set of points in $\mathbb{R}^n$. A set of $n+1$ points yields n n-dimensional simplex in $\mathbb{R}^n$. A simplicial complex is a set of simplices that is closed under taking intersections and faces. Example:

3-simplex, two adjoined 2-simplices



The dimension of a simplicial complex is the greatest integer k such that the complex does not contain simplices containing k' points.

The set of k-chains is given by linear combinations of k-simplices. So

$$C_r(K) = \{\lambda_i v^i \mid v \text{ is an } r\text{-simplex, } \lambda \in \mathbb{Z}\}$$

so it is the free group generated by the r-simplices. Homology is all about finding nice holes in the simplicial complex. In the standard 3-complex, we expect the interior (i.e. the points that are not on any of the three 2-simplices) to form a hole. The homology groups pick those up. So, here we expect $H_2(K)$ to be isomorphic to the integers (as a guide, the rank of the homology group should give the number of holes).

The boundary homomorphism d^r maps r-chains to $r-1$-chains. In particular, given any r-chain x, its image $d^r(x)$ is an $r-1$-boundary. Chains that are mapped to 0 are called cycles. The r-th homology group is the group of r-cycles modulo r-boundaries. So

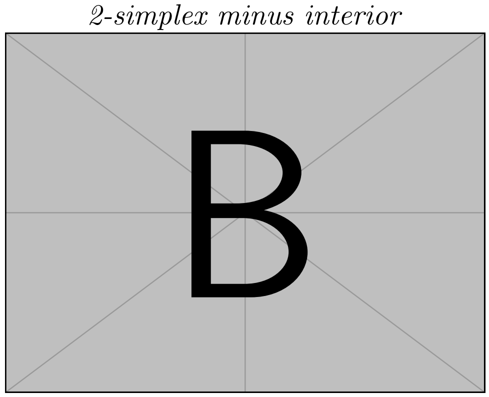
$$H_r(K) = Z_r(K)/B_r(K) = \ker(d^r)/\operatorname{im}(d^{r+1})$$

Thus we get a sequence of the following form:

$$0 \leftarrow C_0(K) \leftarrow C_1(K) \leftarrow C_2(K) \leftarrow \dots$$

with boundary homomorphisms connecting the chain complexes.

2-simplex minus interior



Indeed, by definition $H_1(K) = Z_1(K)/B_1(K)$. Again by definition we have that

$$Z_1(K) = \ker(d^1) = \{x \in C_1(K) \mid d^1(x) = 0\}$$

which we can calculate straight away: write v_{ij} in place of $\langle v_i, v_j \rangle$. Then

$$C_1(K) = \{\alpha v_{01} + \beta v_{12} + \gamma v_{20} \mid \alpha, \beta, \gamma \in \mathbb{Z}\}$$

and so cycles are such combinations satisfying

$$d^1(x) = 0 \Leftrightarrow \alpha(v_1 - v_0) + \beta(v_2 - v_1) = \gamma(v_0 - v_2) = 0$$

and solving for parameters we obtain

$$v_0(-\alpha + \gamma) + v_1(\alpha - \beta) + v_2(\beta - \gamma) = 0)$$

Solving this system we obtain that all solutions are given by tuples

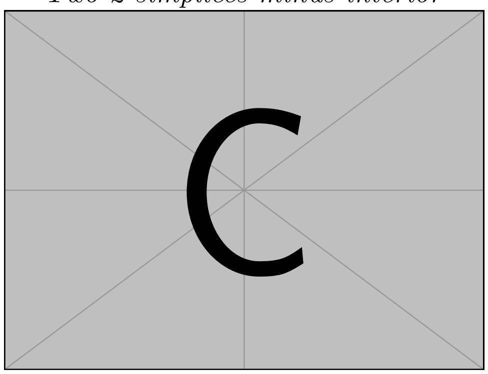
$$(\alpha, \beta, \gamma) = (s, s, s)$$

for $s \in \mathbb{Z}$. Thus

$$Z_1(K) = \{\lambda(v_{01} + v_{12} + v_{20}) \mid \lambda \in \mathbb{Z}\}$$

Since there are no 2-simplices in K we deduce that $B_1(K) \cong 0$ and thus $H_1(K) \cong Z_1(K) \cong \mathbb{Z}$, as expected.

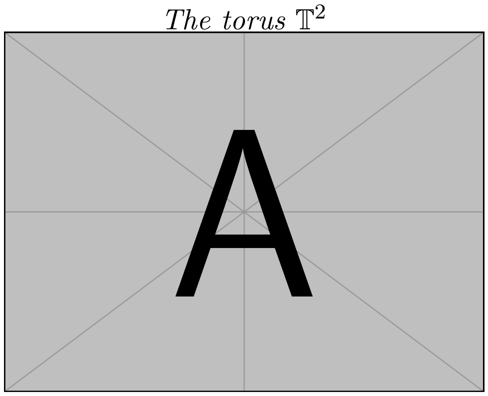
Two 2-simplices minus interior



In this case we go through the motions and show that the first homology group is indeed generated by two cycles (not three), and the group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ (which underlines our intuition about the rank coinciding with the number of holes).

By this reasoning, we would hope that the torus' simplicial homology groups have rank 2 in dimension 1 and one in dimension 2 (since $\mathbb{T}^2$ has two 1-dimensional holes and one open vacuum).

The torus $\mathbb{T}^2$



A space is triangulable if there is a simplex whose underlying space (as a subspace of $\mathbb{R}^n$) is homeomorphic to it. One can show that the torus is triangulable, and thus use simplicial homology to count its holes.

Homology groups are topological invariants.

2 Cohomology

Cohomology is the dual notion of homology. It allows for more complex algebraic structure (and can hence be considered a stronger or *richer* topological invariant) in the sense that the cup product allows us to combine cocycles of differing dimensions.

When considering the ordinals, this yields insights about the combinatorial properties of ordinals ω_n.

We fix as our space X an ordinal δ.

Suppose $\mathcal{U}_\delta$ is the set of initial segments of δ, that is $\{\alpha \mid \alpha < \delta\}$. This is an open cover of δ. (It can be considered a decomposition of the space just like a triangulation). A presheaf $\mathcal{P}$ associates with each $\alpha \in \mathcal{U}_\delta$ a group of functions from α to a fixed abelian group A. We fix $A = \mathbb{Z}$. The set of r-cochains is now given by functions: we write

$$L^j(\mathcal{U}_\delta, \mathcal{P}) = \prod_{\alpha_0 < \ldots < \alpha_j < \delta} \mathcal{P}(\alpha_0)$$

and thus we obtain a sequence of the form

$$0 \to L^0(\mathcal{U}_\delta, \mathcal{P}) \to L^1(\mathcal{U}_\delta, \mathcal{P}) \to \ldots \to L^j(\mathcal{U}_\delta, \mathcal{P}) \to \ldots$$

connected by the boundary homomorphisms.

There are three important presheaves:

- $\mathcal{D}(\alpha) = \bigoplus_\alpha \mathbb{Z}$;
- $\mathcal{E}(\alpha) = \prod_\alpha \mathbb{Z}$;
- $\mathcal{F}(\alpha) = (\prod_\alpha \mathbb{Z})/(\bigoplus_\alpha \mathbb{Z})$;

The cohomology groups w.r.t. the cover $\mathcal{U}_\delta$ are defined as before: cocycles modulo coboundaries. Let's consider the cochains of these complexes:

$$L^0(\mathcal{U}_\delta, \mathcal{E}) = \prod_{\alpha<\delta} \prod_{\alpha} \mathbb{Z}$$

So an element of this cochain complex is a family of functions

$$\Phi = \{\varphi_\alpha \mid \alpha < \delta\}$$

where

$$\Phi(\alpha) = \varphi_\alpha \in \prod_{\alpha} \mathbb{Z}$$

So each φ_α satisfies

$$\varphi_\alpha \colon \alpha \to \mathbb{Z}$$

Similarly, if we replace $\mathcal{E}$ with $\mathcal{D}$ we obtain functions that are finitely supported. The presheaf $\mathcal{F}$ gives equivalence classes of functions that only differ at finitely many points. So we write $\varphi_\alpha \sim \psi_\alpha$ iff $\varphi_\alpha - \psi_\alpha \in \bigoplus_\alpha \mathbb{Z}$. We write

$$\varphi_\alpha =^* \psi_\alpha$$

Suppose Φ satisfies that $\varphi_\beta \restriction \alpha =^* \varphi_\alpha$ for all $\alpha < \beta < \delta$. Then we call Φ *coherent*.

If there is a function $f \colon \delta \to \mathbb{Z}$ such that $f \restriction \alpha =^* \varphi_\alpha$ for all $\alpha < \delta$ then we call Φ *trivial*.

Consider the cohomology group $H^{n-1}(\mathcal{U}_\delta, \mathcal{F}) = \ker(d^1)/\operatorname{im}(d^0)$. One can prove that

$$H^{n-1}(\mathcal{U}_\delta, \mathcal{F}) = \{\Phi \mid \Phi \text{ is } n\text{-coherent}\}/\{\Phi \mid \Phi \text{ is } n\text{-trivial}\}$$

for $n > 0$. At $n = 0$ we have we are left with the group of 0-coherent functions.

These are functions from δ into $\mathbb{Z}$ whose initial segments have finite support – 0-trivial equates to having finite support overall. There are no non-0-trivial 0-coherent functions on ω_1, for example. This is a universal phenomenon.

2.1 Families of Functions and Cohomology Groups

The Čech cohomology group is the direct limit of cohomology groups of δ, under refining covers. Luckily, the cover $\mathcal{U}_\delta$ is fine enought so that we have

$$H^n(\mathcal{U}_\delta, \mathcal{D}) \cong \check{H}^n(\delta, \mathcal{D})$$

for $n > 0$. This is because the cover $\mathcal{V}$ refining $\mathcal{U}_\delta$ contains a subset of nice intervals on some club $C \subset \delta$. If Φ is non-trivial-coherent, then Φ restricted to these intervals is also non-trivial coherent. This Φ can now be extended on the refinement. So non-triviality preserved.

Hence, the Čech cohomology groups vanish iff their family of functions can be trivialised.

- do non-trivial coherent families exist?
- exactly when can non-trivial coherent families be trivialised?

It is a theorem of ZFC that there are non-n-trivial n-coherent families in ω_n (for ω_1 we can use Todorcevic's walks on ordinals to deduce the existence). Equally, the 0-th Čech cohomology group is given by the 0-coherent functions, so this group will never vanish. Hence we have the following picture (for an initial segment of the ordinals):

$\check{H}^3$	?	?	?	nontrivial
$\check{H}^2$	?	?	nontrivial	?
$\check{H}^1$	?	nontrivial	?	?
$\check{H}^0$	nontrivial	nontrivial	nontrivial	nontrivial
	ω	ω_1	ω_2	ω_3

It is a theorem of ZFC that, if δ has cofinality less than ω_n and then the n-th cohomology group vanishes (families can be trivialised via a recursive construction, then the result follows by induction). This settles the cells above the diagonal.

$\check{H}^3$	0	0	0	nontrivial
$\check{H}^2$	0	0	nontrivial	?
$\check{H}^1$	0	nontrivial	?	?
$\check{H}^0$	nontrivial	nontrivial	nontrivial	nontrivial
	ω	ω_1	ω_2	ω_3

Assuming $V = L$ one can show that we can always find non-trivial coherent families (not subject to the limitations of ZFC); so here, every group that need not vanish won't vanish. Vanishing of other groups has been patchy. Todorcevic has shown that, assuming the P-Ideal-Dichotomy, the first cohomology group vanishes iff δ has cofinality not equal to ω_1. This gives us the following:

$\check{H}^3$	0	0	0	nontrivial
$\check{H}^2$	0	0	nontrivial	cons non-0
$\check{H}^1$	0	nontrivial	indep	indep
$\check{H}^0$	nontrivial	nontrivial	nontrivial	nontrivial
	ω	ω_1	ω_2	ω_3

Many of these results have strong ties to combinatorics, and incompactness – a family of functions is coherent iff each initial segment is trivial. So non-trivial coherence is a failure of compactness. Proving this failure is impossible seems to depend heavily on the ordinal in question.

3 Baire space

We can consider a similar setup in the following: consider Baire space ω^ω. We can copy the structure form above: for $f \in \omega^\omega$ define

$$A_f = \bigoplus_{i<\omega} \bigoplus_{j=1}^{f(i)} \mathbb{Z} \text{ (this is } \mathcal{D}\text{)}$$

and

$$B_f = \prod_{i<\omega} \bigoplus_{j=1}^{f(i)} \mathbb{Z} \text{ (this is } \mathcal{E}\text{)}$$

Then we get short exact sequences

$$0 \to A_f \to B_f \to B_f/A_f \to 0$$

for each $f \in \omega^\omega$ (i.e. im = ker, second map is injective, third map is surjective). We can apply the lim-operator to this, which is exact. So we obtain a long exact sequence

$$0 \to \varprojlim \mathcal{A} \to \varprojlim \mathcal{B} \to \varprojlim \mathcal{B}/\mathcal{A} \to \varprojlim{}^1 \mathcal{A} \to \dots$$

It is clear that if q, the quotient map, is surjective, then the first derived limit vanishes.

A family Φ is a family of functions if $\Phi(f) \in B_f$ for all $f \in \omega^\omega$. A family is *coherent* if $\Phi(g) \restriction f =^* \Phi(f)$ for all $f, g \in \omega^\omega$. It is trivial if there is a function $\varphi\colon \omega \times \omega \to \mathbb{Z}$ such that $\varphi =^* \Phi(f)$ Similarly, we can define cochain complexes as follows:

$$K^n(\mathcal{B}) = \prod_{f_0 \neq \ldots \neq f_n \in \omega^\omega} B_{f_0}$$

with boundary homomorphisms and restriction maps.

It is a theorem by Nöbeling and Roos that $H^n(K(\mathcal{A})) \cong \varprojlim^n \mathcal{A}$. Bergfalk showed that $\varprojlim^n \mathcal{A} = 0$ iff every n-coherent family is n-trivial.

Hence we are in exactly the same situation as before: can families be trivialised? This is, in some sense, a local example of the global ordinal case: we only need to consider ZFC properties affecting ω^ω. The dominating number $\mathfrak{d}$, for example, affects the vanishing of cohomology groups: if $\mathfrak{d} = \aleph_2$ then the only cohomology group that vanishes is $H^2(\mathcal{A})$.

What if we consider ω^κ instead of ω^ω? It is true that if a cohomology group vanishes in ω^κ, then it also vanishes in ω^ω. Whether the converse is true is still open.

One can also consider the descriptive set theoretic properties: can a witness to $\varprojlim^n \mathcal{A} \neq 0$ be analytic? Todorcevic showed this is impossible if $n = 1$; the other cases are open.

4 Mittag-Leffler

There are also algebraic conditions that enforce the vanishing of the first derived limit: assume $0 \to A_i \to B_i \to C_i \to 0$ is short exact and $i \in I$ is countable. If the system $\mathcal{A}$ is Mittag-Leffler, then the first derived limit vanishes.

We can apply this to our ordinal case: define

$$A_\alpha = \bigoplus_\alpha A$$

and

$$B_\alpha = \prod_\alpha A$$

and

$$C_\alpha = (\prod_\alpha A)/(\bigoplus_\alpha A)$$

Then the sequence

$$0 \to A_\alpha \to B_\alpha \to C_\alpha \to 0$$

is short exact. Note that $p_{\alpha\beta}\colon A_\beta \to A_\alpha$ is surjective: for any $f \in \bigoplus_\alpha A$, just fill f up with zeroes until it has β-many coordinates. *A directed inverse system with surjective projection functions is Mittag-Leffler if the ground set is countable.* This is clearly true: let δ be an ordinal, and suppose the $p_{\alpha\beta}$ are surjective. In order to be ML we need to prove the following:

For each $\alpha \in \delta$ there is $\beta \geq \alpha$ such that for all $\gamma \geq \beta$ we have $p_{\alpha\beta}[A_\beta] = p_{\alpha\gamma}[A_\gamma]$

Fix $\alpha < \delta$ and pick $\beta = \alpha + 1$. Then $p_{\alpha\beta}\colon A_\beta \to A_\alpha$ is surjective by assumption, i.e. $p_{\alpha\beta}[A_\beta] = A_\alpha$ for any $\beta > \alpha$, or similarly, for any $\beta \geq \alpha + 1$.

Hence, if $\delta = \omega$, for example, the first derived limit vanishes, and so does the first (Čech) cohomology group: by Nöbeling Roos,

$$H^1(K(\mathcal{A})) = \varprojlim{}^1 K(\mathcal{A}) = \varprojlim{}^1 \mathcal{L}(\mathcal{U}_\delta, \mathcal{D})$$

since

$$K(\mathcal{A}) = \mathcal{L}(\mathcal{U}_\delta, \mathcal{D})$$

and thus we actually obtained

$$H^1(\mathcal{U}_\delta, \mathcal{D}) = H^1(\mathcal{L}(\mathcal{U}_\delta, \mathcal{D})) = \varprojlim{}^1 \mathcal{L}(\mathcal{U}_\delta, \mathcal{D}))$$

where the left-hand-side just equals $\check{H}^1(\delta, \mathcal{D})$. The cover $\mathcal{U}_\delta$ gave us the nice characterisation of restriction maps $p_{\alpha\beta}$. By the *nice cover* property of $\mathcal{U}_\delta$ we hence calculated the Čech cohomology group to vanish.

This is a purely algebraic notion, and its extension to uncountable index sets is an open question.