Co-analytic "Counterexamples" to Marstrand's Projection Theorem

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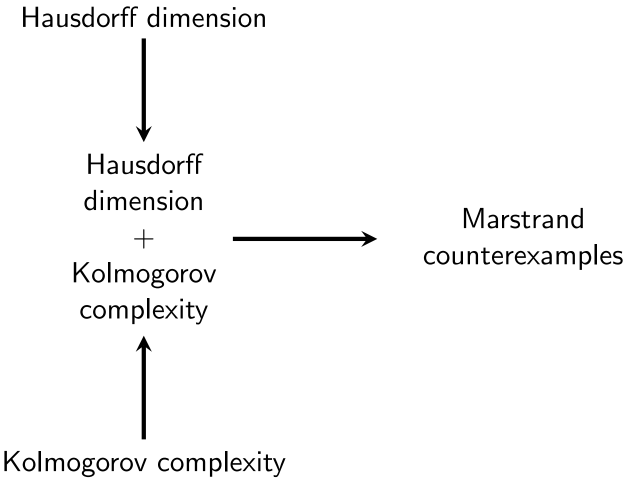
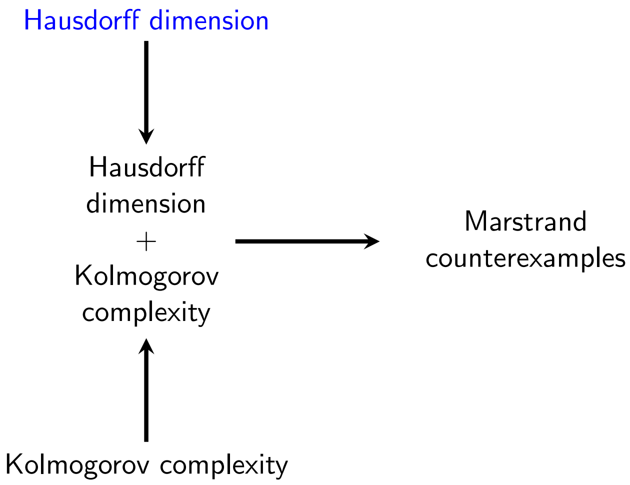
Victoria University of Wellington

~~19~~ 20 September 2023


Hausdorff dimension
Hausdorff dimension + Kolmogorov complexity
Marstrand counterexamples
Kolmogorov complexity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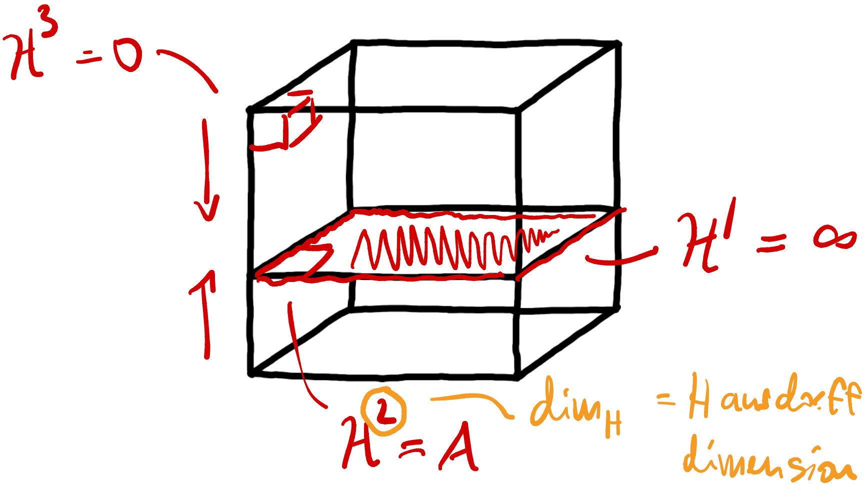
$\mathcal{H}^3 = 0$
$\mathcal{H}^1 = \infty$
$\mathcal{H}^2 = A$
$\dim_H$ = Hausdorff dimension

Definition (Hausdorff dimension)

For $E \subset \mathbb{R}^n$

$$\dim_H(E) = \sup\{s \mid \mathcal{H}^s(E) = \infty\}$$

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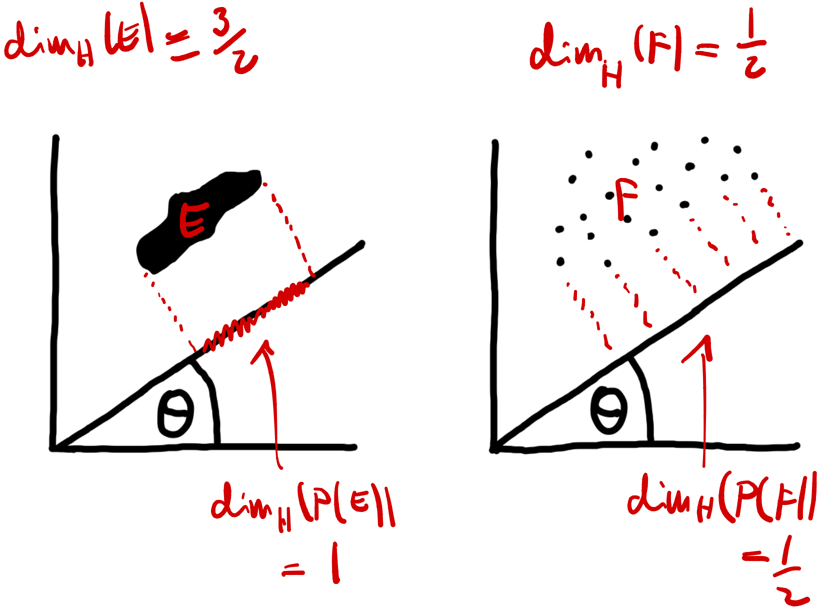
Lemma

$\dim_H$ is invariant under isometries.

Marstrand's Projection Theorem



Marstrand's Projection Theorem (J. Marstrand (1954))

Let $E \subset \mathbb{R}^2$ be analytic. For almost all θ

$$\dim_H(p_\theta(E)) = \min\{\dim_H(E), 1\}$$

where p_θ is the orthogonal projection onto the line θ.

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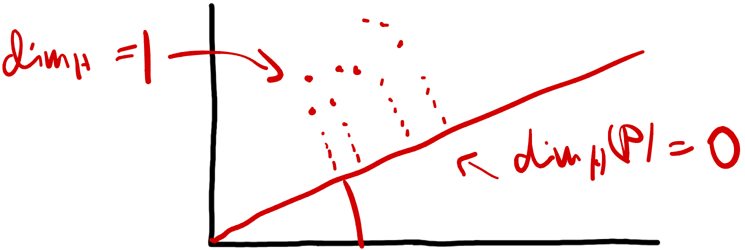
- Also holds if $\dim_H(E) = \dim_P(E)$ (N. Lutz and Stull 2018)
- ...but—assuming CH—does *not* hold for all sets (Davies 1979)



Question

What is the “simplest” set failing Marstrand’s theorem?

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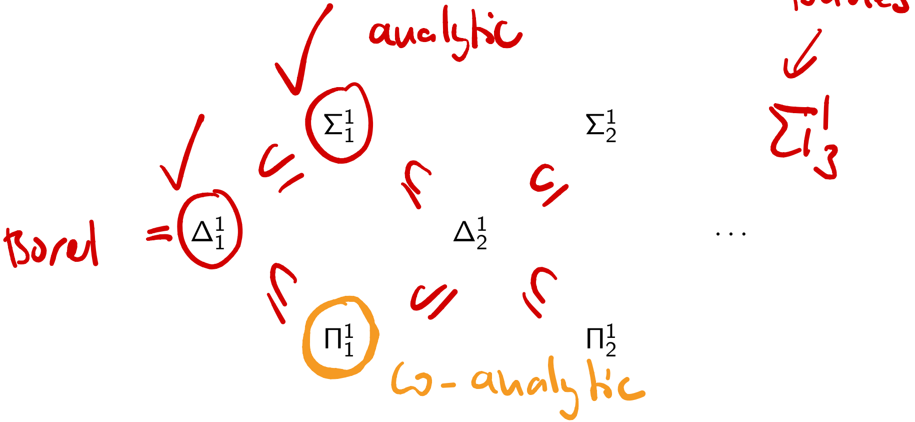
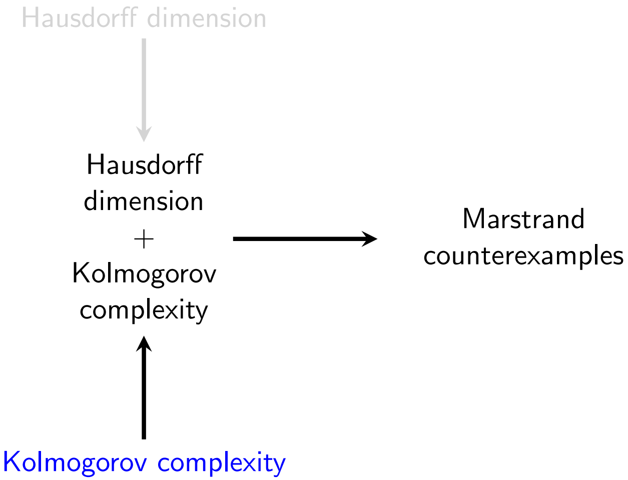
We use descriptive set theory.

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+
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Strings with long descriptions are complicated

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What is the complexity of 01101?

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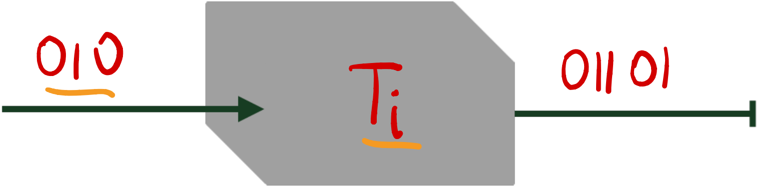
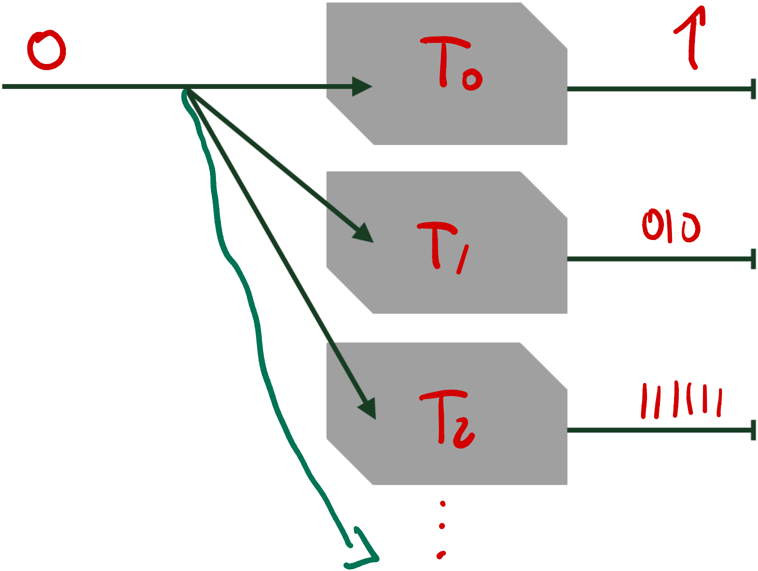
We ask: how long is the shortest description of 01101 in a prefix-free machine?



0 ×
1 ×
⋮
010 ✓

$C_{T_i}(01101) = \ell(010) = 3$

Strings with long descriptions are complicated



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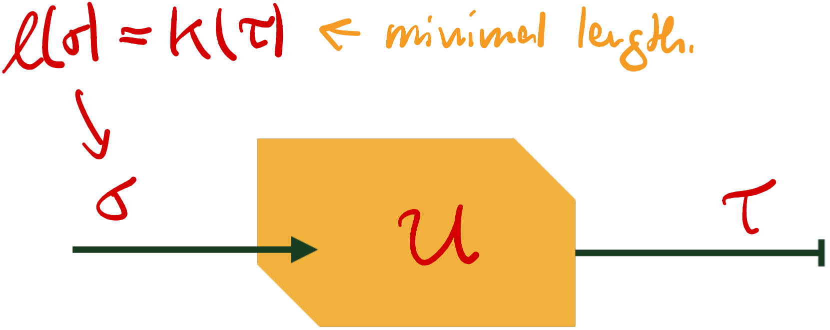

Definition (Levin; Chaitin (1970s))

The prefix-free complexity of a string τ is $K(\tau) = C_U(\tau)$.

$\in 2^{<\omega}$

Definition (Chaitin; Levin (1970s))

A real $f \in 2^\omega$ is Kolmogorov random if $K(f[n]) \geq n - c$ for some constant c.

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The following definitions of randomness are all equivalent!

1. effective open covers $\rightarrow$ Martin-Löf (1966)
2. complexity of strings $\rightarrow$ Chaitin; Levin (1970s)
3. martingales $\rightarrow$ Schnorr (1973)

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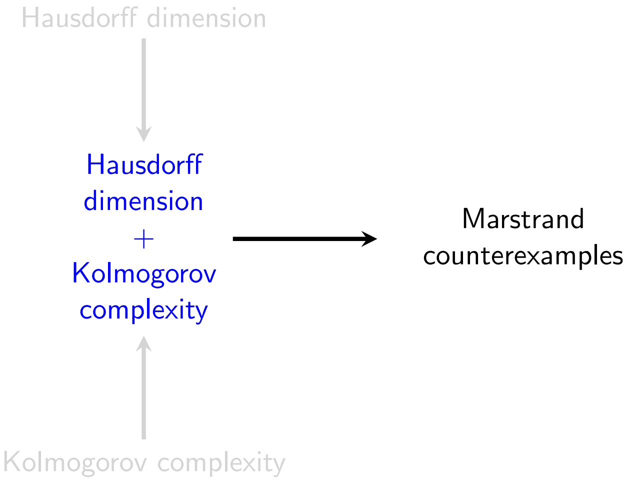
So: random reals do not have (long) patterns!

010110101 0000000000000000...

"print 50000 zeros"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Theorem (J. Lutz; Mayordomo (2003))

There exists dim *on* 2^ω *given by*

$$\dim(f) = \liminf_{n\to\infty} \frac{K(f[n])}{n}.$$

"information density"

This relatives!

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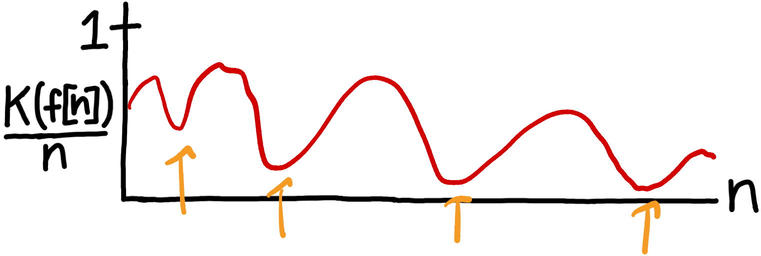
There exists dim *on* 2^ω *given by*

$$\dim(f) = \liminf_{n\to\infty} \frac{K(f[n])}{n}.$$

"information density"

This relativises!

- If $f \in 2^\omega$ is Kolmogorov random then $\dim(f) = 1$.
- If $f \in 2^\omega$ is computable then $\dim(f) = 0$.



$x \in \mathbb{R}$

$$x = \overbrace{x_0 . x_1 x_2 x_3 x_4 \ldots}^{\in 2^\omega}$$

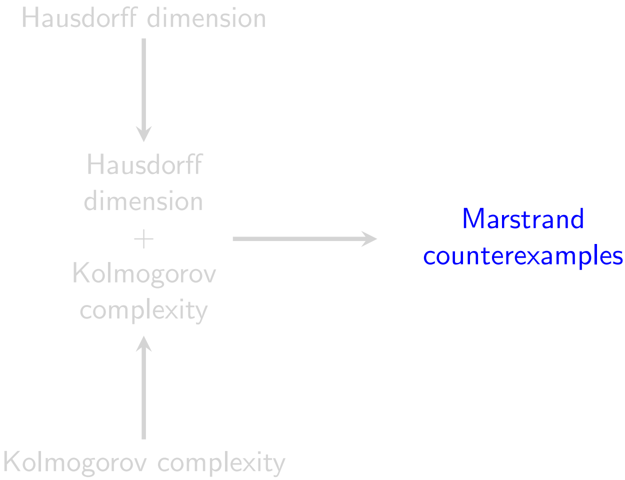
in binary

Point-to-set Principle (J. Lutz, N. Lutz (2018))

For $E \subset \mathbb{R}^n$ we have

$$\dim_H(E) = \min_{A \in 2^\omega} \sup_{x \in E} \dim^A(x).$$

- Extension of result due to Hitchcock (2003) for lightface Π^0_1 classes on 2^ω.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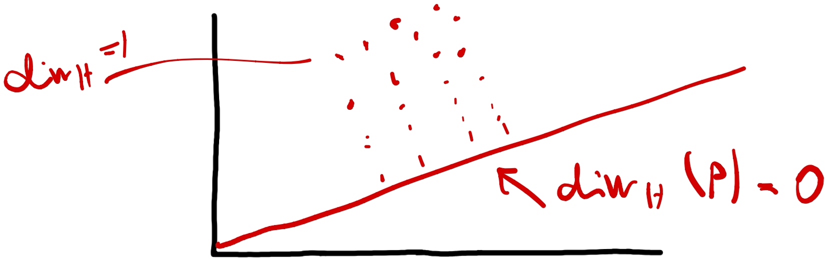
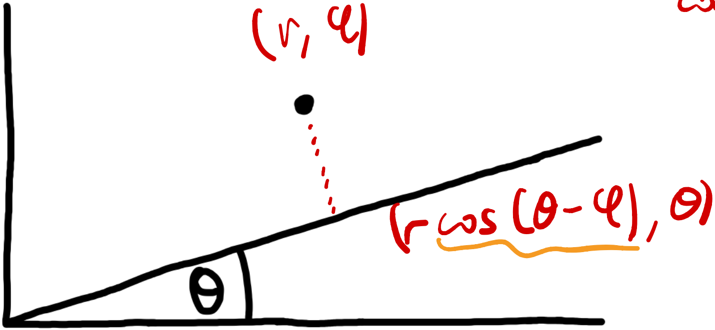
Theorem (R.)

($V\!=\!L$) *There exists a co-analytic* $E \subset \mathbb{R}^2$ *such that* $\dim_H(E) = 1$ *and* $\dim_H(p_\theta(E)) = 0$ *for all* θ.

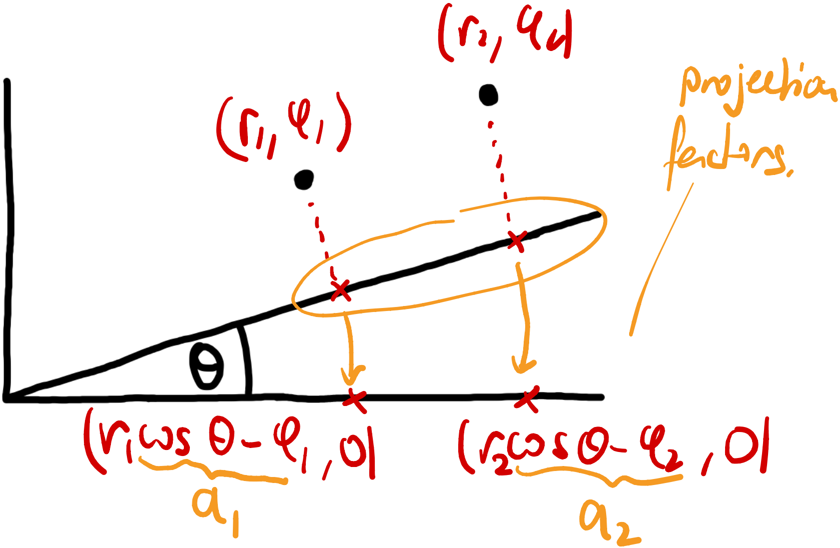


point-to-set principle in polar coordinates

Recall: $\dim_H(E) = \min_{A \in 2^\omega} \sup_{x \in E} \dim^A(x)$



Recall: $\dim_H$ is invariant under isometries.



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How do we **control dimension?**

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Lemma

If $E \subset \mathbb{R}^2$ meets every line through O then $\dim_H(E) \geq 1$.

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Lemma

If $E \subset \mathbb{R}^2$ meets every line through O then $\dim_H(E) \geq 1$.

so $\dim^A(\theta) = 1$

Proof.

Let $A \in 2^\omega$. Take θ random relative to A.

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Let $A \in 2^\omega$. Take θ random relative to A. There exists $r \in \mathbb{R}$ such that $(r, \theta) \in E$.

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$$\dim^A(r, \theta) \geq \dim^A(\theta) = 1.$$

← since random rel. to A.

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A is arbitrary, so PTS completes the argument. □

How do we **construct co-analytic sets?**

By recursion!

By recursion! How we normally do recursion:

Example (Mazurkiewicz 1914)

(AC) There exists a subset $B \subset \mathbb{R}^2$ that intersects every straight line in exactly two points.

Two point set

$$(A, p) \xrightarrow{\text{choose}} x$$

partial sol → A; condition → p; witness → x

points already chosen → A

next line → p

point on line p that satisfies the condition → x

Theorem (Z. Vidnyánszky (2014))

$(V=L)$ Under certain conditions on the set of triples (A, p, x), the above recursion produces a co-analytic set.

p ∈ Borel

partial set → A; current condition → p; witness → x

$$F = \{(A, p, x)\}$$ ← co-analytic

fibers are cofinal in HYP degrees

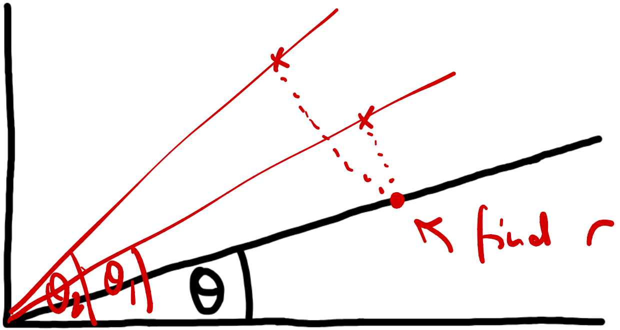
Fix (A, p).

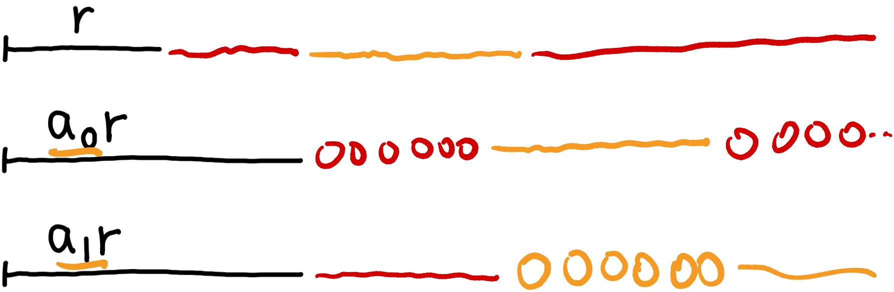
Then $\{x \mid (A, p, x) \in F\}$ cofinal in hyperdegrees

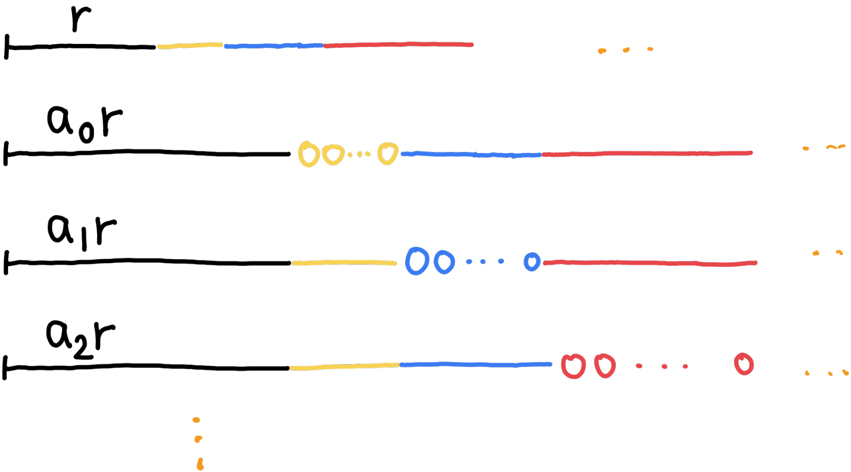
Constructing E by recursion

- do recursion on all lines θ through the origin
- at step θ, take all previous lines $\theta_0, \theta_1, \theta_2, \ldots$
- find r so that $\dim(p_{\theta_i}(r, \theta)) = \dim(a_i r) = 0$
- enumerate (r, θ) into E

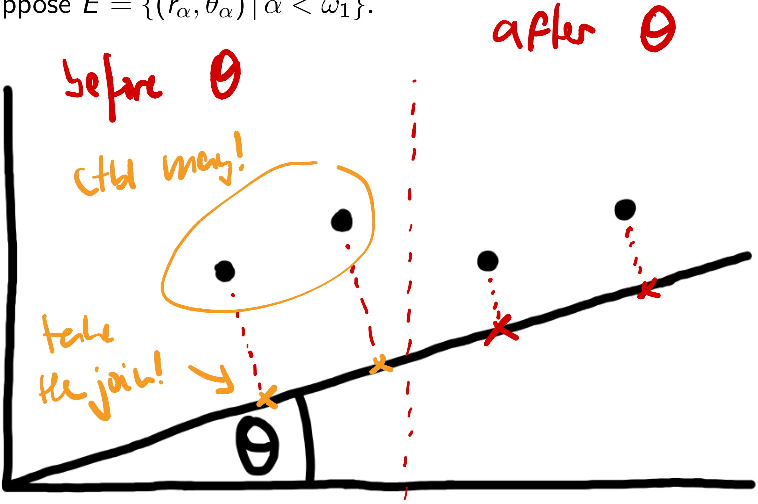
We assumed V=L+CH




r
a0r
a1r


r
a₀r
a₁r
a₂r

Suppose $E = \{(r_\alpha, \theta_\alpha) \mid \alpha < \omega_1\}$.



Theorem (R.)

(V=L) *For every* $\epsilon \in [0,1)$ *there exists a co-analytic* $E_\epsilon \subset \mathbb{R}^2$ *such that* $\dim_H(E_\epsilon) = 1 + \epsilon$ *and* $\dim_H(p_\theta(E_\epsilon)) = \epsilon$ *for all* θ*.*

In fractal geometry:

- What about $\dim_H(E) < 1$?

Open questions

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PTS for packing dimension (J. Lutz, N. Lutz (2018))

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where

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- Is there a co-analytic set failing Marstrand's theorem that is not thin?

Thank you