Strange facts about the marginal distributions of processes based on the Ornstein-Uhlenbeck process

Ray Brownrigg and Estate Khmaladze

School of Mathematics, Statistics and Operations Research,
Victoria University of Wellington,
PO Box 600, Wellington, New Zealand

Abstract

The Ornstein-Uhlenbeck process is particularly useful for modeling stochastic processes in financial applications. Further, functions of such a process can be used to model random volatility of other processes, resulting in more flexible models for financial risk variables. The distribution of such a financial risk variable is of particular interest in Value at Risk analysis. As we know, the far quantiles of the distribution function provide information on the level of capital reserves required to accommodate extreme stress situations. This paper presents an approximation for the distribution function, which in some situations works surprisingly well for even the far tails of the distribution. While theoretically unjustified and strange, it may still be very useful in practice.

keywords: log-normal approximation, high quantiles, VaR, capital reserve, random volatility model.

1 Introduction

Let \( W_t, t \geq 0 \), denote a standard Brownian motion. Then the process defined by the stochastic differential equation

\[
dV_t = -gV_t dt + h dW_t
\]

is called an Ornstein-Uhlenbeck process. The coefficient \( g \) is called the “mean reversion” coefficient – the larger the value of \( g \) the stronger is the tendency of \( V_t \) to return to the zero level. The coefficient \( h \) is just the diffusion coefficient of \( V_t \). The explicit form, or solution, of equation (1) is

\[
V_t = h \int_0^t e^{-g(t-s)} dW_s + V_0 e^{-gt},
\]

where \( V_0 \) is the initial condition of \( V_t \) at \( t = 0 \). Obviously the stochastic integral in equation (2) represents a 0-mean Gaussian process in \( t \) and, once we make the relatively

*Corresponding author.
E-mail address: Estate.Khmaladze@msor.vuw.ac.nz Tel: +64 (4) 463 5652
neutral assumption that $V_0$ is also a Gaussian random variable with mean 0 and variance $h^2/2g$, the process $V_t, t \geq 0$, becomes a stationary Gaussian process with mean 0 and covariance function

$$EV_{t_1}V_{t_2} = \frac{h^2}{2g}e^{-g|t_1-t_2|}.$$ 

The Ornstein-Uhlenbeck process is one of the basic processes used in many problems of probability theory, but it is particularly useful for modeling stochastic processes in financial applications. For example $V$ or its modifications are often used as models for currency exchange rates (see, e.g., [Hull (2009)]). Functions of $V$ are also used to model random volatility of other processes (see, e.g., [Shiryaev (1999)]). For example if $\tilde{W}_t$ is another Brownian motion, then a financial risk variable (such as prices of an asset) can be successfully modeled, see, e.g., [Frishling & Lauer (2006)], as the process

$$dX_t = V^2_t d\tilde{W}_t \quad \text{i.e.} \quad X_t = \int_0^t V^2_s d\tilde{W}_s + X_0 \quad (3)$$

or

$$dX_t = e^{V_t} d\tilde{W}_t \quad \text{i.e.} \quad X_t = \int_0^t e^{V_s} d\tilde{W}_s + X_0. \quad (4)$$

Both of these processes are martingales, but neither is Gaussian. Below we mostly consider the second case and assume $X_0 = 0$. However, at the end of Sec. 2 we show a result for the first case as well.

If $X_t$ is a model for a risk variable, then one of the most important questions is how big it can be and with what probability, i.e. what is the behaviour of the distribution function

$$F_t(x) = P\{X_t \leq x\}.$$ 

In particular, in VaR (value at risk) problems, it is of interest to analyse quantiles of $F_t(\cdot)$ of level $1 - \alpha$ for usually very small $\alpha$. Indeed, this quantile, say $q_t(\alpha)$, i.e. a point such that $F_t(q_t(\alpha)) = 1 - \alpha$, will then be an amount of capital reserve which should be put aside for the case when stress occurs and $X_t$ becomes exceptionally large – larger than $q_t(\alpha)$ although $\alpha$ is small.

In this paper we present an approximation of $F_t(\cdot)$ and show that it works surprisingly well up to $\alpha \sim 10^{-7}$. While theoretically unjustified and strange, it, nonetheless, may be very useful in practice.

2 The case of independent $W$ and $\tilde{W}$.

If the two driving Brownian motions $W$ and $\tilde{W}$ are independent, then it is very natural to consider the conditional distribution of $X_t$ given the trajectory $V_s, s \leq t$. This conditional distribution is simply Gaussian with expected value zero and variance

$$\kappa_t^2 = \int_0^t e^{V_s} ds$$

In notations,

$$P\{X_t \leq x|V_s, s \leq t\} = \Phi\left(\frac{x}{\kappa_t}\right),$$

where $\Phi$ denotes the standard normal distribution function. If $H_t$ is the distribution function of $\kappa_t^2$, i.e.

$$H_t(x) = P\left\{\int_0^t e^{V_s} ds \leq x\right\},$$

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then for $F_t$ we obtain

$$F_t(x) = \int_0^\infty \Phi(\frac{x}{\sqrt{y}})dH_t(y).$$

The strange thing about $H_t$ is that it is extremely close to a log-normal distribution. This is not what one would normally expect: although $e^{V_s}$ is distributed log-normally, it is not true at all that the sum (or integral) of (dependent) log-normal random variables is log-normal.

Sums of log-normal random variables have attracted interest previously, mostly with respect to their behaviour on the tails of the distribution – see e.g. [Goldie & Klüppelberg (1998)], [Asmussen & Rojas-Nandayapa (2008)], [Gao et al. (2009)]. In particular, the recent paper [Gao et al. (2009)] shows asymptotic similarity between the density of a sum of independent log-normal random variables and a log-normal density on the right tail, as well as showing asymptotics of the former on the left tail. However we will see that the distribution function $H_t$ is strikingly close to a log-normal distribution on its whole range.

To illustrate this phenomenon, we present here computational results for various choices of parameters $g$, $h$ and $t$. Namely, Figure 1 shows simulated (empirical) distribution functions of $\kappa^2_t$, based on $n = 10^5$ replications, for two more or less arbitrary choices of the time horizon $t$ and parameters $h$ and $g$ of $V_s$, along with their log-normal approximations. As we see, the graphs are barely distinguishable. As a matter of fact, one approximates another to the third decimal digit. We found the same accuracy throughout many other choices of $g$, $h$ and $t$.

Table 1 shows the values of sup$_x |H_{n,t}(x) - \tilde{H}(x)|$, where $H_{n,t}$ is the empirical distribution function of $\kappa^2_t$ based on $n = 10^6$ replications, while $\tilde{H}$ is its best approximation by a log-normal distribution function. Here and below $\sigma^2 = h^2/2g$ is the variance of $V_s$. Multiplied by $\sqrt{n} = 10^3$ these differences would be the values of Kolmogorov-Smirnov statistics, but with estimated parameters and, therefore, stochastically smaller than usual Kolmogorov-Smirnov statistics (see, e.g., [Khmaladze (1979)]). Therefore, in all cases the hypothesis of log-normality would be rejected, although the absolute difference is very small. Note that the step of approximate integration to calculate values of $V_t$ was quite small, $\delta = 0.001$.

One way of considering the integral $\int_0^t e^{V_s}ds$ is through the mean value theorem. According to this theorem, there is a value $V_{\tau(t)}$, which is a “representative value” for the trajectory $V_s$, $s \leq t$, such that

$$\int_0^t e^{V_s}ds = te^{V_{\tau(t)}}.$$ 

As the moment $\tau(t)$ is random, there is no reason for $V_{\tau(t)}$ to be a Gaussian random variable. Nevertheless, as we see, its distribution is somehow very close to Gaussian. Parameters of this Gaussian distribution, however, have nothing in common with the parameters of the distribution of $V_s$. Moreover, it is more or less clear that as $t$ increases we will have

$$\frac{1}{t} \kappa^2_t \rightarrow \frac{1}{t} \int_0^t e^{V_s}ds \rightarrow Ee^{V_s} = e^{\sigma^2/2}. \quad (5)$$

But since $V_{\tau(t)}$ is approximately Gaussian, with parameters, say, $\mu_t$ and $\nu_t^2$, so that

$$Ee^{V_{\tau(t)}} \approx e^{\mu_t + \nu_t^2/2},$$

convergence (5) would imply that $\nu_t^2 \rightarrow 0$ and, therefore, $\mu_t \rightarrow \sigma^2/2$. Figures 2 and 3 confirm this and show how $\mu_t$ and $\nu_t$ change with $t$ for three different values of $\sigma^2$. As the graphs show, both $\mu_t$ and $\nu_t$ can be well approximated using a power function of $t$. 

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We did not pursue the problem of how the coefficients of these approximations may depend on the parameter $\sigma^2$ any further, because in practical situations the parameters of the approximating log-normal distribution will be estimated through comparison of $F_t$ with the data available on the risk process $X_t$ and not through $H_t$ itself or the distribution of $V_t$ (see Sec. 3). It is much more important to understand how stable the behaviour of $F_t$ is with respect to the parameters of the log-normal approximation of $H_t$.

In Fig 4 we show the graph of $1 - F_t(x)$ for $x > 0$ when $H_t(x)$ is approximated by a log-normal distribution. Then the parameters of the best fitting log-normal distribution were varied by 50% in either direction and replotted. Finally, the distribution of $1 - F_t(x)$ using the empirical distribution function $H_{n,t}$ was also plotted. The resulting four graphs of $1 - F_t(x)$ are all visually almost indistinguishable on the whole range $x > 0$. Further, as Fig 5 shows, even at the very remote tail of $x > 5$, corresponding to $\alpha < 10^{-6}$, they still exhibit relatively very little variation: the quantiles change by no more than 25-30%, while the best log-normal approximation itself is again visually almost indistinguishable from the exact tail. This shows that even very small tail probabilities (of order $10^{-6} - 10^{-7}$), or remote tail quantiles, can be safely calculated using a log-normal approximation for $H_t(x)$. Note that in many financial applications of VaR it is commonly accepted that accuracy of the estimation of remote tail quantiles to within half an order of magnitude is still practically useful. The accuracy we see here is much higher.

Approximations of $H_t$ by log-normal distributions for a range of values of the parameters of the underlying driving process $V_t$ is shown in Figure 6.

As we mentioned at the end of the introduction, it may be useful to consider slightly less variable volatility and thus model the risk process as shown in (3), rather than (4). However, if it was unclear why the log-normal approximation for the distribution $H_t$ of the integral $\kappa^2_t = \int_0^t V_s^4 ds$, which is the conditional variance of the process (4). However, the last graph of this section, Fig.7, shows that again it does.

### 3 Fitting parameters in a practical situation.

So far we have been studying the distribution $H_t$ and its log-normal approximation as such. We also have shown that the resulting mixture distribution $F_t$ is stable with respect to the parameters of this approximation. In dealing with these questions we, naturally, used the empirical distribution function of $\kappa^2_t$. However, in practice, when one uses (4) as a model for a risk variable, the observations of $V_t$ are not directly available - the only process that is observed is $X_t$ itself. Hence, it is useful to consider what happens if we try to estimate the parameters of the log-normal approximation using just the empirical distribution of $X_t$.

It is well-known that an inference about a mixing distribution based on the mixture often forms an ill-posed problem (see, e.g., [Chauveau et al. (1994), Mnatsakanov & Ruymgaart (2003)]). Therefore one should expect that the estimates of the parameters $\mu$ and $\sigma$ obtained in this way will not be very accurate and, as a result, the corresponding estimates of the quantiles of $F_t$ will be less accurate and not as stable as above.

Although this is true, still we found that these estimators are performing quite well. Figure 8 shows a histogram of far quantile values, corresponding to $\alpha = 10^{-7}$, estimated in this way from a series of samples of 1000 values of ‘observed’ $X_t$. Approximately 75%
of the estimated quantile values fall within 25% of the true quantile, which in this context is a commendable result.

4 The case of dependent $W$ and $\tilde{W}$.

In the elementary form, the random variable that has the distribution $F_t(\cdot)$ with log-normal approximation for $H_t$ can be written through two independent Gaussian $(0,1)$ random variables $Z_1$ and $Z_2$ as

$$X = e^{\mu + \nu Z_1} Z_2.$$  

On the other hand, if the two driving Brownian motions $W$ and $\tilde{W}$ are dependent, then the conditioning on the trajectory $\tilde{V}_s, s \leq t,$ does not produce anything simple for the distribution of $X_t$. This dependence, however, may occur in real situations, especially in stressed conditions when movements of prices tend to synchronize (see, e.g., [Frischling & Lauer (2006)]). In the dependent case, the random variables $Z_1$ and $Z_2$ can not be assumed independent, but it is by no means clear that if we assume them to be correlated the distribution of $X$ will still approximate the distribution of $X_t$ well enough. It follows from a very important observation, made in [Lauer (2006)], that for the simple cost of an extra location parameter a very reasonable approximation can be achieved. Below we show in some detail how well the distribution of $X_t$ can be approximated by the distribution $F(x)$ of

$$X = A + e^{B + HZ_1} Z_2,$$

where $Z_1$ and $Z_2$ are correlated standard normal random variables.

Denote $\rho$ the correlation coefficient between $W$ and $\tilde{W}$; $EdW_t d\tilde{W}_t = \rho dt$. It would be reasonable for the approximating random variable $X$ to use some different value for the correlation between $Z_1$ and $Z_2$ to achieve perhaps a better fit. However, we are not doing this and use the same value of $\rho$ as the correlation between $Z_1$ and $Z_2$. Note that in the decomposition $Z_2 = \rho Z_1 + Y$, the Gaussian random variable $Y$ is independent from $Z_1$. Therefore if $\rho \neq 0$ then $Ee^{B + HZ_1} Z_2 = \rho Ee^{B + HZ_1} Z_1 \neq 0$ and the introduction of the location parameter $A$ is unavoidable.

Figures 9 and 10 show for the correlated case the tail and far tail of the ‘true’ distribution of $1 - F_t(x)$, its approximation by $F(x)$, and an indication of how the approximation varies with modifications of the parameters of the approximation. It is clear that not only is the approximation not quite as good as for the uncorrelated case, but the approximation is also a little less robust in terms of changes in the values of the parameters of the approximation.

Finally, Figure 11 shows a histogram of far quantile values, corresponding to $\alpha = 10^{-6}$, estimated in this way from a series of samples of 1000 values of ‘observed’ $X_t$. This indicates that even in the correlated case the use of a log-normal approximation form of $F_t(x)$ can be useful when estimating far tail quantiles.
References


Table 1: Absolute deviations of simulated distribution functions of $\kappa_t^2$ from their log-normal approximations, for $n = 10^6$. The parameter $\sigma^2$ varies from low to high, as do the values of the time horizon $t$. One can think of $t$ as the number of “working days”, e.g. $t = 22$ is one month of calendar time.
Figure 1: Graphs of the empirical distribution function of $\kappa_t^2$ and its log-normal approximations. On the left the variance of $V_s$ is $\sigma^2 = 0.17$ and $t = 1$; on the right $\sigma^2 = 1$ and $t = 10$. The edfs and their approximations are almost indistinguishable.
Figure 2: The values of $2\mu_t/\sigma^2$ and its continuous approximation. A logarithmic scale is used on the $t$-axis. The approximating curves are $1 - 1.00/(t + 1)^{1.17}$ for $\sigma^2 = 0.17$, $1 - 1.13/(t + 1)^{0.96}$ for $\sigma^2 = 1.0$, and $1 - 1.66/(t + 2)^{0.70}$ for $\sigma^2 = 4.5$. 
Figure 3: The values of $\nu_t/\sigma$ and its continuous approximation. A logarithmic scale is used on the $t$-axis. The approximating curves are $0.99/(t+1)^{0.56}$ for $\sigma^2 = 0.17$, $1.06/(t+1)^{0.48}$ for $\sigma^2 = 1.0$, and $1.31/(t+2)^{0.38}$ for $\sigma^2 = 4.50$. 
Figure 4: The tail of $1 - F_t(x)$ using an ‘exact’ $H_t(x)$ (blue or dot-dash), a log-normal approximation (black), and also with the approximating log-normal parameters perturbed by 50% in either direction (green or dotted, red or dashed). Here $t = 1$ and the true parameters of the driving volatility process $V_t$ are $g = 3$ and $h = 1$, so that $\sigma_t^2 = 0.17$. All four curves are visually almost indistinguishable.
Figure 5: The far tail of $1 - F_t(x)$ with $H_t(x)$ approximated by a log-normal distribution. The parameter values are the same as in the previous graph. Even at such distant tail quantiles, the approximating log-normal (black) is almost indistinguishable from the ‘exact’ distribution (blue or dot-dash). The quantiles corresponding to the perturbed parameters vary by a relatively small amount either side of the base values.
Figure 6: Approximations of $H_t$ by log-normal distributions for a range of values of the parameters of the underlying driving process.
Figure 7: The distribution function of $\zeta_t^2$ and its approximation by a log-normal distribution. Here again $t = 1$ and the true parameters of the driving volatility process $V_t$ are $g = 3$ and $h = 1$. 
Figure 8: Histogram of estimated far quantiles ($\alpha = 10^{-7}$) using a log-normal approximation form of the distribution of $F_t(x)$, and samples of 1000 values. The ‘true’ quantile is indicated by the vertical dashed line.
Figure 9: The tail of $1 - F_t(x)$ (blue or dot-dash), its approximation by $F(x)$ (black), and also with the approximating log-normal parameters perturbed by 50% in either direction (green or dotted, red or dashed). Here $t = 1$ and the true parameters of the driving volatility process $V_t$ are $g = 3$ and $h = 1$ and $\rho = 0.5$. 
Figure 10: The far tail of $1 - F_i(x)$ and its approximation by $F(x)$. The parameter values are the same as in the previous graph. Here both the initial approximation and the approximations with perturbed parameters show slightly greater deviations from the ‘true’ distribution than they do for the uncorrelated case.
Figure 11: Histogram of estimated far quantiles ($\alpha = 10^{-6}$) using a log-normal approximation form of the distribution of $F_t(x)$, and samples of 1000 values. The ‘true’ quantile is indicated by the vertical dashed line.