TRANSFORMING TREES INTO ABELIAN GROUPS

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Abstract. We study a coding of trees into torsion-free abelian groups which proved to be useful in effective algebra. We show that this transformation is not injective. Furthermore, we show that for the class of finite trees having all leaves at the same level, the resulting groups are isomorphic if and only if the underlying trees have the same number of nodes at corresponding levels.

Keywords: abelian groups, effective transformations

1. Introduction

1.1. **A coding of trees into abelian groups.** We study a certain (effective) coding trees into torsion-free abelian groups.

One uses infinite divisibility (see Definition 2.3) to distinguish certain elements of a torsion-free abelian group from another. Fuchs [7] used infinite divisibility to construct indecomposable torsion-free abelian groups of large cardinalities. See [7] for more examples of this kind in pure abelian group theory. See also [11] for the study of infinite divisibility and indecomposability of automatic abelian groups.

Hjorth [8] used infinite divisibility to study torsion-free abelian groups from the descriptive set theory point of view. He showed that the isomorphism problem for torsion-free abelian groups is not Borel (see [8] for definitions).

In computable algebra, if a coding is effective then it is usually said to be an effective transformation [9]. Downey and Montalbán [3] applied the ideas of Hjorth and defined an effective transformation of trees to torsion-free abelian groups, as follows. Following Fuchs [7], we denote by $p_n^{-\infty}g$ the collection of generators $\{p_n^{-k}g : k \in \omega\}$. Let (p_n), (q_k) be two disjoint computable sequences of distinct primes. Suppose a tree $T = (V, E)$ with distinguished root r is given. The group $G(T)$ is the subgroup of $\bigoplus_{v\in V} \mathbb{Q}v$ generated by $p_n^{-\infty}v$ for $v \in V$ of height n, and $q_n^{-\infty}(v + w)$ where (v, w) is an edge, v is of height n-1 and w is of height n. Clearly, isomorphic trees give rise to isomorphic groups. It was not clear if the coding preserves the isomorphism type in general (in this case it would be said to be injective). Nonetheless, Downey and Montalbán [3] used this coding to show that the isomorphism problem for computable torsion-free abelian groups is Σ^1_1-complete.

Fokina, Knight et al. [6] showed that the transformation form [3] is injective for the special class of rank homogeneous trees. Although their result was only partial, Fokina, Friedman, Harizanov et al. [5] applied this fact to show that the isomorphism relation on computable torsion-free abelian groups is tc-complete among Σ^1_1 equivalence relations (see [5] for definitions). Andersen, Kach, Melnikov and Solomon [1] used a similar but more general method to study jump degrees of torsion-free abelian groups.

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We show that in general the transformation is not injective and, therefore, the technical problems in the proofs of the main results in [6] and [1] can not be circumvented in an obvious way.

1.2. **Results.** We show that there exist two non-isomorphic trees which give rise to isomorphic groups. As a consequence, for every $n \geq 7$, there exist non-isomorphic finite trees having n nodes which give rise to isomorphic computable groups. It also follows that there exist infinite non-isomorphic computable trees which are mapped to isomorphic computable groups.

For the sake of better understanding of the coding, we show more:

Theorem. Suppose that T and U are finite trees with leaves all at the same level. Then $G(T) \cong G(U)$ if and only if T and U have the same number of vertices at corresponding levels.

We also observe that this theorem fails for finite trees having leaves at different levels.

2. Basic concepts

The results of the paper are motivated by the recent results in the area of computable abelian groups which were mentioned in the introduction. See [4] and [2] for the general background on computable model theory, and [10] for the basics of the theory of computable abelian groups. We assume that the reader is familiar with elementary facts of linear algebra which can be found in any standard textbook.

2.1. **Abelian groups.** Recall that an abelian group G is torsion-free if every non-zero element of G is of infinite order. All the groups below are torsion-free. We will use basic definitions which are well-known and can be found in [7].

Definition 2.1. Elements $x_0, \ldots, x_n$ of a torsion-free abelian group are *linearly independent* if the equality $k_0x_0 + k_1x_1 + \ldots + k_nx_n = 0$ implies that $k_0 = k_1 = \ldots = k_n = 0$, for all $k_0, \ldots, k_n \in Z$.

A maximal linearly independent set is a *basis*. All bases of a torsion-free abelian group G have the same cardinality which is called the *rank* of G. Note that the rank of a torsion-free abelian group is evidently its invariant.

Definition 2.2. An abelian group G is the *direct sum* of groups A_i, $i \in I$, written $G = \bigoplus_{i \in I} A_i$, if the domain of G consists of infinite sequences $(a_0, a_1, a_2, \ldots, a_i, \ldots)$, each $a_i \in A_i$, such that the set $\{i : a_i \neq 0\}$ is finite, and the operation $+$ on the sequences is defined component-wise.

Definition 2.3. Let G be a torsion-free abelian group. We write $k|g$ *in* G and say that k *divides* g if there exists an element $h \in G$ for which $kh = g$, and we denote the element h by $\frac{g}{k}$. If $k^n|g$ in G, for every $n > 0$, the we say that k infinitely divides the element g.

If $k|g$ for every g in G and every integer $k > 0$, then G is said to be *divisible*. Every abelian group is contained in a divisible group. If D is a divisible group which contains G and is the least (under injections preserving G) divisible group with this property, then D is called the *divisible closure* of G. The divisible closure is unique up to isomorphism. We denote the divisible closure of G by $D(G)$.

2.2. **Trees.** As usual, an acyclic connected graph with a specified node called the root is referred as a tree. Given a tree $T = (V, E, r)$ and $n \in \omega$, define

$$T_n = \{v \in V : \text{ the chain from } v \text{ to the root } r \text{ has length } n\}.$$

The nodes belonging to T_n are said to be *at level* n of the tree. Given $n \in \omega$, let

$$E_n = \{(v, w) \in E : v \in T_{n-1} \text{ and } w \in T_n\}.$$

(Note that $E_0 = \emptyset$.)

3. The failure of injectivity

Theorem 3.1. *For each $x \in \{i : i \geq 7\} \cup \{\omega\}$ there exist (computable) trees T^x and U^x having x many nodes, such that $T^x \ncong U^x$ but $G(T^x) \cong G(U^x)$.*

The proof of the theorem is based on the lemma below. This lemma can be generalized (see Theorem 3.6), but the proof of Theorem 3.1 does not need Theorem 3.6 in all its generality.

Lemma 3.2. *Let T and U be trees from Figure 1. Then $G(T) \cong G(U)$. Furthermore, there is an isomorphism $\varphi : G(T) \cong G(U)$ such that $\varphi(y_1) = x_1$.*

The proof of the lemma does not require any knowledge of abelian group theory and is self-contained.

Proof. Let T and U be trees from Figure 1. We need to show that $G(T) \cong G(U)$.

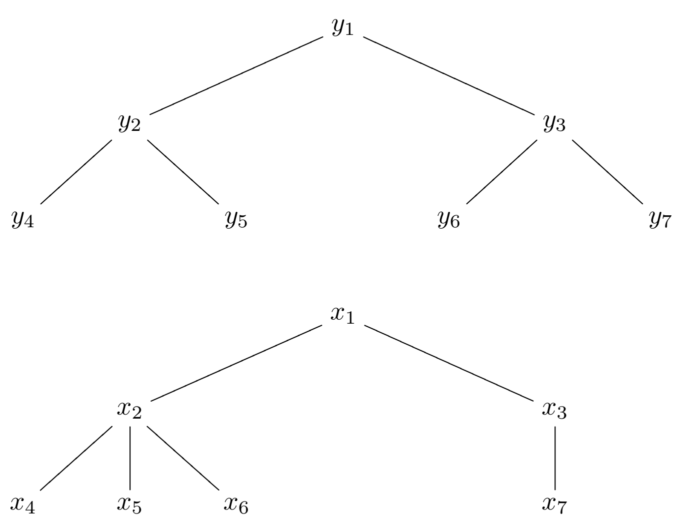

Figure 1: The tree T (top) and the tree U (bottom).

Recall that $D(A)$ stands for the divisible closure of the given abelian group A. We define a homomorphism $\varphi : G(T) \to D(G(U))$. Then we prove that φ is injective and the range of φ is $G(U)$.

We define φ starting with y_j, $j \in \{1, \ldots, 7\}$:

(1) $\varphi(y_j) = x_j$, for $j \in \{1, 2, 3\}$;
(2) $\varphi(y_j) = x_{j+1}$, for $j \in \{4, 5\}$;
(3) $\varphi(y_6) = x_7 + x_4 - x_6$;
(4) $\varphi(y_7) = x_7 + x_5 - x_6$.

For every given linear combination $\sum_{j\in\{1,\dots,7\}} r_j y_j$ with *rational* coefficients, define

$$\varphi(\sum_{j\in\{1,\dots,7\}} r_j y_j) = \sum_{j\in\{1,\dots,7\}} r_j \varphi(y_j).$$

The map $\varphi : D(G(T)) \to D(G(U))$ preserves linear independence and is an isomorphism of vector spaces. We may restrict φ to $G(T)$. Evidently, $\varphi : G(T) \to D(G(U))$ is a homomorphism of $G(T)$ onto an additive subgroup of $D(G(U))$. Furthermore, it is injective, as follows from the claim below.

Claim 3.3. *Let $\psi : A \to C$ be a homomorphism of torsion-free abelian groups which maps a basis B of A into a linear independent subset of C. Then ψ is injective.*

Proof. As can be easily seen,if a set $S \subset B$ is linearly independent, then it is linearly independent over Q (not merely over Z) in $D(B)$. Thus, for every finite subset B_0 of B and every collection of rational coefficients $\{r_b : b \in B_0\}$,

$$\psi(\sum_{b\in B_0} r_b b) = 0 \Rightarrow \sum_{b\in B_0} r_b \psi(b) = 0 \Rightarrow \bigwedge_{b\in B_0} r_b = 0.$$

Note that for every element a of A we can write $a = \sum_{b\in B_0} \frac{n_b}{m} b$, for some finite $B_0 \subseteq B$ and integers m and n_b, $b \in B_0$. Therefore, $\ker \psi = 0$, as required.

□

We have established the injectivity of φ on its domain. We need to check that the range of φ is $G(U)$. This follows from the next two claims.

Claim 3.4. $\varphi(G(T)) \subseteq G(U)$.

Proof. Note that it is enough to show that every generator of $G(T)$ has a φ-image in $G(U)$. Let k be any natural number. We have: $\varphi(\frac{y_1}{p_1^k}) = \frac{\varphi(y_1)}{p_1^k} = \frac{x_1}{p_1^k} \in G(U)$ and $\varphi(\frac{y_7}{p_3^k}) = \frac{\varphi(y_7)}{p_3^k} = \frac{x_7+x_5-x_6}{p_3^k} = \frac{x_7}{p_3^k} + \frac{x_5}{p_3^k} - \frac{x_6}{p_3^k} \in G(U)$. Similar routine check shows that $\varphi(\frac{y_j}{p_2^k}) \in G(U)$, for $j \in \{2, 3, 4, 5, 6\}$.

We may check that φ maps the rest of generators into $G(U)$ as well. Say, $\varphi(\frac{y_3+y_7}{q_2^k}) = \frac{x_3+x_7+x_5-x_6}{q_2^k} = \frac{x_3+x_7}{q_2^k} + \frac{x_5+x_2}{q_2^k} - \frac{x_2+x_6}{q_2^k} \in G(U)$. Thus, the image of every generator of $G(T)$ belongs to $G(U)$. The claim is proved.

□

Claim 3.5. $\varphi(G(T)) \supseteq G(U)$.

Proof. We need to show that every generator of $G(U)$ has a pre-image in $G(T)$. The proof is very similar to the proof of the previous claim. We have $\frac{x_1}{p_1^k} = \varphi(\frac{y_1}{p_1^k})$; the expressions for x_j, $j \in \{2, 3, 5, 6\}$, are similar. We can also see that $\frac{x_4}{p_3^k} = \frac{\varphi(y_6)}{p_3^k} - \frac{\varphi(y_7)}{p_3^k} + \frac{\varphi(y_4)}{p_3^k}$ and $\frac{x_7}{p_3^k} = \frac{\varphi(y_7)}{p_3^k} - \frac{\varphi(y_5)}{p_3^k} + \frac{\varphi(y_4)}{p_3^k}$. The map was defined in a way to make the rest generators of $G(U)$ images of certain elements in $G(T)$. We have: $\frac{x_1+x_j}{q_1^k} = \varphi(\frac{y_1+y_j}{q_1^k})$, for $j \in \{2, 3\}$; $\frac{x_2+x_{j+1}}{q_2^k} = \varphi(\frac{y_2+y_j}{q_2^k})$, for $j \in \{4, 5\}$; $\frac{x_2+x_4}{q_2^k} = \varphi(\frac{(y_2+y_4)+(y_6+y_3)-(y_3+y_7)}{q_2^k})$. Finally, $\frac{x_3+x_7}{q_2^k} = \varphi(\frac{(y_3+y_7)+(y_2+y_5)-(y_2+y_4)}{q_2^k})$.
This finishes the proof of the claim.

□

We have shown that $G(T) \cong G(U)$ via φ. Obviously, $T \not\cong U$. □

Proof of Theorem 3.1. To obtain a pair of trees T^x and U^x add a (finite or infinite, depending on x) chain to the root of each tree from Figure 1, and apply Lemma 3.2. Lemma 3.2 guarantees that the isomorphism $\varphi : G(T) \to G(U)$ can be extended to an isomorphism of $G(T^x)$ onto $G(U^x)$. Indeed, φ maps the root of T to the root of U.

□

Clearly, the upcoming Theorem 3.6 generalizes Lemma 3.2 above. Lemma 3.2 does not require any background in abelian group theory and is sufficient for the proof of Theorem 3.1. Nonetheless, especially if the reader is not a logician, she or he may be interested in further algebraic properties of $G(T)$. We compromise by giving a proof of Theorem 3.6 which does not use Lemma 3.2 but requires a bit more background in abelian group theory.

Consider the following isomorphism invariant of a given tree T which says how many nodes the tree T has at the *n-th* level: $L(T) = \langle |T_n| : n \leq height(T)\rangle$. If T has all leaves at one level, then $L(T)$ is a tuple of the form $\langle n_0, n_1, \ldots, n_h\rangle$, where $1 = n_0 \leq n_1 \leq \ldots \leq n_h$.

Theorem 3.6. *Suppose that finite trees T and U have all leaves at one level. Then $G(T) \cong G(U)$ if and only if $L(T) = L(U)$.*

Proof. Consider the pure subgroup G_n of $G(T)$ generated by the image of T_n. Note that $g \in G_n$ if and only if $p^{\infty}_{n+1}|g$. Observe that the number of nodes in T_n is equal to the rank of G_n. Note also that $n > height(T)$ implies that the rank of G_n is 0. Thus, the number of nodes in T_n depends on the isomorphism type of $G(T)$ only, for every $n \geq 0$. This gives the proof of the "only if" part.

For the "if" part of the theorem, for each $n \leq height(T)$ consider the subgroups H_n and F_n of G_n generated by $\{(v-w) : v, w \in T_n$ have the same predecessor$\}$ and $\{v : v \in T_n\}$, respectively. Observe that $F_{n-1} \cong F_n/H_n$ via a homomorphism ϕ_n which takes each node (element, corresponding to it) to its predecessor. The groups F_n, F_{n-1} and H_n are free, therefore the subgroup H_n detaches as a summand of F_n (see, e.g., [7], Theorem 14.4).

Now assume we are given two trees, T and U, such that $L(T) = L(U)$. Let $G = G(T)$ and $G' = G(U)$. As above, define H_n, F_n and ϕ_n for G, and similarly define H'_n, F'_n and ϕ'_n for G'. Since $L(T) = L(U)$, we have $rk(H_n) = rk(H'_n)$ and $rk(F_n) = rk(F'_n)$, for each $n \leq h = height(T) = height(U)$. Furthermore, by the above observation, we have $F_n \cong F_0 \oplus H_1 \oplus \ldots \oplus H_h$ and $F'_n \cong F'_0 \oplus H'_1 \oplus \ldots \oplus H'_h$. Let $\varphi : F_0 \to F'_0$ be an isomorphism. It is clear how to extend this isomorphism to an isomorphism ψ of F_n onto F'_n so that $\psi(H_n) = H'_n$ for each $n \leq h$.

Using ψ we can define an isomorphism $\theta : \bigoplus_{0\leq n\leq h} F_n \to \bigoplus_{0\leq n\leq h} F'_n$ which maps each summand F_n to the corresponding F'_n and such that $\theta\phi_n = \phi'_n\theta$ on their domain and range, for each n.

It remains to prove:

Claim 3.7. *The map θ can be extended to an isomorphism from $G(T)$ onto $G(U)$.*

Proof. The map can be extended to a map $\theta : \bigoplus_{0\leq n\leq h} G_n \to \bigoplus_{0\leq n\leq h} G'_n$ in the usual way (see the proof of Lemma 3.2). It remains to show that the q divisibility can be preserved as well. Given $a \in F_n$, we have $\theta(a + \phi_n(a)) = \theta(a) + \theta(\phi_n(a)) = \theta(a) + \phi_n(\theta(a))$, and for $b \in F'_n$, we have $\theta^{-1}(a + \phi'_n(a)) = \theta^{-1}(a) + \theta^{-1}(\phi'_n(a)) =$

$\theta^{-1}(a) + \phi_n(\theta^{-1}(a))$. This shows that θ can be correctly extended to the whole group preserving infinite divisibility by q. □

This completes the proof of the theorem.

□

Clearly, Lemma 3.2 follows from Theorem 3.6. It is natural to ask if Theorem 3.6 can be extended to the case of finite trees having leaves not at the same level. The answer is negative:

Proposition 3.8. *There exist finite trees T and U of height 2 such that $L(T) = L(U)$ but $G(T) \ncong G(U)$.*

Proof. Consider the following trees:

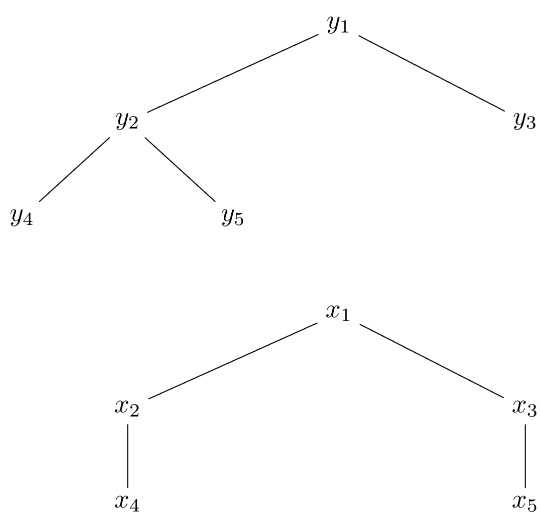

Figure 2: The tree T (top) and the tree U (bottom).

The subgroup of $G(U)$ generated by elements at level 1 having successors and predecessors has rank 2, while the similarly defined subgroup of $G(T)$ has rank 1 (the reader may apply the main technical fact from [8] or may observe it in some other way).

It remains to note that the group generated by elements at level 1 having successors and predecessors can be defined by a first-order formula in the language of abelian groups augmented by predicates $p_1^{\infty}|$, $p_2^{\infty}|$ and $q_1^{\infty}|$. See [6] for more details. □

References

[1] Andersen, B., Kach, A., Melnikov, A., and Solomon, D.R. *Jump degrees of torsion-free abelian groups.* submitted.

[2] Ash, C.J. and Knight, J.F. *Computable structures and the hyperarithmetical hierarchy.* volume 144 of Studies in Logic and the Foundations of Mathematics. North-Holland Publishing Co., Amsterdam, 2000.

[3] Downey, R., and Montalbán, A. *The isomorphism problem for torsion-free abelian groups is analytic complete.* J. Algebra, 320(6), pp. 2291- 2300, 2008.

[4] Ershov, Yu.L, Goncharov, S.S. *Constructive models.* Novosibirsk, Nauchnaya Kniga, 1999.

[5] Fokina, E., Friedman, S.-D., Harizanov, V., Knight, J.F., McCoy, C., and Montalban, A. *Isomorphism and bi-embeddability relations on computable structures.* submitted.

[6] Fokina, E., Knight, J.F., Maher, C., Melnikov, A., and Quinn, S. *Classes of Ulm type, and relations between the class of rank-homogeneous trees and other classes.* To appear in JSL.

[7] Fuchs, L. *Infinite abelian groups. Vol. I.* Academic Press, 1970.

[8] Hjorth, G. *The isomorphism relation on countable torsion free abelian groups.* Fund. Math., 175(3):241 -257, 2002.

[9] Hirschfeldt, D., Khoussainov, B., Shore, R., Slinko, A., *Degree spectra and computable dimensions in algebraic structures.* Annals of Pure and Applied Logic, vol. 115 (2002), pp. 71 - 113.

[10] Khisamiev, N. *Constructive abelian groups.* In: Handbook of recursive mathematics, Vol. 2, Stud. Logic Found. Math., vol. 139, North-Holland, Amsterdam, 1998, pp. 1177–1231.

[11] Nies, A., Semukhin, P. *Finite automata presentable abelian groups.* Ann. Pure Appl. Logic 161(3): 458-467 (2009).

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