SEPARATING NOTIONS IN EFFECTIVE TOPOLOGY

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Abstract. We compare several natural notions of effective presentability of a topological space up to homeomorphism. We note that every left-c.e. (lower-semicomputable) Stone space is homeomorphic to a computable one. In contrast, we produce an example of a locally compact, left-c.e. space that is not homeomorphic to any computable Polish space. We apply a similar technique to produce examples of computable topological spaces not homeomorphic to any right-c.e. (upper-semicomputable) Polish space, and indeed to any arithmetical or even analytical Polish space.

We then apply our techniques to totally disconnected locally compact (tdlc) groups. We prove that every effectively locally compact tdlc group is topologically isomorphic to a computable tdlc group; all notions will be clarified. The result is perhaps unexpected since the hypothesis of the theorem may seem rather weak.

1. Introduction

In the past decade or two there has been an increasing interest in the computability-theoretic aspects of abstract topological spaces. The central questions in such investigations include:

- What does it mean for a space to be computably presented?
- When does a space admit a computable presentation?
- Can we compute the standard topological invariants of a computable space?
- Can we classify computably presentable spaces in a given class? etc.

To attack these and similar questions we will have to depart from classical computable analysis that typically deals with fixed 'natural' computable presentations of Polish and separable Banach spaces. For classical computable analysis, we cite Aberth's [1], Pour-El and Richards [51], Ko [27], Braverman and Yampolsky [6], and Weihrauch [61]. Investigations of computable presentability in abstract topology contribute to the fast developing subject in computable mathematics, namely *computable* topology. There had been many early investigations into effective aspects of abstract topological spaces, to name a few: [20, 21, 22], [45, 46], and [57, 58, 56]. Nonetheless, most of the related work in computable topology is more recent and includes [62, 18, 23, 59, 30, 13, 31]. Some of this work has been motivated by the recently revived systematic research in computable topological groups. Such studies restricted to profinite groups began with Metakides and Nerode [41], La Roche [32, 33] and Smith [55, 54]. After several decades of essentially no activity in this technically challenging subject, computable topological groups (this time, not necessarily profinite) have found new unexpected applications in computable structure theory [39, 15] and, remarkably, in computable topology [38, 34] where they were used to solve problems that seemed unrelated to topological groups.

Melnikov was supported by the Mathematical Center in Akademgorodok under agreement No. 075-15-2019-1613 with the Ministry of Science and Higher Education of the Russian Federation. K.M. Ng was supported by the Ministry of Education, Singapore, under its Academic Research Fund Tier 1 (RG23/19). This work was also partially supported by Rutherford Discovery Fellowship (Wellington) RDF-MAU1905, Royal Society Te Aparangi.

Computable topology is notorious for its 'zoo' of various notions of computability for a topological space, and for a topological group alike. In contrast with effective algebra [11, 2] where all standard notions of computable presentability in common classes had been separated more than half-a-century ago (e.g., Novikov [47], Boone [4], Feiner [12], Khisamiev [24], Odintsov and Selivanov [48]), some of the key notions of computable presentability in topology have been separated only very recently [18, 16, 34, 3]; these results will be discussed in detail later in the paper. In fact, some of the other key notions, such as left-c.e. Polish and computable Polish presentations, have not yet been separated up to homeomorphism. The first main goal of this paper is to fill this apparent gap in the theory by constructing counterexamples.

The second main goal is to apply the techniques used to separate computability notions to obtain a *positive* result about topological groups. In the related work [37, 34], several notions of computable presentability for a totally disconnected locally compact (tdlc) group have been proposed. Remarkably, it has been illustrated in [37, 34] that several seemingly most natural definitions are indeed equivalent, and that the approach in [37, 34] is also equivalent to the well-established notions of computable presentability in the important discrete [35, 52] and profinite [41, 32, 55] cases. In the present paper we connect these investigations to the study of computable Polish groups that are not necessarily tdlc (initiated in [39]). More specifically, we obtain a characterization of computable presentability for a tdlc group in terms of an arbitrary (compatible) computable metric. We believe that our result is unexpected since it uses a seemingly weak hypothesis about the metric.

To state the results formally we need a few basic classical definitions. The notion of a computable Polish space seems to be the most well-established notion of computable presentability for a Polish(able) space. It can be traced back to Ceitin [8] and Moschovakis [42]. A Polish space is *computable* or *computably metrized* if there is a complete, compatible metric d and a dense subset of *special* or *ideal* points $(x_i)_{i\in\omega}$ of the space such that $d(x_i, x_j)$ are computable reals uniformly in i and j. This means that given i, j, n we can calculate $d(x_i, x_j)$ to precision 2^{-n}. This definition can be relativised to an oracle; for instance, a Δ^0_2-presented space can also be defined using algorithms that have access to an oracle for the halting problem.

The following two refinements of Δ^0_2-presentability for spaces are of special importance. If $d(x_i, x_j)$ is merely right-c.e. (right-c.e.), meaning that we can uniformly in i, j list the rationals in the right (resp., left) cut of the real $d(x_i, x_j)$, then we say that the space is *left-c.e.* (resp., *right-c.e.*) presented. In the literature, left- and right-c.e. spaces are also known under the names of lower- and, respectively, upper-semicomputable spaces. The intuition is that right-c.e. and left-c.e. spaces roughly correspond to Π^0_1- and Σ^0_1-presentations, respectively, in computable structure theory. In the context of topological groups this intuition is formally clarified in [28] where it is noted that, e.g., right-c.e. and Σ^0_1-presentability are equivalent for discrete groups. For a Stone space, right-c.e. presentability with a mild additional assumption is equivalent to Σ^0_1-presentability of the dual Boolean algebra [3].

We note that in computable algebra, Δ^0_n-, Π^0_n- and Σ^0_n-presentations had been separated (up to isomorphism) several decades ago; see, e.g., [12, 48, 25]. In contrast, the examples of Δ^0_2-presented (compact) Polish spaces with no computable presentation have been found only very recently; see [16, 18]. One possible explanation of such a delay is that producing such examples typically requires significant effort and new ideas, especially if we want to keep our examples within some natural class of spaces, e.g., compact connected spaces. Recently the notions of a right-c.e. space, computable Polish space, and an 'effectively compact' space (to be defined later) have also been separated in [16, 18, 34, 3]. These results rely on (co)homological methods, the theory of effectively compact spaces, and priority techniques; for a detailed exposition of the methods and the cited results see the technical recent survey [9].

We note that it was not known whether every left-c.e. Polish space must be homeomorphic to a computable one. We will see in Proposition 3.8 that every left-c.e. Stone space has a computable Polish (indeed, effectively compact by [16]) presentation, up to homeomorphism. The first main result of the paper is:

Theorem 1.1. *There exists a left-c.e. Polish space not homeomorphic to any computable Polish space.*

The techniques necessary to prove the result have another peculiar and (we believe) important implication. Namely, we will later define the rather weak notion of a *computable topological presentation* of a space. In this notion, we only assume that there is an effective list of a countable base of topology, and that we can also enumerate the intersections of basic sets, in some weak sense. A standard example of such a presentation is a right-c.e. presented space. It follows from one of the aforementioned separation results ([3]) that there is a right-c.e. (thus, computable topological) Stone space with no computable Polish presentation. However, it seems that the following result is new: *There is a computable topological Polish(able) space not homeomorphic to any right-c.e. Polish space*; see Thm 4.4. In fact, the fairly straightforward construction in Theorem 4.4 shows that, essentially, there is no a priory upper bound on the complexity of the simplest completely metrized presentation of a computably topological space, meaning that it does not have to have an X-computable Polish presentation for however complex fixed oracle X. (It takes some work to push the lower bound on the complexity of the simplest such example down to $X = 0'''$; we do not know if this is sharp.) This result contrasts greatly with one of the main results in the companion paper [28]: *Every computable topological Polish(able)* group *is topologically isomorphic to a right-c.e. metrized one.* Indeed, the metric constructed in [28] is left-invariant (or right-invariant) and gives a right-c.e. presentation of the group in the spacial important cases of locally compact and abelian groups.

We have already discussed that modern computable topology is motivated by its consequences in the study of effective aspects of Polish groups. In fact, the two subjects are so interconnected that no firm line can be drawn between them. For instance, in the recent papers [34, 38, 36] Pontryagin duality between compact and discrete Polish abelian *groups* has been applied to derive corollaries about connected compact *spaces*. In the present paper we also give an application of our methods to groups. More specifically, we apply our machinery to derive the following *positive* result about tdlc groups, which is the second main result of the paper.

A separable group is tdlc (totally disconnected locally compact) if it is either discrete or its domain is homeomorphic to the disjoint union of clopen sets, each homeomorphic to 2^ω. If it is compact, which corresponds to the case when there is only one or finitely many of copies of 2^ω, then the group is profinite. Classically, tdlc groups have been studied in much detail; we cite the recent papers [64, 63, 14, 7, 17].

As we already mentioned above, the notion of a *computable tdlc presentation* [37, 34] generalizes the notions of computability that are well-established in the discrete and proifinite cases. It essentially says that we fix the nicest possible presentation of the domain as a locally compact subset of ω^ω (see above) and assume that the group operations are given by computable functionals acting on strings. It has also been shown in [37, 34] that this is actually equivalent to several other definitions, one involving a Stone-type duality between such a group and the (countable, discrete) groupoid of its cosets, and the other that uses subgroups of the infinite symmetric group S_∞. As shown in [37, 34], the notion enjoys a large number of closure properties. In the abelian case, the notion is equivalent to two more natural notions of computable presentability, and makes Pontryagin - van Kampen duality fully algorithmically effective [34].

Nonetheless, it may seem that this approach to computability for tdlc groups is a bit too strong since it assumes too much about a presentation. For instance, it assumes the metric is a nice ultra-metric induced by ω^ω. How is it related to the general theory of computably metrized Polish groups studied in [39, 50, 16, 38]? There, we merely require that the group has computable Polish presentation compatible with the topology which makes the operations computable. In particular, the metric could be 'unnatural' or could be not an ultra-metric. We believe that our second main result stated below answers this question in an unexpectedly strong way. We first state the result and then we explain the terminology.

Theorem 1.2. *Suppose G is a separable tdlc group. The following are equivalent:*

(1) *G admits a computable locally compact presentation.*
(2) *G admits a computable tdlc presentation.*

Recall the standard classical notion of locally compact space: given a point x there is a (basic) open B and a compact K such that $x \in B \subseteq K$. We will use the most straightforward computable version of this notion (cf. [49, 65, 60]) that involves the robust notion of an effectively compact (sub)space. We say that a group is effectively locally compact if it is computable Polish in the sense of [39] and is additionally effectively locally compact, as defined in Subsection 2.4.

The proofs of our main results are not very long, which is certainly not very typical for a recursion-theoretic paper. Nonetheless, much preliminary analysis is needed to make them work. Our proofs exploit the machinery of effective compactness explained in detail in [9]. We shall make our paper as self-contained as possible, however, within reason. The reader should prepare themselves for a long preliminary section full of little lemmas and propositions.

2. Definitions from computable metric space theory

2.1. **Computable Polish spaces and groups.** The notions below are standard and can be found in, e.g., [9, 19].

Definition 2.1. A Polish space M is *computable Polish* or *computably (completely) metrized* if there is a compatible, complete metric d and a countable sequence of *special points* (x_i) dense in M such that, on input i, j, n, we can compute a rational number r such that $|r - d(x_i, x_j)| < 2^{-n}$.

Remark 2.2. We allow the possibility that $d(x_i, x_j) = 0$. However, it is not difficult at all to exclude repetitions from the dense set if necessary. If we view our spaces up to homeomorphism, then we could computably scale the metric using a computable real, and additionally assume that $d(x_i, x_j) = r$ is a uniformly computable relation (for integer i, j and rational $r = \dfrac{m}{n}$), thus making our approach equivalent (up to homeomorphism) to the one in Moschovakis [43]. The real depends of the fixed presentaiton and can be constructed using a straightforward cantor-style diagonalization. We omit the details since will not need these additional assumptions about equality in our proofs.

Definition 2.3. [39, 38] A computable Polish group is a computable Polish space together with computable group operations $\cdot$ and $^{-1}$.

A *basic open ball* is an open ball having a rational radius and centred in a special point. Let X be a computable Polish space, and (B_i) is the effective list of all its basic open balls, perhaps with repetition. (We also sometimes write $B_r(x)$ for the open ball having radius r and centred in x: $B_r(x) = \{y : d(x,y) < r\}$.)

Definition 2.4. We call

$$N^x = \{i : x \in B_i\}$$

the name of x (in X).

We can also use basic open balls to produce names of open sets, as follows. A *name* of an open set U in a computable topological space X is a set $W \subseteq \mathbb{N}$ such that $U = \bigcup_{i \in W} B_i$, where B_i stands for the i-th basic open set in the basis of X. If an open U has a c.e. name, then we say that U is *effectively open*.

Definition 2.5. A function $f\colon X \to Y$ between two computably metrized Polish spaces is *effectively continuous* if there is a c.e. family $F \subseteq \mathcal{P}(X) \times \mathcal{P}(Y)$ of pairs of (indices of) basic open sets in such that:

(C1): for every $(U,V) \in F$, $f(U) \subseteq V$;

(C2): for every $x \in X$ and basic open $E \ni f(x)$ in Y there exists a basic open $D \ni x$ in X such that $(D,E) \in F$.

Note that a function is continuous if and only if it is effectively continuous relative to some oracle. The lemma below is well-known.

Lemma 2.6. Let $f\colon X \to Y$ be a function between computable Polish spaces. The following are equivalent:

1. f is effectively continuous.
2. There is an enumeration operator Φ that on input a name of an open set Y (in Y), lists a name of $f^{-1}(Y)$ (in X).
3. There is an enumeration operator Ψ, that given the name of $x \in X$, enumerates the name of $f(x)$ in Y.
4. There exists a uniformly effective procedure that on input a fast Cauchy name of $x \in M$ lists a fast Cauchy name of $f(x)$ (note that the Cauchy names need not be computable).

Clearly, computable maps are closed under composition, when it is well-defined.

Definition 2.7. A function $f\colon X \to Y$ is *effectively open* if there is a c.e. family F of pairs of basic open sets such that

(O1): for every $(U,V) \in F$, $f(U) \supseteq V$;

(O2): for every $x \in X$ and any basic open $E \ni x$ there exists a basic open $D \ni f(x)$ such that $(E,D) \in F$.

The lemma below is elementary.

Lemma 2.8. [39] Lef $f\colon X \to Y$ be a function between computable Polish spaces. The following are equivalent:

1. f is effectively open.
2. There is an enumeration operator that given a name of an open set A in X, outputs a name of the open set $f(A)$ in Y.

In particular, if f is a computable and is a homeomorphism, then it is is effectively open if, and only if, f^{-1} is computable. In this case we say that f is a *computable homeomorphism*. We say that two computable metrizations on the same Polish space are effectively compatible if the identity map on the space is a computable homeomorphism when viewed as a map from the first metrization to the second metrization under consideration.

A special kind of self-homeomorphisms are the (left or right) translations of a Polish group by its elements, and also the inverse map.

Fact 2.9. Multiplication and inverse operators are both effectively open in a computable Polish group.

Proof. Given some name for an effectively open set U, in order to enumerate the name for U^{-1}, simply enumerate the preimage of ${}^{-1}$ on U. This must be the name for U^{-1} since $\left(U^{-1}\right)^{-1} = U$. Thus ${}^{-1}$ is effectively open.

Now given names for open U, V, we want to produce computably a name for $U \cdot V$. The map $(x, y) \to x^{-1}y$ is computable. Eenumerate all V' s.t.

$$U^{-1} \cdot V' \subseteq V,$$

which is the same as enumerating all V' with the property $V' \subseteq UV$. □

2.2. **Computable topological spaces.** There are several definitions of a computable topological space that can be found in Kalantari and Weitkamp [20] and Spreen [56]. We will use the following.

Definition 2.10 (see, e.g., Definition 2.1 of [29] of Definition 4 of [62])**.** A *computable topological space* is given by a computable, countable basis of its topology for which the intersection of any two basic open sets ("basic balls") can be uniformly computably listed. More formally, it is a tuple (X, τ, β, ν) such that

- (X, τ) is a topological T_0-space,
- β is a base of τ consisting of non-empty sets,
- $\nu\colon \omega \to \beta$ is a computable surjective map, (i is called an *index* of $\nu(i)$) and
- there exists a c.e. set W such that for any $i, j \in \omega$,

$$\nu(i) \cap \nu(j) = \bigcup \{\nu(k) : (i, j, k) \in W\}.$$

Let (X, τ, β, ν) be a computable topological space. For $i \in \omega$, by B_i we denote the open set $\nu(i)$. As usual, we identify basic open sets B_i and their ν-indices. There are many versions of this notion above in the literature; see, e.g., [56]. Perhaps, the most natural examples of computable topological Polish spaces are right-c.e. spaces; see, e.g., Theorem 2.3 of [29]; we also cite [3, 9] for a detailed proof. For instance, every computably metrized Polish space is a computable topological space. Indeed, some of the basic results from the previous subsection can be proven for computable topological spaces perhaps with some mild additional assumptions. For instance, they should certainly work for right-c.e. spaces as well. We, however, will not need this degree of generality.

2.3. **Effective compactness.** The following definition is equivalent to many other definitions of effective compactness that can be found in the literature.

Definition 2.11. A compact computable Polish space is effectively (computably) compact if there is a (partial) Turing functional that given a countable cover of the space outputs it finite subcover (and is undefined otherwise).

This is equivalent to saying that, or every n, we can uniformly produce at least one finite open 2^{-n}-cover of the space by basic open balls. For several equivalent definitions of effective compactness, see [9] and [19]. It is also well-known that, given a computable Polish space C that is compact (but not necessarily effectively compact) using $0'$ one can produce a sequence of basic open 2^{-n}-covers of the space, thus making it effectively compact relative to $0'$. The following elementary fact is well-known (e.g., [9]):

Lemma 2.12. A computable image of an effectively compact space is itself effectively compact.

To see why the lemma is true, list basic open balls in the image until their preimages finally cover the domain of the computable map. We will also use that the inverse of bijective computable map $f : X \to Y$, where X and Y are effectively compact, is also computable. Also, it is well-known that both the supremum and the infimum of a computable function $f : X \to \mathbb{R}$ is computable provided that X is effectively compact, and this is uniform. We cite [9] for further background on effectively compact spaces.

2.4. **Effective local compactness.** Theorem 1.1 uses notions that need to be formally clarified.

Recall that the notion of an effectively (computably) compact Polish space is robust and admits many equivalent formulations ([9, 19]). The usual definition of an effectively compact space M says that for every $x \in M$ there is an open B and a compact K such that $x \in B \in K$. In the context of computable Polish spaces, it makes sense to adopt the following notion of effective local compactness which is essentially the approach taken in [49, 65, 60], up to notation change[1].

Definition 2.13. A computable Polish space M is effectively (computably) locally compact if there is a uniform procedure which, given (the name N^x of) any point x outputs a basic open B and a computable compact $K \subseteq M$ such that $x \in B \subseteq K$, where K is given by a sequence of finite open 2^{-n}-covers so that each ball in the cover is centred in a (computable) point in K and has a rational radius.

If $B = B(c, r)$ is an open ball, then we write B^c to denote the respective closed ball $\{x : d(c, x) \geq r\}$ which does not have to be equal to the closure of B in general (think about an isolated point at the boundary of B^c). Recall that we say that $B = B(c, r)$ is basic if c is special and r is a rational (given as a fraction). If c and r are merely computable, we say that the ball is computable. The following proposition will be rather useful.

Proposition 2.14. *In the notation of the previous proposition, we can assume K is equal to B^c, where $B \ni x$ is a computable open ball.*

Proof. Proposition 3.27 [9] establishes that we can uniformly produce a system of 2^{-n}-covers of K that consists of closed computable balls each of which is uniformly computably compact. If we have $x \in B \subseteq K$, then wait for such a small enough closed ball D from one of these covers such that $x \in D$ and $D^c \subseteq B$; where the latter is witnessed by formal inclusion (i.e., is deduced from the triangle inequality). This process is uniformly effective. □

Definition 2.15. A computable Polish group is computably locally compact if its domain is computably locally compact and the operations are computable.

We now discuss trees. Our (rooted) trees are viewed as sets of strings in $\omega^{<\omega}$ closed under prefixes. A tree has no dead ends if every finite string is extendible to an infinite path through the tree. The space of paths through a tree T is denoted $[T]$. The space T is an ultra-metric space under the shortest common initial segment metric. A computable tree with no dead ends evidently induces a computable Polish presentation on $[T]$.

Definition 2.16. We say that a computable tree is nicely effectively locally compact if it has no dead ends, only its root is perhaps ω-branching, and for every node we can uniformly compute the number of its successors.

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[1] It appears that the notions in the cited papers, though very similar to each other and to our Def. 2.13, could potentially be non-equivalent (up to homeomorphism) for computable Polish spaces. The differences are subtle, so we leave this as an open problem.

Although we will not use the corollary below, it is important to know that the two definitions given above are actually equivalent for trees that we care about.

Fact 2.17. For a computable locally compact tree T that is infinitely branching only perhaps at the root, TFAE:

(1) T is nicely effectively locally compact;
(2) $[T]$ is computably locally compact.

Proof. The implication (1) $\to$ (2) is obvious. For (2) $\to$ (1), observe that in the metric induced by the tree, $B = B^c$ for any basic open ball B. In Proposition 2.14, the raddii of balls are merely computable. However, the radius of any basic open ball is of the form 2^{-n}, and also any computable open ball in $[T]$ is actually equal to some basic clopen ball. We therefore can compute the radius of the ball from Proposition 2.14 to a sufficient precision and be sure that it represents a certain basic clopen ball extending some fixed string. Clearly, computable compactness of the ball is equivalent to the tree being computably branching. Since such balls/subtrees cover the whole space, we conclude that (1) holds. $\square$

Definition 2.18 ([37, 34])**.** Let G be a Polish t.d.l.c. group. A *computable tdlc presentation* of G is a topological group $\widehat{G} \cong G$ of the form $\widehat{G} = ([T], \cdot, ^{-1})$ such that

(1) T is a nicely effectively locally compact tree;
(2) $\cdot\colon [T] \times [T] \to [T]$ and $^{-1}\colon [T] \to [T]$ are computable (as operators).

We usually omit 'nicely' throughout the rest of the paper. Indeed, it follows from the theorem below that the assumption of 'niceness' (i.e., being ω-branching only at the root) is not necessary in the definition above – this was already established in [37].

3. Stone spaces

In this section we accumulate techniques related to splitting a space into clopen components. The techniques apply not only to Stone spaces and will be crucial in the proofs of both main results of the paper. Some of the elementary technical facts established in this section seem to be new as stated, but most of the rest were included to make the more paper self-contained. We also prove Proposition 3.8 mentioned in the preliminaries; this simple (but perhaps unexpected) result is new.

The fact below (though not as stated) is due to Hoyrup, Kihara, and Selivanov [53]. It can also be recovered from Brattka, le Roux, Miller, Pauly [5] and Harrison-Trainor, Melnikov, and Ng [16].

Theorem 3.1. *Given an effectively compact Stone space M, we can uniformly produce a computable computably branching tree T without dead ends and a computable homeomorphism $f : M \to [T]$.*

Proof. The key step is the following:

Lemma 3.2. Suppose M is effectively compact. Then there is a computable enumeration of all clopen splits of M (perhaps, with repetition).

Proof. Suppose $M = X \sqcup Y$ is a clopen split, and let δ be the infimum-distance between these compact open sets

$$\delta = \inf_{(x,y)\in X\times Y} d(x, y).$$

(Since $X \times Y$ is compact and d is continuous, it attains its infimum at some pair (x_0, y_0). In particular, $\delta > 0$.)

Suppose $0 < \varepsilon < \delta/4$. Then every finite ε-cover will consist of two formally disjoint subsets of basic open balls. Indeed, every ball covering a point in X cannot contain a point in Y, and every ball covering a point in Y cannot contain a point in X. If a basic open B has its centre in X and D has its centre in Y, then the distance between their centres is at least δ, while the sum of their radii is at most $\delta/2 < \delta$, making them *formally disjoint* (this is Σ^0_1).

On the other hand, if a finite open cover of M consists of two formally disjoint subcovers, then these subcovers induce a split of M into clopen components. Since the property of being formally disjoint is a c.e. property, we can effectively list all such clopen splits. □

Suppose X and $\neg X = M \setminus X$ be clopen components represented as finite unions of formally disjoint basic open balls, as in the proof of the lemma above. Given a special point x in M, we can use these finite open names to wait and see whether x in X or x in $\neg X$. This makes bot X and $\neg X$ computable closed subsets of the effectively compact M, and thus then can be viewed as effectively compact spaces. In particular, their diameters are computable reals. (Note this is uniform.)

Lemma 3.3. Given two clopen sets X and Y, as well as their complements $\neg X$ and $\neg Y$ (represented by the strong indices of their finite open names), we can additionally decide whether $X \cap Y$ is empty, and if it is not empty, then output a finite open name of it and its complement.

Proof. For that, recall that a basic open ball $B_r(x)$ is formally contained in a basic open ball $B_q(y)$ if $d(x, y) + q < r$; this is Σ^0_1. Search for an ε-cover of the space M, where ε is so small that every ball in the new cover is formally contained in some ball of each of the two covers that we fixed above (the first for X and $\neg X$, and the second for Y and $\neg Y$). Such a cover must exist. Then $X \cap Y$ is composed of those balls in the cover that are formally contained in balls corresponding to names of X and of Y. If there are no such, then declare the intersection $X \cap Y$ empty. □

To build the tree T, associate the empty string with M. Suppose $\sigma \in T$ of length i has been defined, and suppose σ has been associated with a clopen X. Let $X_i \sqcup \neg X_i$ be the ith clopen split of M in the effective list of all such splits produced above. If both $X \cap X_i$ and $X \cap \neg X_i$ are non-empty, then create two children of σ, $\sigma\widehat{\ }0$ and $\sigma\widehat{\ }1$, and associate $X \cap X_i$ with $\sigma\widehat{\ }0$ and $X \cap \neg X_i$ with $\sigma\widehat{\ }1$. If only one of the $X \cap X_i$ and $X \cap \neg X_i$ is non-empty, say $X \cap X_i \neq \emptyset$, then create only $\sigma\widehat{\ }0$ and associate it with $X \cap X_i$.

It should be clear that $[T]$ is homeomorphic to M. We claim that it is computably homeomorphic to M. For that, not that for every $\xi \in T$ and any $n > 0$, we can compute (uniformly in ξ) an i such that the diameter of the clopen component associated with $\xi \upharpoonright i$ is at most 2^{-n}. We identify $[\sigma]$ with the clopen component of M associated with $\sigma \in T$.

Given a (not necessarily computable) point $x \in M$ and n, search for $\sigma \in T$ such that the component of M associated with σ has diameter at most 2^{-n} and $x \in [\sigma]$. Output (any point in) $[\sigma]$. This gives a computable name of a surjective computable f (that can be viewed as the identity map) between the effectively compact spaces M and $[T]$. Since both spaces are effectively compact, f^{-1} is also computable. □

Remark 3.4. We see that, under a careful choice of notation in the end of the proof above, f can be viewed the identity map on M. In other words, the metric induced by T is *effectively compatible* with the original metric on M. In particular, any operation defined on M that is computable wrt the old metric will also be computable wrt the new ultrametric induced by T.

We shall use the following observation. Write $A \cong_{comp} B$ is A is computably homeomorphic to B.

Remark 3.5. Suppose $M = C \sqcup D$ is effectively compact, where C and D are clopen and effectively compact. Let T_0 and T_1 be computably finitely branching trees with no dead ends such that $C \cong_{comp} [T_0]$ and $D \cong_{comp} [T_1]$. Define a new tree T by adjoining the successors of the root of T_1 to the root of T_0. Then $M \cong_{comp} [T]$, and this is uniform.

The lemma below extends the techniques described above to spaces that are not necessarily compact, but at the const of a few Turing jumps. The lemma will be useful later.

Lemma 3.6. There is an arithmetical procedure that enumerates all connected compact-open components of a given computable Polish space.

Proof. Note that since C is open, it is the union of finitely many open balls, say $B_0, \ldots, B_k$. These balls cannot intersect the complement of C. If δ is the Lebesgue number of this cover, then any ball in any δ-subcover is contained in one for the B_i that does not intersect the complement, which implies $\inf_{x \in C, y \notin C} d(x, y) \geq \delta$.

The main point of the argument above is not to illustrate that the distance between C and is complement is well-defined and is not zero (this is of course well-known), but to explain how to arithmetically list compact open components of the space. This is done as follows.

We can arithmetically search for basic open $B_0, \ldots, B_n$ and a δ such that this infimum (restricted to special points) is at least δ. Once such balls are are fixed, we can define the subspace C generated by the special points that are contained in these B_i. Since a Polish space is compact if, and only if, it is totally bounded, it is arithmetical to tell whether this subspace is compact [40]. Because the subspace C is separated from the rest by δ, it is also open. It is also arithmetical to tell whether a compact space is connected; this follows from, e.g., Lemma 3.2. ☐

Remark 3.7. A more detailed analysis of the argument shows that $0'''$ can list compact connected components of a computable space, and we suspect this is sharp. The upper bound can be improved to $0''$ if the whole space is itself compact; this follows from Lemma 3.2 relativised to $0'$.

3.1. **A byproduct of our technique.** It is known that there exists a right-c.e. Stone space not homeomorphic to any computable Polish space [3]. It is also know that any computable Polish Stone space is homeomorphic to an effectively compact one; [16], and also explained in detail in [9]. Interestingly, we can push this positive result to left-c.e. Stone spaces. The result contrasts with a counter-example (Theorem 1.1) that will be stablished later using homological techniques.

Proposition 3.8. *Every left-c.e. Stone space is homeomorphic to an effectively compact computable Polish space.*

Proof. It is sufficient to argue that the dual Boolean algebra admits a computable copy.

Note that a finite collection of basic *closed* does not cover the space if, and only if, there is a special point x whose distance to the centres of the balls is greater than the radii of the respective balls. This is naturally a c.e. property, and therefore $0'$ can list all finite closed covers of a compact left-c.e. space. The metric is also evidently $0'$-computable. It follows from, the aforementioned result from [16, 9] relativised to $0'$ that we can $0'$-effectively reconstruct the dual Boolean algebra. We claim that this Boolean algebra in fact has a computable presentation.

It is well-known that a Δ^0_2-Boolean algebra with a Δ^0_2 set of atoms has a computable presentation. The elements of the Boolean algebra associated with the given Stone space S are represented by clopen sets. Each clopen set (and its complement) is in turn represented by finitely many basic open (or basic closed if necessary) balls. In other words, each such clopen set is given by its finite (open or closed) name. To see whether a clopen component C is an isolated point, we use $0'$ to calculate the finite name of its clopen complement $S \setminus C$. If $S \setminus C$ is the union of $B_0, \dots, B_k$, then let $x_0, \dots, x_k$ be the centres of these balls, and $r_0, \dots, r_k$ be their rational radii. There is also a rational $\delta > 0$ such that a point y is at distance $> \delta$ from all $x_0, \dots, x_k$ if, and only if, $y \in C$. These finitely many parameters and such a δ can be computed using $0'$ (relative to which the space becomes effectively compact).

If C has more than one point than it does not correspond to an atom. In this case it much also more than one special point. We therefore need to check whether there exist special points z_0 and z_1 such that

$$d(z_0, z_1) > 0 \,\&\, \bigwedge_{i \le k, 0 \le j \le 1} d(z_j, x_i) > \delta,$$

where the parameters x_i, $i = 0, \dots, k$ and the rational δ are already given to us by $0'$. Since the metric is left-c.e., the property above is uniformly Σ^0_1 in these parameters, making the atom relation Δ^0_2 in the Δ^0_2 Boolean algebra of clopen sets. □

4. Cohomology, spheres, and a bad computable topological space

In this section we describe the basic definability techniques which, combined with the methods described in the previous section, will be used in the proofs of our main results.

Our definability methods rely on Čech cohomology of a compact space is defined using nerves of covers. We cite [44] for the formal details. Since we will not actually compute any cohomology directly, we shall omit the tedious details. It well-known that the Čech cohomology of a space homeomorphic to a finite simplicial complex is actually the same that the usual simplical cohomology calculated for the complex. For instance for the n-sphere we have:

$$H^m(S^n; \mathbb{Z}) = \begin{cases} \mathbb{Z}, & \text{if } m = 0, n; \\ 0, & \text{otherwise.} \end{cases}$$

The following result is quite recent and is due to Lupini, Melnikov and Nies [34]; see also [9] for an alternative proof. Recall that a (discrete, countable) group is c.e. presented if it is isomorphic to the factor of a computable free group by its c.e. normal subgroup.

Theorem 4.1. *Suppose C is an effectively compact Polish space. Then the Čech cohomology groups of the space are uniformly c.e. presented.*

Corollary 4.2 (Essentially [9])**.** *Suppose C is a computable Polish space which is homeomorphic to S^n for some n. Then $0''$ can uniformly compute the parameter n.*

Proof. Since the Čech cohomology groups are uniformly Σ^0_2-presentable (by Theorem 4.1 relativized to $0'$), in this case $M \cong S^k$ is equivalent to saying that $\check{H}^k(M)$ contains at least one non-zero element, which is a Σ^0_3 property (the equality in the group is Σ^0_2). It follows that $0''$ can list k such that kth cohomology group of the space is non-trivial. □

Fact 4.3. Suppose M is a right-c.e. presented Polish apace. Then $0^{(4)}$ can list n such that M has a clopen component homeomorphic to S^n.

Proof. If M was computable Polish then the upper bound would be given by Corollary 4.2, Lemma 3.6, and Remark 3.7; specifically, Σ^0_4. Since a right-c.e. space is naturally Δ^0_2, we obtain the bound Σ^0_5 (which we suspect is sharp). □

We can use a relatively simple construction based on n-spheres to establish the following result stated in the introduction.

Theorem 4.4. *There is a Polish space S such that S admits a computable topological presentation but does not have a right-c.e. Polish presentation. Furthermore, there is an S with this property that has a $0'''$-computable Polish presentation.*

As noted in the introduction, the elementary construction of S entails that there is no upper bound on X that would guarantee that S has an X-computable Polish presentation. Indeed, more work is needed to establish the $0'''$ bound which may or may not be sharp.

Proof. We encode an arbitrary set $X \subseteq \mathbb{N}$ into a Polish space S_X so that S_X admits a computable topological presentation $\mathcal{T}$ *that is independent of the choice of* X.

Fix a set $X \subseteq \mathbb{N}$. Let S^n denote the Euclidean n-sphere. For every n, define

$$C_n = \begin{cases} S^n, & \text{if } n \in X \\ \mathbb{R}^n, & \text{otherwise.} \end{cases}$$

Let S_X be the disjoint union of its clopen components C_n. The space is evidently Polish. Furthermore, every connected compact open component of the space has the form S^n for some n.

We assume $n > 0$ throughout. Recall that $\mathbb{R}^n \cong_{hom} S^n \setminus \{p\}$, where p is any point on the sphere. The idea behind the lemma below is that point-free topological presentation cannot possibly know whether p is in or out.

Lemma 4.5. For every $n > 0$, $\mathbb{R}^n$ and S^n share the same computable topological presentation $\mathcal{T}_n$ that can be produced uniformly in n.

Proof. Uniformly in n, fix a computable Polish presentation of S^n. Fix the associated computable topological presentation $\mathcal{T}_n$ given by the balls with rational radii and centred in computable points. Note that this remains a computable topological presentation if we extract one of the special points from the space. □

Let $\mathcal{T}$ be the union of the $\mathcal{T}_n$ over $n \in \mathbb{N}$. When two open sets come from different $\mathcal{T}_n$ and $\mathcal{T}_m$, $m \neq n$, we declare their intersection to be empty. This gives a computable topological presentation of the space S_X which does not depend on X. If we do not care about the $0''$-bound, it remains to note that $X \neq Y$ implies $S_X \not\cong_{hom} S_Y$. Since there are uncountably many subsets of $\mathbb{N}$ and only countably many right-c.e. presentations.

To produce an example that is $0'''$-computably completely metrizable, we also adjoin infinitely many isolated points to the space, and we also duplicate every n-sphere infinitely many times. The claim below is obvious:

Claim 4.6. Suppose Y is Π^0_2. There is a uniform procedure which outputs a computable Polish copy of S^n if $y \in Y$, and outputs finitely many isolated points, otherwise.

We now fix some X that is in $\Sigma^0_6 \setminus \Sigma^0_5$ and view it as a $\Sigma^{0'''}_3$ set. We further split the $\Sigma^{0'''}_3$-predicate into infinitely many $\Pi^{0'''}_2$-predicates, one for each potential existential witness, and relativise the claim above to $0'''$. This way we obtain a $0'''$-computable copy of the space S_X which, by Fact 4.3, cannot have a right-c.e. presentation. □

5. A LEFT-C.E. SPACE NOT HOMEOMORPHIC TO A COMPUTABLE ONE

It is known that there exist a right-c.e. Stone space not homeomorphic to any computable Polish space, but we have seen in Prop. 3.8 that every left-c.e. Stone space has a computable presentation (indeed, an effectively compact one). We prove Theorem 1.1 that states that there exists a left-c.e. Polish space not homeomorphic to any computable Polish space.

Proof of Theorem 1.1. For a set $X \subseteq \omega$, let $C(X)$ be the one-point compactification of the disjoint union of n-spheres S^n and n-discs $D^n \cong [0,1]^n$, with infinitely many copies for each $n \in X$. Then proposition below is reminiscent of the similar coding results from [53, 9]. We will need only the 'if' part of it, but we state it in full generality.

Proposition 5.1. *X is Σ^0_3 if, and only if, $C(X)$ is homeomorphic to a computable Polish space.*

Proof. The 'if' part is based on calculating the cohomology of connected compact-open components. Note that the cohomologies $H^i(D^n, \mathbb{Z})$ of D^n are trivial when $i > 0$ (folklore). Thus, Corollary 4.2 combined with the proof of Lemma 3.6 which (as noted in Remark 3.7) gives the upper bound Σ^0_3.

For the other implication (that we only sketch since we do not really need it) note that the spaces D^n are evidently uniformly computable. The construction from [9] can be easily extended to incorporate these spaces into $C(X)$ while running the construction of $CP(X)$ on the other components. □

Define L_X to be the space consisting of infinitely many compact open components, one of which is $C(X)$ (see Prop. 5.1), and the rest are isolated points. We will need only the 'if' part of the lemma below, but we state it in full.

Lemma 5.2. X is Σ^0_3 if, and only if, L_X has a computable Polish copy.

Proof. Suppose L_X has a computable Polish presentation. Fix finitely many open basic balls isolating the component $C(X)$ from the rest of the space. Define the subspace of L_X by listing the special points contained in these finitely many balls. This gives a computable Polish presentation of $C(X)$; it remains to apply Prop. 5.1.

On the other hand, by Prop. 5.1 if X is Σ^0_3 then we can produce a computable Polish presentation of $C(X)$. It is easy to extend it to a computable presentation of L_X by adjoining infinitely many points and declaring them to be at distance (say) 1 between each other and the component homeomorphic to $C(X)$. □

Fix a Π^0_3-complete X. By the lemma above, to prove the theorem it is sufficient to argue that L_X has a left-c.e. presentation.

Lemma 5.3. There is a uniform procedure which, given $n > 0$, produces a left-c.e. space $U(n)$ of the form:

$$U(n) \cong \begin{cases} \tilde{S}^n, & \text{if } n \in X \\ \tilde{D}^n & \text{otherwise,} \end{cases}$$

where $\tilde{S}^n$ is the disjoint union of the n-sphere S^n and infinitely many isolated points (with no limit point), and $\tilde{D}^n$ is the disjoint union of the n-disc $D^n \cong [0,1]^n$ and infinitely many isolated points (with no limit point).

Proof. We begin our construction with the standard computable and effectively compact presentation of S^n. We also begin with a sequence of isolated points disjoint from S^n so that they have no accumulation point.

Fix a special $p \in S^n$. We represent $S^n \setminus \{p\}$ as the union of a nested uniformly computable sequence of subspaces

$$D(0) \subset D(1) \subset \ldots D(k) \subset \ldots,$$

where each $D(i)$ is homeomorphic to D^n, and the boundaries of $D(i)$ approach the point p in the limit.

We also fix a Π^0_3 predicate $P(n)$ for X. We represent it via $\forall x \exists^{<\infty} y R(x, y, n)$, where R is a computable predicate. We say that the predicate 'fires' on x if we discover a fresh y never seen before such that $\exists y R(x, y, n)$.

The construction of $U(n)$ proceeds as follows. We shall begin with approximating S^n naturally, but we exclude the special point (the 'pole') p from its presentation. This is done by putting more and more special points into more and more of the nested discs $D(i)$. If the predicate never fires for any x then we will end up with S^n in which p is put as a limit point. We also will gradually add more and more isolated points outside S^n. We can use an ultra-metric between all clopen components to ensure that the triangle inequality holds, or we can opt to work within a copy of $\mathbb{R}^{n+1}$ and during the construction 'move' the points by redefining their interpretation in $\mathbb{R}^{n+1}$; the exact set-up does not matter since we are working with spaces up to homeomorphism.

If at some stage the predicate fires on some x, then assume x is least such.Without loss of generality, we can assume $x > 0$. We then take the finitely many points that have been introduced so far into $D(k)\setminus D(x)$, $k > x$, and we 'move these points away' from the component, as follows. Recall that the metric has to be merely left-c.e. Declare that the distance from these points to any other point in the construction is much larger than any number seen so far in the construction. We then return to the natural approximation of the nested discs and wait for the predicate to fire again (if ever), etc.

If the Π^0_3 predicate holds on n, then we eventually end up with a stable disc $D(k)$ for every k, and thus with a sphere. Otherwise, for some x all discs $D(k)$ for $k > x$ will be 'blown away' infinitely many times. Since we assumed that $x > 0$, we will end up with $D(x) \cong D^n$. In both cases, we will also have infinitely many isolated points with no accumulation point.

The space is evidently left-c.e., by construction. This is because the the distances between points can only potentially increase, making the left cut of $d(\alpha, \beta)$ c.e. for any special points α and β of the space. $\square$

We return to the proof of the theorem. Recall the definition of L_X, and recall that X is Π^0_3.

Lemma 5.4. The space L_X has a left-c.e. presentation.

Proof. We iterate (the proof of) Lemma 5.3. We combine the constructions described in the proof of Lemma 5.3 inside a copy of $C(X)$. We associate one compact open component of the space with each n, and we also initially have infinitely many compact open components, infinitely many for each n-disc D^n. When we move points away according to the instructions of the main diagonalization module in the proof of Lemma 5.3, we also move it away from all points in the whole construction introduced so far, not only from the points that are listed in our specific component working with n. The rest is repeated verbatim. In particular, we will end up with infinitely many isolated points with no limit point, the resulting space is left-c.e., and it is homeomorphic top L_X. $\square$

Theorem 1.1 now follows from Lemma 5.4 and Lemma 5.2. $\square$

6. Totally disconnected locally compact groups

Recall that Theorem 1.2 states that, for a separable tdlc group, Def. 2.15 and Def. 2.18 are equivalent, up to topological group-isomorphism. We note that this result implies the earlier result in [9] that works only in the special case of a profinite group.

Proof of Theorem 1.2. We will need the following auxiliary definition:

Definition 6.1. A separable tdlc group G *computably splits* if the underlying space can be represented as a disjoint union of uniformly effectively compact and uniformly c.e. open sets $G = \bigcup_i C_i$, such that, for every n and i, we can uniformly compute a 2^{-n} cover whose union is *equal* to C_i.

Note that the definition above is stronger then effectively locally compact computable Polish; given a point we can search for C_i which is both effectively compact and open. Suppose G is computable tdlc, $G = [T]$ for an effectively compact T. The desired C_i are induced by the compact open sets associated by the compact subtrees of T. Recall that the latter can be computably listed. It follows that any computable tdlc computably splits, and thus is effectively locally compact and computable Polish.

Now suppose G is computable Polish and effectively locally compact.

Lemma 6.2. G computably splits.

Proof. By the well-known van Dantzig's theorem, there is a compact open subgroup H of G (which is however not necessarily normal). Every point in H is contained in a basic open B which is itself included in a compact $K \subseteq G$.

By Proposition 2.14, every point of the compact open H is contained in an open B so that the respective closed basic ball B^c is effectively compact. Since H is open, there are finitely many such B that cover the whole H. Since H is also closed, and since we can (non-uniformly) assume the radii of the balls are smaller than the distance from H to $G \setminus H$, we can assume that H is also equal to the union of finitely many closed balls B^c, each of which is computable compact. By slightly increasing the radii of the balls (using the distance to the complement again), we can also produce the finite open name of H.

In summary, we have produced both a finite open and a finite computable compact name of H. The latter evidently implies that H is effectively compact.

For any special point ξ, the left translation operation $x \to \xi x$ is both computable and effectively open. (The latter holds because ξ^{-1} is evidently a computable point, and the left-translation by ξ^{-1} is the inverse of the left translation by ξ.)

Since the left cosets of H are open, each such coset contains a special point. In other words, all H-cosets have the form ξH for some special ξ. Since the computable image of an effectively compact set is effectively compact, all these cosets are effectively compact with all possible uniformity. Further, the image of any finite cover of H under the effectively open map $x \to \xi x$ gives an open cover of ξH. Since ξH is (uniformly) effectively compact, we can get a finite subcover of this cover. Fixing one such finite cover (uniformly), we can conclude that ξH is equal to the union of this cover. Since ξH is (uniformly) effectively compact, we can search for covers that (formally) refine this cover. The union of any such cover will also be equal to the whole coset ξH.

It remains to remove receptions from the sequence of cosets defined above. This is done as follows. Non-uniformly fix a rational δ that bounds the distance between

H and $G \setminus H$ from below. To see whether $\xi H \cap \zeta H = \emptyset$, calculate the distance from the effectively compact H to the computable point $\zeta^{-1}\xi$ to precision $\delta/4$. If this approximation is $> \delta/2$ then $\xi H \cap \zeta H =$, and if $< \delta/2$ then $\xi H = \zeta H$; note these are exclusive cases.

In summary, we can list the compact open cosets without repetition in the nicest possible way. In particular, the group computably splits into these cosets. This gives us the lemma. □

Suppose our tdlc G computably splits into compact open D_i. For instance, these D_i could be the cosets by a compact open subgroup; but this assumption is not really necessary below. Let $C_i = \bigcup_{j<i} D_i$.

Observe that each C_i is an effectively compact Stone space. By Theorem 3.1, there is a uniformly computable procedure that, given an effectively compact Stone space C_i, outputs a computably branching, computable tree T_i with no dead ends such that $C_i \cong_{somp} [T_i]$.

Observe also that, for every i, C_i is computable clopen subset of C_{i+1}. Its complement in C_{i+1} is also computable compact open; this follows from the proof of Lemma 3.2 with all possible uniformity. (List open finite covers of C_i and search for a cover of the whole C_{i+1} that is composed of the cover of C_i and finitely many balls formally disjoint from it.) We are therefore in the position to apply Remark 3.5.

We see that the tree T_i that can be uniformly produced for C_i (by Theorem 3.1) can be viewed as a subtree of T_{i+1} corresponding to C_{i+1}; this is Remark 3.5. It follows that we can define T to be $\bigcup_{i\in\omega} T_i$. By Remark 3.4 the original metric on C_i is uniformly effectively compatible with the new ultrametric induced by the tree T_i. It follows that the group operations remain computable and effectively open on $[T]$. Since T_i were (uniformly) computably branching, T satisfies the conditions of computable tdlc presentation. □

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