

TORSION-FREE ABELIAN GROUPS WITH OPTIMAL SCOTT FAMILIES

ALEXANDER G. MELNIKOV

ABSTRACT. We prove that for any computable successor ordinal of the form $\alpha = \delta + 2k$ (δ limit and $k \in \omega$) there exists computable torsion-free abelian group (TFAG) that is relatively Δ_α^0 -categorical and not $\Delta_{\alpha-1}^0$ -categorical. Equivalently, for any such α there exists a computable TFAG whose initial segments are uniformly described by Σ_α^c infinitary computable formulae up to automorphism (i.e., has a c.e. uniformly Σ_α^c -Scott family), and there is no syntactically simpler (c.e.) family of formulae that would capture these orbits. As far as we are concerned, the problem of finding such optimal examples of (relatively) Δ_α^0 -categorical TFAGs for arbitrarily large α was first raised by Goncharov at least 10 years ago, but it has resisted solution (see, e.g., Problem 7.1 in survey [Mel14]). As a byproduct of the proof, we introduce an effective functor that transforms a $0''$ -computable *worthy* labeled tree (to be defined) into a computable TFAG. We expect this technical result will find further applications not necessarily related to categoricity questions.

1. INTRODUCTION

We follow Mal'cev [Mal61] and Rabin [Rab60] and use Turing machines to define computability upon countable algebraic structures. We say that a countably infinite algebraic structure $\mathcal{A} = \langle A; f_1, \dots, f_k \rangle$ is *computable* (constructive) if $A = \omega$ and the operations/relations f_1, \dots, f_k are uniformly Turing computable. This definition generalizes several earlier notions that had been used by algebraists in the specific contexts of fields [FS56, vdW30] and (typically finitely generated) groups [Hig61]. Mal'cev [Mal61, Mal62] immediately tested his new general definition on the class of torsion-free abelian groups. Among other results Mal'cev showed that the divisible group $\bigoplus_{i \in \omega} \mathbb{Q}$ has two isomorphic computable copies that are *not computably isomorphic*. This observation led him to the general problem of characterising structures in natural algebraic classes that possess a unique computable presentation *up to computable isomorphism*. Such structures that possess a unique computable isomorphic copy are usually called *computably categorical* or *autostable*. Similar questions were also independently raised by researchers in the USA and Australia (e.g., [MN79, MN77, Ash86, Smi81]).

In all early examples (such as [Nur74, LaR77, Smi81]) computable categoricity was fully relativizable, in the following sense.

Definition 1.1. A computable algebraic structure \mathcal{A} is *relatively computably categorical* if for any (not necessarily computable) $\mathcal{B} \cong \mathcal{A}$ there exists an isomorphism $\phi : \mathcal{B} \rightarrow \mathcal{A}$ computable relative to the open diagram of \mathcal{B} .

In other words, if we could compute \mathcal{B} , then we would be able to compute an isomorphism $\phi : \mathcal{B} \rightarrow \mathcal{A}$. It is well-known and easy to prove that relative computable categoricity is equivalent to having a computably enumerable family of first-order

existential formulae with finitely many parameters, such that the automorphism orbits of initial segments of \mathcal{A} are described by the formulae (see e.g. survey []). A family of formulae with such properties is usually called a (c.e.) *Scott family* [AK00]. It is also well-known that computable categoricity does not imply relative computable categoricity in general (see survey [DHK03]), but in many natural algebraic classes these two notions are equivalent [AK00]. As a consequence, algebraically non-trivial computable algebras in standard algebraic classes tend to be *not* computably categorical (but not in general, see [DKL+15]). If a computable \mathcal{A} is not computably categorical, it makes sense to ask:

How far is \mathcal{A} from being computably categorical?

The question above is usually formally clarified using (relative) Δ_α^0 -categoricity which will be defined shortly.

1.1. Relative Δ_α^0 -categoricity. Recall that the Turing jump can be iterated over computable ordinals, and this leads to the Hyperarithmetical Hierarchy ([AK00, Rog87]) and in particular to classes Δ_α^0 . For example, Δ_2^0 consists of the sets computable in the Halting Problem. If we replace “computable isomorphism” by “ Δ_α^0 -computable isomorphism” in the definition of computable categoricity, we obtain the notion of a Δ_α^0 -categorical computable structure. Similarly, if we relax Definition 1.1 by allowing \mathcal{B} to range over arbitrary copies of the structure and the isomorphism to be merely $\Delta_\alpha^0(\mathcal{B})$, we get the notion of relative Δ_α^0 -categoricity. It is well-known that relative Δ_α^0 -categoricity is equivalent to having a computably enumerable Scott family consisting of infinitary computable Σ_α^c -formulae [AKMS89].

When $\alpha \geq 2$, we usually cannot hope for a nice algebraic characterization of (even relatively) Δ_α^0 -categorical structures in a given class K (but see [DM13, McC03, Mk02, Baz14]). Thus, we usually seek for *examples* of Δ_α^0 -categorical structures (in K) that are not Δ_β^0 -categorical for $\beta < \alpha$. In contrast to computable categoricity, already for $\alpha = 2$ relative and plain Δ_α^0 -categoricity differ in many standard classes, e.g. quite surprisingly in the class of equivalence structures [KT10]. Thus, it is clearly more satisfying to have a *relatively* Δ_α^0 -categorical example that is not Δ_β^0 -categorical for $\beta < \alpha$. We call such examples *optimal* for the given α . We summarize several classical and recent optimal examples of relatively Δ_α^0 -categorical structures in Theorem 1.2 below.

Theorem 1.2. In each of the listed below classes, we indicate the form of computable ordinals $\alpha > 1$ for which optimal examples of relative Δ_α^0 -categoricity exist in the class (according to the cited result). In each case δ is either 0 or a limit ordinal, and $k \in \omega$.

- (1) Linear orders, each α of the form $\delta + 2k$ (Ash [Ash86]).
- (2) Boolean algebras, each α of the form $\delta + 2k + 2$ or δ (Knight, see [AK00]).
- (3) Abelian p-groups, any α (Barker [Bar95]).
- (4) Ordered abelian groups, each α of the form $1 + \delta + 2k$ (M., [Mela]).
- (5) Real closed fields, each α of the form $1 + \delta + 2k$ (Ocasio-Gonzalez [Oca14]).
- (6) Completely decomposable groups and any $\alpha \leq 5$ (Downey and M., [DM13, DM14]).
- (7) Effectively universal classes including graphs, fields, 2-step nilpotent groups, and any computable α (folklore combined with [HKSS02, MPSS]).

Also, similar examples can be found in the class of differentially closed fields (follows from [MM16]) and many other classes. The results listed above are quite

satisfying since they provide us with a plethora of *arbitrarily complex* but at the same time *sufficiently well understood* examples in each class. Indeed, each of these results gives a finger-grained and optimal syntactical analysis of the back-and-forth procedure used to construct an isomorphism between arbitrary copies of the respective structures. Furthermore, most of these results provide us with a deep analysis of a reach enough natural subclass of the respective class. For example, (1) and (2) of Theorem 1.2 are witnessed by well-orders and superatomic Boolean algebras, respectively.

Various techniques introduced to construct optimal examples of relative Δ_α^0 -categorical structures usually find further applications. Such applications are often unrelated to categoricity questions, and sometimes are not even restricted to one class of structures. For instance, α -systems of Ash [Ash86] have become a standard tool of computable structure theory [AK00]. Also, in his search for optimal examples of Δ_α^0 -categoricity, Ocasio-Gonzales [Oca14] has recently defined an *effective* functor that transforms/effectively encodes an arbitrary Δ_2^0 linear order into a computable real closed field. The result can be used to translate various results on linear orders [Dow98] into theorems on real closed fields, but typically “loosing one jump” in the process. See also [Mela] for similar result and its consequences for ordered abelian groups, and see [HTMMM16] for a recent general framework on effective functors.

On the other hand, in some classes we may *fail* to produce examples for a large enough α , but instead we show that all structures in the class are relatively Δ_β^0 -categorical for some fixed β , see e.g. [DM13, DM14, Cal05]. Such a uniform upper bound gives a lot of structural information about the class. For example, a result of this sort usually can be used to measure the complexity of the isomorphism problem in the class (e.g., [DM14]). See [AK00] for a further discussion of Δ_α^0 -categoricity, and see also the recent paper [DKL+15].

1.2. The class of torsion-free abelian groups. As we noted above, Mal'cev initiated the study of computable abelian groups in the early 1960's, and indeed he was especially interested in the torsion-free ones (see his early paper [Mal38]). Over the past 50 years the topic has accumulated many results and techniques (see surveys [Khi98, Mel14]). In the case of computable torsion-free abelian groups (TFAGs) all results can be sub-divided into three categories:

I. General results that exploit only local properties of TFAGs.

For example, the classical result of Nurtazin [Nur74] says that every computable TFAG has a computable copy with a computable (Prüfer) basis. One would expect the proof to go through several different cases, each case potentially telling us much about the structure of computable TFAGs. However, the proof exploits only some local properties of Prüfer independence in TFAGs. Many other notions of independence possess such properties (e.g. δ -independence in DCF_0 [HTMM15]), so there is even nothing special about TFAGs here. As a corollary, a TFAG is (relatively) computable categorical iff its (Prüfer) rank is finite. See [Khi98, Mel14] for more results of this sort.

II. Theorems showing how complicated TFAGs can be in general.

For example, Downey and Montalban [DM08] proved that *the isomorphism problem* (in the sense of [GK02]) for TFAGs is analytic complete. The groups produced in the proof are complex enough to witness the theorem, but at the

same time the groups are perhaps too complex to be fully understood. For instance, what are the automorphism orbits of elements in such groups? We cite [Hjo02, FKM⁺11, Rig, FFH⁺12] for other results of this nature.

III. Results on the special class of completely decomposable groups.

A group is *completely decomposable* [Bae37] if it is of the form $\bigoplus_{i \in I} H_i$, where $H_i \leq \mathbb{Q}$ for each i . Classically these groups are very well understood [Fuc70, Bae37], however computable members of the class may possess remarkably complex algorithmic properties (e.g., Khisamiev and Krykpaeva [KK97]). In their recent work [DM14], Downey and M. showed that all such groups are relatively Δ_5^0 -categorical. As a consequence, the isomorphism problem for such groups is Σ_7^0 [DM14], in contrast to TFAGs in general [DM08]. Although it is open whether the isomorphism problem is Σ_7^0 -complete, the upper bound $\alpha = 5$ for Δ_α^0 categoricity is optimal [DM14].

We conclude that none of these three categories of results provides us with *arbitrarily complicated* but at the same time *algebraically and effectively well-understood* examples of computable torsion-free abelian groups. There are only two results on TFAGs which do not quite fall into the three categories above, namely the recent works [AKMS12, Melb] on *degree spectra* of TFAGs (see [AKMS12, Melb] for a definition). These two papers *do* provide us with complex enough examples, but similarly to the results of type **II** these examples are only partially understood. Although this partial information was enough to derive consequences on degree spectra, it is unknown (for instance) whether the families of groups constructed in [AKMS12, Melb] give optimal examples of relative Δ_α^0 -categoricity for the respective large α , and the answer could in fact be negative. Thus, these results are more of type **II** rather than of any new type.

Thus, optimal examples of Δ_α^0 -categoricity would give a result of a new type (not in **I** - **III**). One would expect further applications of the apparatus required to construct and (especially) verify such examples. In fact, a significant portion of the techniques used in [AKMS12, FKM⁺11, Melb, DM13, DM14] was initially designed to build optimal examples of relatively Δ_α^0 -categorical structures for large α . Although these early attempts were unsuccessful, even these insufficient techniques have already proved to be quite useful beyond categoricity questions.

It may seem quite surprising that the extensively studied class of TFAGs is not already listed among the other standard classes in Theorem 1.2. What are the difficulties we will need to face?

The obvious complication is that (in contrast to abelian p -groups), no convenient system of invariants for TFAGs is known, and indeed we have a strong evidence there cannot be one [DM08, FFH⁺12, Hjo02]. On the other hand, one would hope that some natural well-understood subclass of TFAGs could do the job, just like superatomic BAs in the case of arbitrary BAs in (2) of Theorem 1.2. Nonetheless, the most natural candidate, namely the class of completely decomposable groups, fails to have the desired property (see **III** above). Therefore, optimal examples for $\alpha > 5$ must have non-trivial indecomposable summands. Indecomposable groups are not very well understood even classically ([Fuc73], see also [Rig]), so we should expect significant technical difficulties which are most likely unavoidable.

1.3. The result. We are ready to state and discuss the main result of the paper. We say that an ordinal is *even* if it is a successor ordinal of the form $\delta + 2k$, where $\delta = 0$ or is a limit ordinal, and $k \in \omega$.

Theorem 1.3. *For every computable even ordinal α there exists a torsion-free abelian group \mathcal{G}_α which is relatively Δ_α^0 -categorical, but not $\Delta_{\alpha-1}^0$ -categorical.*

The case when $\alpha \leq 4$ follows from [DM14, DM13], so we may assume $\alpha > 4$. The proof combines several new ideas with techniques from [AKMS12, DM13, DM14]. We briefly discuss the main ingredients of the proof.

First of all, we will be using a (folklore) method of constructing indecomposable groups that goes back to Pontryagin [Pon34] and Levi [Lev]. These techniques and their extensions had been used to construct “large” indecomposable groups, see e.g. [Fuc73]. More recently, the basic technique has been applied in [Hjo02, DM08, FFH⁺12] to illustrate how complicated the class of TFAGs is. In [AKMS12, Melb] the method has been further extended, and a certain definability technique on top of it has been developed. We will be using the definability technique, but it will have to be adjusted.

The groups in [AKMS12, Melb] are (in some sense) more complicated than EPLAGs [Hjo02, DM08, FKM⁺11]. (Here “complicated” means that the automorphism orbits of elements seem harder to describe.) This complication was necessary to make the definability line up with the recursion-theoretic complexity of certain algebraic invariants. Indeed, in [DM08, FKM⁺11] the recursion-theoretic complexity tends to be seemingly “off by two jumps” when compared with the natural syntactical complexity. As one of the key ingredients of the proof, we will define an effective functor that given a Δ_3^0 *worthy* labeled tree T (to be defined) produces a computable copy of a TFAG naturally coding the tree (but not quite in the same way as in [DM08]). This transformation allowed us to significantly simplify the proof, since our coding will use only infinite divisibility, just as in [Hjo02, DM08]. The corresponding class of *worthy tree-groups* (to be defined) is perhaps the most natural class of TFAGs that could potentially contain witnesses for Theorem 1.3.

The other ingredient of the proof is \mathcal{P} -*independence* introduced in [Mel09] and used in [DM13, DM14]. This notion captures automorphism orbits in free abelian groups and, if generalized appropriately, in completely decomposable groups [DM13, DM14]. Our groups will be not completely decomposable, but remarkably all their definable subgroups that we need will be free abelian.

The technical core of the proof of Theorem 1.3 is contained in Section 9. The proof exploits a certain vague analogy between worthy tree-groups and reduced abelian p -groups. A reader familiar with the technique of stripping [DMN14, Mel14] will perhaps see the (vague) analogy of this technique with the proof of Claim 9.6. We note that it is generally unusual to see even a slight technical connection between TFAGs and reduced abelian p -groups.

We conclude the introduction with a discussion of the natural question: *Can Theorem 1.3 be extended to all computable α , or at least to some other computable α ?* Although we conjecture that the answer is affirmative, our techniques seem to be insufficient to cover all α . The exact obstacle will become clear in Section 10 (where it will be discussed). We suspect that a significantly new idea might be necessary already to cover all odd finite $\alpha \in \omega$, we leave this as an open problem.

1.4. The plan of the paper. We discuss the key definitions in the brief preliminary Section 2. The rest of the paper is devoted to the proof of Theorem 1.3 which is split into several major parts. In Section 3 we will introduce some specific terminology and notation that will be used throughout the proof. Section 4 contains the definition of the group \mathcal{G}_α witnessing Theorem 1.3 for a given even α . In Section 5 we isolate $\mathcal{L}_{\omega_1\omega}^c$ -definitions of certain nice linear combinations of special elements in \mathcal{G}_α . The next Section 6 contains the definition of a functor taking a $0''$ worthy tree to a computable copy of $G(T) \oplus \mathbb{F}$, where \mathbb{F} is free of rank ω (notation and terminology will be clarified). The facts proved in Section 6 and Section 5 will be used in Section 7 to show that \mathcal{G}_α is not $\Delta_{\alpha-1}^0$ -categorical (recall α has a predecessor). In Section 8 we give conditions that isolate the automorphism orbit of a finite tuple taken from some specifically chosen nice basis of \mathcal{G}_α . In Section 8 we also prove that having a description of orbits of such elements is enough to run a back-and-forth construction on \mathcal{G}_α . In Section 9 we verify that the conditions indeed describe the automorphism orbit of the respective finite tuple. Finally, in Section 10 we apply results from Section 5 to show that our definition of orbits is of complexity at most Σ_α^c , as desired.

Apart from formal proofs some intuitive explanations will be provided. Most of these discussions appear as remarks throughout the paper. An impatient reader may safely skip these remarks.

2. BASIC NOTIONS AND NOTATION

We assume that the reader is familiar with the standard terminology of computable structure theory, see [AK00]. We will also follow the notation of [AK00] in the definition of the Hyperarithmetical hierarchy, the classes Σ_α^0 , and the transfinite iterate $H(\alpha)$ the Turing jump. For example, $H(\omega) = 0^{(\omega)}$ (the uniform join of arithmetical jumps) but $H(4) = 0^{(3)}$. We also follow [AK00] in our definitions of classes Σ_α^c of computable infinitary $\mathcal{L}_{\omega_1\omega}$ formulae over a computable language \mathcal{L} . For example, Σ_1^c consists of computably enumerable disjunctions of first-order existential formulae over a (common) finite tuple of indeterminates. Further notations and conventions will be introduced when necessary.

We shall also be using the standard notation and terminology of abelian group theory, see [Fuc70]. Abelian groups are naturally \mathbb{Z} -modules, and we will be using module-theoretic notation throughout the paper. Recall that a subgroup H of an abelian group A is *pure* if for every $h \in H$ and any $m \in \mathbb{Z}$,

$$A \models (\exists u) mu = h \implies H \models (\exists u) mu = h.$$

In a TFAG A , if $(\exists u) mu = a$ for some $a \in A$ then such an element u is unique. We also write $m \mid a$ if such element u exists. Thus, for a TFAG A and $S \subseteq A$ we can define the least pure subgroup of A that contains S . This subgroup is called the *pure closure* of S in A and will be denoted $[S]_A^*$ or $[S]^*$ if A is clear from the context.

Recall also that non-zero $a_1, \dots, a_k \in A$ are (Prüfer) independent if for any choice of $m_i \in \mathbb{Z}$,

$$\sum_i m_i a_i = 0 \implies \bigwedge_i (m_i a_i = 0),$$

and the element 0 is assumed to be dependent on itself. In the case when A is a TFAG, $m_i a_i = 0$ is clearly equivalent to saying that $m_i = 0$. The definition

above leads to the respective notion of a (Prüfer) basis. Prüfer basis should not be confused with a basis of a free abelian group, since a Prüfer basis does not have to generate the group. To distinguish these two notions, we will use the self-explanatory term a *generating basis* of (a free) A .

We will also be defining groups by divisibility conditions. For example, for p, q (distinct) primes,

$$\langle v, w : p^5 | v, q^\infty | (2w - v) \rangle$$

stands for the subgroup of $\mathbb{Q}v \oplus \mathbb{Q}w$ generated by elements of the form $\frac{v}{p^5}$ and $\frac{2w - v}{q^n}$, $n \in \omega$. Here $\mathbb{Q}v \oplus \mathbb{Q}w$ stands for the additive group of the \mathbb{Q} -vector space upon the basis $\{v, w\}$.

A group is divisible if $m | g$ for any $m \in \mathbb{Z}$ and arbitrary g of the group. Recall that any TFAG is contained in a divisible TFAG. Therefore, we can always consistently adjoin witnesses for arbitrary divisibility conditions to a TFAG without destroying the property of being a TFAG. One way to find the divisible hull of a TFAG is to fix a maximal linearly independent set of the group and consider the \mathbb{Q} -vector space upon the same basis. Although finding a basis usually requires a jump, this process will be usually effective when necessary.

2.1. \mathcal{P} -independence. We need the following special case of the more general notion of \mathcal{S} -independence from [DM13]. Let \mathcal{P} be the set of all primes.

Definition 2.1. Let A be a TFAG, and suppose $S \subseteq A$. We say that S is \mathcal{P} -independent if for any $p \in \mathcal{P}$, each $s_1, \dots, s_k \in S$ and $m_1, \dots, m_k \in \mathbb{Z}$,

$$p \mid \sum_i m_i s_i \implies \bigwedge_i p \mid m_i.$$

In other words, $[S]_A^* = \bigoplus_{s \in S} \mathbb{Z}s$, and thus the notion naturally corresponds to the well-known criterion of freeness due to Pontryagin [Pon34]. It is clear that any finite subset of a \mathcal{P} -independent S generates a free summand of A , but when S is itself infinite this may be no longer true for the whole S , and even if A is itself free (folklore, see [Fuc70]). Nonetheless, if A is computable free abelian and we are *building* S (say, using $0'$), then we can easily ensure $[S]^* = A$ by forcing the i 'th element to be generated by the i 'th initial segment of S [DM13]. This way we can produce a generating basis of the free abelian group A . We conclude that \mathcal{P} -independence captures the automorphism orbits of finite initial segments of any generating basis of the free A . It takes a lot more effort to extend these ideas to arbitrary sets of primes and to all completely decomposable groups [DM13, DM13], but we will not need this technique in its full power.

3. ROOTED GROUPS

3.1. Rooted groups. A *rooted group* [AKMS12, Melb] is a pair (G, g) , where G is a non-trivial torsion-free abelian group and g is a distinguished non-zero element of G .

We define a special class \mathcal{TG} of rooted groups that we call *tree-groups*, as follows. Fix any countable graph-theoretic tree T with a distinguished root whose edges and vertices (nodes) are labeled by not necessarily distinct primes. Suppose $l(x)$ is the

prime labelling $x \in V(T) \cup E(T)$. Define

$$G(T) = \left\langle \frac{v}{l(v)^m}, \frac{v+w}{l((v,w))^m} : v \in V(T), (v,w) \in E(T), m \in \omega \right\rangle \cong \bigoplus_{v \in V(T)} \mathbb{Q}v,$$

with root equal to the root of T . Let \mathcal{TG} consist of all such $G(T)$, where T ranges over all (rooted) trees labeled by primes. The tree T will be called a structural tree of $G(T)$.

Convention 3.1. A structural tree of $G \in \mathcal{TG}$ is *not* uniquely determined by the isomorphism type of G [Mel11]. Nonetheless, all our rooted groups will be defined simultaneously with some *fixed* structural tree representing the group. This tree from the definition of G will be called *the canonical structural tree of G* , or just the structural tree of G , and will be denoted $T(G)$.

3.2. Operations on tree-groups. Let \mathcal{P} be the set of all primes.

Definition 3.2 (Prime substitution). Suppose $\phi : \mathcal{P} \rightarrow \mathcal{P}$ is an injective function. Given any $G(\Gamma) \in \mathcal{TG}$, define $[G(\Gamma)]_\phi$ to be the group $G(T)$, where T is obtained from Γ by replacing any label p by $\phi(p)$.

The operation of prime substitution can be shown to not depend on the choice of a tree T in all natural cases that are important for this paper. Nonetheless, since our tree T will be fixed (see Convention 3.1) in the definition above we may assume that T is this fixed tree. Also, we will be using only computable $\phi : \mathcal{P} \rightarrow \mathcal{P}$. We may also assume that ϕ is defined only on the primes used as labels in Γ .

Definition 3.3 (Branching operation). Given rooted (H, h) and $(G_j, g_j)_{j \in J}$ and a sequence of primes $(q_j)_{j \in J}$ ($J \neq \emptyset$), let

$$H \left(\frac{h+g_j}{q_j^\infty} \right)_{j \in J} G_j$$

be the rooted group

$$\left\langle H \oplus \bigoplus_{j \in J} G_j; \frac{h+g_j}{q_j^m} : h \in H, g_j \in G_j, j \in J, m \in \omega \right\rangle$$

with root h . If $J = \emptyset$ then $H \left(\frac{h+g_j}{q_j^\infty} \right)_{j \in J} G_j = H$.

If it is clear from the context what J is, we may suppress J and write $H \left(\frac{h+g_j}{q_j^\infty} \right) G_j$. Furthermore, if J is a singleton then we suppress j as well and write $H \left(\frac{h+g}{q^\infty} \right) G$. We also say that $A = H \left(\frac{h+g_j}{q_j^\infty} \right) G_j$ is obtained from H and the G_j using *the branching operation* \square with the leading term (H, h) , and we say sometimes $|J|$ -branching (e.g., 3-branching) to indicate how many rooted groups are involved in the operation.

4. THE DEFINITIONS OF P_β , $S_\beta[m]$, AND \mathcal{G}_α

Fix an even computable ordinal $\alpha > 4$ and a path through Kleene's \mathcal{O} that gives a uniformly computable system of notations for ordinals below α . We identify each such ordinal with its notation in the system. We also assume that every limit ordinal below α is the supremum of a computable sequence of *even* ordinals. Thus, according to our conventions, Σ_ω^0 sets are the ones c.e. relative to $0^{(\omega)} = \bigoplus_{n \in \omega} 0^{(2n)}$.

We also fix infinitely many infinite computable disjoint arrays of primes that will be denoted by p, q, \dots with various subscripts. Let $[\mathbb{Z}]_p$ denote the additive group of the localisation of \mathbb{Z} by a prime p :

$$[\mathbb{Z}]_p = \left\langle \frac{1}{p^n} : n \in \omega \right\rangle \leq \mathbb{Q}.$$

In $[\mathbb{Z}]_p$, the usual multiplicative unity will be denoted by $1_{[\mathbb{Z}]_p}$.

Definition 4.1. Let $\beta \leq \alpha$ be either even or a limit ordinal, and $m \in \omega$. We define rooted groups P_β and $S_\beta[m]$ by recursion as follows.

($\beta = 0$) Define

$$P_0 = [\mathbb{Z}]_{p_0}$$

with root $r_0 = 1$.

($\beta = 2$) Define

$$P_2 = [\mathbb{Z}]_{p_2} \left(\frac{1_{[\mathbb{Z}]_{p_2}} + 1_{[\mathbb{Z}]_{p_0}}}{q_2^\infty} \right) [\mathbb{Z}]_{p_0},$$

i.e. adjoin a copy of P_0 to $[\mathbb{Z}]_{p_2}$ using the 1-branching operation with a new prime q_2 .

($\beta = 4$) Define

$$P_4 = [\mathbb{Z}]_{p_4} \left(\frac{1_{[\mathbb{Z}]_{p_4}} + 1_{[\mathbb{Z}]_{p_2,i}}}{q_{4,i}^\infty} \right) [\mathbb{Z}]_{p_2,i},$$

i.e. adjoin infinitely many re-labelled copies of P_2 to $[\mathbb{Z}]_{p_4}$ using the ω -branching operation with primes $q_{4,i}$. All primes are new/never used so far.

For any integer $m \geq 1$, also define

$$S_{4,m} = [\mathbb{Z}]_{p_4} \left(\frac{1_{[\mathbb{Z}]_{p_4}} + 1_{[\mathbb{Z}]_{p_2,i}}}{q_{4,i}^\infty} \right)_{i \in \{1, \dots, m\}} [\mathbb{Z}]_{p_2,i},$$

i.e. use branching to adjoin m distinctly re-labelled copies of P_2 to $[\mathbb{Z}]_{p_4}$ using *the same* primes as in P_4 .

($\beta > 4$) First, we fix a uniformly computable family of injective $(\phi_{\beta,i})_{i \in \omega}$ that range over disjoint computable sets of new primes never used so far in the definition. We shall also require that given a prime p we can effectively reconstruct β and i s.t. $p \in \text{rng}(\phi_{\beta,i})$ (if such β, i exist). If β is a limit ordinal, then let $(\gamma_i)_{i \in \omega}$ be a computable sequence of even ordinals whose supremum is β . Otherwise, let $\gamma_i = \beta - 2$ for all i .

Given $i \in \omega$ and $\lambda \in \omega \cup \{\omega\}$, define $H_{i,\lambda} = [S_{\gamma_i,\lambda}]_{\phi_i}$ if $\lambda \in \omega$ and otherwise $H_{i,\omega} = [P_{\gamma_i}]_{\phi_i}$. Let $r_{i,\lambda}$ be the root of $H_{i,\lambda}$. Now define

$$P_\beta = [\mathbb{Z}]_{p_\beta} \left(\frac{1_{[\mathbb{Z}]_{p_\beta}} + r_{i,\lambda}}{q_{\beta,i}^\infty} \right)_{i \in \omega, \lambda \in \omega \cup \{\omega\}} H_{i,\lambda},$$

i.e. adjoin infinitely many distinctly labelled sections of successors to the new root, with the i 'th section containing a new re-labelled copies of P_{γ_i} and $S_{\gamma_i,m}$, one of each. Note that for a fixed i different λ -s share the same $q_{\beta,i}$.

Similarly, for any $m \in \omega$, define $S_{\beta,m}$ just as P_β but allowing only the first m sections to have a re-labelled copy of P_{γ_i} . More formally,

$$S_{\beta,m} = [\mathbb{Z}]_{p_\beta} \left(\frac{1_{[\mathbb{Z}]_{p_\beta}} + r_{i,\lambda}}{q_{\beta,i}^\infty} \right) H_{i,\lambda},$$

where $i \in \omega$ and if $i \leq m$ then $\lambda \in \omega \cup \{\omega\}$, and otherwise $\lambda \in \omega$.

Note that in each case we also have a natural definition of a structural tree that will be called *the* structural tree of the respective (rooted) group H and will be denoted $T(H)$. At this point the reader may also notice that P_4 looks more like a group coding a Π_2^0 -outcome, not a Π_4^0 -outcome as its name suggests. Nonetheless, this naive intuition fails and these groups *will code* the claimed complexity in the construction.

Remark 4.2. We also further clarify the choice of ϕ in the definition for $\beta > 4$. It serves two goals. First, it ensures that different components of the group are defined using distinct primes. This property simplifies the definability analysis contained in Section 5. Second, we are able to fully and effectively reconstruct all primes used in the definition of the respective re-labeled copies of $T(P_\gamma)$ or $T(S_\gamma[m])$ at every iteration of the definition. The case $\beta = 4$ was suggested as an explicit elementary example of such a labelling, and the reader should have no difficulties in extending this idea to any computable $\beta < \alpha$. The early write-up of the paper contained an explicit choice of prime labels, but as long as the two goals above are accomplished we do not need to know what exactly the labels are.

4.1. The definition of \mathcal{G}_α . We have defined P_β and $S_\beta[m]$ for even or limit $\beta < \alpha$, but this will not be enough. Recall that α is computable and even (thus successor).

Definition 4.3. For $\lambda \in \omega \cup \{\omega\}$, define $H_\lambda = P_{\alpha-2}$ if $\lambda = \omega$, and let $H_\lambda = S_\alpha[\lambda]$ if $\lambda \in \omega$. Also, let r_λ be the root of H_λ . Define

$$S_{\alpha-1} = [\mathbb{Z}]_{p_{\alpha-1}} \left(\frac{1_{[\mathbb{Z}]_{p_{\alpha-1}}} + r_\lambda}{q_{\alpha-1}^\infty} \right)_{\lambda \in \omega \cup \{\omega\}} H_\lambda,$$

and also define

$$P_{\alpha-1} = [\mathbb{Z}]_{p_{\alpha-1}} \left(\frac{1_{[\mathbb{Z}]_{p_{\alpha-1}}} + r_\lambda}{q_{\alpha-1}^\infty} \right)_{\lambda \in \omega} H_\lambda.$$

Notice the “successors” below $r_{\alpha-1}$ are not subdivided into distinctly labelled sections (in contrast with Definition 4.1). So the difference between $S_{\alpha-1}$ and $P_{\alpha-1}$ is that the former uses a (unique) copy of $P_{\alpha-2}$, and the latter uses only $S_{\alpha-2}[\lambda]$ (one for each λ).

In the definition below, we view $P_{\alpha-1}$ and $S_{\alpha-1}$ as groups (not as groups with a distinguished element).

Definition 4.4. For any α even, let

$$\mathcal{G}_\alpha = \bigoplus_{i \in \omega} S_{\alpha-1} \oplus \bigoplus_{i \in \omega} P_{\alpha-1} \oplus \bigoplus_{i \in \omega} \mathbb{Z}.$$

Thus, \mathcal{G}_α is the direct sum of infinitely many copies of $S_{\alpha-1}$, infinitely many copies of $P_{\alpha-1}$, and the free abelian group of rank ω . Clearly, \mathcal{G}_α is torsion-free. According to their respective definitions, the groups P_β and $S_\beta[m]$ have computable copies with all possible uniformity in $m \in \omega$ and (the notation for) $\beta < \alpha$. Furthermore, in each of these computable copies we have a computable basis consisting of the vertices of the underlying computable canonical structural trees. Therefore, we have:

Fact 4.5. \mathcal{G}_α has a computable copy with a computable basis consisting of a generating basis of the free summand and the nodes of the structural trees of the non-free summands.

The disjoint union of the structural trees of the non-free summands of \mathcal{G}_α will be called *the structural forest of \mathcal{G}_α* and denoted $T(\mathcal{G}_\alpha)$.

5. DEFINABILITY

Recall that \mathcal{G}_α was built from smaller groups using direct sum and the chain operation. Since we were using prime substitution, different “locations” of \mathcal{G}_α were using different versions of the same P- and S-subcomponents, depending on their position in the structural forest $T(\mathcal{G}_\alpha)$ of \mathcal{G}_α . In fact, given any prime that was used as a label in $T(\mathcal{G}_\alpha)$, we can effectively reconstruct whether the prime labels a node or an edge (of $T(\mathcal{G}_\alpha)$), and more importantly we can effectively extract γ such that the respective subcomponent is (a relabelled version of) P_γ or S_γ .

Theorem 5.1. *Suppose p was used in the definition of \mathcal{G}_α as a label for the roots of a certain relabelled versions of P_β and $S_\beta[m]$, $m \in \omega$ ($\beta \geq 4$). We can uniformly (in p) produce a Π^c_β formula Ψ_β such that, for any non-zero $g \in \mathcal{G}_\alpha$:*

$$(1) \quad \mathcal{G}_\alpha \models \Psi_\beta(g) \text{ iff } g = \sum_i m_i a_i,$$

where the sum ranges over a finite set, all $m_i \in \mathbb{Z}$, and each a_i is a root of P_β -type component whose root is labelled by p .

Proof of Theorem 5.1. We need a lemma from [Melb] that can be viewed as an extension of the main technical proposition in [Hjo02]. For the sake of complete exposition, we give a detailed proof (while [Melb] contains merely an extended sketch).

Lemma 5.2. Let $I \subseteq \omega$ be a non-empty set, and let

$$A = \bigoplus_{i \in I} \left(H_i \left(\frac{h_i + g_{i,k}}{p^\infty} \right)_{k \in K_i} G_{i,k} \right),$$

where H_i and $G_{i,k}$ having roots h_i and $g_{i,k}$. Assume also that H_i and $G_{i,j}$ are localizations of \mathbb{Z} by non-empty disjoint sets of primes P and P_1 respectively, and assume neither of P, P_1 contains p . Assume a is a non-zero element of A such that $p^\infty | a$. Then

$$a = \sum_i (c_i h_i + \sum_{f \in F_i} d_{i,f} g_{i,f})$$

where $c_i, d_{i,f}$ are rationals and $F_i \subseteq K_i$ are finite index sets such that

$$c_i = \sum_{f \in F_i} d_{i,f}$$

for every i .

Remark 5.3. The group in the lemma is not a tree-group, neither it is a direct sum of tree groups. The lemma will be used to localize certain divisibility conditions that hold in \mathcal{G}_α to smaller “chunks” of the structural forest of \mathcal{G}_α . The process of localization will lead to groups similar to A . Notice that for some (possibly, for all) indices i , the coefficient c_i in $a = \sum_i (c_i h_i + \sum_{f \in F_i} d_{i,f} g_{i,f})$ may be zero. No extra assumption about the index sets K_i is needed.

Proof of Lemma 5.2. First, we verify the seemingly obvious claim:

Claim 5.4. *Suppose $p^\infty \mid a$ in A . Then $a = \sum_{i,j} r_{i,j}(h_i + g_{i,j})$ for some $r_{i,j} \in \mathbb{Q}$.*

Proof of Claim. Any element of A having the claimed form is clearly infinitely divisible by p . Our task is to show that *no other elements have this property*. This necessary direction is not that trivial and goes through a sequence of reductions. The idea is that if there is a counterexample then it must be “very local”.

Fix finitely many h_i and $g_{i,j}$ such that $a = \sum_i b_i h_i + \sum_{i,j} a_{i,j} g_{i,j}$. Such a finite set of roots and such coefficients must exist since A is generated by elements of this form. Since direct summands are pure, we may consider a projection of a and assume that

$$a = bh_i + \sum_j a_{i,j} g_{i,j},$$

where $p^\infty \mid (h + g_j)$ for each of the finitely many $g_{i,j}$. We write h and g_j suppressing the common i . Since multiplying by an integer does not change the property of being infinitely divisible by p , we may assume that the coefficients are integers.

We then “accumulate” infinite divisibility below a single g_j as follows. Since $p^\infty \mid h + g_j$, for each $k \neq j$ we may subtract $a_k(h + g_k)$ from a and still get an element infinitely divisible by p . We will be left with

$$p^\infty \mid (nh + mg_j),$$

where $n, m \in \mathbb{Z}$. If $m = n$ then we have nothing to prove since in this case $b = \sum_j a_j$, while the case when $b \neq \sum_j a_j$ corresponds to $m \neq n$. Assuming $m \neq n$ and taking into consideration $p^\infty \mid h + g_j$, we conclude that $p^\infty \mid g_j$ and $p^\infty \mid h$. Our task is to verify that it cannot be.

If $p^\infty \mid g_j$ (equivalently, $p^\infty \mid h$), then for each $m > 0$ the element u_m witnessing $p^m \mid g_j$ already occurs in some finitely generated subgroup of the group A , and furthermore the station can again be restricted to one direct summand of the group. Thus, fixing an arbitrarily large m , without loss of generality all generators needed to span the m 'th root may be taken from a subgroup of A of the form $H\left(\frac{h+g_k}{p^\infty}\right)_{k \in K} G_k$ where K is finite. It is fairly easy to see that, under our assumptions on the isomorphism types of H and G_k we have

$$H\left(\frac{h+g_k}{p^\infty}\right) G_k \cong H\left(\frac{h+g_j}{p^\infty}\right) G_j \oplus \bigoplus_{k \neq j} [\mathbb{Z}]_{\{p\} \cup P_1} (g_k - g_j),$$

(as TFAGs) where \mathbb{Z}_S stands for the localization of \mathbb{Z} by S . Indeed, each element of $H\left(\frac{h+g_k}{p^\infty}\right) G_k$ can be expressed as a linear combination of elements in the suggested summands (note each of the $g_k - g_j$ is infinitely divisible by p and by each $\hat{p} \in P_1$). Such an expression is also unique since $\{h, g_j, g_k - g_j : j \neq k\}$ is a basis of the group. We suppress j and write g and G . We conclude that it is sufficient to verify that neither $p^\infty \mid g$ nor $p^\infty \mid h$ may occur in a group of the form

$$H\left(\frac{h+g}{p^\infty}\right) G \cong [\mathbb{Z}]_P \left(\frac{1_{[\mathbb{Z}]_P} + 1_{[\mathbb{Z}]_{P_1}}}{p^\infty} \right) [\mathbb{Z}]_{P_1} = \left\langle \frac{g}{\hat{p}^n}, \frac{h}{\hat{p}^n}, \frac{g+h}{p^n} : \tilde{p} \in P, \hat{p} \in P, n \in \omega \right\rangle,$$

where $P \cap P_1 = \emptyset$ and $p \notin P \cup P_1$. Fix some large enough n such that

$$\frac{g}{p^m} \in U_n = \left\langle \frac{g}{\hat{p}^n}, \frac{h}{\hat{p}^n}, \frac{g+h}{p^n} : \tilde{p} \in P, \hat{p} \in P_1 \right\rangle.$$

Let $x = \prod_{\hat{p} \in P} \hat{p}^n$ and $y = \prod_{\hat{p} \in P_1} \hat{p}^n$, and define $\xi = \frac{g}{x}$, $\eta = \frac{g}{y}$, and $\tau = \frac{g+h}{p^n}$. We have $U_n = \langle \xi, \eta, \tau \rangle$. Embed U_n isomorphically into $\mathbb{Z} \oplus \mathbb{Z}$ so that

$$\xi \rightarrow (p^n, 0), \eta \rightarrow (0, p^n), \tau \rightarrow (x, y).$$

Note that $p^n \tau \rightarrow (p^n x, p^n y) = x(p^n, 0) + y(0, p^n)$ which corresponds to $g + h$, so the relation $p^n \xi = x\xi + y\eta$ upon the generators is respected, and the map can be extended to a homomorphism ϕ . Also, since a basis is mapped to a basis, the embedding is injective. We identify U_n with its image under the embedding ϕ . Since $g = (xp^n, 0)$, it will be sufficient to show that

$$(xp^{n-1}, 0) \notin \langle (p^n, 0), (0, p^n), (x, y) \rangle,$$

and this will be equivalent to saying that g (thus, h) is not divisible by p in U_n , contradicting the choice of $m > 0$ and n . Suppose we have

$$(xp^{n-1}, 0) = s(p^n, 0) + s'(0, p^n) + t(x, y),$$

where $t, s, s' \in \mathbb{Z}$. In particular, $0 = s'p^n + ty$, and since $p \nmid y$ this implies $p^n \mid t$. But we also have $xp^{n-1} = sp^n + tx$ and therefore $p^n \mid xp^{n-1}$ and $p \mid x$, contradicting the choice of x . \square

The lemma now easily follows from the claim above. Suppose $g_{i,j}$ is so that $r_{i,j} \neq 0$ in $a = \sum_{i,j} r_{i,j}(h_i + g_{i,j})$. The element $a'_i = a_i - r_{i,j}(h_i + g_{i,j})$ is either 0 or has less roots of A mentioned in its respective expression (with non-zero coefficients). In either case, by the inductive hypothesis, we can find $a'_i = c'_i h_i + \sum_{f \in F'_i} d'_{i,f} g_{i,f}$ such that $c'_i = \sum_{f \in F'_i} d'_{i,f}$. Then clearly $a_i = (c'_i + r_{i,j})h_i + \sum_{f \in F'_i} d'_{i,f} g_{i,f} + r_{i,j} g_{i,j}$, and $c'_i + r_{i,j} = \sum_{f \in F'_i} d'_{i,f} + r_{i,j}$. \square

We return to the proof of Theorem 5.1.

Basic case. First, assume $\beta = 4$, in which case we are dealing with P_4 and $S_4[m]$ for various m . Recall p labels roots of these subcomponents, and let q be the prime used in the branching operation involving these roots as leading components (i.e., q labels edges between the root and its immediate successors in the tree), and assume p_i are the primes that labels the roots of various P_2 -successors in such components. Note the primes q, p_i can be found with all possible uniformity.

Now the formula $\Psi_4(g)$ says that:

- $p^\infty \mid g$ and $g \neq 0$ (which is Π_2^g), and
- $\bigwedge_{i \in \omega} (\exists w)[p_i^\infty \mid w \wedge q^\infty \mid (g + w)]$ (which is Π_4^g).

Note that if g is of the desired form (see the statement of the theorem) then $\Psi_4(g)$ holds trivially. Now suppose the formula holds.

Assuming $i = 0$, $p_0^\infty \mid w$ implies w is a linear combination of roots of P_2 -components, with rational coefficients. To see why, we shall use a certain trick, which we will call *the extension trick* which will be repeated several times throughout the rest of the paper. Let q' be the prime labelling the edge between the root of a P_2 -component (having the right labelling) and its unique P_0 -successor. Then extend the group by making each root of a P_2 -successor infinitely divisible by q and q' . The pure closure generated by the roots of such enriched P_2 -components detach in the extended group, and furthermore no element of their natural direct complement will be divisible by p_0 . Since infinite divisibility still holds in the extended group, we get that w must be of the claimed form. Similarly, the first condition implies that

$g = \sum_j c_j h_j$, where c_j are *rational* numbers and h_j are roots of P_4 - and $S_4[m]$ -type components.

One would hope to apply Lemma 5.2 to derive information from condition $q^\infty|(g+w)$, but this would not work. Indeed, the lemma can only be applied to a direct sum of groups of a certain form, which is not quite the case here (the direct sum of the P_4 - and $S_4[m]$ -components does not detach in \mathcal{G}_α). We use the extension trick again. In the structural forest of \mathcal{G}_α , the roots of the P_4 and $S_4[m]$ (labelled by p) will perhaps have predecessors, and the edge between these roots and their predecessors will be labelled by some (new) prime ξ . Similarly, the roots of P_2 -components (labelled by p') will have P_0 -successors, and the edges connecting these roots to their successors will be labelled some new prime η . Extend the group \mathcal{G}_α by declaring the roots of these P_4 and $S_4[m]$ -components infinitely divisible by ξ , and the roots of the P_2 -components infinitely divisible by η . Call the new group \mathcal{G}'_α . In \mathcal{G}'_α , the least pure subgroup H containing the roots of P_4 and $S_4[m]$ -components and their immediate successors (roots of various P_2 -components) detaches as a direct summand. Furthermore, this group H is of the form as in Lemma 5.2, since ξ, η we different from q . Notice the formula Ψ_4 claims only positive information about g , and thus its validity is preserved under expansions of \mathcal{G}_α . Note the conditions $p_0^\infty|w$ and $p^\infty|g$ imply that $w, g \in H$, and by Lemma 5.2, $q^\infty|(g+w)$ guarantees

$$g + w = \sum_j (c_j h_j + \sum_f d_{j,f} g_{j,f})$$

where for every j

$$c_j = \sum_f d_{j,f},$$

where h_j are roots of P_4 or $S_4[m]$ -components labelled by p and $g_{j,f}$ are roots of the respective immediate successors of h_j of type P_2 labelled by p_0 . Note for any j the root h_j has at most one successor labelled by p_0 , and thus if $c_i \neq 0$ then h_i the sum $\sum_f d_{j,f} g_{j,f}$ is really equal to just a single summand of the form $d_j g_j$. Since $c_j = d_j$ we conclude that $g = \sum_j c_j h_j$ and $w = \sum_j c_j g_j$. This observation brings us to two conclusions. First, each h_j involved in $g = \sum_j c_j h_j$ with a non-zero coefficient must have a successor labelled by p_0 . Repeating this argument for p_1, p_2, \dots we conclude that all such h_j must be roots of P_4 -components of the right labelling type. Second, the (in general) rational coefficients c_j and d_j belong to copies \mathbb{Z} localized by distinct, non-intersecting sets primes. Since we have $c_j = d_j$ we must have $c_j \in \mathbb{Z}$. Since these conclusions can be made in the larger group \mathcal{G}'_α , the same must be true in \mathcal{G}_α as well. Thus, we have Theorem 5.1 for the simplest case $\beta = 4$.

Inductive step. If $\beta > 4$ is a limit ordinal, then let $\beta = \sup_i \gamma_i$ where $(\gamma_i)_{i \in \omega}$ is a computable sequence of even ordinals, and otherwise let $\gamma_i = \beta - 2$ for all i . Let $(\Psi_{\gamma_i})_{i \in \omega}$ be the respective uniformly computable formulae whose existence is given by the IH. Let Ψ'_{γ_i} be the formula Ψ_{γ_i} in which the primes are (uniformly, effectively) replaced by the respective primes used in the definition of the version of P_β determined by label p . (Recall p labels the roots of some versions of $P_\beta, S_\beta[m]$.) Suppose, as in the case $\beta = 4$, primes p_i, q label the immediate successors of such roots and the edges in-between, respectively. Then $\Psi_\beta(g)$ says:

- $p_\beta^\infty|g$ and $g \neq 0$ which is Π_2^c , and

- $\bigwedge_{i \in \omega} (\exists w) [\Psi_{\gamma_i}^i(w) \wedge p_i^\infty | w \wedge q^\infty | (g + w)]$.

According to our conventions, the second conjunct is Π_β^c . Indeed, if β is a successor even ordinal, then we simply have a conjunction of $\Sigma_{\beta-1}^c$ -formulae. Otherwise, we have a conjunction of $\Sigma_{\gamma_i+1}^c$ -formulae, with $\sup_i \gamma_i = \beta$. We call such formulae Π_β^c . (Recall that, according to our conventions, Π_ω^0 consists of sets co-c.e. in $0^{(\omega)}$.)

Again, if g is of the desired form then it satisfies the formula trivially. We now verify that Ψ_β implies that g is as claimed. We use the extension trick and Lemma 5.2 to derive that $g = \sum_j m_j r_j$, where r_j are roots of either $S_\beta[m]$ or P_β -trees in the respective direct components, and the coefficients m_j are integers. Now for every fixed i the first conjunct in $p_i^\infty | w \wedge q^\infty | (g + w)$ guarantees that w is a sum of roots of $S_{\gamma_i}[m]$ or P_{γ_i} -subtrees (relabelled accordingly). Furthermore, by the IH, $\Psi_{\gamma_i}^i(w)$ implies that $w = \sum_j n_j v_j$, where v_j are roots of P_{γ_i} -subcomponents of the right labelling type and n_j are integers. Recall that the i 'th section of successors in P_β or $S[m]_\beta$ contains *at most one* copy of P_{γ_i} . By Lemma 5.2, in $w = \sum_j n_j v_j$ and $g = \sum_j m_j r_j$ we have $m_j = n_j$. This means, in particular, that for every i each of the r_j has *at least one* P_{γ_i} -successor within the i 'th section, showing that all the r_j are roots of P_β of the right labelling type, as desired. \square

Recall α is even (thus, successor) ordinal, and $\mathcal{G}_\alpha = \bigoplus_{i \in \omega} S_{\alpha-1} \oplus \bigoplus_{i \in \omega} P_{\alpha-1} \oplus \bigoplus_{i \in \omega} \mathbb{Z}$.

Corollary 5.5. *There exists a computable $\Sigma_{\alpha-1}^c$ -formula $\Phi_{\alpha-1}$ such that for any non-zero $g \in \mathcal{G}_\alpha$*

$$\mathcal{G}_\alpha \models \Phi_{\alpha-1}(g) \text{ iff } g = \sum_i m_i a_i,$$

where the sum ranges over a finite set, each $m_i \in \mathbb{Z}$, and every a_i is the root of a $S_{\alpha-1}$ -summand.

Proof. The formula $\Phi_{\alpha-1}(g)$ says:

- $p_{\alpha-1}^\infty | g$ and $g \neq 0$ which is Π_2^c , and
- $(\exists w) [q_{\alpha-1}^\infty | (g + w)] \wedge p_{\alpha-2}^\infty | w \wedge \Psi_{\alpha-2}(w)$,

where $\Psi_{\alpha-2}$ uses the same prime labels as the standard definition of $P_{\alpha-2}$ (i.e., without any prime substitution). In other words, the formula claims the most naive property that distinguishes the root of a $S_{\alpha-1}$ from roots of $P_{\alpha-1}$ -summands, i.e., existence of a $P_{\alpha-2}$ -successor. But these divisibility conditions still hold for any integer linear combination of such roots, so we have the ‘if’ part of the corollary. For the ‘only if’ direction, suppose $\mathcal{G}_\alpha \models \Phi_{\alpha-1}(g)$. We use the extension trick and Lemma 5.2 throughout. The condition $p_{\alpha-1}^\infty | g$ in conjunction with $(\exists w) [q_{\alpha-1}^\infty | (g + w)] \wedge p_{\alpha-2}^\infty | w$ implies (as in the proof of Theorem 5.1) that $g = \sum_i m_i a_i$, where $m_i \in \mathbb{Z}$ and a_i are roots of $S_{\alpha-2}$ - and $P_{\alpha-2}$ -summands. Furthermore, by Theorem 5.1, the witness w must be a linear combination of roots of $P_{\alpha-2}$ -components,

$$w = \sum_{i,j} n_{i,j} d_{i,j},$$

where $d_{i,j}$ are below a_j . But $m_i = \sum_j n_{i,j}$ (Lemma 5.2) and thus $m_i \neq 0$ implies $n_{i,j_i} \neq 0$ for some j_i . In other words, any a_i that appears in $g = \sum_i m_i a_i$ with a non-zero coefficient must be the root of some $S_{\alpha-1}$ -summand, since $P_{\alpha-1}$ -summands have no $P_{\alpha-2}$ -components in their definition. \square

5.1. Vertex-like elements. Although there is no direct way to apply techniques from [FKM⁺11], we at least adopt a similar terminology.

We will identify vertices of T with the respective elements of $G(T)$. We call such elements *vertices*, *nodes*, or *vertex-elements*.

Definition 5.6. Say that a non-zero element x of \mathcal{G} is *vertex-like* if it is a linear combination *with integer coefficients* of vertices of the computable structural forest of \mathcal{G} that are labelled by the same prime.

Fact 5.7. *Vertex-like elements with a fixed label are Σ_3^c -definable (with all possible uniformity).*

Proof. The proof is rather elementary (compared to Theorem 5.1). The desired formula with argument g says that there exist distinct primes p, q, s and an element w such that $p^\infty | g$, $s^\infty | w$, and $q^\infty | (g+w)$. We clearly require the primes p, s to label vertices on consequent levels, and s labels the respective edges. Then Lemma 5.2 (and the extension trick) implies that g is vertex-like. There is one subtlety here that needs to be discussed separately. In particular, if g is a linear combination of roots of P_0 -subtrees (those are terminal nodes in the structural forest of \mathcal{G}_α), then the prime s should be chosen equal to the label of the *predecessor* of such a P_0 -node, while in all other cases (i.e., $\beta > 0$) s should be chosen to be the label of a *successor*. In the former case, every P_2 -root has a unique P_0 -successor, and therefore in this case Lemma 5.2 can be applied “upside-down” to get the desired property (i.e., with P_0 being H_i in the notation of Lemma 5.2). \square

Later in the proof we will use Theorem 5.1 and Corollary 5.5 to define special subgroups of vertex elements. Having defined vertex-like elements, it makes sense to define some analogy of the successor relation.

Definition 5.8. We say that a vertex-like element v is a successor of another vertex-like element u if for some $n \geq 0$ we have:

- (1) u is a linear combination of vertex elements at level n ,
- (2) v is a linear combination of vertex elements at level $n + 1$, and
- (3) $q^\infty | (u + v)$, where q labels edges between levels n and $n + 1$.

Remark 5.9. The main difficulty we will have to face is that the vertices that appear in the linear combination of a successor v of u may be *not* below the vertices that generate u . In fact, already in the case $\alpha = 4$ there exist an automorphism of \mathcal{G}_α showing the “bad” situation is unavoidable.

6. CODING Π_β^0 vs. Σ_β^0 .

Recall that we fixed a computable $\alpha = \delta + 2k$, $k \in \mathbb{Z}^+$.

Theorem 6.1. *There exists a uniform procedure that on input $i \in \omega$ builds $C_i \cong S_{\alpha-1} \oplus \mathbb{F}$ if $i \in H(\alpha)$ and $C_i \cong P_{\alpha-1} \oplus \mathbb{F}$ otherwise, where \mathbb{F} is the free abelian group of rank ω .*

Recall that the natural complexity of the structural tree of (say) P_4 is Π_2^0 , not Π_4^0 . As a consequence, the proof requires a new idea.

Proof of Theorem 6.1. We first state the easy:

Lemma 6.2. *There exists a uniform procedure that produces a $0''$ -computable copy of the structural tree of $S_{\alpha-1}$ if $i \in H(\alpha)$ and a $0''$ -computable copy of the structural tree of $P_{\alpha-1}$, otherwise. In this sequence, we may assume that the roots of the $S_{\alpha-1}$ and $P_{\alpha-1}$ form a computable set.*

Proof Sketch. The proof is a straightforward induction based on the fact that $\Pi_\beta^{0''}$ can be viewed as $\Pi_{2+\beta}^0$, and that Π_β^0 and Σ_β^0 -outcomes are naturally reflected in the definitions of the structural trees of $P_{2+\beta}$ and $S_{2+\beta}[m]$ -outcomes. Note that we were always using at most one successor of each type in every section of $P_{2+\beta}$ (and $S_{2+\beta}[m]$), but one can easily see that any predicate may be uniformly effectively transformed to a predicate in which witnesses of quantifiers naturally line-up with the definition of the trees. \square

Remark 6.3. We explicitly say what the “tricks” are. As usual, we may assume that there exists at most one existential witness. As a consequence, at most one P-component will appear at the right section of the group (see Definition 4.1), and this feature of the groups will be useful in the proof. To remove repetition of the $S_\beta[m]$ -components for a fixed m , we may use another standard trick. For simplicity, we will explain what happens at the recursive level and with a Π_2^0 predicate R . If R “fires” on x then grow the respective component (currently attempting to code $\Sigma_2^0[m]$) to the one that codes $\Sigma_2^0[n]$, where $n > m$ is very large. Also, introduce a fresh $\Sigma_2^0[k]$ components for each $k < n$ (if such a component is not already there). In our context, $S_4[m]$ plays the role of $\Sigma_2^0[m]$. This allows us to have no repetition in the $S_4[m]$ -components (for each m), as in the definition. This idea can be relativized and recursively iterated throughout computable ordinals $< \alpha$. We note that the uniqueness of a $S[m]$ -component for each fixed m is merely a *notational convenience* and will not be important in the proof. If the reader is uncomfortable with the above explanation, they may safely assume that there are infinitely many $S[m]$ -components for each m in each section of the larger component. In fact, under this seemingly more relaxed assumption we will end up with a group isomorphic to \mathcal{G}_α (it can be derived from Section 9).

6.1. Saving two jumps. In this subsection we develop a certain general technique and then apply the technique to prove the theorem. We note that “saving two jumps” (to be stated formally in Proposition 6.6) is vital only when α is finite.

Definition 6.4. We call a countable rooted tree T labelled by primes *worthy* if its prime labelling satisfies the properties:

- (1) nodes at different levels of T are labelled by distinct primes, and
- (2) edges at distinct levels of T are labelled by distinct primes,
- (3) labels of edges and nodes are always distinct.

We say that $G \in \mathcal{TG}$ is worthy if it possesses a worthy structural tree T .

Remark 6.5. We note that nodes (or edges) at the same level of a worthy tree can be labeled by distinct primes. Each worthy tree-group is an EPLAG in the sense of Hjorth [Hj02]. Groups from [DM08, FKM⁺11] are worthy tree-groups, but the main coding components in [AKMS12, Melb] are not even EPLAGs.

Note that the structural trees of P_β , S_β and $S_\beta[m]$ are worthy. Recall that $G(T)$ denotes the group built using a labelled tree T .

Proposition 6.6 (Saving two jumps). *There exists a uniform procedure that, given a worthy Δ_3^0 -tree T , produces a computable group $A_T \cong G(T) \oplus \mathbb{F}$, where \mathbb{F} is the free abelian group of rank ω .*

Furthermore, if the root of T is known (say, is the first element in the domain of T) then we can effectively distinguish the vertex-element representing the root in the $G(T)$ -summand of A_T .

Proof. The goal of the first part of the proof is to reduce the complexity of the labelled tree T to Π_2^0 (under the right definition of what Π_2^0 means in this context). Throughout this proof, we fix the standard computable listing p_0, p_1, \dots of all primes.

An *embedding* of a labelled tree T into a labelled tree Γ is an injective map that sends the root of T into the root of Γ and respects labels and the immediate predecessor relation. We claim that, under this notion of embedding, there exists a universal object \mathbb{U}_i among all countable prime-labelled trees with a specified root-labelling p_i .

Definition 6.7. Fix $i \geq 0$. Define \mathbb{U}_i to be $\omega^{<\omega}$ labelled as follows. Start with e labelled by p_i . Then for each prime p_k adjoin infinitely many edges (e, v) labelled by p_k , repeat for all $k \in \omega$. Among all edges (e, v) labelled by p_k reserve infinitely many edges for each $j \in \omega$, and label the corresponding v by p_j . Then repeat the process for each such successor v , with v and p_j playing the roles of e and p_0 , then for successors of the successors, etc.

One could view the above definition as a computable Fraisse limit of finite labelled trees with root label p_i , and under the right notion of embedding. Clearly, \mathbb{U}_i has a computable presentation, for any i . We identify \mathbb{U}_i with this computable presentation. Then \mathbb{U}_i is universal in the following strong sense.

Lemma 6.8. *Fix i . There exists a uniform procedure that given a Δ_2^0 (labelled) tree T whose root is labelled by p_i produces a Δ_2^0 embedding ϕ of T into \mathbb{U}_i such that $\phi(T)$ is a Π_1^0 labelled subtree of \mathbb{U}_i .*

Furthermore, if the root of T is known (say, the first element of its domain), then its ϕ -image will be uniformly computable.

Proof. First of all, note that there exists a Δ_2^0 -embedding ψ of T into \mathbb{U}_i . This embedding can be chosen so that “lots of” nodes of \mathbb{U}_i are outside $\phi(T)$. Reserve exactly one successor labelled by p_j (among the infinitely many such successors of some w , and for any j) to potentially have a non-empty intersection with the range of ψ .

Now we adjust ψ and define a new Δ_2^0 embedding ϕ so that $\phi(T)$ is actually a Π_1^0 -subset of \mathbb{U}_i . The root is sent to the root of \mathbb{U}_i under ψ non-uniformly (and effectively uniformly if the prime label and the ω -name of the root are known). Now, using the Limit Lemma, express $\psi = \lim_s \psi_s$. The labels in T and the successor-predecessor relations in T can also be guessed in a Δ_2^0 -way.

We now explain a typical action performed at a stage of the construction. Suppose on input $v \in T$, $\psi_{s+1}(v) \neq \psi_s(v)$ but $\psi_{s+1}(w) = \psi_s(w)$ for any ancestor w of v in T (our trees grow downwards). Let v^- be the predecessor of v . Then choose a successor x of $\phi_s(v^-)$ in \mathbb{U}_i among the infinitely many reserved nodes with the same prime label that must be outside the range of ψ . We choose x that has never been used so far, and such that the labels of x and $(\phi_s(v), x)$ match the (current guess on) the labels of v and (v^-, v) in T . Define $\phi_s(v) = x$. Also, redefine ϕ_s on all nodes extending v if necessary, again using fresh nodes in \mathbb{U}_i . Then wait for ψ_s to change on v again (if ever). While waiting, declare more nodes below $\phi_t(v)$, $t \geq s$, to be out of the range of ϕ to make the range Π_1^0 .

The construction is organised so that any node once it is thrown away from $\phi_s(T)$ will never appear in (our approximation to) the range of ϕ again. Furthermore, we will never run out of potential images at any finite stage, and below any node. Thus, $\phi(T) = [\lim_s \phi_s](T)$ forms a Π_1^0 -subset of \mathbb{U}_i , as desired. \square

Clearly, the lemma above can be relativized to $0'$.

Remark 6.9. In this relativization, (the index for) \mathbb{U}_i can be viewed as a parameter and will stay computable, but the tree T and the embedding ϕ will be $0''$ -computable, thus the range of ϕ will be a Π_2^0 -subset of \mathbb{U}_i . (One could surely think of a brute-force full approximation Π_2^0 -proof of the relativized lemma as well, which is not hard.)

6.1.1. *Dispersing leaves of T .* We return to the proof of Proposition 6.6. By Lemma 6.8, we may assume T is a Π_2^0 labelled subtree of the respective computable universal prime-labelled copy of $\omega^{<\omega}$, say of \mathbb{U}_0 . Given a Π_2^0 -worthy tree $T \subseteq \mathbb{U}_0$, we will produce a computable copy of $G(T) \oplus \bigoplus_{i \in \omega} \mathbb{Z}$.

The proof of Proposition 6.6 will be given by a construction. At stage s of the construction we will have a finite subtree $T_s \subseteq \mathbb{U}_0$ labelled by primes, and the associated (partial) free abelian group $G(T_s)[s]$. The group $G(T_s)[s]$ will be defined by divisibility conditions of the form $p^s|x$ and $q^s|(x+y)$, where p, q are primes, x is a vertex of T_s and (x, y) is an edge of T_s . It will be isomorphic to a finite subgroup of $G(T_s)$, under the natural isomorphism taking nodes to nodes, plus a direct summand which is isomorphic to a finite initial segment \mathbb{F}_s of the free abelian group upon finitely many generators. At stage s we will not impose infinite divisibility conditions, but merely use $p^s|$ and $q^s|$, where s is the stage.

Remark 6.10. Note that $G(T_s)[s]$ will be a *finite partial group*. Although we will be dealing with partial subgroups, we really have in mind potentially unbounded initial segments of actual groups upon the same generators. Thus, we may wait for a long enough segment so that a certain local property of the group holds already within this segment (i.e., an element splits into its projections onto direct components of the group, etc.).

We write $T_s \subseteq \mathbb{U}_0$ for the (finite) current Π_2^0 -approximation to $T \subseteq \mathbb{U}_0$ at stage s :

$$v \in T \iff \exists^\infty s \ v \in T_s.$$

The key technical definition of this part of the proof is:

Definition 6.11. We say that a leaf v of T_s in $G(T_s)[s]$ is *dispersible* if there exists a (partial) subgroup X of $G(T_s)[s]$ such that $G(T_s \setminus \{v\})[s] \oplus X = G(T_s)[s]$.

The main point of the definition above is to keep $G(T_s \setminus \{v\})[s]$ untouched while “dispersing” v .

Remark 6.12. In the notation of the definition above, we do not require $v \in X$, and indeed in the construction it will almost never be the case. This “dispersing” will correspond to v leaving the current approximation of T (which is Π_2^0 within \mathbb{U}_0). We will argue that when the node v enters T again due to the respective Π_2^0 -outcome becoming active at a later stage, we can “resurrect” v without any effect on the previously imposed divisibility conditions (to be clarified). On the other hand, if v never resurrects, then X will contribute to the free summand \mathbb{F} .

The formal construction and its verification will be given shortly. The key group-theoretic ingredient that will make the construction work is that *any leaf v of T_s is dispersible in $G(T_s)[s]$* (to be proven in Lemma 6.13). For now, we take this property as granted.

6.1.2. *Construction.* At stage 0 we start by setting T_0 equal to the root of \mathbb{U}_0 labelled by p_0 . We also declare $G(T_0)[0]$ equal to the group consisting of the root of \mathbb{U}_0 , its additive inverse, and 0.

At every stage $s+1$ of the construction we perform one (and only one) of the following elementary actions:

- (1) If the Π_2^0 predicate does not fire on $x \in T_s$ but has fired on all its predecessor y , then *disperse* x in $G(T_s)$ by forming a new (partial) free direct complement of $G(T_s \setminus \{x\})[s]$ within $G(T_s)[s]$ as described above and setting $T_{s+1} = T_s \setminus \{x\}$. Never add new prime roots to the generators of the new free direct summand unless the element is *resurrected* at a later stage.
- (2) If the predicate fires on $x \notin T_s$, and all its predecessors (which are assumed to be currently in T_s), and x has never appeared in T_t for $t < s$. Then set $T_{s+1} = T_s \cup \{x\}$.
- (3) Same as (2) but x has already appeared in T and was dispersed at some stage $t < s$. Then *resurrect* the element x as follows. Let $X[t]$ be such that $X[t] \oplus G(T_t)[t]$ witnessed that x was dispersed at stage t . Let $X[s]$ be the free partial group upon the same generators as $X[t]$ that has been produced in the construction. (Note $X[s]$ properly extends $X[t]$ in general). Choose $F' \leq F_s$ to be the natural complement of $X[t]$ in F_s that was formed by various earlier dispersed elements that are still dispersed,

$$G(T_s)[s] \oplus F_s = X[t] \oplus F' \oplus G(T_s)[s].$$

Also, let x^- be the predecessor of x in T_s . Set $T_{s+1} = T \cup \{x\}$. Assuming p and q were prime labels of x and (x^-, x) , respectively, choose $\xi, \eta \in G(T_s)[s] \oplus F_s$ such that $p^t \xi = x$ and $q^t \eta = x^- + x$. Introduce new generators ζ, κ (if necessary) that witness $p^{s-t} | \xi$ and $q^{s-t} | \eta$. Set $G(T_{s+1}) = \langle g_{m_s}, \zeta, \kappa \rangle$ and $F_{s+1} = F'$.

In all cases, finish the stage by extending both $G(T_{s+1})$ and F_s by extending the definition of the group operation. Also, if currently $v, w, \dots \in T_s$, declare v, w, \dots and all edges between these elements divisible by the $(s+1)$ -th power of the respective prime. We do so by introducing more generators to $G(T_{s+1})$ witnessing these divisibility conditions.

(We may artificially adjoin the free group of rank ω to A_T at the end, just to make sure that the free summand of the output is always of rank ω .)

6.1.3. *Verification.* We follow the same notation as in the construction. Fix $s > 0$ and assume T_s has more than one node.

Lemma 6.13. *Any leaf y of T_s is dispersible in $G(T_s)[s]$.*

Proof. Recall T_s is finite, and $G(T_s)[s]$ is a finite partial group. More specifically, $G(T_s)[s]$ is a finite partial subgroup of the group defined by finitely many divisibility conditions that are restrictions of the infinitary divisibility conditions in $G(T_s)$.

If T consists of only two nodes x, y , then the lemma is reduced to:

Claim 6.14. *Suppose we have*

$$B = \langle x, y : p^n | x, q^m | y, r^s | (x + y) \rangle,$$

where p, q, r are distinct primes and $n, m, s \in \mathbb{N}$. Let α be the unique solution of $p^n \alpha = x$ in B . Then $\langle \alpha \rangle$ detaches as a direct summand of B .

Remark 6.15. According to our terminology and notation, the claim says that y is dispersible.

Proof. We give a self-contained proof that exploits nothing but linear algebra. For simplicity, embed B isomorphically into $\mathbb{Z} \oplus \mathbb{Z}$ as follows. Take $\alpha, \beta, \gamma \in B$ such

that $p^n\alpha = x$, $q^m\beta = y$, and $r^s\gamma = x + y$. Similarly to the proof of Lemma 5.2, we claim that the partial map

$$\begin{aligned}\alpha &\rightarrow (r^s, 0) \\ \beta &\rightarrow (0, r^s) \\ \gamma &\rightarrow (p^n, q^m)\end{aligned}$$

can be (uniquely) extended to an isomorphic embedding ψ of B into $\mathbb{Z} \oplus \mathbb{Z}$. We have $x \rightarrow (p^n r^s, 0)$ and $y \rightarrow (0, q^m r^s)$ under ψ . In particular, ψ is well-defined on γ . Note that $\{x, y\}$ is a basis of B , and clearly $\{\psi(x), \psi(y)\}$ is linearly independent. Since ψ maps a basis into a basis, ψ can be uniquely extended to an injective homomorphism of B into $\mathbb{Z} \oplus \mathbb{Z}$.

We put the row-vectors $\psi(\alpha), \psi(\beta), \psi(\gamma)$ together into a matrix:

$$(2) \quad \begin{pmatrix} r^s & 0 \\ 0 & r^s \\ p^n & q^m \end{pmatrix}.$$

Using (integer) row elementary operations, run the Euclidean algorithm within the second column of the matrix, i.e., for the q^m and r^s (ignoring the 0 on top). Since q^m and r^s are co-prime, we will eventually arrive at a matrix of the form

$\begin{pmatrix} r^s & 0 \\ x & 1 \\ y & 0 \end{pmatrix}$ or of the form $\begin{pmatrix} r^s & 0 \\ y & 0 \\ x & 1 \end{pmatrix}$. First, assume we arrive at the former kind of matrix. Since the first entry of the second row in (2) is zero, at every stage we

will have a matrix of the form $\begin{pmatrix} r^s & 0 \\ \rho p^n & \tau r^s + \rho q^m \\ \xi p^n & \xi q^m + \eta r^s \end{pmatrix}$, where σ, ρ, τ, ξ are integers.

As a consequence, there exist integers ξ and η such that $\xi p^n = y$ and $\xi q^m + \eta r^s = 0$. If $\xi \neq 0$ then $\eta \neq 0$. Also, $\xi q^m + \eta r^s = 0$ and therefore $r^s | \xi q^m$, which implies $r^s | \xi$ and $r^s | y$. Using another elementary (integer) row transformation, this time applied

to the first row and the row with y in it, we obtain $\begin{pmatrix} r^s & 0 \\ x & 1 \\ 0 & 0 \end{pmatrix}$. On the other

hand, if $\xi = 0$ then $\eta = 0$, and if this were the case we would've arrived at the latter matrix immediately¹. Now, suppose we arrive at $\begin{pmatrix} r^s & 0 \\ y & 0 \\ x & 1 \end{pmatrix}$. But then

$\tau r^s + \rho q^m = 0$ and $\rho p^n = y$ imply, as above, that $r^s | \rho$ and thus $r^s | y$. Therefore, we

again can arrive at $\begin{pmatrix} r^s & 0 \\ x & 1 \\ 0 & 0 \end{pmatrix}$ using only elementary transformations. Note the

elementary transformations that we used may be represented by matrices (over \mathbb{Z}) having inverses (over \mathbb{Z}). At this stage the reader can perhaps already see that the claim is proved, but we give more explanation below.

We claim that the constructive argument above witnesses the existence of the desired decomposition. Since $(x, 1)$ is an integer linear combination of $\psi(\alpha), \psi(\beta), \psi(\alpha)$, we have $(x, 1) \in \phi(B)$. On the other hand, all our elementary transformations were invertible, and thus $\psi(\alpha), \psi(\beta), \psi(\alpha)$ can be generated by $\psi(\alpha), (x, 1)$. Since

¹The case $\xi = 0$ is in fact impossible, but this fact has no value for us.

$\psi(\alpha) = (r^s, 0)$ and $(x, 1)$ are clearly linearly independent and generate $\psi(B)$, we have

$$\psi(B) = \langle \psi(\alpha) \rangle \oplus \langle (x, 1) \rangle.$$

Since ψ is an isomorphism,

$$B = \langle \alpha \rangle \oplus C,$$

where $C = \langle \psi^{-1}((x, 1)) \rangle$. □

Using the claim above we prove the general case. Suppressing s , the lemma states that $A_s = Y \oplus G(T \setminus \{y\})$, but this time the finite worthy T may contain more than two vertices. Let $x = y^-$, i.e., x is the predecessor of the fixed above leaf y . Take the subgroup H of A_s generated by the elements witnessing prime divisibility of x, y and $(x + y)$ that appear in the definition of $G(T)$. By the claim, the group H splits into $X \oplus Y$, with X being the pure subgroup of H generated by x ,

$$X = [x]_{G(T)}^*.$$

Let $U = G(T \setminus \{y\})$. We claim that $G(T) = Y \oplus U$. Embed $G(T)$ into its divisible hull. The basis of $G(T)$ consisting of the vertices of T is also a basis of the hull. In the hull, the generator y' of Y is a linear combination of x and y (with rational coefficients), say $y' = l_1x + l_2y$. If $z \in Y \cap U$ then we have

$$my' = ml_1x + ml_2y = \sum_v r_v v,$$

where v range over the vertices of $T \setminus \{y\}$. Since the vertices of T form a basis of $G(T)$ and y is not among the v , we must have $ml_2 = 0$. We arrive at

$$my' = ml_1x.$$

But y' is a generator of Y , and Y has only trivial intersection with the pure subgroup generated by x . Thus $l_1 = 0$ and $Y \cap U = \mathbf{0}$.

It remains to show that $Y + U = G(T)$. Recall the definitions of Y, X and H . Note that $X \subseteq U$ and also that all prime roots of x, y and $(x + y)$ that appear in the definition of $G(T)$ are contained already in $H = Y \oplus X \subseteq Y + U$. Furthermore, all other generators of $G(T)$ are induced by various divisibility conditions imposed on the vertices in $T \setminus \{y\}$ and thus are contained in U . Therefore $G(T) = Y + U$ and $G(T) = Y \oplus U$. □

According to the construction, the resurrected element (if it can be resurrected) must become a new leaf of the tree. Let this element be v , and w.l.o.g. assume that its predecessor v^- has already been resurrected. We argue that the resurrection of v is always successful, i.e. that nothing can block us from returning to the previously declared divisibility conditions upon v and the other elements, and that no new conditions were imposed on v while it was dispersed.

At stage s we have $G(T_s)[s] \oplus F_s$, with a direct summand of F_s containing the free partial X that was used to disperse v at stage t :

$$v \in T_t \text{ but } v \notin T_{t+1}.$$

Note that X could be further extended towards building the free abelian group of finite rank upon the same basic generators, but no further divisibility conditions on v could be imposed. In fact, no new nontrivial relation among the generators have been imposed in this extension. Also, in $G(T_s)$ we perhaps have introduced some further divisibility conditions towards building a larger piece of $G(T)$, and this was

done by introducing new generators but *not* by declaring any new relations among the already existing generators. These new generators witnessing the conditions cannot interfere with the elements in F_s (and U in particular) as they were taken in the direct complement of F_s . This means that we may adjoin v to the current T_s and then safely extend $G(T_s \cup \{v\})[s]$ naturally to a better approximation of $G(T_s \cup \{v\})$, just as it is done in the construction.

Now, if $v \notin T$, meaning that it was dispersed and will never be resurrected, we will end up building a free direct summand of the group that will forever disperse v (and its roots) without touching the rest of the generators. On the other hand, if v is resurrected infinitely often (which implies that all its predecessors in \mathbb{U}_0 are also like that) we will end up imposing the infinite divisibility conditions on the respective element of the group, as dictated by the label of v and its predecessor in \mathbb{U}_0 . Thus, we will end up constructing a group isomorphic to $G(T) \oplus \mathbb{F}$, as desired. We have completed the proof of Proposition 6.6 \square

Remark 6.16. The reader may notice that in the proof of Proposition 6.6 we actually do not have to do the “dispersing” explicitly. Instead, we could merely just stop adding divisibility conditions involving v if v leaves T . Nonetheless, we believe that the explicit dispersing/resurrecting makes the proof more intuitively approachable.

6.2. Finishing the proof of Theorem 6.1. Lemma 6.2 implies that there exists a uniform way of constructing $0''$ -computable trees T_i such that $T_i \cong S_{\alpha-1}$ iff $i \in H(\alpha)$ (and $C_i \cong P_{\alpha-1}$ otherwise). As we noted in Lemma 6.2, we may assume that the ω -names of the roots of C_i form a computable set. Now, Proposition 6.6 implies that we can uniformly transform these $0''$ -computable trees into computable groups of the desired isomorphism type. \square

7. NOT $\Delta_{\alpha-1}^0$ -CATEGORICAL.

Proposition 7.1. *The group $\mathcal{G}_\alpha = \bigoplus_{i \in \omega} S_{\alpha-1} \oplus \bigoplus_{i \in \omega} P_{\alpha-1} \oplus \bigoplus_{i \in \omega} \mathbb{Z}$ is not $\Delta_{\alpha-1}^0$ -categorical.*

Proof. Using Theorem 6.1, produce a sequence $(C_i)_{i \in \omega}$ such that $C_i \cong S_{\alpha-1} \oplus \mathbb{F}$ if $i \in H(\alpha)$ and $C_i \cong P_{\alpha-1} \oplus \mathbb{F}$ otherwise, where \mathbb{F} is the free abelian group of rank ω . Furthermore, we may assume that the roots of the $P_{\alpha-1}$ - and $S_{\alpha-1}$ -components of the C_i form a computable set. Without losing the uniform effectiveness, pass to the direct sum

$$\bigoplus_{i \in \omega} C_i \cong \mathcal{G}_\alpha,$$

in which the mentioned above roots form a computable set.

Recall that the natural computable copy \mathcal{A} of \mathcal{G}_α has an algorithm which decides the isomorphism type of all direct summands/components of \mathcal{A} . If ψ is an isomorphism from \mathcal{B} onto \mathcal{A} then ψ can decide $H(\alpha)$, as follows. For the i 'th root a_i (among the roots of the $P_{\alpha-1}$ and $S_{\alpha-1}$ -components in \mathcal{B}), compute $\psi(a_i)$. It follows that $\psi(a_i)$ must be a linear combination of roots of the $P_{\alpha-1}$ and $S_{\alpha-1}$ -components in \mathcal{A} . Indeed, the infinite divisibility conditions must be preserved at the respective levels (in particular, this immediately excludes the free summand from consideration). Say, $a_i = \sum_j m_j b_j$. Clearly, the m_j must be integers since ψ preserves divisibility conditions, and this property can be expressed using only infinite divisibility by primes (see Section 5).

We claim that a_i is the root of a $S_{\alpha-1}$ -component if and only if all of the b_j in $a_i = \sum_j m_j b_j$ are also roots of the $S_{\alpha-1}$ -component (in \mathcal{A}). Indeed, if a_i is the root of a $S_{\alpha-1}$ -component, then it must satisfy $\Phi_{\alpha-1}$, and thus its isomorphic image must satisfy the formula as well. But we know from Corollary 5.5 (Section 5) that $\mathcal{A} \models \Phi_{\alpha-1}(\sum_j m_j b_j)$ iff all b_j (with $m_i \neq 0$) are roots of $S_{\alpha-1}$ -components. Since we can decide the isomorphism types of components in \mathcal{A} , we see that the above procedure gives a ψ -computable way of deciding $H(\alpha)$. \square

8. BUILDING AN ISOMORPHISM

Theorem 8.1. *For every even $\alpha \geq 6$ the group $\mathcal{G}_\alpha = \bigoplus_{i \in \omega} S_{\alpha-1} \oplus \bigoplus_{i \in \omega} P_{\alpha-1} \oplus \bigoplus_{i \in \omega} \mathbb{Z}$ is relatively Δ_α^0 -categorical.*

Proof. The proof will be split into several parts. Throughout the proof, we fix a “good” copy of G_α (which we also denote \mathcal{G}_α). In \mathcal{G}_α the structural forest $T(\mathcal{G}_\alpha)$ and the isomorphism types of all building blocks are computable, and the free summand has a computable generating, see Fact 4.5.

Definition 8.2. An element $a \in \mathcal{G}_\alpha$ is *special* if it is either a node of $T(\mathcal{G}_\alpha)$ or is an element of the free summand of \mathcal{G}_α taken from the computable linearly independent set of its generators. A tuple (finite or infinite) $\bar{g} \in \mathcal{G}_\alpha$ is *special* if it consists of distinct special elements. Clearly, any special tuple consists of linearly independent elements. The *standard special basis* is a basis of \mathcal{G} that consists of all special elements, and a *special basis* is any automorphic image of the standard one.

We describe the general plan of the proof. Given a finite tuple \bar{g} of special elements in the nice copy \mathcal{G} , we will uniformly produce a $\mathcal{L}_{\omega_1\omega}^c$ formula $\Theta_{\bar{g}}$ (\mathcal{L} is the language of additive groups) of complexity at most Σ_α^c that describes the automorphism orbit of the tuple. Roughly speaking, if we succeed in defining $\Theta_{\bar{g}}$, then it will imply that G_α is (relatively) Δ_α^0 -categorical. Indeed, we will argue shortly that using $\Theta_{\bar{g}}$ we can build a maximal linearly independent set B' that is automorphic to the special basis B of \mathcal{G}_α , and then we can extend the natural matching of B and B' to an automorphism of \mathcal{G}_α . We will give a more detailed explanation in due course.

To simplify the rather involved (but natural) description of $\Theta_{\bar{g}}$, we need some notation. Recall that a vertex element is an integer linear combination of actual nodes coming from the structural forest of \mathcal{G}_α .

Notation 8.3. We will later show that the following groups are either $\mathcal{L}_{\omega_1\omega}^c$ -definable in \mathcal{G}_α , or are factors of \mathcal{G}_α by a $\mathcal{L}_{\omega_1\omega}^c$ -definable direct summand of \mathcal{G}_α . At this stage we need only their names, the complexity analysis will be given later.

- (1) V stands for the free subgroup of \mathcal{G}_α generated by all vertex elements.
- (2) $R \leq V$ is generated by the roots of $S_{\alpha-1}$ - and $P_{\alpha-1}$ -subcomponents.
- (3) $S \leq R$ is generated by the roots of $S_{\alpha-1}$ -subcomponents.

Before we proceed, note that $\mathbb{F} \cong \mathcal{G}_\alpha/[V]_{\mathcal{G}_\alpha}^*$, i.e. the free summand of \mathcal{G}_α is isomorphic to the factor of \mathcal{G}_α by the pure closure of V . Indeed, $[V]_{\mathcal{G}_\alpha}^* = \bigoplus_{i \in \omega} S_{\alpha-1} \oplus \bigoplus_{i \in \omega} P_{\alpha-1}$.

8.1. Defining orbits. Let $\bar{g} = (g_1, g_2, \dots, g_k)$ be a non-empty special tuple in \mathcal{G}_α .

Convention 8.4. For our purposes it will be sufficient to assume that \bar{g} is non-empty and closed upwards under the successor relation, i.e., such that each g_i in

\bar{g} has a predecessor g_j in \bar{g} ($j < i$) unless g_i is the root of a $S_{\alpha-1}$ - or $P_{\alpha-1}$ -subcomponent or comes from the free summand. Indeed, knowing orbits of such tuples will be sufficient to run a back-and-forth procedure (to be explained).

Notation 8.5. We assume that vertex elements in $\bar{g} = (g_1, \dots, g_k)$ are listed level-by-level and in the order of decreasing complexity of the subcomponents whose roots are the g_i . More specifically,

$$g_1, \dots, g_{l_1}$$

are roots of some $S_{\alpha-1}$ - and $P_{\alpha-1}$ -components of \mathcal{G}_α , and then

$$g_{l_1+1}, \dots, g_{l_2}$$

are roots of various $S_\beta(k)$ - and P_β -subcomponents for the largest $\gamma_2 < \alpha - 1$ that appears among the components rooted in g_i , etc. Finally, when all vertices are exhausted after $d-1$ steps (and thus l_{d-1} has been defined), we assume that

$$g_{l_{d-1}+1}, \dots, g_{l_d}$$

(thus $g_{l_d} = g_k$) are elements of the fixed computable generating set of the free abelian summand \mathbb{F} of \mathcal{G}_α .

In the definition below, the successor relation is understood as the relation between vertex-like elements, see Subsection 5.1. Recall also that \mathcal{P} denotes the set of all primes.

Definition 8.6 (Definition of $\Theta_{\bar{g}}$). Let \mathcal{P} be the set of all primes. Fix a special tuple \bar{g} in \mathcal{G} . Let $\Theta_{\bar{g}}(\bar{x})$ be the infinitary computable formula (whose complexity will be analyzed later) that is a conjunction of 7 conditions o1-o7:

- o1. The elements $x_1, \dots, x_{l_{d-1}}$ are vertex-like elements, with the same prime labels as the respective $g_1, \dots, g_{l_{d-1}}$.
- o2. The successor relation among $x_1, \dots, x_{l_{d-1}}$ is the same as the successor relation among $g_1, \dots, g_{l_{d-1}}$, and with the same primes witnessing the relation.
- o3. The elements $x_{l_{d-1}+1}, \dots, x_{l_d}$ are \mathcal{P} -independent within $\mathcal{G}_\alpha/[V]_{\mathcal{G}_\alpha}^*$.
- o4. If g_{m_1}, \dots, g_{m_k} are roots of $S_{\alpha-1}$ -subcomponents, then $x_{m_1}, \dots, x_{m_k} \in S$ are \mathcal{P} -independent within S (see Notation 8.3).
- o5. If g_{m_1}, \dots, g_{m_k} are roots of $P_{\alpha-1}$ -subcomponents, then $x_{m_1}, \dots, x_{m_k} \in R$ are \mathcal{P} -independent within R/S .
- o6. Suppose g_{m_1}, \dots, g_{m_k} are roots of P_γ -subcomponents ($\gamma < \alpha - 1$) all labelled by the same primes. Then x_{m_1}, \dots, x_{m_k} are integer linear combinations of the roots of P_γ -subcomponents with the same prime labelling.
- o7. Suppose g_{m_1}, \dots, g_{m_k} are roots of $S_\gamma[m]$ -subcomponents ($\gamma < \alpha - 1$, and m is fixed) all labelled by the same primes. Let C_n be the subgroup of \mathcal{G}_α generated by the roots of P_γ - and $S_\gamma[k]$ -subcomponents, $k \geq n$, with the same choice of prime labels. Then the formula says that $x_{m_1}, \dots, x_{m_k} \in C_n$ and x_{m_1}, \dots, x_{m_k} are \mathcal{P} -independent in C_m/C_{m+1} .

Remark 8.7. The first two conditions o1.-o2. come from the most straightforward attempt to express the graph-theoretic structure on \bar{g} (in $T(\mathcal{G}_\alpha)$) using only infinite divisibility by primes. Conditions o3-o5 exploit the most straightforward analogy with the case of free abelian groups where \mathcal{P} -independence fully describes automorphism orbits of generating sets (Section 2.1). Note all groups involved in o3-o5 are free abelian, so \mathcal{P} -independence without any extra assumptions is again the most straightforward property we may hope for. In o3 and o5 are stated in factor-groups since (say) the direct complement of R in S is not stable under automorphisms of \mathcal{G}_α , and similarly for o3. Conditions o6-o7 generalize o4-o5 to deeper levels of $T(\mathcal{G}_\alpha)$ in the most straightforward

way. Again, all the groups mentioned in $o6 - o7$ are free, thus \mathcal{P} -independence (combined with the right choice of prime labels and Theorem 5.1) is the simplest property we can hope for. Note in $o6$ we can even suppress \mathcal{P} -independence (this was not a typo) for reasons that will be revealed in the verification.

We will illustrate that property $\Theta_{\bar{g}}$ (which is the conjunction of $o1 - o7$) describes the automorphism orbit of \bar{g} . Then we will show that $\Theta_{\bar{g}}$ can be expressed as an infinitary computable formula of complexity at most Σ_α^c , with all possible uniformity. But first of all we give a detailed explanation of why having such Σ_α^c -formulae is sufficient to prove the theorem.

8.2. Back-and-forth using $\{\Theta_{\bar{g}}\}$. Suppose we already know that the $\Theta_{\bar{g}}$ have the desired properties (i.e. are uniformly Σ_α^c and describe orbits of \bar{g}). We explain how we can use this c.e. set of Σ_α^c -formulae to build an isomorphism from \mathcal{G}_α to any other copy \mathcal{A} of \mathcal{G}_α . We explain a back-and-forth procedure first, and then we illustrate that it actually produces a Δ_α^0 -isomorphism from \mathcal{G}_α to \mathcal{A} (the latter will not be totally obvious).

We list finite nested initial segments $\bar{g}_0, \bar{g}_1, \dots$ of the computable special basis in \mathcal{G}_α , and also list elements of $\mathcal{G}_\alpha = \{d_0, d_1, \dots\}$.

Forth. At stage $2i$ we search for a tuple \bar{g}_{2i} in the special basis of \mathcal{G}_α that extends \bar{g}_{2i-1} and such that

$$d_i = \sum_j r_j g_j,$$

where $r_j \in \mathbb{Q}$ and g_j range over basic elements in \bar{g}_{2i} . Such \bar{g}_{2i} and r_j must exist because the special basis is a maximal linearly independent set of \mathcal{G}_α . Once we have found such a \bar{g}_{2i} , we look for a tuple \bar{h}_{2i} in \mathcal{A} extending \bar{h}_{2i-1} such that $\mathcal{A} \models \Theta_{\bar{g}_{2i}}(\bar{h}_{2i})$, and we also search for an element $a \in \mathcal{A}$ such that

$$a = \sum_j r_j h_j,$$

where h_j range over elements on \bar{h}_{2i} , and r_j are the same as above. We then extend our isomorphism to map d_i to a and \bar{g}_{2i} to \bar{h}_{2i} . Note that the properties of $\theta_{\bar{g}_{2i}}$ ensure that such an a and \bar{h}_{2i} can always be found.

Back. To ensure the map is onto, at stage $2i + 1$ we also pick the i 'th element a_i of \mathcal{A} and search for an extension \bar{h}_{2i+1} of \bar{h}_{2i} that satisfies some formula $\Theta_{\bar{g}_{2i+1}}$ for some special extension \bar{g}_{2i+1} of the tuple \bar{g}_{2i} , and such that

$$a_i = \sum_j r_j h_j,$$

where r_j are rational and h_j range over elements in \bar{h}_{2i+1} . We then fix a $g \in \mathcal{G}_\alpha$ such that

$$g = \sum_j r_j g_j,$$

with g_j ranging over \bar{g}_{2i+1} and extend the isomorphism to map g to a_i and \bar{g}_{2i+1} to \bar{h}_{2i+1} .

The map ϕ constructed by the back-and-forth procedure above is indeed $\Delta_\alpha^0(\mathcal{A})$, since by our assumption the family $\{\Theta_{\bar{g}}\}$ is a c.e. family of Σ_α^c -formulae (with no parameters). The map is also total and onto since the properties of Θ will ensure we are never stuck in our definition. Also, we claim that ϕ is a homomorphism.

Indeed, suppose we have defined $\phi(g)$ and $\phi(d)$, we claim $\phi(d + g) = \phi(d) + \phi(g)$. Let \bar{g}_k be the initial segment of the special basis of \mathcal{G}_α that spans both d and g (allowing rational coefficients). Then both g and d can be uniquely expressed as reduced linear combinations of the finitely many elements in \bar{g}_k , and the definition of ϕ ensures that $\phi(g)$ and $\phi(d)$ correspond to the same linear combinations but over $\phi(\bar{g}_k) = \bar{h}_k$. Linear independence of both \bar{g}_k and \bar{h}_k implies that $\phi(d + g)$ and $\phi(d) + \phi(g)$ will result in identical linear combinations of \bar{h}_k .

Finally, ϕ is injective since it maps the special basis B to the linearly independent set $\phi(B)$. (To see why $\phi(B)$ is linearly independent, note that linear independence is a local property, and if it fails then it must fail for some initial segment of $\phi(B)$, contradicting the properties of Θ . To see that $\text{Ker } \phi = 0$, suppose $0 = \phi(\sum_i r_i g_i) = \sum_i r_i \phi(g_i)$ which implies $r_i = 0$ for all i and thus $\sum_i r_i g_i = 0$.)

Therefore, *assuming that $\Theta_{\bar{g}}$ capture the orbits of the respective \bar{g} and are of complexity Σ_α^c* , we have proved Theorem 8.1. These properties of $\Theta_{\bar{g}}$ are verified in the next two sections. \square

9. $\Theta_{\bar{g}}$ DESCRIBES THE ORBIT OF \bar{g} .

First, we observe that any special tuple $\bar{g} = (g_1, \dots, g_k)$ itself satisfies $\Theta_{\bar{g}}$. Indeed, note that *o1* – *o2* hold trivially, and the \mathcal{P} -independence conditions from *o3* – *o7* also hold by the definition of \mathcal{G}_α .

Now assume $\Theta_{\bar{g}}(\bar{x})$ holds. We will construct an automorphism ψ of \mathcal{G} that takes \bar{g} to \bar{x} . The constructive definition of ψ will be split into finitely many phases. First, we shall work with the elements x_1, \dots, x_{l_1} and prove that g_1, \dots, g_{l_1} are automorphic to x_1, \dots, x_{l_1} . We then look at $g_{l_1+1}, \dots, g_{l_2}$, etc. We will make sure that the *i*'th phase defines an automorphism that does not move $x_1, \dots, x_{l_{i-1}}$. As we will further clarify, this will allow us to take the composition of all these automorphisms to get the desired automorphism taking \bar{g} to \bar{x} .

9.1. Phaze 1. We argue that (*o1*) – (*o7*) imply that (g_1, \dots, g_{l_1}) is automorphic to (x_1, \dots, x_{l_1}) . Recall that, by our convention, g_1, \dots, g_{l_1} are roots of $S_{\alpha-1}$ - and $P_{\alpha-1}$ -subcomponents. Without loss of generality, let the first s among them come from different $S_{\alpha-1}$ -subcomponents, and the rest from $P_{\alpha-1}$ -subcomponents. Then (*o4*) implies that x_1, \dots, x_s can be extended to a generating basis B'_1 of S (Section 2.1), while (*o5*) guarantees that x_{s+1}, \dots, x_{l_1} can be extended to a generating basis B'_2 of R/S . We put these bases together to get a linearly independent generating set of R . Let B_1 be the “standard” generating basis of S consisting of special elements, and let B_2 be the “standard” generating basis of R/S consisting of S -cosets of special elements. To see why Phaze 1 gives an automorphism, it is sufficient to prove:

Lemma 9.1. *Let*

$$\psi : B_1 \cup B_2 \rightarrow B'_1 \cup B'_2$$

be any bijection such that $\psi(B_1) = B'_1$ and $\psi(B_2) = B'_2$. Then ψ can be extended to an automorphism of \mathcal{G}_α .

Proof of Lemma 9.1. The proof is quite elementary but somewhat tedious. We introduce the notion of a *natural successor* that will be also useful later in the

paper. Let $v_0, \dots, v_k \in \mathcal{G}_\alpha$ be vertices of $T(\mathcal{G}_\alpha)$ and let

$$g = \sum_i m_i v_i$$

be a vertex-like element generated by v_0, \dots, v_k ($m_i \in \mathbb{Z}$). We say that a vertex-like element h is a *natural successor* of g if

$$h = \sum_i m_i w_i,$$

where w_0, \dots, w_k are vertices of $T(\mathcal{G}_\alpha)$ that are successors of v_0, \dots, v_k , respectively (in $T(\mathcal{G}_\alpha)$). Note that natural successors of $T(\mathcal{G}_\alpha)$ -vertices are exactly their successors in $T(\mathcal{G}_\alpha)$.

Informally, we simply extend ψ to natural successors carefully enough (i.e., respecting the isomorphic types of subcomponents) and observe that the resulting map induces an automorphism of \mathcal{G}_α . Such an extension is possible since $T(S_{\alpha-1})$ contains $T(P_{\alpha-1})$ as a subtree. We details below.

By the choice of B'_1, B'_2 , the map ψ can be (uniquely) extended to an automorphism of R that furthermore also has the property $\psi(S) = S$. Since the free \mathbb{F} is a direct summand in \mathcal{G}_α , we can set ψ to be identical on \mathbb{F} . First, we extend ψ to the whole summand of \mathcal{G}_α consisting of $S_{\alpha-1}$ -subgroups. To do that, introduce a numbering the natural successors of every root of $S_{\alpha-1}$. Define a numbering of vertices below the root of every $S_{\alpha-1}$ -component so that numberings between distinct roots are isomorphic, i.e. so that the natural successors with the same index are themselves roots of isomorphic subcomponents. Now suppose x is a root of a $S_{\alpha-1}$ -component, and let z be the i 'th successor of the vertex x in $T(\mathcal{G}_\alpha)$, according to the fixed numbering. Suppose $\psi(x) = \sum_j m_j y_j$. Note each y_j must itself be the root of a $S_{\alpha-1}$ -summand. Choose the i 'th successor z_j below each of the y_j and define $\psi(z) = \sum_j m_j y_j$.

Once ψ has been defined on all immediate successors of $S_{\alpha-1}$ -roots, we may repeat the procedure but now using a natural numbering of the immediate successors of the successors, etc. It is now fairly routine to check that this inductive level-by-level definition induces a unique extension to the whole group $[S]_{\mathcal{G}_\alpha}^* = \bigoplus_{i \in \omega} S_{\alpha-1}$, and it is also quite obvious that this extension will be an automorphism of $[S]_{\mathcal{G}_\alpha}^*$.

Remark 9.2. We perhaps owe the reader a sketch. The divisibility conditions upon vertices of $T([S]_{\mathcal{G}_\alpha}^*)$ are naturally preserved under ψ , thus ψ can be extended to the whole $[S]_{\mathcal{G}_\alpha}^*$. Also, B'_1 is a generating basis of S , and thus the natural successors chosen this way will generate their respective free subgroups which are generated by vertices at the respective level of $T([S]_{\mathcal{G}_\alpha}^*)$. For the same reason any generator of $[S]_{\mathcal{G}_\alpha}^*$ has a ψ pre-image, and thus ψ is onto. It is 1-1 since it maps a basis into a basis. We believe that the reader should have no problem reconstructing these details. Furthermore, in the next phase (see 9.2.2. Subphase 2) we will have a much more intricate proof of this sort which will be explained in full detail.

Extending ψ further requires a bit more care. Recall B'_2 consists of elements that generate R/S , i.e., generate the roots of $P_{\alpha-1}$ but perhaps only modulo the roots of $S_{\alpha-1}$. Thus, for x the root of a $P_{\alpha-1}$ -component, we will have $\psi(x) = \sum_j m_j y_j + \sum_k n_k u_k$, where y_j are roots of $P_{\alpha-1}$ -components and u_k are roots of $S_{\alpha-1}$ -components. But this is not a problem though. Again, introduce a canonical numbering of the nodes in $P_{\alpha-1}$ and recall that $T(S_{\alpha-1})$ contains an isomorphic copy of $T(P_{\alpha-1})$ in it. (Recall $S_{\alpha-1}$ has a $P_{\alpha-2}$ -subcomponent but $P_{\alpha-1}$ does not, and this was the only difference.) Thus we can index only those nodes of $T(S_{\alpha-1})$ that are involved in the isomorphic copy of $T(P_{\alpha-1})$ within $T(S_{\alpha-1})$.

Now using this new numbering of the successors of y_j and u_k , and then successors of their successors etc., extend ψ as it was done for $[S]_{\mathcal{G}_\alpha}^*$, and then extend ψ naturally to any linear combinations (with rational coefficients). It is again fairly easy to see that this extension gives an automorphism of \mathcal{G}_α . Indeed, by the choice of B'_1 and B'_2 and by the new extended definition of ψ , we have that ψ maps a basis of \mathcal{G}_α to a basis of \mathcal{G}_α . Since the divisibility conditions upon vertices of $T(\mathcal{G}_\alpha)$ are respected, any linear combination of special elements in \mathcal{G}_α with rational coefficients can be safely sent to the same linear combination of the respective images. It remains only to argue that ψ is onto. But the natural successors below u_k (and their successors etc.) can be generated already in $[S]_{\mathcal{G}_\alpha}^*$, and thus their rational linear combinations are in the range as well. But then we are left only with rational combinations of nodes within the $P_{\alpha-1}$ -components. By the choice of the coefficients, the assumption on B'_2 , and the choice of natural successors in the definition of ψ , we have that the direct complement of $[S]_{\mathcal{G}_\alpha}^*$ can be generated as well. \square

9.2. Phaze t , $1 < t < d$. We now get to $(x_{l_{t-1}+1}, \dots, x_{l_t})$. Form Phazes $z < t$, we may assume that $(g_1, \dots, g_{l_{t-1}})$ is automorphic to $(x_1, \dots, x_{l_{t-1}})$. Thus, without loss of generality, we may assume that $(x_1, \dots, x_{l_{t-1}})$ is a special tuple, having in mind some special basis of \mathcal{G} whose initial segment is $(x_1, \dots, x_{l_{t-1}})$. We also clearly have, by induction, that for each $i \leq l_{t-1}$ the element x_i is a root of the same type of a components and with the same choice of prime labels as the corresponding g_i .

Recall that the tuple \bar{g} was closed upwards under the predecessor relation, thus it is necessary that each node among the $(g_{l_{t-1}+1}, \dots, g_{l_t})$ has a predecessor among $(g_1, \dots, g_{l_{t-1}})$. The phase is split into finitely many further subphazes.

9.2.1. Subphaze 1. First, assume without loss of generality that the first s elements among the $(g_{l_{t-1}+1}, \dots, g_{l_t})$ are roots of P_γ subcomponents for some fixed γ and with a fixed choice of labelling (recall we have different choices of prime labellings, depending on the location of a component). Let g_{m_1}, \dots, g_{m_s} be the predecessors of $g_{l_{t-1}+1}, \dots, g_{m_s}$, respectively. Recall that $x_{l_1}, \dots, x_{l_{t-1}}$ can be assumed equal to elements of some special basis, and furthermore they are roots of the same kind of components as the respective g_i are, $i \leq l_{t-1}$. Recall that in (o1)-(o2) we used infinite divisibility to express successor/predecessor relation and prime labelling. Also, (o6) says that $x_{l_{t-1}+1}, \dots, x_s$ are integer linear combinations of P_γ -components, with the fixed above labelling. But each of the x_{m_1}, \dots, x_{m_s} must have at most one successor of this kind, and therefore Lemma 5.2 implies each of the $x_{l_{t-1}+1}, \dots, x_s$ is indeed special (i.e, a vertex) under the same automorphism that makes the respective x_{m_1}, \dots, x_{m_s} special. Furthermore, since g_{m_1}, \dots, g_{m_s} were distinct and thus were coming from different subcomponents, we must have no repetition in the sequence g_{m_1}, \dots, g_{m_s} and thus in x_{m_1}, \dots, x_{m_s} as well. Condition (o6) guarantees that these special elements are roots of P_γ -components carrying the right labels. Since x_{m_1}, \dots, x_{m_s} come from distinct components, then $x_{l_{t-1}+1}, \dots, x_s$ also come from different components and represent the unique predecessors of type P_γ (with the fixed labelling) of the respective x_{m_i} . Thus $x_{l_{t-1}+1}, \dots, x_s$ are in particular \mathcal{P} -independent in the free subgroup of \mathcal{G}_α generated by all roots of P_γ (with the

fixed above labelling)². In fact, the same automorphism that maps $(g_1, \dots, g_{l_{t-1}})$ to $(x_1, \dots, x_{l_{t-1}})$ will also map $(g_{l_{t-1}+1}, \dots, g_s)$ to the special $(x_{l_{t-1}+1}, \dots, x_s)$.

Now repeat the instructions above for other choices of the labelling (but for the same P_γ -type component), until all roots of the P_γ -type components among $(g_{l_{t-1}+1}, \dots, g_t)$ are exhausted. This finishes Subphase 1.

Remark 9.3. Note we have not really done much in Subphase 1, all the job was done at the previous phases. At this point it should also become clear why we did not use \mathcal{P} -independence in condition o6.

9.2.2. *Subphase 2.* We assume that no P_γ -components are left among $g_s \dots, g_t$. We fix some m and some labelling of $S_\gamma[m]$ -type components so that some of the $g_s \dots, g_t$ are roots of such components. We assume that m is largest such number. Suppose g_{m_s}, \dots, g_r are roots of such components, for a fixed labelling. Condition (o7) says that x_s, \dots, x_r are \mathcal{P} -independent in the free group C_m/C_{m+1} , where C_m is generated by vertices that are roots of $S_\gamma[k]$ -type subcomponents with $k \geq m$ and of P_γ -type subcomponents, all having the fixed above labelling type.

Conventions. We adopt two conventions that will significantly simplify notation. Let x_{n_i} be the predecessor of x_i , $s \leq i \leq r$ (perhaps $n_i = n_j$ for some $i \neq j$). Recall that x_{n_i} appears sufficiently early in the list x_1, x_2, \dots , and thus (up to an automorphism) x_{n_i} is special. We claim that, *having in mind the automorphism of \mathcal{G} constructed above, we may identify x_i and g_i if $i < s$.* In particular, we assume $g_{n_i} = x_{n_i}$ for $s \leq i \leq r$ (up to the automorphism).

First, we explain what may go wrong. The automorphism that was used to identify g_i and x_i for small i may move g_j to a non-vertex for some $j > s$ (but the image will be vertex-like). Nonetheless, further adjusting the automorphism *we may assume that for any $j > s$ the element g_j is mapped to a vertex.* Indeed, we may assume that x_j ($j < s$) are vertices of some other structural forest of \mathcal{G}_α which is given by the automorphism constructed above. Now, given $s \leq j \leq r$, choose some vertex a_j below the predecessor x_{m_j} of x_j that is also the root of the same type $S_\gamma[m]$ -component (as g_j is), and map g_j to this a_j . Make sure $a_j \neq a_i$ if $i \neq j$. Then repeat this process for successors of such g_j , etc. The resulting map can be clearly extended to an automorphism of \mathcal{G}_α in the most straightforward way (just match the respective nodes in the two structural trees, the standard one and the one in which x_j are nodes). Furthermore, this automorphism still maps g_j to x_j for $j < s$. Thus, having in mind this further adjusted automorphism, we may identify g_j with x_j ($j < s$) and g_i ($i \geq s$) with their respective images a_i under this automorphism. *Note the a_i are nodes.* Now our task is to show that the automorphism can be further adjusted to map the g_i (aka a_i) to the respective x_i , $s \leq i \leq r$, where the latter are not necessarily nodes.

We now return to Subphase 2. Each element x_i , $s \leq i \leq r$, will be an integer linear combination of roots of P_γ -type and $S_\gamma[k]$ -type components (with $k \geq m$)

²As we noted in the discussion after (o6), we did not have to include \mathcal{P} -independence in (o6).

having the right labelling:

$$(3) \quad x_i = \sum_j m_{ij} u_j.$$

It will be convenient to assume that all the x_i share the same finite collection of roots u_j of subcomponents that they are spanned by, and we do so by allowing some of the m_{ij} to be null. For each i at least one root among the special u_j with $m_{ij} \neq 0$ is on top of a $S_\gamma[m]$ -type subcomponent. Indeed, otherwise the element would've been zero in the factor-group. Fix some special successor v_{n_i} of g_{n_i} (the predecessor of x_i , see the convention above) that is the root of a $S_\gamma[k]$ -component (with the same labelling as above) with k very large. In particular, there are no roots of $S_\gamma[k]$ -components mentioned among the u_j (as defined above) in any of the $x_i = \sum_j m_{ij} u_j$. Let q be the prime that labels the edge between g_{n_i} and v_i (all $s \leq i \leq r$ will share the same prime with this property; the same prime also labels the edge (g_{n_i}, g_i)). Then $g_{n_i} + x_i - (g_{n_i} + v_{n_i})$ is infinitely divisible by q , i.e.

$$q^\infty | (x_i - v_{n_i}),$$

for each i , $s \leq i \leq r$. Let U be the subgroup of \mathcal{G} generated by all u_j mentioned in (3) that are roots of $S_\gamma[m]$ -components. Extending U if necessary to a larger free group of finite rank, we may assume that $g_s, \dots, g_r \in U$ (see the convention above). Pick a finite sequence $w_1, \dots, w_z \in U$ that spans, together with the elements x_j and modulo C_{m+1} , the group U . Such an extension exists since the x_j are \mathcal{P} -independent modulo C_{m+1} . Adding/subtracting an appropriate linear combination of various v_{n_i} , replace w_1, \dots, w_z by elements c_1, \dots, c_z in the same C_{m+1} -congruence classes so that

$$q^\infty | c_j, \quad j = 1, \dots, z.$$

Recall that we assumed $g_s, \dots, g_r \in U$, and by the convention these elements are vertices of $T(\mathcal{G}_\alpha)$. Let g'_1, \dots, g'_z be the roots of various $S_\gamma[m]$ -subcomponents that, together with g_s, \dots, g_r , form a generating basis of U (not merely mod C_{m+1}). Such an extension exists since g_s, \dots, g_r are distinct vertices that are roots of $S_\gamma[m]$ (as we discussed above). Let also h_{t_i} be the predecessor of g'_i . Note that not all of the h_i have to be among g_1, \dots, g_{s-1} , but some perhaps are. Also, we may have $t_i = t_j$ for some $i \neq j$. Fix also some successor v'_{t_i} of h_{t_i} which is a root of $S_\gamma[k]$ with k the same (large) as in the definition of v_{n_i} . Furthermore, if $h_{t_i} = g_{n_j}$ for some i, j then we require $v'_{t_i} = v_{n_j}$.

We are ready to make the first steps towards the definition of an automorphism ψ . Let

- i. $\psi(g_i) = x_i$, $s \leq i \leq r$ (as desired);
- ii. $\psi(g'_j) = c_j + v'_{t_j}$, $j = 1, \dots, z$.

We also set ψ to be the identity on all the other vertices we defined in the previous paragraph, in particular

- iii. $\psi(g_j) = g_j$, $j < s$;
- iv. $\psi(v_{n_i}) = v_{n_i}$, $i = s, \dots, r$;
- v. $\psi(v'_{t_j}) = v'_{t_j}$, $j = 1, \dots, z$;
- vi. $\psi(h_{t_i}) = h_{t_i}$, $j = 1, \dots, z$.
- vii. $\psi(u) = u$ for any other $u \notin U$ mentioned in the decompositions of x_i in (3).

Claim 9.4. ψ respects the successor relation on its domain.

Proof. In other words, it is sufficient to show that $\psi(g_i)$ is the successor of $\psi(g_{n_i})$ and $\psi(g'_j)$ is the successor of $\psi(h_{t_j})$, for any choice of i, j . The former follows immediately from (o2). For the latter, note that $q^\infty|c_j$ and $q^\infty|(h_{t_j} + v'_{t_j})$ ($j = 1, \dots, z$). Thus, $q^\infty|(h_{t_j} + v'_{t_j} + c_j)$, and by ii. we have $q^\infty|(h_{t_j} + \psi(g'_j))$. Finally, vi. implies $q^\infty|(\psi(h_{t_j}) + \psi(g'_j))$, as desired. \square

Let U' be the free group generated by all the u_j involved in the expressions for x_i (see (3)) and also by the auxiliary v_{n_i}, v'_{t_j} ($i = s, \dots, r$ and $j = 1, \dots, z$). We have that $U \cong U'$ and in fact U detaches in U' . Since ψ injectively maps a generating basis of U' to another generating basis of U' , it (uniquely) induces an automorphism of the free group U' that we also denote ψ .

Now let H be the least pure subgroup of \mathcal{G}_α that is generated by U' and also $g_1, \dots, g_{s-1}, h_{n_1}, \dots, h_{n_z}$,

$$H = [U' + \langle g_1, \dots, g_{s-1}, h_{n_1}, \dots, h_{n_z} \rangle]_{\mathcal{G}_\alpha}^*$$

The generators of H are of the form $\frac{a}{\tilde{p}^n}$ and $\frac{a+b}{\tilde{q}^m}$, where a, b are vertices of $T(\mathcal{G}_\alpha)$ that are mentioned in U' or are among $g_1, \dots, g_{s-1}, h_{n_1}, \dots, h_{n_z}$, and \tilde{p}, \tilde{q} are primes.

Claim 9.5. ψ (uniquely) induces an injective self-homeomorphism of H that fixes g_1, \dots, g_{s-1} .

Proof. Recall that ψ is already defined on the basis of H consisting of vertices of $T(\mathcal{G}_\alpha)$ (including in particular $g_1, \dots, g_r, h_{n_1}, \dots, h_{n_z}$; note this sequence may have repetitions). To extend ψ onto the whole H , pick any linear combination with rational coefficients $\sum_i r_i a_i \in H$ of such vertices and set

$$\psi\left(\sum_i r_i a_i\right) = \sum_i r_i \psi(v_i).$$

First, we need to argue that this definition makes sense, i.e., that the element of the form $\sum_i r_i \psi(v_i)$ exists in H . Recall that the generators of H are of the form $\frac{a}{\tilde{p}^n}$ and $\frac{a+b}{\tilde{q}^m}$, where a, b are vertices of $T(\mathcal{G}_\alpha)$, \tilde{p}, \tilde{q} are primes, and m a positive integer. But ψ was chosen so that it respects infinite divisibility by the primes that label vertices in $T(\mathcal{G}_\alpha)$, and by Claim 9.4 ψ also respects infinite divisibility by the primes used to label edges in $T(\mathcal{G}_\alpha)$. Thus, every generator of the form $\frac{a}{\tilde{p}^n}$ can be

safely mapped to an element of the form $\frac{\psi(a)}{\tilde{p}^n}$, and similarly we may map $\frac{a+b}{\tilde{q}^m}$ to $\frac{\psi(a) + \psi(b)}{\tilde{q}^m}$. Since in H every linear combination of the form $\sum_i r_i a_i$ can be

re-expressed using generators $\frac{\psi(a)}{\tilde{p}^n}$ and $\frac{a+b}{\tilde{q}^m}$, we conclude that the definition of ψ is sound. We now argue that ψ is an injective self-homeomorphism of H . So that it is a homeomorphism is obvious. It is injective since it (homeomorphically) maps a maximal linearly independent set of H to another maximal linearly independent set of H . A standard argument shows that this property implies injectivity (see e.g. the last paragraph of Section 8). We also observe that, for the same reason, the suggested extension of ψ is indeed the only possible extension of ψ to an injective self-homeomorphism of H . \square

We identify ψ with the injective endomorphism of H given by Claim 9.5.

Claim 9.6. ψ is an automorphism of H .

Proof. It is sufficient to argue that all generators of H are in the range of ψ . Note that, according to its definition, ψ respects infinite divisibility by the primes used to label vertices of $T(\mathcal{G}_\alpha)$. Since ψ is an automorphism of U , every generator of the form $\frac{a}{\tilde{p}^n}$ will be in the range of ψ (where \tilde{p} was used to label vertices of $T(\mathcal{G}_\alpha)$).

As we already observed above, ψ is an automorphism of U , and it does not move vertex elements mentioned in H if these vertices are outside U . Therefore, it will be sufficient to show that for every vertex element u in U and its predecessor z_u , the elements $(u + z_u)/q^k$ have ψ -pre-images for all k . Indeed, all other generators of the form $(a + b)/q^k$ will stay fixed under ψ .

Recall $x_s, \dots, x_r, c_1 + v'_{t_1}, \dots, c_z + v'_{t_z}$ form a generating basis of the free U/C_{m+1} (modulo C_{m+1}). They are also successors of $g_{n_1}, \dots, g_{n_s}, h_{t_1}, \dots, h_{t_z}$, respectively (the latter sequence may have repetitions). Recall also $q^\infty | (x_i - v_{n_i})$ and $q^\infty | c_j$, and ψ is identity on $g_{n_i}, v_{n_j}, h_{t_j}, v'_{t_j}$.

Fix $s \leq i \leq r$, and consider the element $x_i - v_{n_i}$. Note that, by our assumption $x_{n_j} = g_{n_j}$ up to some fixed automorphism, and thus we may assume g_i, v_{n_i} and x_i share the same predecessor g_{n_i} . Therefore $q^\infty | (x_i - v_{n_i})$ (consider $x_i + g_{n_i} - (v_{n_i} + g_{n_i})$). Also, according to the definition of ψ ,

$$\psi(g_i - v_{n_i}) = x_i - v_{n_i},$$

and thus for every positive integer d

$$\psi^{-1} \left(\frac{x_i - v_{n_i}}{q^d} \right) = \frac{g_i - v_{n_i}}{q^d}$$

is well-defined. Similarly, $\psi(g'_j) = c_j + v'_{t_j}$ and $\psi(v'_{t_j}) = v'_{t_j}$ imply that $\psi(g'_j - v'_{t_j}) = c_j$, and since $g'_j - v'_{t_j}$ is again infinitely divisible by q (as both g'_j and v'_{t_j} share the same predecessor h_{t_j}), we have that

$$\psi^{-1} \left(\frac{c_j}{q^d} \right) = \frac{g'_j - v'_{t_j}}{q^d}$$

is well-defined for every j and any positive integer d .

Recall also that $x_s, \dots, x_r, c_1, \dots, c_z$ form a generating basis of U/C_{m+1} , and also that $v_{n_j} \in C_{m+1}$ since we chose v_{n_j} to be the root of $S_\gamma[k]$ with k very large. Thus, taking integer linear combinations of the (q -infinitely divisible) $x_i - v_{n_i}$ and c_j we may obtain, for any vertex $u \in U$, an element of the form

$$u + c_u,$$

where $c_u \in C_{m+1}$. Since we have a linear combination of elements infinitely divisible by q , necessarily

$$q^\infty | (u + c_u).$$

Taking a linear combination with the same coefficients, but of the ψ -preimages of $x_i - v_{n_i}$ and c_j (which exist from above), we see that each root of the form $\frac{u + c_u}{q^d}$ has a ψ -preimage.

Let a be the vertex of $T(\mathcal{G}_\alpha)$ that is the predecessor of the vertex $u \in U$. Evidently,

$$a + u = (a - c_u) + (u + c_u),$$

and we have just showed that all q -roots of $a + c_u$ have ψ -preimages. Thus, to show that $(a + u)/q^d$ has a pre-image under ψ , it is sufficient to show that $(a - c_u)/q^d$ has a pre-image. Note that $a - c_u$ is infinitely divisible by q . Also, note that $c_u \in C_{m+1}$, and ψ was set equal to the identity on C_{m+1} as well as on all elements-predecessors of vertices in U (including a). But this means that ψ does not move $(a - c_u)$ and thus does not move its q -roots as well. Therefore, the element $(a - c_u)/q^d$ has a pre-image under ψ which is indeed equal to $(a - c_u)/q^d$ itself. \square

The hard part of the Subphase 2 is over now, the rest of it is routine. Recall the forest $T(\mathcal{G}_\alpha)$ grows downwards. We need to extend ψ further to an automorphism of the whole \mathcal{G}_α . The plan for the rest of Subphase 2 is:

- a. If $v \in T(\mathcal{G}_\alpha)$ is not below some node from U or $v \in \mathbb{F}$ (the free summand), then we set $\psi(v) = v$.
- b. If $v \in T(\mathcal{G}_\alpha)$ has an ancestor $z \in U$, then we use the technique of natural successors introduced in the proof of Lemma 9.1 to define ψ on v .
- c. Finally, we will extend ψ to all rational linear combinations of vertices in $T(\mathcal{G}_\alpha)$ naturally and argue that the resulting map is an automorphism of \mathcal{G}_α .

(If the reader can readily see that *a.-c.* can be performed successfully, they may safely skip the rest of Subphase 2 and go to the description of the next Phase.)

We proceed according to the plan above. According to (a.), if a node $v \in T(\mathcal{G}_\alpha)$ comes *not* from below the vertices in U , then we set $\psi(v) = v$. This definition clearly respects the successor relation because only nodes listed in U can be moved by ψ , and in particular their predecessors/ancestors stay fixed under ψ . Thus, even if v does not come from a different direct summand disjoint from U (i.e., shares an ancestor with some node $u \in U$), the successor relation is still preserved trivially. Therefore, we can safely extend the new definition of ψ further to arbitrary rational linear combinations of such nodes. Note, so far ψ is a homomorphism, injective and onto (when restricted to its domain). Also, since the free summand \mathbb{F} detaches in \mathcal{G}_α , we may extend ψ further to \mathbb{F} by setting it equal the identity map on \mathbb{F} (just as (a.) suggests).

Now suppose $v \in T(\mathcal{G}_\alpha)$ is below some $u \in U$. Recall the definition of a natural successor (proof of Lemma 9.1). We extend ψ level-by-level and first assume that U contains the predecessor u of v . Exactly as we did in the proof of Lemma 9.1, for each $w \in U$ choose a numbering of successors of w , as follows³. Make sure that the successors with the same number coming from below different w, w' are themselves roots of components of the same type, i.e., make sure that the numberings of different successors are consistent (with respect to the the labelling and the isomorphism types of components). Recall $U \leq U'$, and the former contained only vertices that were roots of $S_\gamma[m]$, while U' was generated by vertices of types P_γ -type and $S_\gamma[n]$ for various $n > m$. The rest of Subphase 2 is devoted to an argument similar to the one in the proof of Lemma 9.1.

We observe that $T(S_\gamma[m])$ is isomorphically contained in $T(P_\gamma)$ and in $T(S_\gamma[n])$ for any $n > m$. Therefore, we may number the respective successors of a vertex $u' \in U' \setminus U$ that represent the isomorphic copy of $T(S_\gamma[m])$ in the respective $T(S_\gamma[n])$ rooted in u' .

³Surely we are dealing with two instances of a certain more general fact about automorphism extensions in worthy tree-groups. The fact is unfortunately hard enough to state.

Suppose v is the i' th successor of $u \in U$ according to the numbering of all successors of u fixed above, and suppose

$$\psi(u) = \sum_j m_j z_j,$$

where z_j are vertices contained in U' . For each z_j fix the i' th successor s_j of z_j and define

$$\psi(v) = \sum_j m_j s_j,$$

where m_j are exactly the same coefficients as in the expression for $\psi(u)$. Then we repeat this process of defining ψ for successors of successors of vertices in U using the definition of ψ on the successors, then repeat for the next level, etc. Each time we introduce a coherent numbering of the next lair of vertices below U , and we repeat until all descendants of nodes in U are exhausted. Note we are never stuck in our definition of ψ . Indeed, this follows from the fact that $S_\gamma[m]$ is embeddable into $S_\gamma[n]$ and into P_γ (for any fixed labelling).

Now, extend ψ to arbitrary (rational) linear combinations of special elements that occur in \mathcal{G}_α :

$$\psi\left(\sum_i r_i v_i\right) = \sum_i r_i \psi(v_i),$$

for any special v_i and $r_i \in \mathbb{Q}$ such that $\sum_i r_i v_i \in \mathcal{G}_\alpha$. The definition of ψ respects the prime labels of vertices and edges in $T(\mathcal{G}_\alpha)$, and therefore (just as in Claim 9.5), the above extension of ψ exists. It is also a self-homomorphism of \mathcal{G}_α , and since ψ maps the special basis to a linearly independent set, it is also injective.

We argue that ψ (as defined above) is also onto, i.e., is an automorphism of \mathcal{G}_α . Since \mathbb{F} is in the range of ψ , we need to show that any generator of the form $\frac{a}{\tilde{p}^n}$

and of the form $\frac{a+b}{\tilde{q}^m}$ (see the definition of \mathcal{G}_α) is in the range. The generators that appear in H are already in the range by Claim 9.6. If the generators correspond to vertices or edges *not* below U , then these generators are also in the range trivially (since ψ does not move such generators).

It remains to consider the case when these generators come from the subforest rooted in the vertices contained in U . Recall that ψ induces the automorphism of the free group U' that extends U by several auxiliary vertices of $T(\mathcal{G}_\alpha)$ which are not moved by ψ . According to the definition of ψ , it naturally induces an automorphism of the group generated by the successors of vertices in U' . Indeed, we used the same coefficients for the natural successors/descendants, and the matrix of these coefficients witnesses that the induced linear transformation is bijective.

We are done with $S_\gamma[m]$. Now repeat the instructions above for the same choice of labelling but for $S_\gamma[m']$ for the next largest m' , etc., there are no S_γ -components left with the same fixed choice of labelling. Notice that the next iteration for $m' < m$ does not move any roots of $S_\gamma[m]$ among $(g_{l_{t-1}+1}, \dots, g_{l_t})$. Also, we never move any S_γ -root whose labelling is distinct from the one we are currently working with. Then switch to the next possible labelling, etc., until there are no roots of S_γ -components left among $(g_{l_{t-1}+1}, \dots, g_{l_t})$. This finishes both Subphase 2 and Phase t , $t < d$.

9.3. Phaze d . We now finally consider $(x_{l_{d-1}+1}, \dots, x_{l_d})$ which must be \mathcal{P} -independent in $\mathbb{F}/[V]^*$. As before, we may identify $g_1, \dots, g_{l_{d-1}}$ with the respectively automorphic $x_1, \dots, x_{l_{d-1}}$. Furthermore, \mathbb{F} detaches as a summand, and so far we did not permute \mathbb{F} . Thus, under the same automorphism that we have already defined, $(g_{l_{d-1}+1}, \dots, g_{l_d})$ are still members of some fixed generating basis of \mathbb{F} . We now define an automorphism that maps $(x_{l_{d-1}+1}, \dots, x_{l_d})$ to $(g_{l_{d-1}+1}, \dots, g_{l_d})$ and furthermore fixes the natural, definable direct complement of \mathbb{F} in \mathcal{G}_α . Taking the composition of all automorphisms defined at the previous phases, we get the desired automorphism witnessing that \bar{x} is automorphic \bar{g} .

To define the automorphism, we extend \bar{x} to a generating basis of $\mathbb{F}/[V]^*$. Fix some representatives of the new extended basis, and form the set B' . Let also B be the basis of \mathbb{F} that we fixed and called standard.

Define

$$\phi(x_i) = g_i, \quad i = l_{d-1} + 1, \dots, l_d,$$

and then extend ϕ to an injection of B' onto B . Extend ϕ further to all linear combinations with integer coefficients. Define ϕ to be the identity on the direct complement $D = [V]^*$ of \mathbb{F} .

Since every element of \mathcal{G}_α can be written as $\sum_i m_i b'_i + d$, where $d \in D$, $b'_i \in B'$ and $m_i \in \mathbb{Z}$, we have that ϕ is well-defined and is a homomorphism. Furthermore, every element of \mathcal{G}_α can be (uniquely) expressed as $\sum_i m_i b_i + d$, where $d \in D$, $b_i \in B$ and $m_i \in \mathbb{Z}$. Let $b'_i \in B'$ be such that $\phi(b'_i) = b_i$. We have

$$\phi\left(\sum_i m_i b'_i + d\right) = \sum_i m_i b_i + d,$$

for an arbitrary choice of m_i, d , thus ϕ is onto. Finally, ϕ is injective since for any integers m_i, n_i and $d, f \in D$,

$$\phi\left(\sum_i m_i b'_i + d\right) = \sum_i m_i b_i + d = \sum_i n_i b_i + f = \phi\left(\sum_i n_i b'_i + f\right)$$

implies $m_i = n_i$ for all i . Indeed, B is independent mod D . But then $d = f$.

9.4. Gluing all phazes together. We have defined a nested sequence of automorphisms of \mathcal{G}_α with the property that each consequent automorphism was sending the longer initial segment of \bar{x} to the respective initial segment of \bar{g} . As we noted in the description of each phase, in our definition we could, without loss of generality, assume that the previously defined automorphism had already been applied to \bar{x} and thus we could identify the respective initial segments of \bar{x} and \bar{g} . In fact, the automorphism defined at the next phase (or the next subphaze of Phase $t < d$) would not move the previous initial segment. Thus, taking the composition of all these nested maps defined at the Phazes $t \leq d$ is equivalent to taking the automorphism constructed at the end of Phaze d .

10. COMPLEXITY ANALYSIS

Most of the hard work was done in Section 5, but we must put everything together.

Proposition 10.1. *For any special \bar{g} the formula $\Theta_{\bar{g}}$ is Σ_α^c , with all possible uniformity in \bar{g} .*

Proof. We first prove several useful facts about various ingredients used in $o1 - o7$, and then we carefully analyse the complexity of $o1 - o7$.

10.1. **Auxiliary facts.** Recall $[V]_{\mathcal{G}_\alpha}^*$ (or simply $[V]^*$) was the least pure subgroup of \mathcal{G}_α generated by $T(\mathcal{G}_\alpha)$ -vertices, and recall also

$$\mathcal{G}_\alpha = [V]^* \oplus \mathbb{F},$$

where \mathbb{F} is free abelian of rank ω .

Claim 10.2. $[V]^* \leq \mathcal{G}$ is isolated in \mathcal{G} by a Σ_3^c formula.

Proof. We claim that $a \in [V]^*$ iff $a = \sum_i g_i$ where g_1, \dots, g_k are non-zero elements infinitely divisible by some primes $p_1 \dots p_k$ (which do not have to be distinct). The latter can be expressed by a Σ_3^c formula in the language of additive groups.

Indeed, suppose $a \in [V]^*$. Then it is a linear combination of vertices in $T(\mathcal{G}_\alpha)$, and it follows from the definition of \mathcal{G}_α that every special element is infinitely divisible by some prime.

Now suppose there exists such a linear combination $a = \sum_i g_i$ where g_1, \dots, g_k are non-zero and infinitely divisible by some primes. Then each such g_j must have zero projection onto the free summand \mathbb{F} of \mathcal{G}_α , for otherwise we would get a contradiction with infinite divisibility. Thus, we must have $a \in [V]^*$. \square

Recall R is the free abelian group generated by the roots of $S_{\alpha-1}$ and $P_{\alpha-1}$ -components, and $S \leq R$ is generated by the roots of $S_{\alpha-1}$ -components.

Claim 10.3. The group R is Σ_3^c -definable, and S is $\Sigma_{\alpha-1}^c$ -definable.

Proof. The first statement follows from the discussion after Definition 5.6, and the second statement is exactly Corollary 5.5. \square

Claim 10.4. The groups C_n used in (o7) are Σ_γ^c -definable, with all possible uniformity (i.e., in $\gamma < \alpha - 1$, $n \in \omega$, and the choice of labelling).

Proof. This can be derived from Theorem 5.1. Recall that $S_\gamma[m]$ and P_γ have sections of successors, each section labelled differently. Also, each section of P_γ contains a largest possible P-component (suppressing the subscript), while only m sections of $S_\gamma[m]$ have this property. Thus, in particular $T(S_\gamma[m])$ is embeddable into $T(P_\gamma)$, for every m . Suppose that having a P-successor in the i 'th section is expressed by a formula Φ'_i , of complexity $\Sigma_{\beta_i}^c$ for some $\beta_i < \gamma$, such that Ψ_γ from Theorem 5.1 is equal to the conjunction $\bigwedge_{i \in \omega} \Phi'_i$.

The desired formula with argument g says that:

- g is a linear combination of various roots of S_γ - and P_γ -components, of the right labelling type.
- $\bigwedge_{i \leq n} \Phi'_i(g)$ holds, i.e., at least n sections have a P-component.

The former is (this is Σ_3^c , since we just say it is vertex-like of the right labelling type), and the latter is at most Σ_γ^c (we could do better here). Since $T(S_\gamma[m])$ is embeddable into $T(P_\gamma)$, we have that every element of C_n satisfies the formula. On the other hand, an element satisfying the formula must be generated by roots of $S_\gamma[m]$ and P_γ , for otherwise we would get a contradiction with disjointness of prime labelling and Theorem 5.1. \square

Recall \mathcal{P} stands for the set of all primes.

Claim 10.5. Let A be a computable abelian group, and let $U \leq W \leq A$ be Σ_γ^c -definable subgroups of A . Then for elements of W , \mathcal{P} -independence mod U is Π_γ^c -definable in A .

Proof. In other words, suppose we are given $g_1, \dots, g_n \in W$. Then g_1, \dots, g_n are \mathcal{P} -independent in the factor-group U/W iff for all $p \in \mathcal{P}$ and any $m_1, \dots, m_n \in \mathbb{Z}$,

$$\left[(\exists h)(\exists u) \ h \in W \ \& \ u \in U \ \& \ ph + u = \sum_i m_i g_i \right] \implies \bigwedge p | m_i.$$

This is a condition of the form $\bigwedge_{c.e.} (\Sigma_\gamma^c \implies \text{computable})$, thus it can be expressed as a computable infinitary formula of complexity at most Π_γ^c [AK00]. \square

Remark 10.6. We note that W in the claim above is of complexity Σ_γ^c , thus overall we can hope only for a $\Sigma_{\gamma+1}^c$ definition of tuples in W that are \mathcal{P} -independent mod U . Also, notice that if U is Π_γ^c then the formula becomes $\Pi_{\gamma+1}^c$ increasing the description of tuples by one jump. Unfortunately, our coding techniques allow us to (meaningfully) factor only by Σ_α^c -subgroups. This is the obvious complication that blocks us from extending the result to arbitrary computable ordinals (see the Introduction).

10.2. The complexity of $\Theta_{\bar{g}}$. We are ready to give a detailed complexity analysis of $\Theta_{\bar{g}}$. Recall $\alpha \geq 4$.

Properties (o1) – (o2) state that \bar{x} are vertex-like elements (Σ_3^c) that also satisfy several infinitary divisibility conditions that naturally reflect the successor-relation (Π_2^c). Property (o3) uses \mathcal{P} -independence in $\mathcal{G}_\alpha/[V]^*$, by Claim 10.2 and Claim 10.5 this property is at most Σ_4^c .

Condition (o4) uses \mathcal{P} -independence within $S \leq G$ generated by $S_{\alpha-1}$. By Claim 10.3, S is $\Sigma_{\alpha-1}^c$ -definable. Claim 10.5 (and the discussion after the claim) guarantees the complexity is Σ_α^c .

Property (o5) uses \mathcal{P} -independence of elements in R modulo S . By Claim 10.3, R is Σ_3^c -definable and S is $\Sigma_{\alpha-1}^c$ -definable in \mathcal{G}_α . Claim 10.5 (and the discussion after the claim) guarantees the overall complexity is Σ_α^c .

Recall (o6) was not using \mathcal{P} -independence and was referring to some $\gamma < \alpha-1$. It was also using the definition of the subgroup of \mathcal{G}_α generated by the roots of P_γ -components, with a fixed labelling. This subgroup is Π_γ^c -definable, by Theorem 5.1.

Finally, (o7) used merely $\gamma < \alpha - 1$ and claimed \mathcal{P} -independence of certain elements of C_m mod C_{m+1} . By Claim 10.4, these subgroups of \mathcal{G}_α are Σ_γ^c -definable. By Claim 10.5, \mathcal{P} -independence of tuples in C_m is uniformly $\Sigma_{\gamma+1}^c$ -definable (recall $\gamma < \alpha - 1$).

We conclude that $\Theta_{\bar{g}}$ is of complexity at most Σ_α^c . The proof of Theorem 1.3 is now complete. \square

Remark 10.7. We return to the question raised in the Introduction: *What about odd/limit computable α ?* See Remark 10.6 for the exact problem with our particular set-up. The less obvious blockage is related to the very nature of our definability technique which relies on special positive infinitary formulae (e.g., Theorem 5.1). The properties of elements reflected by the formulae intuitively (but not technically) resemble the definitions of ordinal p -height in abelian p -groups and α -atomicity in Boolean algebras. We use only *infinite* divisibility throughout. Recall that infinite divisibility is a $\forall\exists$ -property, and thus we go two quantifiers up every time we increase the complexity of the “height”. Thus, at least the basis case (e.g., say $\alpha = 3$) should perhaps involve a new idea, and maybe this idea must be using finite divisibility. We note that finite divisibility is often hard to work with in TFAGs, see e.g. examples of pathological decompositions [Fuc73]. Removing some effects related to finite divisibility smoothens many pathologies (see [Fuc73] for the notion of quasi-isomorphism and its applications). Removing finite divisibility (Proposition 6.6) helped us in our analysis. Nonetheless, we strongly suspect that all limit α might be in principle doable using our methods, but the most natural/naive attempts unfortunately fail for reasons similar to the one discussed in Remark 10.6.

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