Computability and Structure

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Preface

This dissertation contains results in the area of constructive mathematics with emphasis to computable algebra and computable analysis. Mal'cev [66] and Rabin [86] initiated the study of computable groups, and Turing [96, 95] started the investigation of effective procedures in analysis. The thesis in hand is divided into two parts.

Part I contains results on computable abelian groups. More specifically, we introduce a new computably-theoretic concept of limitwise monotonic sequence and apply this notion to study effectively presentable torsion abelian groups and other structures. We completely describe higher computable categoricity in the class of homogeneous completely decomposable groups. For this description we need new computably-theoretic and algebraic methods. We show that a functor from the class of countable trees into the class of abelian groups defined in [50] is injective on a certain subclass of trees. This fact has recently found an application in computable group theory [35]. We also study α jump degrees of torsion-free abelian groups, and show that for every computable α there exist a torsion-free abelian group having a proper α jump degree.

Part II is devoted to the study of computable separable metric and Banach spaces, with a strong influence of certain ideas from computable model theory and algorithmic randomness. We consider computable metric spaces associated to Banach spaces and show that every separable Hilbert space possesses a unique computable structure, up to a computable isometry, and $C[0,1]$ and l_1 possess more than one. We study computable metric spaces which are not associated to Banach spaces and show that Cantor space and the Urysohn space have a unique computable structure, up to a computable isometry, and also describe computable subspaces of $\mathbb{R}^n$ having a unique computable structure. Finally, we generalize the concept of K-triviality [80] to an arbitrary computable metric space, and show that two possible adequate generalizations of K-triviality actually coincide.

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Contents

Chapter 1

Introduction

Computable (constructive) mathematics was born quite early in the XX'th century, and its birth pre-dates the formal clarification of what is a computable process. For instance, Brouwer [13], [14] used intuitively effective procedures; Max Dehn in 1911 [22] studied word, conjugacy and isomorphisms of finitely presented groups using intuitive algorithms. Modern computable mathematics goes back to 1930s to the fundamental papers of Turing [96, 95], and the Russian school of constructive mathematics founded by Markov in the late 1940s (see, e.g., Kusner [62]) who explicitly used computability theory.

This dissertation is devoted to the study of effective procedures in infinitely generated groups and separable metric spaces. These results belong to effective algebra and computable analysis, respectively. Both fields are branches of computable mathematics and have many similarities. Nonetheless, they have been developing quite independently in the recent years and have certain distinctions. As a consequence, the thesis is divided in two independent parts.

The first part of the thesis is devoted to infinitely generated *computably presented abelian groups*. By computable groups, we mean groups where the domain is computable and the algebraic operation is computable upon that domain. A group is computably presented if it has a computable isomorphic copy. The study of computably presentable groups was initiated by Rabin [86] and, independently, Mal'cev [67] in the early 1960's. Such studies can be generalized to other algebraic structures, a tradition going back to Grete Herrmann [44], van der Waerden [99], and explicitly using computability theory, Rabin [86], Mal'cev [66] and Frölich and Shepherdson [38]. The fundamental works of Mal'cev and Rabin mentioned above gave rise to computable model theory which studies effective procedures in abstract algebraic structures [33, 3].

The second part of the thesis studies *computable separable metric* and *Banach spaces*. A separable metric space is computable if it contains a dense computable sequence of points. A computable normed space is defined similarly. The spaces are usually assumed to be complete. The study of effective procedures in uncountable spaces goes back to Turing [96, 95]. These studies can be generalized to topological spaces [101]. Methods of modern computable analysis have various applications in algorithmic randomness [80, 26, 81]. There are certain interactions of computable analysis and theory of numberings [101]. Standard references for computable analysis are [101] and [85].

1.1 Overview of Part I: Computable abelian groups

As its name suggests, computable abelian group theory combines methods of computability theory and commutative algebra. The systematic study of computable *abelian* groups was initiated by Mal'cev [67]. The standard references for computable abelian group theory are [58, 23].

We assume that the reader is familiar with basic notions of computability theory [92]. Further notions will either be given when needed, or can be found in [92, 33, 3].

1.1.1 Computable algebraic structures.

The definition below is central to the first part of the thesis.

Definition (Mal'cev, Rabin). An infinite countable algebraic structure (e.g., a group) is *computable* if its universe can be identified with the natural numbers so that the functions and relations become uniformly computable. This numbering of the universe is called a *computable copy* of the structure.

For instance, a group is computable if its domain and the operation are both computable. Computable copies are also called computable presentations or constructivizations [33, 3].

Computable algebraic structures are the main objects of study in computable algebra. Nonetheless, non-computable structures appear naturally in many cases. For instance, every finitely presented algebra is computably enumerable. Such algebras are also called Σ^0_1-algebras, see e.g. [33, 60]. It is well-known that there exist finitely presented algebras with undecidable word problem, whence non-computable. For a group-theoretical description of computably enumerable groups see the famous work of Higman [45]; see also Feiner [34] for an application of computably enumerable Boolean algebras in degree theory. Another example of non-computable but (in some sense) effective structures are finitely generated subgroups of computable permutations of the natural numbers. Such a group may not be computable in general, yet it is Π^0_1 or co-c.e., because if two elements are unequal we will eventually see it. These Σ^0_1 and Π^0_1 structures share several common properties with computable ones (see, e.g., [60]).

The natural examples discussed above motivate the study of structures computable relative to an oracle. A countably infinite algebraic structure $\mathcal{A}$ is **d**-*computable* if its universe can be identified with the natural numbers ω in such a way that the operations become (uniformly) **d**-computable. For instance, one may speak of structures computable in the halting problem, natural examples being Σ^0_1 and Π^0_1 structures.

Classically algebraic structures are considered up to isomorphisms. It is rather natural to consider computable structures up to computable isomorphisms:

Definition (Mal'cev, Rabin). Two computable structures are *autoequivalent* if they are isomorphic via a computable isomorphism. A computable structure is said to be *computably categorical* or *autostable* if every two computable copies of this structure have a computable isomorphism between them.

The terms "autoequivalent" and "autostable" are mostly used by Russian-speaking mathematicians [33], while most of English-speacking authors use the terms "computably isomorphic" and "computably categorical" [3]. If a structure is not computably categorical, it is natural to ask the question of how close to being computably categorical the structure is. For instance, a linear ordering of order type $\mathbb{N}$ is not computably categorical since there is the canonical example where the successor relation is computable, and another where the successor relation is not. But if we are given an oracle for the successor relation, then the structure is computably categorical *relative* to that. The halting problem would be enough to decide whether y is the successor of x in such an ordering. This motivates the following definition.

We say that a structure $\mathcal{A}$ is Δ^0_n-*categorical* if every two computable presentations of $\mathcal{A}$ have an isomorphism between them which is computable with oracle $\emptyset^{(n-1)}$, where $\emptyset^{(n-1)}$ is the (n-1)-th iteration of the halting problem.

1.1.2 The Two Problems

Most of the results included in the first part of the thesis are related to Problem A and Problem B below, restricted to the class of infinite abelian groups.

Problem A. Given an infinite countable structure (e.g., a group or a field), determine if this structure has a computable copy. More generally, describe all Turing degrees **d** such that the structure has an **d**-computable copy.

There is no hope of finding a general necessary and sufficient condition for countable structures to be computably presentable. For instance, there is a graph having an **d**-computable copy for every non-computable **d**, which yet has no computable copy [100, 89]. Thus, probably the strongest condition one can think of fails for structures in general. Furthermore, even for certain sufficiently broad classes of algebraic structures such as Boolean algebras and abelian groups there is a little hope to obtain a satisfactory solution to this problem. In context of infinitely generated abelian groups, it is not even known which reduced abelian p-groups can be computably presented. Khisamiev [58] obtained a satisfactory solution in the special case of p-groups of small Ulm length. However, the case of arbitrary (constructive) Ulm length has remained undiscovered for over 20 years, and the problem seems to be difficult. We refer to [3] for a more detailed discussion and more partial results concerning Problem A.

The second problem addressed in the first part of the thesis is:

Problem B. Given two computable structures, determine if there is a computable homomorphism (embedding, isomorphism etc.) between the structures. Describe computably and Δ^0_n-computably categorical structures (for positive $n \in \omega$ or n a non-empty constructive ordinal).

In contrast to Problem A, a lot more is known about computably categorical structures. For instance, there are characterizations of computably categorical algebraic structures in the classes of Boolean algebras [42], [64], linear orders [87], torsion-free abelian groups [41], [83], abelian p-groups [33], and other structures [3]. There are notions similar to computable categoricity such as relative computable categoricity [3]. Although computable categoricity has been central to computable algebra for over 50 years, there is still a lot to be understood. For instance, not much is known about computably categorical fields [77]. Also, it is not known if the index set of computably categorical structures is Π^1_1-hard [102]. For more recent results on computably categorical countable structures see [25].

Once computably categorical structures in a given class are characterized, it is natural to ask which members of this class are Δ^0_2-categorical. Here the situation usually becomes more complex. There are only few results in this area, most of them are partial. For instance, McCoy [68] characterizes Δ^0_2-categorical linear orders and Boolean algebras under some extra effectiveness conditions. Also it is known that in general Δ^0_{n+1}-categoricity does not imply Δ^0_n-categoricity in the classes of linear orders [4], Boolean algebras [3], abelian p-groups [7], and ordered abelian groups [72].

Problems A and B in general suggest to view computable (**d**-computable) structures of a certain language as a category in which computable (**d**-computable) isomorphisms are morphisms. This idea has recently developed to the study of computable functors between classes of computable structures (see, e.g., [59]). We emphasize that Problems A and B are not the only important themes of computable model theory and, more generally, effective mathematics. For other problems such as spectra of relations or computable dimension see [33], [3].

Part I is organized so that the algebraic content of the results increases from chapter to chapter. It is assumed that the reader has a sufficient background in computability theory [92] and knows basics of computable model theory [3]. We, however, except that the reader may not have a sufficient background in abelian groups. Thus, we give some classical definitions of the theory which will be used without expect reference in the text. After we give the basics, we review the upcoming chapters, stating results formally and providing more background along the way.

1.1.3 Basics of abelian group theory

Following the tradition of abelian group theory, we use + to denote the group operation [39, 40]. Recall that if we replace a filed by a commutative ring in the definition of a vector space, we obtain the definition of a module over this ring. An abelian group H can be viewed as a Z-module, as follows. We write kh to denote $\underbrace{h+h+\ldots+h}_{k \text{ times}}$, and we write $(-k)h$ to denote $-(\underbrace{h+h+\ldots+h}_{k \text{ times}})$, where k is a positive integer and $h \in H$. We also set $0h = 0$, for every $h \in H$. Basics of module theory can be found in [63].

An abelian group H is *torsion-free* if $kh \neq 0$ for every nonzero $h \in H$ and every positive integer k. An abelian group is *torsion* if for every $h \in H$ there exists a positive integer k such that $kh = 0$. There are abelian groups which are neither torsion nor torsion-free. For h and element of an abelian group, if there exists k such that $kh = 0$, then the least k with this property is called the order of h. For a prime p, an abelian torsion group is a p-group if every element of it has order p^k for some k. We usually restrict ourselves to torsion-free or p-groups, but there will be one theorem about torsion groups which are not p-groups.

In algebraic lemmas we may view abelian groups as modules. We will frequently use the notion of linear independence over Z taken from module theory:

Definition. Elements $g_0, \ldots, g_n$ of a torsion-free abelian group G are *linearly independent* if, for all $c_0, \ldots, c_n \in Z$, the equality $c_0 g_0 + c_1 g_1 + \ldots + c_n g_n = 0$ implies that $c_0 = c_1 = \ldots = c_n = 0$. An infinite set is *linearly independent* if every finite subset of this set is linearly independent. A maximal linearly independent set is a *basis*. All bases of G have the same cardinality. This cardinality is called the *rank* of G.

We write $A \leqq B$ to denote that A is a subgroup of B. It is not hard to see that a torsion-free abelian group A has rank 1 if and only if $A \leqq \langle Q, + \rangle$.

Definition. An abelian group G is the *direct sum* of groups A_i, $i \in I$, written $G = \bigoplus_{i\in I} A_i$, if G can be presented as follows:

1. The domain consists of infinite sequences $(a_0, a_1, a_2, \dots, a_i, \dots)$, each $a_i \in A_i$, such that the set $\{i : a_i \neq 0\}$ is finite.

2. The operation $+$ is defined component-wise.

The groups A_i are the *direct summands* or *direct components* of G (with respect to the given decomposition). Note that there may be lots of different ways to decompose the given subgroup. One can check that $G \cong \bigoplus_{i\in I} A_i$, where $A_i \leqq G$, if and only if (1) $G = \sum_{i\in I} A_i$, i.e. $\{A_i : i \in I\}$ generates G, and (2) for all j we have $A_j \cap \sum_{i\in I, i\neq j} A_i = \{0\}$.

This thesis studies effective content of algebraic structures, abelian groups in particular, therefore we have to agree on the signature (formal syntactical language). We will see that for our purposes we do not need to use the more complicated two-sorted signature of modules (see, e.g., Proposition 3.2.2). By convention, our signature contains only $+$. Thus, $-$ and the module multiplication kh should be understood as abbreviations. For example, $x - y = z$ stands for $(\forall h)[(h + y = 0) \to x + h = z]$. Furthermore, if $+$ is represented by a computable function, then both $-$ and kh can be computed effectively and uniformly.

We will use one more abbreviation. For k a positive integer and $g \in G$, we write $k|g$ *in* G (or simply $k|g$ if it is clear from the context which group is considered) and say that k *divides* g *in* G if there exists an element $h \in G$ for which $kh = g$, and we say that h is a *k-root* of g. Note that $k|g$ is simply an abbreviation for the formula $(\exists h)(\underbrace{h + h + \ldots + h}_{k \text{ times}} = g)$ in the signature of abelian groups. If for every $k \neq 0$ and every $x \in G$, we have $k|x$, then the group is called divisible. It is well-known that every abelian group G can be embedded into a divisible group. The least subgroup of the divisible group containing G is uniquely determined up to an isomorphism, and is called the divisible hull or the divisive closure of G. There is no standard notation for the divisible closure of G, some authors use $D(G)$.

If the group G is torsion-free then every $g \in G$ has at most one k-root, for every $k \neq 0$. Assume there were two distinct k-roots, h_1 and h_2, of an element g. Then $k(h_1 - h_2) = 0$ would imply $h_1 = h_2$, a contradiction.

Definition. Let G be a torsion-free abelian group. A subgroup A of G is called *pure* if for every $a \in A$ and every n, $n|a$ in G implies $n|a$ in A. For any subset X of G we denote by $[X]$ the least pure subgroup of G that contains X.

For instance, every direct summand of a given group G is pure in G, while the converse is not necessarily the case. Also, every pure subgroup of a divisible group is divisible itself.

1.1.4 Summary of Chapter 2: Limitwise monotonic sequences

In this chapter we introduce and study a purely computably-theoretic concept of uniform limitwise monotonicity for sequences of sets. Then we apply the computably-theoretic results to investigate degree spectra (to be defined) of certain algebraic structures including abelian groups.

A set S is *limitwise monotonic* relative to a given degree $\mathbf{a}$ if there is an $\mathbf{a}$-computable function $g : \omega \times \omega \to \omega$ such that (1) $\max_s g(x, s)$ exists, for every x, and (2) $S = \operatorname{rng}(\lambda x[\max_s g(x, s)])$. The second condition can equivalently be replaced by (2′) g is total, $g(x, s) \leq g(x, s+1)$ for all $x, s \in \omega$, and $\lim_s g(x, s)$ exists. We say that the function $f(x) = \max_s g(x, s)$ is *limitwise monotonic*.

Khisamiev was the first to introduce the notions of computable monotonic approximation and the notions of s-function and s_1-function [58]. As we have already mentioned, Khisamiev used computable monotonic approximations of Σ^0_2 sets to study computable abelian p-groups of small Ulm length [58]. He also showed that there is a Δ^0_2 set which is not limitwise monotonic (Proposition 3.8 of [58]). Independently, Khoussainov, Nies and Shore in [59] introduced limitwise monotonic functions in the study of computable models of $\aleph_1$-categorical theories. See [21, 43, 54, 15, 49, 19, 29] and [46] for further applications, and [47, 55] for certain generalizations of this notion.

We introduce the following notion:

Definition. An ordered sequence of sets $\mathcal{S} = \{S_n\}_{n\in\omega}$ is *uniformly limitwise monotonic* (relative to $\mathbf{a}$, in $\mathbf{a}$) if there is a computable ($\mathbf{a}$-computable) function g such that

- $\max_x g(n, y, x)$ exists for every $n, y \in \omega$, and
- $S_n = \operatorname{rng}(\lambda y[\max_x g(n, y, x)])$, for every $n \in \omega$.

We say that g is a *uniform limitwise monotonic (relative to* $\mathbf{a}$*, in* $\mathbf{a}$*) approximation* of $\mathcal{S}$.

This is clearly a generalization of the concept of limitwise monotonic sets. It follows that a set A is limitwise monotonic if and only if the sequence $\{S_n\}_{n\in\omega}$, where $S_n = A$ for all n, is uniformly limitwise monotonic. We study uniformly limitwise monotonic sequences from the computably-theoretic point of view, and then apply these results to computable model theory. We prove:

Theorem 2.2.1. There is a sequence of infinite sets which is uniformly limitwise monotonic relative to every hyperimmune degree (in particular, relative to every nonzero Δ_2^0-degree), but is not uniformly limitwise monotonic.

The proof of the next theorem uses the ideas of the proof of the theorem above. However, we note that in the theorem above we are able to handle uncountably many (hyperimmune) degrees while in the theorem below, in the case of a single set, we are able to deal with only countably many degrees.

Theorem 2.3.1. There is a Σ_2^0 set S such that S is limitwise monotonic in every nonzero Δ_2^0-degree, but is not limitwise monotonic.

A natural question arises if the theorems above can be extended to all non-zero degrees. This question is answered in the next theorem:

Theorem 2.4.1. If a sequence of sets $\{S_n\}_{n\in\omega}$ is uniformly limitwise monotonic in all degrees except perhaps countably many, then $\{S_n\}_{n\in\omega}$ is uniformly limitwise monotonic.

We apply the above theorems to investigate degree spectra of structures. If an algebraic structure is not computable, then it is natural to ask how close to computable the structure is. This property is reflective in the collection of all Turing degrees relative to which a given structure possesses a computable presentation. More formally, the degree spectrum of $\mathcal{A}$ is

$$\mathrm{DegSp}(\mathcal{A}) := \{\mathbf{d} : \mathcal{A} \text{ is } \mathbf{d}\text{-computable}\}.$$

We apply our computably-theoretic results to abelian groups and obtain the following result:

Theorem 2.5.1.

1. There is a torsion abelian group G such that (a) G has no computable copy, and (b) G has an $\mathbf{a}$-computable copy, for every hyperimmune degree $\mathbf{a}$.

2. There is an abelian p-group A such that $DegSp(A)$ contains a Δ_2^0 degree $\mathbf{a}$ if and only if $\mathbf{a} > 0$.

In the first part of this theorem, the group G is of the form $G = \bigoplus_{p\in X}\left(\bigoplus_{n\in S_p} Z_{p^n}\right)$, where X is a set of prime numbers and $S_p \subseteq \omega$ for each $p \in X$. In the second part, the group A is a p-group and is of the form $\bigoplus_{n\in S} Z_{p^n}$, where $S \subseteq \omega \setminus \{0\}$. The result resembles a similar fact for linear orders [76] , but the proof is quite different.

It is natural to ask whether the theorem above can be strengthened for groups of the form $G = \bigoplus_{p \in X}(\bigoplus_{n \in S_p} Z_{p^n})$. For instance, one would like to know if there exists a group G of such form such that the group has an $\mathbf{x}$-computable copy if and only if $\mathbf{x} > 0$ (or if and only if $\mathbf{x}$ is not $\mathrm{low_n}$ for some n). We apply our computably-theoretic result to show that such groups do not exist:

Theorem 2.5.2. For any group G of the form $G = \bigoplus_{p \in X}\left(\bigoplus_{n \in S_p} Z_{p^n}\right)$, where X is a set of prime numbers and $S_p \subseteq \omega \setminus \{0\}$ for each $p \in X$, we have the following: If the group G has an $\mathbf{x}$-computable copy for every degree $\mathbf{x}$, except perhaps countably many, then G has a computable copy.

Then we illustrate similar applications of the computably-theoretic results to equivalence structures and $\aleph_1$-categorical theories.

This chapter is based on the paper [56].

1.1.5 Summary of Chapter 3: Completely decomposable groups

In this chapter we give a higher level classification of effective categoricity for a certain basic class of torsion-free abelian groups. Recall that an abelian group is torsion-free if every nonzero element of this group is of infinite order.

Question. *Which computably presentable torsion-free abelian groups are Δ^0_n-categorical, for $n \geq 2$?*

As with the classical theory of torsion-free abelian groups, general questions about isomorphism classes are often rather difficult. The main difficulty is the absence of satisfactory invariants for computable torsion-free abelian groups which would characterize these groups up to isomorphism [32].

There are better understood subclasses of the torsion-free abelian groups such as the rank one groups, the additive subgroups of the rationals. As we remind the reader in the next section, these groups have a nice structure theory via Baer's theory of *types* (Baer [5]). This theory can be extended to groups that are of the form $\oplus_i H_i$ where each H_i has rank 1, a class called the *completely decomposable* groups. As is well-known, Baer's theory extends to this class so we would have some hope of understanding the computable algebra in this setting.

We restrict ourselves to a natural subclass, the *homogeneous completely decomposable* groups which are countable direct powers of a subgroup of the rationals. More formally, we consider the groups of the form $\bigoplus_{i \in \omega} H$, where H is an additive subgroup of $(Q, +)$. These groups in the classical setting were first studied by Baer [5]. The class of homogeneous completely decomposable groups of rank ω

is certainly the simplest and most well-understood class of torsion-free abelian groups of infinite rank. Note that, from the computability-theoretic point of view, this is the simplest possible non-trivial case we may consider: every torsion-free abelian group of finite rank is computably categorical. As we will see, even in this classically simplest case the complete answer to the problem does not seem to be straightforward. To understand the effective categoricity of these groups, we will need both new uses of computability theory in the study of torsion-free abelian groups, and some new algebraic structure theory.

More specifically, we introduce a new purely algebraic notion of *S-independence*, where S is a *set* of primes. This is a generalization of the well-known notion of p-independence for a single prime p. In the theory of primary abelian groups, p-independence plays an important role. See Chapter VI of [39] for the theory of p-independent sets and p-basic subgroups. We establish several technical facts about S-independent subsets of homogeneous completely decomposable groups. These facts are of independent interest from the purely algebraic point of view. For instance, our results essentially show that S-independence and free modules over a localization of Z play a similar role in the theory of completely decomposable groups as p-independence and p-basic subgroups do in the theory of primary abelian groups. The notion of S-independence seems to be an adequate replacement of linear independence in the case of free modules over a localization of Z.

We apply the algebraic techniques developed for S-independent sets to establish an upper bound on the complexity of isomorphisms.

Theorem 3.4.1. Every homogeneous completely decomposable group is Δ^0_3 - categorical.

This result is sharp: there exist homogeneous completely decomposable groups which are not Δ^0_2-categorical so that we cannot replace Δ^0_3 by Δ^0_2. Also, a homogeneous completely decomposable group of rank ω is never computably categorical (folklore). It is natural to ask for a necessary and sufficient condition for a homogeneous completely decomposable group to be Δ^0_2-categorical. Remarkably, there is a natural condition on the group classifying exactly when this happens.

We characterize the case where a computable completely decomposable homogeneous group is Δ^0_2-categorical via a combination of an algebraic (the group must be of the form $\bigoplus_{i\in\omega} Q^{(P)}$, where $Q^{(P)} = \langle\{1/p^n : p \in P, n \in \omega\}\rangle$) and a mild effectiveness consideration (the complement of the corresponding set P is *semi-low*). That is, P must resemble a computable set in the sense that it has a weak guessing procedure for membership, called *semi-lowness*.

We say that a set S is *semi-low* if the set $H_S = \{e : W_e \cap S \neq \emptyset\}$ is computable in the halting problem. As the name suggests (for co-c.e. sets) this is weaker than being low (meaning that $A' \equiv_T \emptyset'$, since every low c.e. set is one with a semi-low complement, but not conversely, see Soare [91, 92]). Semi-low sets are connected with the ability to give a fastest enumeration of a computably enumerable set as discovered by Soare [91]. In that paper, Soare showed that if **a** is a c.e. degree which is nonlow, then it contains a c.e. set whose complement is not semi-low. Semi-low sets also appear naturally when one studies automorphisms of the lattice $\mathcal{E}$ of computably enumerable sets under set-theoretical inclusion. Soare (see, e.g., [92], Theorem 1.1 on page 375) showed that if a c.e. set S has a semi-low complement, then the lattice of all c.e. sets is isomorphic to the principal filter $\mathcal{L}(S)$ of c.e. supersets of S. Furthermore, if a c.e. set S has a semi-low complement, then $\mathcal{L}(A)/\mathcal{F}$ is effectively isomorphic to $\mathcal{E}/\mathcal{F}$, where $\mathcal{F}$ stands for the ideal of finite sets. We mention that a c.e. degree is low if and only if it contains a semi-low$_{1.5}$ co-c.e. set [28]. It is rather interesting that semi-lowness appears in the characterization of Δ^0_2-categorical abelian groups:

Theorem 3.5.1. A computable homogeneous completely decomposable group A of rank ω is Δ^0_2 - categorical if and only if A is isomorphic to $\bigoplus_{i\in\omega} Q^{(P)}$, where P is a c.e. set of primes such that $\{p : p \text{ prime and } p \notin P\}$ is semi-low.

In particular, if P is c.e. and low, then G_P is Δ^0_2 categorical. As far as we know, this is the first application of semi-low sets in effective algebra. Also, the proof of the theorem above is of some technical interest as it splits into several cases depending on the manner by which the type of the group A is enumerated. The flavour of this proof is that of the "limitwise monotonic" proofs in the literature but is a lot more subtle. The method has a number of new ideas which would seem to have other applications.

The chapter also contains further results on bases of certain homogeneous completely decomposable groups viewed as free modules over a localization of integers. This chapter is based on the paper [31].

1.1.6 Summary of Chapter 4: An effective transfromation

There are many known functors between classes of structures, used in different ways. For instance, Mal'cev [65] considered a functor taking rings to their Heisenberg groups. He showed that there is a copy of the input ring, defined with parameters, in the output group. Mal'cev used this idea to obtain, from the ring of integers, a group whose elementary first order theory is hereditarily undecidable. In computable algebra functors are often called effective transformations to emphasize their constructive nature [48].

In this chapter we study a transformation of trees into torsion-free abelian groups which proved to be useful in effective algebra. One uses infinite divisibility to distinguish certain elements of a torsion-free abelian group from other elements. Fuchs [39] used infinite divisibility to construct indecomposable torsion-free abelian groups of large cardinalities. See [39] for more examples of this kind in pure abelian group theory. See also [82] for the study of infinite divisibility and indecomposability of automatic abelian groups.

Hjorth [50] used infinite divisibility to study torsion-free abelian groups from the descriptive set theory point of view. He showed that the isomorphism problem for torsion-free abelian groups is not Borel (see [50] for definitions).

Downey and Montalbán [32] applied the ideas of Hjorth and defined an effective transformation of trees to torsion-free abelian groups, as follows. Following Fuchs [39], we denote by $p_n^{-\infty} g$ the collection of generators $\{p_n^{-k} g : k \in \omega\}$. Let (p_n), (q_k) be two disjoint computable sequences of distinct primes. Suppose a tree $T = (V, E)$ with distinguished root r is given. The group $G(T)$ is the subgroup of $\bigoplus_{v \in V} \mathbb{Q}v$ generated by $p_n^{-\infty} v$ for $v \in V$ of height n, and $q_n^{-\infty}(v + w)$ where (v, w) is an edge, v is of height n-1 and w is of height n. Clearly, isomorphic trees give rise to isomorphic groups. It was not clear if the coding preserves the isomorphism type in general (in this case it would be said to be injective). Nonetheless, Downey and Montalbán [32] used this coding to show that the isomorphism problem for computable torsion-free abelian groups is Σ_1^1-complete. We show:

Theorem 4.3.1. The transformation form [32] is injective for the special class of rank-homogeneous trees (to be defined).

Although the class of rank-homogeneous trees is rather specific, Fokina, Friedman, Harizanov et al. [35] recently applied this fact to show that the isomorphism relation on computable torsion-free abelian groups is tc-complete among Σ_1^1 equivalence relations (see [35] for definitions). We will not discuss this application in the chapter.

The proof of the theorem above is purely algebraic. We conclude the chapter by a proposition which states that in general the transformation is not injective and, therefore, the statement of the theorem can not be extended to the class of all trees.

The chapter is based on the paper [36]. The non-injectivity proposition was published in [73].

1.1.7 Summary of Chapter 5: Jump degrees of torsion-free abelian groups

Recall that the degree spectrum of an algebraic strucutre $\mathcal{A}$ is

$$\mathrm{DegSp}(\mathcal{A}) := \{\mathbf{d} : \mathcal{A} \text{ is } \mathbf{d}\text{-computable}\}.$$

The degree spectrum may have no least element (for example, see [88]). As a result, there has been a line of study into the jump degrees of structures. If $\mathcal{A}$ is a countable structure, α is a computable ordinal, and $\mathbf{a} \geq \mathbf{0}^{(\alpha)}$ is a degree, then $\mathcal{A}$ has α^{th} *jump degree* $\mathbf{a}$ if the set

$$\{\mathbf{d}^{(\alpha)} : \mathbf{d} \in \mathrm{DegSp}(\mathcal{A})\}$$

has $\mathbf{a}$ as its least element. In this case, the structure $\mathcal{A}$ is said to have α^{th} *jump degree*.

A structure $\mathcal{A}$ has *proper* α^{th} *jump degree* $\mathbf{a}$ if $\mathcal{A}$ has α^{th} jump degree $\mathbf{a}$ but not β^{th} jump degree for any $\beta < \alpha$. In this case, the structure $\mathcal{A}$ is said to have *proper* α^{th} *jump degree*.

For a computable ordinal α, it is well-known that an arbitrary structure may not have α^{th} jump degree (for example, see [30]). The existence or nonexistence of a structure with proper α^{th} jump degree $\mathbf{a}$ for $\mathbf{a} \geq \mathbf{0}^{(\alpha)}$ depends heavily on the class of algebraic structures considered. Within the context of linear orders, if an order type has a degree, it must be $\mathbf{0}$; if an order type has first jump degree, it must be $\mathbf{0}'$; and yet for each computable ordinal $\alpha \geq 2$ and degree $\mathbf{a} \geq \mathbf{0}^{(\alpha)}$, there is a linear order having proper α^{th} jump degree $\mathbf{a}$ (see [30], culminating results in [2], [52], [61] and [88]). Within the context of Boolean algebras, if a Boolean algebra has n^{th} jump degree (for any $n \in \omega$), it must be $\mathbf{0}^{(n)}$; yet for each $\mathbf{a} \geq \mathbf{0}^{(\omega)}$, there is a Boolean algebra with proper ω^{th} jump degree $\mathbf{a}$ (see [53]).

The subject of this chapter is the existence of jump degrees of torsion-free abelian groups. For $\alpha \in \{0, 1, 2\}$, it is known that every possible proper α^{th} jump degree is realized.

Theorem (Downey [23]; Downey and Jockusch [23])**.** *For every degree* $\mathbf{a} \geq \mathbf{0}$*, there is a (rank one) torsion-free abelian group having degree* $\mathbf{a}$*. For every degree* $\mathbf{b} \geq \mathbf{0}'$*, there is a (rank one) torsion-free abelian group having proper first jump degree* $\mathbf{b}$*.*

Indeed, every finite rank torsion-free abelian group has first jump degree as a consequence of a computability-theoretic result of Coles, Downey, and Slaman [20]. In contrast, not every infinite rank torsion-free abelian group has first jump degree as a consequence of the following theorem. Recall that a nonzero degree $\mathbf{a}$ is *low* if $\mathbf{a}' = \mathbf{0}'$ and *nonlow* otherwise.

Theorem (Melnikov [71])**.** *There is a torsion-free abelian group $\mathcal{G}$ such that DegSp($\mathcal{G}$) = $\{\mathbf{a} : \mathbf{a}$ is nonlow$\}$. Consequently, there is a torsion-free abelian group with proper second jump degree $\mathbf{0}''$.*

The proof of the theorem above will not be included in the chapter and can be found in [71]. Results of Ash, Jockusch, and Knight (see [2]) and two observations of Melnikov (see Theorem 3 and Proposition 10 of [71]) have implications about proper second jump degrees and proper third jump degrees.

Theorem (Melnikov [71])**.** *For every degree $\mathbf{a} > \mathbf{0}''$, there is a torsion-free abelian group having proper second jump degree $\mathbf{a}$. For every degree $\mathbf{b} > \mathbf{0}'''$, there is a torsion-free abelian group having proper third jump degree $\mathbf{b}$.*

In this chapter we generalize this result to an arbitrary computable ordinal α.

Theorem 5.0.1. For every computable ordinal α and degree $\mathbf{a} > \mathbf{0}^{(\alpha)}$, there is a torsion-free abelian group having proper α^{th} jump degree $\mathbf{a}$.

Fixing α, we prove this theorem by coding sets $S \subseteq \omega$ into groups $\mathcal{G}_S^\alpha$ in such a way that $\mathcal{G}_S^\alpha$ is X-computable if and only if $S \in \Sigma_\alpha^0(X)$. The coding method is based on techniques in Fuchs (see Section XIII, Chapter 88 and Chapter 89, of [40]). In particular, given torsion-free abelian groups $\mathcal{A}$ and $\mathcal{B}$ of a certain type and elements $a \in \mathcal{A}$ and $b \in \mathcal{B}$, Fuchs adds elements of the form $p^{-n}(a+b)$ for $n \in \omega$ to $\mathcal{A} \oplus \mathcal{B}$ to build an indecomposable group containing $\mathcal{A} \oplus \mathcal{B}$. This method can be also viewed as a generalization of the methods discussed in the previous section (the latter really being a simplification of the former).

The chapter is based on the paper [1].

1.2 Overview of Part II: Computable metric spaces

Computable analysis combines methods of recursion theory and classical analysis [12, 85]. In contrast to computable algebra, a computable metric or normed space generally contains non-computable points. Another difference is that many effective procedures in analysis output an effectively fast converging sequence, while a procedures in effective algebra typically gives a single element at once. For instance, a computable isomorphism (embedding, homomorphism, etc.) in algebra is a computable function, and a computable isomorphism (homeomorphism, isometry, quisi-isometry, embedding, etc.) in analysis is usually a Turing functional.

1.2.1 Basics of computable metric space theory

Given a classical result from analysis, one may ask for its effective versions. For instance, one may prove the effective analogue of the classical Weierstrass theorem for computable functions [85] or study derivatives of computable differentiable functions (see Myhill [78], Pour-El and Richards [84] and Nies [81]). However, results may depend on the initial definition of a computable function (see, e.g., the recent paper [10]). To develop a meaningful theory we have to choose a definition to work with. We choose an approach which is common for separable metric spaces and, more specifically, for separable Banach spaces [12]:

Definition. Let (M, d) be a complete separable metric space, and let $(q_i)_{i\in\omega}$ be a dense sequence without repetitions. The triple $\mathcal{M} = (M, d, (q_i)_{i\in\omega})$ is a *computable metric space* if $d(q_i, q_k)$ is a computable real uniformly in i, k. We say that $(q_i)_{i\in\omega}$ is a *computable structure* on M.

We refer to the elements of the sequence $(q_i)_{i\in\omega}$ as the *special points*. A *Cauchy name* for a point x is a sequence $(q_{f(s)})_{s\in\omega}$ of special points converging to x such that $d(q_{f(s)}, q_{f(t)}) \leq 2^{-s}$ for each $t > s$. An element x of $\mathcal{M}$ is *computable* if there exists a computable function f such that $(q_{f(s)})_{s\in\omega}$ is a Cauchy name for x. An X-computable space is defined similarly. It is well-known that a point x from $\mathcal{M} = (M, d, (q_i)_{i\in\omega})$ is computable if, and only if, from a positive rational δ one can compute p such that $d(x, q_p) \leq \delta$. To emphasize which computable structure on M is considered, we say that x is computable with respect to $(q_i)_{i\in\omega}$ (written w.r.t. $(q_i)_{i\in\omega}$). We usually identify a special point α_i with its number i and say "find a special point such that ..." in place of "find a number i such that α_i ...".

Example. The following metric spaces possess computable strucutres:

(i) The unit interval $[0,1]$ with the usual distance metric.

(ii) Cantor space $\{0,1\}^\omega$, consisting of the functions $f\colon \omega \to \{0,1\}$ with the distance function $d(f,g) = \max\{2^{-n}\colon\ f(n) \neq g(n)\}$, (where $\max \emptyset = 0$).

(iii) The space $C[0,1]$ of the continuous functions on the unit interval with the pointwise supremum metric.

Definition. Let $\mathcal{M}$ and $\mathcal{N}$ be computable metric spaces. A map $F\colon \mathcal{M} \to \mathcal{N}$ is *computable* if there is a Turing functional Φ such that, for each x in the domain of F and for every Cauchy name χ for x, the functional Φ enumerates a Cauchy name for $F(x)$ using χ as an oracle[1].

To emphasize which computable structures we consider, we say that a map F is computable with respect to $(\alpha_i)_{i\in\omega}$ and $(\beta_i)_{i\in\omega}$ (written w.r.t. $(\alpha_i)_{i\in\omega}$ and $(\beta_i)_{i\in\omega}$). The composition of two computable maps is computable.

In the special case of isometric (more generally, bi-Lipschitz) maps, Definition 6.1.1 is equivalent to saying that for every special point α_i in $\mathcal{M}$ the point $F(\alpha_i)$ is computable uniformly in i. *We will use this observation without explicit reference to it.*

Problems A and B discussed in the overview of Part I have natural analogs for computable metric spaces. For instance, one may raise the question of existence or uniqueness of computable structures on a given space. This leads to the notions of equivalent structures [12] and isometric structures [85]. Also, in contrast to computable algebra, a computable space may well contain a non-computable point. As a result, one may study non-computable points which are close to being computable [74], or ask how close they are to being computable [75]. See [101] for more on computable analysis.

[1] That is, $(\Phi^\chi(n))_{n\in\omega}$ is a Cauchy name for $F(x)$.

Part II is organized so that it should better be read in order. For instance, definitions and certain facts from Chapter 6 will be used in Chapter 7, and Chapter 8 has an application of results contained in Chapter 7.

1.2.2 Summary of Chapter 6: Computably isometric Banach spaces

In this chapter we study computable isometries between computable metric spaces associated to Banach spaces.

Recall that a computable structure on a separable metric space is defined via a computable dense substructure of it, a typical example being the space of the reals with the computable subspace of the rationals. Clearly, a metric space may have more than one computable structure. For example, on the space of continuous functions $C[0,1]$, the collection of piecewise linear functions with rational breakpoints is a computable structure, and so is the collection of all polynomials with rational coefficients. These structures are different, but given a Cauchy name of a function f in one structure, one can uniformly pass to a Cauchy name for f in another structure, and vice versa. This leads to the notion of equivalent computable structures which has been intensively studied [12, 85].

Pour-El and Richards [85] were probably the first to observe that in many natural settings the notion of equivalent structures seems too restricted. For instance, consider the reals $\mathbb{R}$ with the usual computable structure given by an effective listing of rationals $(q_i)_{i\in\omega}$. Let γ be a non-computable real. The collection $(q_i+\gamma)_{i\in\omega}$ is a computable structure on $\mathbb{R}$ not equivalent to $(q_i)_{i\in\omega}$. However, the structures $(q_i+\gamma)_{i\in\omega}$ and $(q_i)_{i\in\omega}$ may be represented by the same algorithm computing the distances between the special points. Also, there is an isometry $x \to x+\gamma$ which preserves computability of points in an algorithmically uniform way. Classically metric spaces are often considered up to surjective isometries. This example suggests to consider *computable* structures up to *computable* surjective isometries. Computable structures $(\alpha_i)_{i\in\omega}$ and $(\beta_i)_{i\in\omega}$ on a complete separable metric space (M,d) are *equivalent up to computable isometry*, or *computably isometric*, if there exists a surjective self-isometry ϕ of M and an effectively uniform algorithm which on input i outputs a Cauchy name for $\phi(\alpha_i)$ in $(\beta_i)_{i\in\omega}$. Similar notions have already appeared in literature in a different terminology (Pour-El and Richards [85] for Banach spaces, recently and independently Iljazoviĉ [51] for metric spaces).

We may think of the collection of all computable structures as of a category in which computable isometries are the morphisms. The following definition is central to the chapter:

Definition. A metric space (M, d) is *computably categorical* if every two computable structures on M are computably isometric.

It is natural to ask:

Question. Which classical metric spaces are computably categorical?

Computable categoricity of countable algebraic structures typically depends on the signature of a given structure. This basic idea turns to be useful in the study of *uncountable* metric spaces associated to Banach spaces. For instance, if a metric space $\mathcal{M}$ is associated to a Banach space, than we may ask if the addition operation is computable with respect to every computable structure on $\mathcal{M}$. We will show that if the answer is "no", then (under certain extra conditions) it implies $\mathcal{M}$ is not computably categorical. If the answer is "yes", then it is interesting on its own right. Also, in the case of Hilbert spaces, the positive answer implies computable categoricity (to be shown in Theorem 6.3.2).

Using this basic idea and the classical theorem of Mazur and Ulam, we prove several technical facts about computably categorical Banach spaces. As a consequence of these facts and a result form [85], the space $l_1 = \{(c_i)_{i \in \omega} : \sum_i |c_i| < \infty\}$ with the metric induced by the l_1-norm is not computably categorical.

We prove that for every computable structure on a Hilbert space $\mathbb{H}$, *if* 0 *is computable point then the vector space operations are computable as well.* Together with the results from [85], it implies:

Theorem 6.3.2. Every separable Hilbert space is computably categorical as a metric space.

In contrast to Hilbert spaces, we prove that the space $C[0, 1]$ of continuous functions on the unit interval has a computable structure with respect to which the operation $x \to (1/2)x$ is not computable. As we will show, this implies:

Theorem 6.4.2. The space $C[0, 1]$ with the pointwise supremum metric is not computably categorical as a metric space.

The chapter is based on the paper [70].

1.2.3 Summary of Chapter 7: Computably categorical metric spaces

This chapter continues the study of computably categorical spaces. In this chapter we mainly consider metric spaces which are not necessarily associated to Banach spaces.

One uses an effective version of the usual back-and-forth technique to show that the countable dense linear order is computably categorical as a countable algebraic structure (folklore, see also [24]). In the case of uncountable metric spaces the situation is generally more complex. Nonetheless, using a variant of the back-and-forth technique we prove:

Theorem 7.1.1. Cantor space $\{0,1\}^{\omega}$ with the metric $\max\{2^{-n} \colon f(n) \neq g(n)\}$ is computably categorical.

Cantor space with the (ultra)metric $\max\{2^{-n} \colon f(n) \neq g(n)\}$ is the central to the modern theory of computably random reals [80], [26].

The Urysohn space [97] is the Fraisse limit of finite metric spaces. It is the unique ultrahomogeneous universal separable space [57]. It is known that Urysohn space is homeomorphic to a Hilbert space (Uspenskij [98]). Remarkably, the original construction of Urysohn [97] was effective. As a consequence, the Urysohn space is computable. We show:

Theorem 7.2.1. The Urysohn space is computably categorical.

It is unknown if one can define the Urysohn space "explicitly" without using variations of the Fraisse construction or a random process. Our theorem essentially shows that the original *effective* construction due to Urysohn is the unique way one can effectively define the Urysohn space.

We characterize computably categorical subsets of $\mathbb{R}^n$, where $n \in \omega$. We introduce the notion of an intrinsically computable base which is essentially a linearly independent set computable in every computable structure, up to an isometry. We show:

Theorem 7.3.1. A computable metric space isometric to a subset of $\mathbb{R}^n$ is computably categorical if, and only if, it contains an intrinsically computable base.

This theorem resembles results on countable Boolean algebras, linear orders and other countable structures mentioned in the previous subsections. We also give an alternative characterization of computably categorical subspaces of $\mathbb{R}^n$ which does not implicitly involve the geometry of $\mathbb{R}^n$. This leads to a sufficient condition for an arbitrary space to be computably categorical. The methods developed for $\mathbb{R}^n$ and its subspaces have an application interesting on its own right. More specifically, one can show that every two computable strucutres on the unit interval (and on many other rigid subspaces of $\mathbb{R}^n$) are equivalent. As a consequence of this fact and results from the next chapter, the usual notion of K-triviality is invariant under the change of computable structure on the unit interval.

The chapter is based on the paper [70].

1.2.4 Summary of Chapter 8: K-triviality in metric spaces

Chaitin [18] and Solovay [94] were the first to study K-trivial infinite sequences of bits. In the last decade this notion has turned out to be of key importance for the interactions of computability and randomness.

Let $K(x)$ denote the prefix-free Kolmogorov complexity of a binary string x. This is a variant of the usual plain Kolmogorov complexity $C(x)$ based on a universal machine such that no string in the domain can be an initial segment of another.

We identify infinite sequences of bits with subsets of ω. A set $A \subseteq \omega$ is Martin-Löf random iff $\forall n\, [K(A \upharpoonright_n) \geq n - O(1)]$, namely, the complexity in the sense of K of its initial segments $A \upharpoonright_n$ is close to maximal. By definition, a K-trivial set A is far from random: the complexity of its initial segments is minimal up to a constant, namely

$$K(A \upharpoonright_n) \leq K(n) + O(1), \tag{1.1}$$

where the number n is identified with the string given by its binary expansion. The notion of C-triviality is defined in an analogous way. Chaitin [18] showed that each C-trivial set is computable, and that each K-trivial set is Δ^0_2. Solovay built a K-trivial set that is incomputable [94]. A much simpler construction of such a set that it also computably enumerable (c.e.) was given in [27]. The coincidence of the K-trivial sets with several other classes was shown in [79]. For instance, a set $A \subseteq \omega$ is called *low for K* if $K(y) \leq K^A(y) + O(1)$ for each string y. Nies and Hirschfeldt (see [79]) proved that K-triviality is equivalent to lowness for K. These and other results show that the K-trivial sets are very close to being computable.

We are interested in extending the notion of K-triviality to settings more general than subsets of ω. Recall that a computable strucutre on a metric space is a sequence $(\alpha_p)_{p\in\omega}$ of points dense in the space for which the distances between points are uniformly computable. We say that a point $x \in M$ is K-trivial if for each positive rational δ there is $p \in \omega$ such that

$$d(x, \alpha_p) \leq \delta \text{ and } K(\langle p, \delta\rangle) \leq K(\delta) + O(1). \tag{1.2}$$

(Here we fix some effective encoding of the positive rationals by natural numbers, and hence binary strings; by $K(\delta)$ we mean the complexity of the string encoding δ.) Note that the pair $(\langle p, \delta\rangle)$ determines an elementary closed ball $\{y \colon d(y, p) \leq \delta\}$. The intuition is that for each δ, the point x is contained in such a ball that is highly compressible as measured by δ and p. We give fourfold evidence that this is the right generalization of K-triviality to a computable metric space $\mathcal{M}$.

This class of points coincides with the one defined above in the case of Cantor space. For the unit interval, our definition of a K-trivial point yields the class of points with a K-trivial binary expansion. This establishes that the class is actually independent of the choice of base 2. Barmpalias et al. [8] introduced a notion of K-triviality for compact subsets of Cantor space. We show that their notion coincides with ours for the metric space of compact sets with Hausdorff distance. We show in Theorem 8.4.1 that every computable complete metric space without isolated points contains a dense set of K-trivial incomputable points. We show that K-triviality of a point is invariant under the change of computable structure to an equivalent one. As a consequence of the results of the previous chapter, it implies that K-triviality on the unit interval $[0,1]$ is independent of the standard definition of a computable real.

We also discuss other possible definitions of K-triviality in metric spaces. At first sight one may think that our definition should be replaced by

$$d(x, \alpha_p) \leq \delta \text{ and } K(p) \leq K(\delta) + O(1). \tag{1.3}$$

However, this is not the right generalization of K-triviality for various reasons, as we will see. For instance, this weaker definition does not imply the usual K-triviality in Cantor space. On the other hand, each K-trivial set A satisfies our initial condition in Cantor space: for each n let p_n be the number so that $\alpha_{p_n} = A \cap \{0, \ldots, n-1\}$. But K-triviality in Cantor space seems to be much stronger: since we can compute the tuple $(p_0, \ldots, p_{n-1})$ from $A \upharpoonright_n$, we have in fact $K(p_0, \ldots, p_{n-1}) \leq K(n) + O(1)$. This condition says that the point has a K-trivial Cauchy name: we say that a *Cauchy name* for a point x is a sequence $(p_s)_{s \in \omega}$ of special points converging to x such that $d(p_s, p_t) \leq 2^{-s}$ for each $t > s$ (the notion of K-triviality has a natural generalization to functions from ω to ω, as we will discuss in detail).

It may seem that our initial "local condition" , which talks about each n separately, is inadequate, because it relies on the particular structure of Cantor space. In this case there would be no reasonable way to extend this notion to the general setting of a computable metric space. However, this is not the case, as we show:

Theorem 8.3.1. Every K-trivial point in a computable metric space has a K-trivial Cauchy name.

The theorem above shows that in fact for each K-trivial point we can find a sequence $(p_n)_{n \in \omega}$ in such that $K(p_0, \ldots, p_{n-1}) \leq K(n) + O(1)$.

The chapter is based on the paper [74].

Part I

Computable abelian groups

Chapter 2

Limitwise monotonic sequences

In this chapter we study uniformly limitwise monotonic sequences of sets, and then apply our computably-theoretic results to abelian groups, and other structures.

2.1 Preliminaries

Recall the following definition:

Definition 2.1.1. *An ordered sequence of sets* $\mathcal{S} = \{S_n\}_{n\in\omega}$ *is* uniformly limitwise monotonic *(relative to* $\mathbf{a}$*, in* $\mathbf{a}$*) if there is a computable (*$\mathbf{a}$*-computable) function* g *such that*

- $\max_x g(n, y, x)$ *exists for every* $n, y \in \omega$*, and*
- $S_n = \operatorname{rng}(\lambda y[\max_x g(n, y, x)])$*, for every* $n \in \omega$*.*

We say that g *is a uniform limitwise monotonic (relative to* $\mathbf{a}$*, in* $\mathbf{a}$*) approximation of* $\mathcal{S}$*.*

Here we prove two properties of sequences of sets. The first property states that uniformly limitwise monotonic sequences of infinite sets possess injective enumerations. This is a uniform statement of the result of Harris [43] for limitwise monotonic sets. The second property gives a necessary and sufficient condition for a uniformly Σ^0_2–sequence of sets to be uniformly limitwise monotonic.

Proposition 2.1.1. (Harris [43]) Suppose $\mathcal{S} = \{S_n\}_{n\in\omega}$ is a uniformly limitwise monotonic sequence of infinite sets. Then there is a limitwise monotonic approximation $g(n, y, x)$ of $\mathcal{S}$ such that $\lambda y[\max_x g(n, y, x)]$ is injective for every $n \in \omega$.

The following proof is a uniform version of the proof for the case of a single set (see [43] or [29]).

Proof. Let $f(n, y, x)$ be a uniform limitwise monotonic approximation for $\mathcal{S}$. Without loss of generality we can assume that $\lambda x[f(n, y, x)]$ is non-decreasing for every $n, y \in \omega$.

First we set $g(n, y, x) = 0$ if $x \leq y$. To define $g(n, y, x)$ for $x > y$, suppose that for every $y' < y$ and $x' < x$ the values of $g(n, y', x)$ and $g(n, y, x')$ have already been defined. Choose $\langle y_0, x_0 \rangle$ least such that $x_0 > x$, $f(n, y_0, x_0) \geq \max_{x'<x} g(n, y, x')$ and $f(n, y_0, x_0) \notin \{g(n, y', x) \mid y' < y\}$. Set $g(n, y, x) = f(n, y_0, x_0)$.

It is not hard to check that $\lambda y[\max_x g(n, y, x)]$ is total and injective. It is also evident that $\operatorname{rng}(\lambda y[\max_x g(n, y, x)]) \subseteq S_n$, for every $n \in \omega$. To see that $S_n \subseteq \operatorname{rng}(\lambda y[\max_x g(n, y, x)])$, we use an inductive argument. The definition of g may be viewed as a construction, where at each stage we compute the value of g for exactly one new pair of arguments, and the value of f for a new pair of arguments as well. We denote our current guess about $\max_x g(n, y, x)$ at stage s by $[\max_x g(n, y, x)]_s$, and similarly for f. Suppose there are y and n such that $\max_x f(n, y, x) \notin \operatorname{rng}(\lambda y[\max_x g(n, y, x)])$, and y is least with this property.

Then there should be a stage s such that for every stage $t \geq s$ and every $y' < y$, $[\max_x f(n, y', x)]_s = [\max_x f(n, y', x)]_t \in \operatorname{rng}(\lambda y[\max_x g(n, y, x)])$. We may further assume that s satisfies $[\max_x f(n, y, x)]_s = \max_x f(n, y, x)$. By the definition of g, there should be a stage $t_0 \geq s$ and an argument y_0 such that $[\max_{x'<x} g(n, y_0, x')]_{t_0} = \max_x f(n, y, x)$, since we always start with $g(n, y, x) = 0$ for $x \leq y$. If there is no stage $t_1 \geq t_0$ and $y_1 < y_0$ such that $[\max_x g(n, y_1, x)]_{t_1} = [\max_x g(n, y_0, x)]_{t_0}$ then $\max_x g(n, y_0, x) = \max_x f(n, y, x)$. Therefore there should exist y_1 and a stage t_1 such that $[\max_x g(n, y_1, x)]_{t_1} = [\max_x g(n, y_0, x)]_{t_0}$. We can use the same argument to find y_2 and t_2 which play the same role for y_1 and t_1 as the latter arguments do for y and t. Since $y = y_0 > y_1 > \ldots$ we will find the least y_i in this sequence. But then $\max_x g(n, y_i, x) = \max_x f(n, y, x)$, contrary to the hypothesis. □

We will need the following lemma that gives a necessary and sufficient condition for a Σ^0_2–set to be limitwise monotonic.

Lemma 2.1.1. (Folklore; see, e.g., [29]). An infinite Σ^0_2–set is limitwise monotonic if and only if it contains an infinite limitwise monotonic subset.

Proof. Let S be a Σ^0_2–set and $U \subset S$ be limitwise monotonic and infinite. Since S is a Σ^0_2–set, there exists a computable function h such that for every $z \in \omega$ we have

$$z \in S \iff W_{h(z)} \text{ is finite.}$$

Let $f(y, s)$ be a limitwise monotonic approximation of U. We shall define g so that g is a limitwise monotonic approximation of S.

Fix a computable list $\{(z_k, y_k, s_k)\}_{k\in\omega}$ of all triples with the property that $f(y_k, s_k) \geq z_k$ for all $k \in \omega$. Define

$$g(k, s) = \begin{cases} z_k & \text{if } W_{h(z_k),s} \subseteq W_{h(z_k),s_k}, \\ f(y_k, s) & \text{otherwise.} \end{cases}$$

We claim that g is a limitwise monotonic approximation of S. Indeed, first note that for all $k, s \in \omega$, we have $g(k, s) \leq g(k, s+1)$. It is also easy to see that $\max_s g(k, s)$ exists for all $k \in \omega$, as $\max_s g(k, s) \leq \max_s f(k, s)$.

Assume that $z \in S$. Since U is an infinite set, there exists a k such that $f(y_k, s_k) \geq z_k$ where $z = z_k$ and $W_{h(z_k)} = W_{h(z_k),s_k}$. Thus, for all s we have $W_{h(z_k),s} \subseteq W_{h(z_k),s_k}$. Therefore $\lim_s g(k, s) = z_k = z$. Now assume that $z \notin S$. Then the set $W_{h(z)}$ must be infinite. Let k, s be such that $g(k, s) = z$. There is an $s' > s$ such that $W_{h(z_k),s'} \nsubseteq W_{h(z_k),s_k}$. Then $g(k, s') = f(y_k, s')$ for all $s' > s$. We conclude that $z \neq \lim_s g(k, s)$ for all $k \in \omega$. □

We will need a uniform version of the lemma above. For a set $A \subseteq \omega$, we define $\sup A = \max A$ if A is finite, and $\sup A = \infty$ otherwise.

Proposition 2.1.2. Suppose $\mathcal{S} = \{S_n\}_{n\in\omega}$ is uniformly Σ^0_2. Assume that there is a uniformly limitwise monotonic sequence $\mathcal{U} = \{U_n\}_{n\in\omega}$ such that $U_n \subseteq S_n$ and $\sup U_n = \sup S_n \in \omega \cup \{\infty\}$, for every $n \in \omega$. Then $\mathcal{S}$ is uniformly limitwise monotonic.

Proof. We carry out the proof of the lemma above uniformly in n. There is a computable function h such that for all $z, n \in \omega$ we have

$$z \in S_n \iff W_{h(n,z)} \text{ is finite.}$$

Let $f(n, y, s)$ be a uniform limitwise monotonic approximation for the sequence $\mathcal{U} = \{U_n\}_{n\in\omega}$.

For each $n \in \omega$, let $\{(z^n_k, y^n_k, s^n_k)\}_{k\in\omega}$ be a uniformly computable listing of all triples such that $f(n, y^n_k, s^n_k) \geq z^n_k$. Define

$$g(n, k, s) = \begin{cases} z^n_k & \text{if } W_{h(n,z^n_k),s} \subseteq W_{h(n,z^n_k),s^n_k}, \\ f(n, y^n_k, s) & \text{otherwise.} \end{cases}$$

Taking into account that $\sup U_n = \sup S_n$, one proves, as in the lemma above, that $\lambda n, k[\max_s g(n, k, s)]$ is total and $S_n = \operatorname{rng}(\lambda k[\max_s g(n, k, s)])$ for $n \in \omega$. $\square$

2.2 Relative to every hyperimmune degree.

In this section we prove:

Theorem 2.2.1. *There is a sequence of infinite sets which is uniformly limitwise monotonic relative to every hyperimmune degree (in particular, relative to every nonzero Δ^0_2-degree), but is not uniformly limitwise monotonic.*

Proof. In order to prove Theorem 2.2.1 we need to build a sequence $\mathcal{S} = \{S_n\}_{n\in\omega}$ of infinite sets such that $\mathcal{S}$ satisfies the following conditions:

(a) $\mathcal{S}$ is not uniformly limitwise monotonic, and

(b) $\mathcal{S}$ is uniformly limitwise monotonic relative to every hyperimmune degree.

The definition of the family $\mathcal{S}$ is simply a diagonalization construction. We define the *n-th* set S_n as follows:

$$S_n = \begin{cases} \omega - \{\max_x \varphi_n(x)\} & \text{if } \max_x \varphi_n(x) \text{ exists,} \\ \omega & \text{otherwise.} \end{cases}$$

We do not assume that φ_n is total; we simply say that $\max_x \varphi_n(x)$ is defined if $\{y : \varphi_n(x) \downarrow = y\}$ not empty and, furthermore, has a maximal element. We assume for a contradiction that the sequence $\mathcal{S}$ is uniformly limitwise monotonic.

Then there exists a computable function g such that (1) for all $n, y \in \omega$ the value $\max_x g(n, y, x)$ exists, and (2) $S_n = \operatorname{rng}(\lambda y[\max_x g(n, y, x)])$ for all $n \in \omega$. Let f be a computable function such that $\varphi_{f(n)} = \lambda x[g(n, 0, x)]$ for every n. Then $a_n = \max_x \varphi_{f(n)}(x)$ exists and so $a_n \notin S_{f(n)}$. On the other hand, we have $a_n = \max_x g(n, 0, x) \in S_n$ for every n. Hence, $\varphi_{f(n)} \neq \varphi_n$ for every $n \in \omega$. We have a contradiction with the Recursion Theorem. Thus, $\mathcal{S}$ is not uniformly limitwise monotonic.

Now we show that $\mathcal{S}$ is uniformly limitwise monotonic relative to every hyperimmune degree $\mathbf{x}$. Fix a function $r \leq_T \mathbf{x}$ such that no computable function dominates r. We define an $\mathbf{x}$-computable function g as follows. First, set $g(n, \langle m, u\rangle, 0) = m$ for all n, m and u. Supposing $g(n, \langle m, u\rangle, s)$ has been defined with value k, we set

$$g(n, \langle m, u\rangle, s+1) = \begin{cases} k+1 & \text{if } u < s \text{ and } \max_{x<t} \varphi_{n,t}(x) = k, \\ k & \text{otherwise,} \end{cases}$$

where $t = \max\{r(k), s\}$. Fixing n and m, one can see that $m \neq \max_x \varphi_n(x)$ if and only if $\max_s g(n, \langle m, u\rangle, s) = m$ for some u. Furthermore, if $k = \max_s g(n, \langle m, u\rangle, s)$ exists, then $k \neq \max_x \varphi_n(x)$.

Now assume that $\sup_s g(n, \langle m, u\rangle, s) = \infty$ for some n, m and u. It must be the case that $\sup_x \varphi_n(x) = \infty$. For each $k \in \omega$, define $h(k)$ to be the least integer such that

$$k < \max_{x<h(k)} \varphi_{n,h(k)}(x).$$

By our hypothesis, the computable function h fails to dominate r. Therefore there must be an integer $k > m$ such that $h(k) < r(k)$. But the definition of g ensures that $\max_s g(n, \langle m, u\rangle, s) \leq k$, contrary to our assumption. Indeed, suppose $g(n, \langle m, u\rangle, s) = k$, for some s. We show that $g(n, \langle m, u\rangle, s+1) = k$. Let $t = \max\{r(k), s\}$. We have $t > h(k)$ and $\max_{x<t} \varphi_{n,t}(x) \geq \max_{x<h(k)} \varphi_{n,h(k)}(x) > k$. By the definition of g, $g(n, \langle m, u\rangle, s+1) = k$. This is a contradiction. Thus, $\max_s g(n, y, s)$ exists for every $n, y \in \omega$, and $S_n = \operatorname{rng}(\lambda y[\max_s g(n, y, s)])$ for all $n \in \omega$. □

2.3 Relative to every non-recursive Δ^0_2 degree

The main result of this section is:

Theorem 2.3.1. *There is a Σ^0_2 set S such that S is limitwise monotonic in every nonzero Δ^0_2-degree, but is not limitwise monotonic.*

Preliminary remarks. The proof is similar to the proof of the previous theorem. The difference is that now we work within columns of a single set, not within different sets in a sequence, and our strategies will interact. In the situation of a single set, we need a finite injury argument combined with the permitting strategy described in detail in the previous paragraph. We give a formal proof below.

Proof. Recall that we have to build an infinite Σ^0_2 set S which is not limitwise monotonic, but is limitwise monotonic in every nonzero Δ^0_2 degree. It is well-known that every nonzero Δ^0_2 degree is hyperimmune.

Let $\{\psi_n\}_{n\in\omega}$ be a computable listing of all partial $\emptyset'$-computable functions. To be more specific, we define the n'th partial Δ_2^0 function to be $\lim_k \varphi_n(x,k)$, where φ_n is the n'th partial computable function of two arguments. Thus, the limit and even $\varphi_n(x,k)$ for some or all (n,k) may be undefined. The listing, however, cover all total Δ_2^0 functions, by Limit Lemma. We need to build a set $S \in \Sigma_2^0$ that satisfies the following requirements:

$N_i : \lambda y[\max_x \varphi_i(y,x)]$ is total and injective $\Longrightarrow S \neq \mathrm{rng}\,(\lambda y[\max_x \varphi_i(y,x)])$;

$R_n : \psi_n$ is total and ψ_n is not computably dominated $\Longrightarrow S$ is limitwise monotonic relative to ψ_n.

Note that by Proposition 2.1.1, the requirements N_i guarantee that S is not limitwise monotonic. To satisfy R_n we define a (trace) function $\lambda m, s[g_n(m,s)]$ with the following properties:

1. The function $\lambda m, s[g_n(m,s)]$ is total and computable in ψ_n if ψ_n is total.
2. For each n and m, the value of $g_n(m,s)$ is equal to $\langle n,m,k\rangle$, for some k.
3. The function $\lambda m[\max_s g_n(m,s)]$ is injective on its domain, and satisfies the following sub-requirements for all $m \in \omega$:

$R_{n,m} : \psi_n$ is total and ψ_n is not computably dominated $\Longrightarrow \max_s g_n(m,s)$ exists, and $\max_s g_n(m,s) \in S$.

Note that, by Proposition 2.1.2, if the requirements $R_{n,m}$ are met for every m, then the requirement R_n is met. We order the requirements effectively in such a way that N_i is of a higher priority than $R_{n,m}$ if $i \leq \langle n,m\rangle$.

The strategy for N_i is to keep $\max_x \varphi_i(j,x)$ for at least one j outside S, where $j \leq i$. All these values are restrained for the $R_{n,m}$-requirements of lower priority. The strategy can be injured by at most i many traces $\max_s g_n(m,s) \in S$ that are of higher priority than N_i (that is, $\langle m,n\rangle < i$). Thus, N_i wins by keeping $\max_x \varphi_i(j,x)$ outside of S for some $j \leq i$.

The strategy for each $R_{n,m}$ is to define functions $g_n(m,s)$ which we call "traces", for $s \in \omega$, avoiding the numbers restrained by N_i-requirements of higher priorities. The definition will be similar to the one we had in the proof of Theorem 2.2.1. We need to keep in S the value of $\max_s g_n(m,s)$. Here we have to be more careful because it may happen that ψ_n is not total. In this case the naive definition of S (e.g., as the collection of final traces for all n and m) can cause $S \notin \Sigma_2^0$. To circumvent this problem we give a more accurate definition of S (see below).

For each $n \in \omega$, define a partial function g_n by induction as follows. Set $g_n(m,0) =$

$\langle n, m, 0\rangle$ and

$$g_n(m, s+1) = \begin{cases} \langle n, m, k{+}1\rangle & \text{if } g_n(m,s)\downarrow = \langle n,m,k\rangle,\ \ \psi_n(k)\downarrow, \text{ and} \\ & \langle n,m,k\rangle \in \{\max_{x<t}\varphi_{i,t}(j,x) \mid j \le i \le \langle n,m\rangle\}, \\ & \text{where } t = \max\{\psi_n(k), s\}, \\ \langle n,m,k\rangle & \text{if } g_n(m,s)\downarrow = \langle n,m,k\rangle,\ \ \psi_n(k)\downarrow, \text{ and} \\ & \langle n,m,k\rangle \notin \{\max_{x<t}\varphi_{i,t}(j,x) \mid j \le i \le \langle n,m\rangle\}, \\ & \text{where } t = \max\{\psi_n(k), s\}, \\ \uparrow & \text{otherwise.} \end{cases}$$

By its definition, $\{g_n\}_{n\in\omega}$ is a uniformly computable sequence of partial $\emptyset'$-computable functions. Now set

$$S = \{\langle n,m,k\rangle \mid (\exists s)[g_n(m,s) = \langle n,m,k\rangle \ \& \\ (\forall u \ge s)(\forall i \le \langle n,m\rangle)(\forall j \le i)[\langle n,m,k\rangle \ne \max_{x<u}\varphi_{i,u}(j,x)]]\}.$$

By its definition, S is Σ^0_2. Furthermore, if $\langle n,m,k\rangle \in S$ then $\langle n,m,k\rangle = \max_s g_n(m,s)$. Indeed, let $\langle n,m,k\rangle \in S$. Then $g_n(m,s) = \langle n,m,k\rangle$ and

$$(\forall u \ge s)(\forall i \le \langle n,m\rangle)(\forall j \le i)[\langle n,m,k\rangle \ne \max_{x<u}\varphi_{i,u}(j,x)],$$

for some s. It follows that $g_n(m,u) = \langle n,m,k\rangle$ or $g_n(m,u)\uparrow$, for all $u \ge s$. Thus, for every n, m there exists at most one k such that $\langle n,m,k\rangle \in S$.

First, we show that S is limitwise monotonic relative to every $A \in \Delta^0_2 - \Delta^0_1$. Fixing $A \in \Delta^0_2 - \Delta^0_1$, since A is hyperimmune, there is an $n \in \omega$ such that ψ_n is total, $\psi_n \le_T A$, and ψ_n is not dominated by any computable function. Thus, g_n is a total function, and $g_n \le_T A$.

We claim that if $\max_s g_n(m,s) = \langle n,m,k\rangle$ then $\langle n,m,k\rangle \in S$. Fix an $s > \psi_n(k)$ such that $g_n(m,u) = \langle n,m,k\rangle$ for all $u \ge s$. Then we have

$$(\forall u \ge s)(\forall i \le \langle n,m\rangle)(\forall j \le i)[\langle n,m,k\rangle \ne \max_{x<u}\varphi_{i,u}(j,x)].$$

Therefore, $\langle n,m,k\rangle \in S$.

For the sake of contradiction, suppose $\sup_s g_n(m,s) = \infty$ for some $m \in \omega$. Then the finite set $I = \{\langle i,j\rangle \mid j \le i \le \langle n,m\rangle \ \& \ \sup_x \varphi_i(j,x) = \infty\}$ is not empty. Choose an integer $s_0 \in \omega$ such that

$$\text{if } j \le i \le \langle n,m\rangle \text{ and } \langle i,j\rangle \notin I \text{ then } \max_x \varphi_i(j,x) < s_0.$$

For every $k \in \omega$, define $h(k)$ to be the least integer such that

$$\langle n, m, k\rangle < \min_{\langle i,j\rangle \in I} \max_{x<h(k)} \varphi_{i,h(k)}(j, x).$$

By the choice of n, there must be an integer $k > s_0$ such that $h(k) < \psi_n(k)$.

The definition of g_n ensures that $\max_s g_n(m, s) \leq \langle n, m, k\rangle$, contrary to our assumption. Indeed, assume $g_n(m, s) = \langle n, m, k\rangle$ for some s. We show that $g_n(m, s + 1) = \langle n, m, k\rangle$.

Let $t = \max\{\psi_n(k), s\}$. Note that $t \geq \psi_n(k) > h(k)$. If $j \leq i \leq \langle n, m\rangle$ and $\langle i, j\rangle \in I$ then

$$\max_{x<t} \varphi_{i,t}(j, x) \geq \max_{x<h(k)} \varphi_{i,h(k)}(j, x) > \langle n, m, k\rangle.$$

If $j \leq i \leq \langle n, m\rangle$ and $\langle i, j\rangle \notin I$, then

$$\langle n, m, k\rangle \geq k > s_0 > \max_x \varphi_i(j, x) \geq \max_{x<t} \varphi_{i,t}(j, x).$$

Thus, whether $\langle i, j\rangle \in I$ or $\langle i, j\rangle \notin I$,

$$\langle n, m, k\rangle \notin \{\max_{x<t} \varphi_{i,t}(j, x) \mid j \leq i \leq \langle n, m\rangle\}.$$

By the definition of g_n, we have $g_n(m, s + 1) = \langle n, m, k\rangle$.

Thus, $\max_s g_n(m, s)$ exists for every $m \in \omega$, $\lambda m[\max_s g_n(m, s)]$ is injective, and $\operatorname{rng}(\lambda m[\max_s g_n(m, s)]) \subseteq S$. By Lemma 2.1.1, the infinite set S is limitwise monotonic in A.

We prove that S is not limitwise monotonic. Suppose S is limitwise monotonic. Then, by Proposition 2.1.1, there is an $i \in \omega$ such that

$$S = \operatorname{rng}(\lambda j[\max_x \varphi_i(j, x)]),$$

where $\lambda j[\max_x \varphi_i(j, x)]$ is total and injective. Hence, the finite set

$$\{\max_x \varphi_i(j, x) \mid j \leq i\} \subseteq S$$

has cardinality $i + 1$. Observe that for every n and m there exists at most one k such that $\langle n, m, k\rangle \in S$. Therefore, there are integers n, m, k and $j \leq i$ such that $i \leq \langle n, m\rangle$ and $\max_x \varphi_i(j, x) = \langle n, m, k\rangle \in S$. But by the definition of S we have $\langle n, m, k\rangle \neq \max_x \varphi_i(j, x)$, for $j \leq i \leq \langle n, m\rangle$. Thus, S is not limitwise monotonic. □

2.4 Relative to every degree except perhaps countably many

We prove:

Theorem 2.4.1. *If a sequence of sets $\{S_n\}_{n\in\omega}$ is uniformly limitwise monotonic in all degrees except perhaps countably many, then $\{S_n\}_{n\in\omega}$ is uniformly limitwise monotonic.*

Recall that a tree $T \subseteq 2^{<\omega}$ is *splitting* if for every $\sigma \in T$ there exist incomparable strings $\rho_0, \rho_1 \in T$, such that $\sigma \subseteq \rho_0$ and $\sigma \subseteq \rho_1$. Theorem 2.4.1 follows from the technical lemma below:

Lemma 2.4.1. Suppose a sequence $\{S_n\}_{n\in\omega}$ is not uniformly limitwise monotonic. Let Φ be a Turing operator and $T \subseteq 2^{<\omega}$ be a non-empty computable splitting tree. Then there is a non-empty computable splitting subtree $T_1 \subseteq T$ such that the condition

$$(\forall n)(\forall k)[\max_s \Phi^X(n,k,s) \text{ exists }] \;\&\; (\forall n)[S_n = \mathrm{rng}\,(\lambda k[\max_s \Phi^X(n,k,s)])]$$

(that is, Φ^X is a uniform limitwise monotonic approximation of $\{S_n\}_{n\in\omega}$) fails for every infinite path X through T_1.

First we prove the theorem using Lemma 2.4.1, and then we prove the lemma.

Proof of Theorem 2.4.1. Let $\{S_n\}_{n\in\omega}$ be uniformly limitwise monotonic in all degrees except perhaps countably many. We prove that $\{S_n\}_{n\in\omega}$ has to be uniformly limitwise monotonic.

Assume that $\{S_n\}_{n\in\omega}$ is not uniformly limitwise monotonic. We will build uncountably many sets X satisfying the requirements

$$P_m^X : \Phi_m^X \text{ is not a uniform limitwise monotonic approximation of } \{S_n\}_{n\in\omega},$$

for a fixed effective list of all Turing operators $\{\Phi_m\}_{m\in\omega}$.

To make sure that each requirement P_m^X is met for X, we apply Lemma 2.4.1. More specifically, for each computable splitting tree T and each Turing operator Φ, we fix a subtree $P(\Phi, T) \subseteq T$ such that Φ^X is not a uniform limitwise monotonic approximation of $\{S_n\}_{n\in\omega}$, for every infinite path X through $P(\Phi, T)$.

For a non-empty computable splitting tree $T \subseteq 2^{<\omega}$, define

$$R_0(T) = \{\sigma \in T \mid \sigma \subseteq \rho_0 \text{ or } \rho_0 \subseteq \sigma\}$$

and

$$R_1(T) = \{\sigma \in T \mid \sigma \subseteq \rho_1 \text{ or } \rho_1 \subseteq \sigma\},$$

where $\rho_0, \rho_1 \in T$ are incomparable. We have $R_0(T) \subseteq T$, $R_1(T) \subseteq T$, $[R_0(T)] \cap [R_1(T)] = \emptyset$, and $R_0(T)$ and $R_1(T)$ are non-empty (computable) splitting trees.

Let $h : \omega \longrightarrow \{0,1\}$ be any (not necessarily computable) function. Let X_h be an infinite path through $\bigcap_{k\in\omega} T_k$, where $\{T_k\}_{k\in\omega}$ is the family of computable splitting trees defined recursively as follows:

1. $T_0 = 2^{<\omega}$;
2. $T_{2m+1} = R_{h(m)}(T_{2m}),\ m \in \omega$;
3. $T_{2m+2} = P(\Phi_m, T_{2m+1}),\ m \in \omega$.

Then the set X_h satisfies P_m, for all $m \in \omega$, and the map $h \mapsto X_h$ is injective. Thus, $\{S_n\}_{n\in\omega}$ is not uniformly limitwise monotonic relative to $2^{\aleph_0}$ many different oracles. This is a contradiction. □

Proof of Lemma 2.4.1. For each n consider the set

$$\begin{aligned} M_n = \{y \mid (\exists k)(\exists s)(\exists \sigma \in T)[\Phi^\sigma(n,k,s)\downarrow = y\ \& \\ (\forall \tau \in T)(\forall s')(\forall y')[\sigma \subseteq \tau\ \&\ \Phi^\tau(n,k,s')\downarrow = y' \to y' \leq y]]\}. \end{aligned}$$

We have following cases:

Case 1. There exists $y \in M_n - S_n$, for some y, n.

Then we can choose $\sigma \in T$ such that $y = \Phi^\sigma(n,k,s)\downarrow$ and

$$(\forall \tau \in T)(\forall s')(\forall y')[\sigma \subseteq \tau\ \&\ \Phi^\tau(n,k,s')\downarrow = y' \to y' \leq y]\},$$

for some k, s. We can set $T_1 = \{\tau \in T \mid \tau \subseteq \sigma \text{ or } \sigma \subseteq \tau\}$, since $y = \max_s \Phi^X(n,k,s) \notin S_n$, for every infinite path X through T_1.

Case 2. There exists $y \in S_n - M_n$, for some y, n.

Define a computable function

$$f : 2^{<\omega} \longrightarrow T$$

by recursion as follows:

1. Define $f(\lambda) = \lambda$, where λ is the empty string.

2. Suppose we have $f(\alpha) = \sigma$, and the length of α has the form $\ell = \langle k, t\rangle$ (it is important for us to make sure that for each k we will have infinitely many corresponding lengths of the form $\ell = \langle k, i\rangle$). Suppose $\Phi^{\sigma}(n, k, s)[t] \downarrow= y$, for some $s < t$. Since $y \notin M_n$ we can find a string $\tau \in T$, $\sigma \subseteq \tau$, such that $\Phi^{\tau}(n, k, s') \downarrow> y$, for some s'. If $\Phi^{\sigma}(n, k, s)[t] \uparrow$ or $\Phi^{\sigma}(n, k, s)[t] \downarrow\neq y$ for every $s < t$, then set $\tau = \sigma$. Then we choose incomparable strings $\rho_0, \rho_1 \in T$, such that $\tau \subseteq \rho_0$ and $\tau \subseteq \rho_1$, and define $f(\alpha * i) = \rho_i$, for $i \in \{0, 1\}$.

Let $T_1 = \{\sigma \mid (\exists \beta \in 2^{<\omega})[\sigma \subseteq f(\beta)]\}$. Suppose X is an infinite path through T_1. For every $k \in \omega$ we have only two possibilities. First, there is a string $\tau \subseteq X$ such that $\Phi^{\tau}(n, k, s) \downarrow> y$, for some s. Second, there are no $\sigma \in T_1$ and s such that $\Phi^{\sigma}(n, k, s) \downarrow= y$. Hence,

$$y \in S_n - \mathrm{rng}\,(\lambda k[\max_s \Phi^X(n, k, s)]),$$

for every infinite path X through T_1.

Case 3. $S_n = M_n$, for every n.

In this case, the sequence $\{S_n\}_{n\in\omega}$ is uniformly Σ^0_2 (see the definition of M_n).

For every $m \in \omega$, let $\psi(n, m) = \langle \sigma_{n,m}, k_{n,m}, y_{n,m}\rangle$ be the first found triple $\langle \sigma, k, y\rangle$ such that $\sigma \in T$ and $\Phi^{\sigma}(n, k, s) \downarrow= y \geq m$, for some s. Set $\psi(n, m)$ to be undefined if such a triple does not exist. Clearly, ψ is a partial computable function.

Note that if $\psi(n, m)$ is defined, then $\psi(n, m')$ is defined for every $m' < m$.

Suppose firstly that $\psi(n, m)$ is undefined, for some $m \in S_n$. We have

$$m \in S_n - \mathrm{rng}\,(\lambda k, s[\Phi^X(n, k, s)]) \subseteq S_n - \mathrm{rng}\,(\lambda k[\max_s \Phi^X(n, k, s)]),$$

for every infinite path X through T. Then we can set $T_1 = T$. Hence we assume that $\psi(n, m)$ is defined for every $n, m \in \omega$ such that $m \leq \sup S_n$.

The intuition behind the formal argument below is as follows. We attempt to define a uniform limitwise monotonic approximation of some subset of $\{S_n\}_{n\in\omega}$ using the fact that $\{S_n\}_{n\in\omega} = \{M_n\}_{n\in\omega}$. Since by Proposition 2.1.2 we cannot succeed, we will have to have an infinite splitting subtree of T witnessing this failure.

Define partial computable functions

$$f : \omega \times \omega \times 2^{<\omega} \longrightarrow T \text{ and } g : \omega \times \omega \times 2^{<\omega} \longrightarrow \omega$$

by recursion, as follows:

1. With λ the empty string, define

$$f(n, m, \lambda) = \sigma_{n,m} \text{ and } g(n, m, \lambda) = y_{n,m},$$

   if $\psi(n, m) \downarrow= \langle \sigma_{n,m}, k_{n,m}, y_{n,m} \rangle$, and set $f(n, m, \lambda) \uparrow$ and $g(n, m, \lambda) \uparrow$ if $\psi(n, m)$ is undefined. Note that $f(n, m, \lambda)$ and $g(n, m, \lambda)$ are defined if $m \leq \sup S_n$.

2. Suppose $f(n, m, \alpha)$ and $g(n, m, \alpha)$ have already been defined. Let $\langle \tau, y \rangle$ be the first found pair such that $\tau \in T$, $f(n, m, \alpha) \subseteq \tau$, $g(n, m, \alpha) < y$, and $\Phi^{\tau}(n, k_{n,m}, s) \downarrow= y$, for some s. Then we choose incomparable strings $\rho_0, \rho_1 \in T$ such that $\tau \subseteq \rho_0$, $\tau \subseteq \rho_1$. We set

$$f(n, m, \alpha * i) = \rho_i \text{ and } g(n, m, \alpha * i) = y,$$

   for $i \in \{0, 1\}$. If $f(n, m, \alpha), g(n, m, \alpha)$ are undefined or $\langle \tau, y \rangle$ does not exist, then $f(n, m, \alpha * i)$ and $g(n, m, \alpha * i)$ remain undefined.

It is crucial that if $g(n, m, \alpha)$ is defined and $g(n, m, \alpha * i)$ is undefined then $g(n, m, \alpha) \in M_n = S_n$.

Suppose that for every $n, m \in \omega$ there is a string $\beta \in 2^{<\omega}$ such that $g(n, m, \beta)$ is undefined. By our assumption, β is not the empty string if $m \leq \sup S_n$. Hence, if $m \leq \sup S_n$ then there is a string $\alpha \in 2^{<\omega}$, such that $g(n, m, \alpha)$ is defined, but $g(n, m, \alpha * 0)$ is undefined (recall that $g(n, m, \alpha * 0) \downarrow$ iff $g(n, m, \alpha * 1) \downarrow$).

Consider the sequence $\{U_n\}_{n\in\omega}$, where $U_n = \{h(n, m) \mid g(n, m, \lambda) \downarrow\}$, and $h(n, m) = \min\{y \mid (\exists \alpha \in 2^{<\omega})[g(n, m, \alpha) \downarrow= y \text{ and } g(n, m, \alpha * 0) \uparrow]\}$.

We show that $\{U_n\}_{n\in\omega}$ is uniformly limitwise monotonic. By the definition of h, $\mathrm{dom}(h) = \mathrm{dom}(\psi)$ is c.e., and $g(n, m, \alpha * i) \downarrow$ implies $g(n, m, \alpha) < g(n, m, \alpha * i)$, for each $i \in \{0, 1\}$. Thus, given m and n, we can monotonically and uniformly in n and m approximate $h(n, m)$, as follows. Let g_s be the (finite) part of g computed at stage s. As usual, we may assume that $g_s(n, m, \alpha) \downarrow$ implies that the length of α is less than s. Therefore, given $n, m, u, s \in \omega$ such that $g_u(n, m, \lambda) \downarrow$ and $s \geq u$, we can effectively choose $k(n, m, u, s)$ least such that

$$g_s(n, m, \alpha) \downarrow= k(n, m, u, s) \text{ and } g_s(n, m, \alpha * i) \uparrow,$$

for some $i \in \{0, 1\}$ and $\alpha \in 2^{<\omega}$. Suppose $g_s(n, m, \alpha) \uparrow$, $g_u(n, m, \lambda) \downarrow$ and $s \geq u$. Let $\gamma \subset \alpha$ be the $\subseteq$-maximal substring of α such that $g_s(n, m, \gamma) \downarrow$. Such γ exists because $g_s(n, m, \lambda) \downarrow= g_u(n, m, \lambda)$. Fix $i \in \{0, 1\}$ so that $\gamma * i \subseteq \alpha$. By the choice of γ, we have

$g_s(n,m,\gamma)\downarrow$, $g_s(n,m,\gamma * i)\uparrow$, and, therefore,

$$g(n,m,\alpha)\downarrow \implies g(n,m,\alpha)\geq g(n,m,\gamma)\geq k(n,m,u,s).$$

Thus, $k(n,m,u,s+1)\geq k(n,m,u,s)$ for every $n,m,u,s\in\omega$ such that $g_u(n,m,\lambda)\downarrow$ and $s\geq u$. Furthermore, for every α we have $g(n,m,\alpha * 0)\downarrow$ if and only if $g(n,m,\alpha * 1)\downarrow$. This implies $\max_{s\geq u} k(n,m,u,s) = h(n,m)$. It remains to define a total limitwise monotonic approximation of $h(n,m)$ using the partial approximation $k(n,m,u,s)$, as follows. Let $M(n,z)$ be a total computable function such that

$$h(n,m)\downarrow \iff g(n,m,\lambda)\downarrow \iff \psi(n,m)\downarrow \iff (\exists z)[M(n,z)=m].$$

Define $u(n,z)$ to be equal to the least stage u such that $g_u(n,M(n,z),\lambda)\downarrow$. Set

$$H(n,z,s) = k(n,M(n,z),u(n,z),s+u(n,z)).$$

The function $H(n,z,s)$ is total and non-decreasing in s. Furthermore,

$$\max_s H(n,z,s) = \max_s k(n,M(n,z),u(n,z),s+u(n,z)) = h(n,M(n,z)),$$

for every n and z. Hence, H is a uniform limitwise monotonic approximation of $\{U_n\}_{n\in\omega}$. We have $\sup S_n = \sup U_n$, since $h(n,m)\downarrow\geq m$ for every $m\leq \sup S_n$ and $U_n\subseteq M_n = S_n$. By Proposition 2.1.2, $\{S_n\}_{n\in\omega}$ is uniformly limitwise monotonic, contrary to the hypothesis.

Thus, there exist $n,m\in\omega$ such that $f(n,m,\beta)$ and $g(n,m,\beta)$ are defined for every $\beta\in 2^{<\omega}$. Set

$$T_1 = \{\sigma \mid (\exists\beta\in 2^{<\omega})[\sigma\subseteq f(n,m,\beta)]\}\subseteq T.$$

We have $\lim_{\alpha: f(n,m,\alpha)\subset X} g(n,m,\alpha) = \infty$ for every infinite path X through T_1, and $\Phi^{f(n,m,\alpha)}(n,k_{n,m},s_\alpha) = g(n,m,\alpha)$ for some s_α. Therefore $\max_s \Phi^X(n,k_{n,m},s)$ does not exist for every infinite path X through T_1. □

2.5 Applications

Our goal is to to apply the results obtained in the previous sections to study degree spectra of structures in the classes of abelian groups, equivalence structures, and models of $\aleph_1$-categorical theories. For a background on the general theory of computable structures, see [33].

2.5.1 Abelian groups

Let $p_0, p_1, \ldots$ be the sequence of prime numbers listed in increasing order. For a prime p and integer n, the cyclic group of order p^n is denoted by Z_{p^n}. For an infinite set S ($0 \notin S$) and prime p we denote by $A_p(S)$ the group $A_p(S) = \bigoplus_{n \in S} Z_{p^n}$. We need the following uniform version of a well-known result of Khisamiev ([58], Theorem 3.4):

Lemma 2.5.1. A sequence $\mathcal{S} = \{S_n\}_{n\in\omega}$ of infinite sets of positive integers is uniformly limitwise monotonic if and only if the abelian group $G = \bigoplus_{n\in\omega} A_{p_n}(S_n)$ has a computable copy.

Proof. Assume $G = \bigcup_{s\in\omega} G_s$ is computable, where G_s is the part of G enumerated at stage s. By the definition of G, for every element a of G we can effectively choose a positive integer m_a least such that $m_a a = 0$. We have $a \in A_{p_n}(S_n)$ if and only if m_a is a power of p_n. Therefore, given $n \in \omega$ and a computable index for G, we can uniformly compute an index for the computable subgroup $A_{p_n}(S_n)$ of G. Also, given an index for $A_{p_n}(S_n)$, we can uniformly pass to a computable index for $C_n = \{c \in A_{p_n}(S_n) \mid p_n c = 0\} = \{c_i\}_{i\in\omega}$. Let $f(n,i,s) = \max\{h \mid h = 1 \vee (\exists b \in G_s)[(\forall k < h)p_n^k b \neq 0 \wedge p_n^{h-1} b = c_i]\}$. By the definition of C_n, we have $S_n = \operatorname{rng}(\lambda i[\max_s f(n,i,s)])$. The function $f(n,i,s)$ is a uniform limitwise monotonic approximation of $\mathcal{S}$.

Now suppose $\mathcal{S} = \{S_n\}_{n\in\omega}$ is uniformly limitwise monotonic. By Proposition 2.1.1, we can choose a uniform limitwise monotonic approximation $f(n,i,s)$ of $\mathcal{S}$ such that $\lambda i[\max_s f(n,i,s)]$ is injective for every n. We have $G = \bigcup_{t\in\omega} \bigoplus_{n\leq t} \left(\bigoplus_{i \leq t} Z_{p_n^{h(n,i,t)}} \right)$, where $h(n,i,t) = \max_{s\leq t} f(n,i,s)$ and $\bigoplus_{n\leq t} \left(\bigoplus_{i\leq t} Z_{p_n^{h(n,i,t)}} \right)$ is a naturally-defined subgroup of the group $\bigoplus_{n\leq t+1} \left(\bigoplus_{i\leq t+1} Z_{p_n^{h(n,i,t+1)}} \right)$. □

Theorem 2.5.1.

1. *There is a torsion abelian group G such that (a) G has no computable copy, and (b) G has an* **a***-computable copy, for every hyperimmune degree* **a***.*

2. *There is an abelian p-group A such that $DegSp(A)$ contains a Δ^0_2 degree* **a** *if and only if* **a** > 0*.*

Proof. The first part of the theorem follows from Lemma 2.5.1 (relativized) and Theorem 2.2.1, and the second part of the theorem follows from Theorem 3.4 (relativized) of [58] and Theorem 2.3.1. □

Theorem 2.5.2. *For any group G of the form $G = \bigoplus_{p\in X}\left(\bigoplus_{n\in S_p} Z_{p^n}\right)$, where X is a set of prime numbers and $S_p \subseteq \omega \setminus \{0\}$ for each $p \in X$, we have the following: If the group G has an $\mathbf{x}$-computable copy for every degree $\mathbf{x}$, except perhaps countably many, then G has a computable copy.*

Proof. The theorem follows from Lemma 2.5.1 (relativized) and Theorem 2.4.1. □

2.5.2 Equivalence relations

To apply our computably-theoretic results we need the following observation (see, e.g., [17] or [15] for a proof):

Lemma 2.5.2 ([15])**.** Suppose Θ is an equivalence structure in which all equivalence classes are finite and have distinct cardinalities $c_0, c_1, \ldots$. Then Θ has a computable copy if and only if $C = \{c_0, c_1, \ldots\}$ is limitwise monotonic.

Now the proof of the following theorem follows from Lemma 2.5.2 and Theorem 2.3.1:

Theorem 2.5.3. *There exists an equivalence structure Θ such that $DegSp(\Theta)$ contains a Δ^0_2 degree $\mathbf{a}$ if and only if $\mathbf{a} > 0$.* □

2.5.3 $\aleph_1$–Categorical theories

Recall that a first order complete theory T is *$\aleph_1$-categorical* if all models of T of cardinality $\aleph_1$ are isomorphic. There are many natural examples of $\aleph_1$-categorical theories: the theory of algebraically closed fields of a given characteristic, the theory of vector spaces over a given countable field, and the theory of one successor function on the natural numbers. Baldwin and Lachlan [6] showed that all models of a given $\aleph_1$-categorical theory T with more than one countable model (up to isomorphism) form an elementary chain $\mathcal{A}_0 \preceq \mathcal{A}_1 \preceq \ldots \preceq \mathcal{A}_\omega$ of length $\omega + 1$, where $\preceq$ stands for an elementary embedding. In this chain $\mathcal{A}_0$ is the prime model of T, and $\mathcal{A}_\omega$ is the saturated model of T. A natural question arises for a given $\aleph_1$-categorical theory T: Which models in the corresponding elementary chain are computable? This is known as the spectra problem for $\aleph_1$-categorical theories [59]. Goncharov [33] asked whether the prime model $\mathcal{A}_0$ of T is computable, given that one of the models $\mathcal{A}_i$ in the elementary chain is computable.

The problem of Goncharov was resolved in [59]. Recall that a model $\mathcal{M}$ of a theory T is minimal if there is no formula $\phi(x)$ such that the sets $\{m \mid \mathcal{M} \models \phi(m)\}$

and $\{m \mid \mathcal{M} \models \neg\phi(m)\}$ are infinite. A theory T is *strongly minimal* if all models of T are minimal. A theory is *algebraically trivial* if the algebraic closure of every set X in every model of the theory equals to the union of the algebraic closures of elements of X.

Theorem 2.5.4 ([59]). *For every given set S there exists an $\aleph_1$-categorical but not $\aleph_0$-categorical theory T_S with the following properties:*

1. *The theory T_S is strongly minimal and algebraically trivial,*
2. *Every (countable) non-prime model of T_S has a computable copy if and only if $S \in \Sigma^0_2$,*
3. *The prime model of T_S has a computable copy if and only if S is limitwise monotonic.*

Theorem 2.5.4 and the existence of a Σ^0_2 set which is not limitwise monotonic (see, e.g., [59] or Proposition 3.8 of [58]) implies that there is a strongly minimal and algebraically trivial $\aleph_1$-categorical theory such that every (countable) non-prime model of T is computable, but the prime model of T is not computable. This is a negative solution to Goncharov's problem in a strong form. However, Theorem 2.3.1 combined with Theorem 2.5.4 has an even stronger consequence:

Theorem 2.5.5. *There exists an $\aleph_1$-categorical but not $\aleph_0$-categorical theory T with the following properties:*

1. *The theory T is strongly minimal and algebraically trivial.*
2. *Each (countable) non-prime model of T has a computable copy.*
3. *The degree spectrum $DegSp(\mathcal{A}_0)$ of the prime model of T contains a Δ^0_2 degree $\mathbf{a}$ if and only if $\mathbf{a} > 0$.*

□

Chapter 3

Completely decomposable groups

This chapter contains a complete characterization of Δ^0_n-categorical homogeneous completely decomposable groups, for $n > 1$.

3.1 Algebraic preliminaries: torsion-free groups

Let us fix the canonical listing of the prime numbers:

$$p_0, p_1, \ldots, p_n, \ldots.$$

Definition 3.1.1 (Characteristic and h_i)**.** *Suppose G is a torsion-free abelian group. For $g \in G$, $g \neq 0$, and a prime number p_i, set*

$$h_i(g) = \begin{cases} \max\{k : p_i^k | g \textit{ in } G\}, \textit{if this maximum exists}, \\ \infty, \textit{otherwise}. \end{cases}$$

The sequence $\chi_G(g) = (h_0(g), h_1(g), \ldots)$ is called the characteristic of the element g in G.

Thus, for a torsion-free groups G, a subgroup H of G is a pure subgroup of G if and only if $\chi_H(h) = \chi_G(h)$ for every $h \in H$.

Definition 3.1.2. Let $\alpha = (k_0, k_1, \ldots)$ and $\beta = (l_0, l_1, \ldots)$ be two characteristics. Then we write $\alpha \leq \beta$ if $k_i \leq l_i$ for all i, where ∞ is greater than any natural number.

Definition 3.1.3 (Type)**.** *Two characteristics, $\alpha = (k_0, k_1, \ldots)$ and $\beta = (l_0, l_1, \ldots)$, are* equivalent, *written $\alpha \simeq \beta$, if $k_n \neq l_n$ only for finitely many n, and k_n and l_n are finite for these n. The equivalence classes of this relation are called* types.

We write $\mathbf{t}(g)$ for the type of an element g. If $G \leq \langle Q, +\rangle$ (equivalently, if G has rank 1) then all non-zero elements of G have equivalent types, by the definition of rank. Hence, we can correctly define the type of G to be $\mathbf{t}(g)$ for a non-zero $g \in G$, and denote it by $\mathbf{t}(G)$. The following theorem classifies torsion-free abelian groups of rank 1:

Theorem 3.1.1 (Baer [5])**.** *Let G and H be torsion-free abelian groups of rank 1. Then G and H are isomorphic if and only if $\mathbf{t}(G) = \mathbf{t}(H)$.*

The proof essentially uses that every rational group of type $\mathbf{t}$ has an element of characteristic χ, where χ is (any) characteristic of the type $\mathbf{t}$. The proof is easy and can be left to the reader. The next simplest class of torsion-free abelian groups is the class of *homogeneous completely decomposable* groups.

Definition 3.1.4 (Completely decomposable group)**.** *A torsion-free abelian group is called* completely decomposable *if G is a direct sum of groups each having rank 1. A completely decomposable group is* homogeneous *if all its elementary summands are isomorphic.*

It is known that any two decompositions of a completely decomposable group into direct summands of rank 1 are isomorphic. It means that every decomposition has the same number of elementary summands of every isomorphism type. For instance, two homogeneous completely decomposable groups of the same rank are isomorphic if and only if these groups have the same type [5]. We will refer to this fact by citing Theorem 3.1.1 since it is a straightforward consequence of this theorem [39].

Definition 3.1.5. *Suppose G is a torsion-free abelian group, g is an element of G, and $n|g$ some n. If $r = \dfrac{m}{n}$ then we denote by rg the (unique) element mh such that $nh = g$.*

Notation 3.1.1. *Let G be an abelian group and $A \subseteq G$. Suppose $\{r_a : a \in A\}$ is a set of (rational) indices. If we write $\sum_{a \in A} r_a a$ then we assume that $r_a a \neq 0$ for at most finitely many $a \in A$, and every element $r_a a$ is well-defined in G, according to Definition 3.1.5. We will use this convention without explicit reference to it.*

Now suppose $R \leqq \langle Q, +\rangle$, and $A \subseteq G$. We denote by $(A)_R$ the subgroup of G (if this subgroup exists) generated by $A \subset G$ over $R \leqq Q$, i.e. $(A)_R = \{\sum_{a \in A} r_a a : r_a \in R\}$. Finally, for $R \leqq Q$ and $a \in G$, we denote by Ra the subgroup $(\{a\})_R$ of G.

Let $R \leqq Q$. If a set $A \leqq G$ is linearly independent then every element of $(A)_R$ has the unique presentation $\sum_{a \in A} r_a a$. Otherwise we would have $\sum_{a \in A} r_a a = 0$ for some set of rational indices $\{r_a : a \in A\}$, and thus $m \sum_{a \in A} r_a a = \sum_{a \in A} m r_a a = 0$, for some integer m such that $m r_a \in Z$ for all $a \in A$, contrary to our hypothesis. Therefore, $(A)_R = \bigoplus_{a \in A} Ra$ for every linearly independent set A.

3.2 Computable abelian groups and modules

The notion of a c.e. characteristic is one of the central notions of computable abelian group theory.

Definition 3.2.1. *Let $\alpha = (h_i)_{i \in \omega}$, where $h_i \in \omega \cup \{\infty\}$ for each i, be a characteristic. We say that α is c.e. if the set $\{\langle i, j \rangle : j \leq h_i, h_i > 0\}$ is c.e. (see [71]). .*

The definition above is equivalent to saying that there is a non-decreasing uniform computable approximation $h_{i,s}$ such that $h_i = \sup_s h_{i,s}$, for every i. Observe that this is a type-invariant property. Thus, a type $\mathbf{f}$ is c.e. if α is c.e., for every α in $\mathbf{f}$ (equivalently, for some α in $\mathbf{f}$) Theorem 3.2.1 below was rediscovered several times by various mathematicians including Knight, Downey, and others (see, e.g., [23]).

Theorem 3.2.1 (Mal'tsev [67]). *Let G be a torsion-free abelian group of rank 1. Then the following are equivalent:*

(1) The group G has a computable presentation.

(2) The type $\mathbf{t}(G)$ is c.e.

(3) The group G is isomorphic to a c.e. additive subgroup R of a computable presentation of the rationals $(Q, +, \times)$. Furthermore, we may assume that $1 \in R$.

Furthermore, each c.e. type corresponds to some computably presented subgroup of the rationals. See [71] for a proof. If a group G is homogeneous completely decomposable then $\mathbf{t}(G)$ is also well-defined. The (1) $\leftrightarrow$ (2) part of Theorem 3.2.1 can be easily generalized to the class of homogeneous completely decomposable groups:

Proposition 3.2.1. A homogeneous completely decomposable group G has a computable presentation if and only if $\mathbf{t}(G)$ is c.e.

See [71] for more details.

Definition 3.2.2. *We say that C is a computable presentation of a module M over a ring R if*

(1) the ring R is isomorphic to a c.e. subring R_1 of a computable ring R_2,

(2) C is a computable presentation of M as an abelian group, and

(3) there is a (total) computable function $f : R_2 \times C \to C$ which maps (r, m) to $r \cdot m \in C$, for every $m \in C$ and $r \in R_1$.

Recall that $Q^{(P)}$ is the subgroup of the rationals $(Q, +)$ generated by the set of fractions $\{\frac{1}{p^k} : k \in \omega$ and $p \in P\}$. recall also:

Notation 3.2.1. *For a given set of primes P, the group G_P is the countably infinite direct sum of isomorphic copies of $Q^{(P)}$.*

Remark 3.2.1. *According to Definition 3.1.5, for every $r = \dfrac{m}{n} \in Q^{(P)}$ and a an element of the* group *G_P, the element $ra \in G_P$ is definable by the formula $\Phi_r(x, a) \leftrightharpoons mx = na$ in the language of abelian groups (recall that mx and na are abbreviations).*

Proposition 3.2.2. The following are equivalent:

1. P is c.e.
2. $Q^{(P)}$ is a c.e. subring of a computable presentation of $(Q, +, \times)$.
3. G_P is computably presentable as an abelian group.
4. G_P is computably presentable as a module over $Q^{(P)}$.

Proof. The implications (1) → (2) and (2) → (3) are obvious.

(3) → (4). By Proposition 3.2.1, the characteristic α of G_P is c.e. By Theorem 3.2.1, $Q^{(P)}$ is isomorphic to a c.e. additive subgroup A of $(Q, +, \times)$. Observe that $Q^{(P)}$ may be considered as a c.e. *subring* of Q, because we can ensure that $1 \in A$. It remains to observe that for each element $g \in G_P$ and each rational $r \in Q^{(P)}$, the element rg can be found effectively and uniformly.

(4) → (1). Pick an element g of G_P which is divisible by a prime p if and only if $p \in P$. Thus, $p \in P$ if and only if $(\exists x \in G_P)\, px = g$, proving that P is c.e. □

Remark 3.2.2. *Actually we have shown that every computable presentation of G_P is already a computable presentation of G_P as a module over $Q^{(P)}$.*

Lemma 3.2.1. For a c.e. set of primes P, the following are equivalent:

1. Every computable presentation of the group G_P has a Σ^0_n basis which generates this presentation as a module over $Q^{(P)}$.

2. The group G_P is Δ^0_n-categorical.

3. The $Q^{(P)}$-module G_P is Δ^0_n-categorical.

Proof. By Proposition 3.2.2, the *ring* $Q^{(P)}$ is a c.e. subring of a computable presentation of $(Q, +, \times)$.

(1) → (2). Let A and B be computable presentations of the group G_P. Both A and B have Σ^0_n bases which generate these groups over $Q^{(P)}$. We map these bases one into another using $0^{(n-1)}$. By Remark 3.2.1, we can extend this map to an isomorphism effectively, using the c.e. subring $Q^{(P)}$ of Q.

(2) → (3). Observe that every computable *group-isomorphism* between two computable module-presentations of G_P is already a computable *module-isomorphism.*

(3) → (1). Pick a computable presentation H of G_P such that the basis which generates H over $Q^{(P)}$ is computable. If G_P is Δ^0_n-categorical then every computable presentation of G_P has a Σ^0_n basis which is the image of the computable one in H. ◻

Thus, from the computability-theoretic point of view, G_P may be alternatively considered as an abelian group or a $Q^{(P)}$-module.

3.3 S-independence and excellent S-bases.

The notion of p-independence (for a single prime p) is a fundamental concept in abelian group theory (see [39], Chapter VI). We introduce a certain generalization of p-independence to *sets* of primes:

Definition 3.3.1 (S-independence and excellent bases)**.** *Let S be a set of primes, and let G be a torsion-free abelian group. If $S \neq \emptyset$, then we say that elements $b_1, \dots, b_k$ of G are S-independent in G if $p | \sum_{i\in\{1,\dots,k\}} m_i b_i$ in G implies $\bigwedge_{i\in\{1,\dots,k\}} p|m_i$, for all integers $m_1, \dots, m_k$ and $p \in S$. If $S = \emptyset$, then we say that elements are S-independent if they are simply linearly independent.*

Every maximal S-independent subset of G is said to be an S-basis of G. We say that an S-basis is excellent if it is a maximal linearly independent subset of G.

It is easy to check that S-independence in general implies linear independence. However, an S-basis does not have to be excellent. Lemma 35.1 in [39] implies that *the free abelian group of rank ω contains a $\{p\}$-basis which is not excellent.*

The main reason why we introduce the notion of S-independence is reflected in the example and the lemma below.

Example 3.3.1. Let Z^2 be the free abelian group of rank 2, and let e_1 and e_2 be such that $Z^2 = Ze_1 \oplus Ze_2$. Suppose that we need to *test*, given a pair of elements g_1 and g_2, if $Zg_1 + Zg_2 = Z^2$. That is, we wish to be able to say "no" if g_1 and g_2 do not generate Z^2. If g_1 and g_2 together generate the group, then $\{g_1, g_2\}$ should be linearly independent. But this is not sufficient: suppose that $g_1 = 2e_0 + e_1$ and $g_2 = e_1$; then $2|g_1 - g_2$, but the element $h = \dfrac{g_1 - g_2}{2}$ is not in the span of $\{g_1, g_2\}$.

Now we make each Z-component of Z^2 infinitely divisible by 2 and consider the group $Q^{(2)}e_1 \oplus Q^{(2)}e_2$. Note that $2|g_1 - g_2$ in $Q^{(2)}e_1 \oplus Q^{(2)}e_2$, but it is not a problem: it is easy to check that $\{g_1, g_2\}$ generates $Q^{(2)}e_1 \oplus Q^{(2)}e_2$ over $Q^{(2)}$. In contrast, the elements $h_1 = 3e_0 + e_1$ and $h_2 = e_1$ fail to generate $Q^{(2)}e_1 \oplus Q^{(2)}e_2$ over $Q^{(2)}$.

More generally, in $Q^{(P)}e_1 \oplus Q^{(P)}e_2$, *the existence of p-roots for $p \in P$ can not be used to test if two given elements generate the whole group over $Q^{(P)}$ or not.*

Notation 3.3.1. *In this section P stands for a set of primes and $\widehat{P}$ for the complement of P within the set of* all *primes:*

$$\widehat{P} = \{p : p \textit{ is prime and } p \notin P\}.$$

Lemma 3.3.1. Suppose $G \cong \bigoplus_{i \in I} Q^{(P)}$, and let $B \subseteq G$. Then B is an excellent $\widehat{P}$-basis of G if and only if B generates G as a free module over $Q^{(P)}$.

Let $\mathcal{P}$ be the set of *all* primes. Then $\widehat{\mathcal{P}} = \emptyset$. Recall that $\emptyset$-independence is simply linear independence, and $G_{\mathcal{P}} \cong D(\omega) = \bigoplus_{i \in \omega} Q$. It is well-known that every maximal linearly independent set generates the vector space $D(\omega)$ over Q. If $P = \emptyset$ then $G_{\emptyset} \cong FA(\omega) = \bigoplus_{i \in \omega} Z$ is the free abelian group of the rank ω. As a consequence of the lemma, every excellent $\mathcal{P}$-basis of $FA(\omega)$ generates $FA(\omega)$ as a free abelian group.

Proof. ($\Rightarrow$). Let B be an excellent $\widehat{P}$-basis of G. Suppose $g \in G$. By our assumption, B is a basis of G. Therefore, there exist integers m and m_b, $b \in B$, such that $mg = \sum_b m_b b$. Suppose $m = pm'$ for some $p \in \widehat{P}$. By Definition 3.3.1, $p|m_b$ for all $b \in B$. Therefore, without loss of generality, we can assume that $(m, p) = 1$, for every $p \in \widehat{P}$. By the definition of G, we have:

$$g = \sum_b \frac{m_b}{m} b \in (B)_{Q^{(P)}} \leqq G.$$

The set B is linearly independent, therefore $(B)_{Q^{(P)}} = \bigoplus_{b \in B} Q^{(P)} b$ (see the discussion after Notation 3.1.1). We have $g \in (B)_{Q^{(P)}} \leqq G$ for every $g \in G$. Thus, $G = (B)_{Q^{(P)}}$.

($\Leftarrow$). Let $G = \bigoplus_{b\in B} Q^{(P)}b$ for $B \subseteq G$, and $ph = \sum_{b\in B} m_b b$, where m_b is integer for every $b \in B$, and $p \in \widehat{P}$. We have $h \in G_P$ and thus $h = \sum_{b\in B} h_b$, where $h_b \in Q^{(P)}b$ for each $b \in B$ (recall that $h_b = 0$ for a.e. b).

Therefore $ph = p\sum_{b\in B} h_b = \sum_{b\in B} ph_b = \sum_b m_b b$, and $ph_b = m_b b$ for every b (by the uniqueness of the decomposition of an element). Each direct component of G in the considered decomposition has the form $Q^{(P)}b$. In other words, the element b plays the role of $\mathbf{1}$ in the corresponding $Q^{(P)}$-component of this decomposition. Now recall that $p \notin P$. Thus, $m_b \neq 0$ implies $p|m_b$ for every b, by the definition of $Q^{(P)}$. $\square$

In later proofs we will have to approximate an *excellent* basis stage-by-stage, using a certain oracle. Recall that not every maximal $\widehat{P}$-independent set is an excellent basis of G_P. Therefore, we need to show that, for a given finite $\widehat{P}$-independent subset B of G_P and an element $g \in G_P$, there exists a finite extension $B^\star$ of B such that $B^\star$ is $\widehat{P}$-independent and the element g is contained in the $Q^{(P)}$-span of $B^\star$.

Proposition 3.3.1. Suppose $B \subset G_P$ is a finite $\widehat{P}$-independent subset of G_P. For every $g \in G_P$ there exists a finite $\widehat{P}$-independent set $B^\star \subset G_P$ such that $B \subseteq B^\star$ and $g \in (B^\star)_{Q^{(P)}}$.

Proof. Pick $\{e_i : i \in \omega\} \subseteq G_P$ such that $G_P = \bigoplus_{i\in\omega} Q^{(P)}e_i$. Let $\{e_0, e_1, \dots, e_n\}$ be such that both $B = \{b_0, \dots, b_k\}$ and g are contained in $(\{e_0, e_1, \dots, e_n\})_{Q^{(P)}}$. We may assume $k < n$.

Lemma 3.3.2. Suppose $B = \{b_0, \dots, b_k\} \subseteq \bigoplus_{i\in\{0,\dots,n\}} Q^{(P)}e_i$, is a linearly independent set. There exists a set $C = \{c_0, \dots, c_n\} \subseteq \bigoplus_{i\in\{0,\dots,n\}} Q^{(P)}e_i$, and coefficients $r_0, \dots, r_k \in Q^{(P)}$ such that

(1) $\bigoplus_{i\in\{0,\dots,n\}} Q^{(P)}e_i, = \bigoplus_{i\in\{0,\dots,n\}} Q^{(P)}c_i$, and

(2) $(\{r_0c_0, \dots, r_kc_k\})_{Q^{(P)}} = (B)_{Q^{(P)}}$.

Proof. It is a special case of a well-known fact ([63], Theorem 7.8) which holds in general for every finitely generated module over a principal ideal domain (note that $Q^{(P)}$ is a principal ideal domain). $\square$

We show that if B is $\widehat{P}$-independent (not merely linearly independent) then we can set $B^\star = \{b_0, \dots, b_k\} \cup \{c_{k+1}, \dots, c_n\}$, where $C = \{c_0, \dots, c_n\}$ is the set from Lemma 3.3.2. Suppose $p| \sum_{0\le i\le k} n_ib_i + \sum_{k+1\le i\le n} n_ic_i$ for a prime $p \in \widehat{P}$. We have

$$\bigoplus_{i\in\{0,\dots,n\}} Q^{(P)}e_i = \bigoplus_{1\le i\le k} Q^{(P)}c_i \oplus \bigoplus_{k+1\le i\le n} Q^{(P)}c_i,$$

and $\sum_{1\le i\le k} n_i b_i \in \bigoplus_{1\le i\le k} Q^{(P)} c_i$. By the purity of direct components, we have $p|\sum_{1\le i\le k} n_i b_i$ within $\bigoplus_{1\le i\le k} Q^{(P)} c_i$ and $p|\sum_{k+1\le i\le n} n_i c_i$ within $\bigoplus_{k+1\le i\le n} Q^{(P)} c_i$. But the former implies $p|n_i$ for all $1\le i\le k$ by our assumption, and the latter implies $p|n_i$ for all $k+1\le i\le n$ by the choice of C and Lemma 3.3.1.

The set $B^\star$ is actually an excellent $\widehat{P}$-basis of $\bigoplus_{i\in\{0,\dots,n\}} Q^{(P)} e_i$, since the cardinality of $B^\star$ is $n+1 = rk(\bigoplus_{i\in\{0,\dots,n\}} Q^{(P)} e_i)$. Therefore, the set $B^\star = \{b_0,\dots,b_k\}\cup\{c_{k+1},\dots,c_n\}$ is a $\widehat{P}$-independent set with the needed properties. □

Suppose G is a torsion-free abelian group, and $a, b\in G$. Recall that $\chi(a)\le\chi(b)$ iff $h_i(a)\le h_i(b)$ for all i. In other words, $p^k|a$ implies $p^k|b$, for all $k\in\omega$ and every prime p.

Definition 3.3.2. *Let G be a torsion-free abelian group. For a given characteristic α, let $G[\alpha] = \{g\in G : \alpha\le\chi(g)\}$.*

We have $h_i(a) = h_i(-a)$ and $\inf(h_i(a), h_i(b)) \le h_i(a+b)$, for all i. Furthermore, $\chi(0)\ge\alpha$, for every characteristic α. Therefore, $G[\alpha]$ is a subgroup of G.

Definition 3.3.3. *Let $\alpha = (\alpha_0, \alpha_2, \dots)$. Then $Q(\alpha)$ is the subgroup of $(Q, +)$ generated by elements of the form $1/p_k^x$ where $x\le\alpha_k$.*

Example 3.3.2. *Let $\alpha = (\infty, 1, \infty, 1, \dots, \alpha_{2k} = \infty, \alpha_{2k+1} = 1, \dots)$. Consider*

$$\beta = \alpha + (0, 1, 0, -1, 0, 0, 0, 0, 0, \dots, 0, \dots).$$

By Definition 3.1.3, $\beta\cong\alpha$. Consider the group $H = Q(\alpha)$. We have $1\in Q(\alpha)$ and $\chi(1) = \alpha$ within $Q(\alpha)$. Note that the characteristic of $a = 3/7$ in $H(\alpha)$ is β. Observe that a/p_{2k}^j belongs to $H[\beta]$, for every $k, j\in\omega$. In contrast, a/p_{2k+1} does not belong to $H[\beta]$. Indeed, the characteristic $\chi_H(a/13)$ is

$$(\infty, 2, \infty, 0, \infty, 0, \infty, 1, \infty, 1, \infty, 1, \dots)$$

and

$$(\infty, 2, \infty, 0, \infty, 0, \infty, 1, \infty, 1, \infty, 1, \dots) \not\geq \beta = (\infty, 2, \infty, 0, \infty, 1, \infty, 1, \infty, 1, \dots).$$

Recall that the type is an equivalence class of characteristics. Thus, the type of $H\leqq Q$ is simply the type of any nonzero element of H. We are ready to state and prove the main result of this section.

Theorem 3.3.1. *Let $\mathcal{G} = \bigoplus_{i\in\omega} H$, where $H \leqq Q$, $\mathbf{t}(H) = \mathbf{f}$ and $\alpha = (\alpha_0, \alpha_1, \ldots)$ is of type $\mathbf{f}$. Then $\mathcal{G}[\alpha] \cong G_P$, where $P = \{p_i : h_i = \infty \text{ in } \alpha\}$. Furthermore, if B is an excellent $\widehat{P}$-basis of $\mathcal{G}[\alpha]$, then $\mathcal{G}$ is generated by B over $Q(\alpha)$.*

Informally, this theorem says that each homogeneous completely decomposable group of rank ω has a subgroup isomorphic to G_P, for some P. Furthermore, every excellent $\widehat{P}$-basis of this subgroup generates the whole group G over a certain rational subgroup $Q(\alpha)$ taken as a domain of coefficients. The group $Q(\alpha)$ is not necessarily a ring (recall Notation 3.1.1). The idea of the technical proof below was essentially illustrated in Example 3.3.2.

Proof. We prove that $\mathcal{G}[\alpha] \cong G_P$.

Let g_i be the element of the i'th presentation of H in the decomposition $\mathcal{G} = \bigoplus_{i\in\omega} H$ such that $\chi(g_i) = \alpha$. The collection $\{g_i : i \in \omega\}$ is a basis of $\mathcal{G}$. Therefore, $\{g_i : i \in \omega\}$ is a basis of $\mathcal{G}[\alpha]$. By the definition of P, $(\{g_i : i \in \omega\})_{Q^{(P)}}$ is a subgroup of $\mathcal{G}[\alpha]$. Furthermore, since $\{g_i : i \in \omega\}$ is linearly independent,

$$(\{g_i : i \in \omega\})_{Q^{(P)}} \cong \bigoplus_{i\in\omega} Q^{(P)} g_i.$$

Thus, we have

$$\bigoplus_{i\in\omega} Q^{(P)} g_i \subseteq \mathcal{G}[\alpha].$$

We are going to show that every element $g \in \mathcal{G}[\alpha]$ is generated by $\{g_i : i \in \omega\}$ over $Q^{(P)}$. This will imply $\mathcal{G}[\alpha] \cong G_P$.

Pick any nonzero $g \in \mathcal{G}[\alpha]$. The set $\{g_i : i \in \omega\}$ is a basis of $\mathcal{G}[\alpha]$, therefore $ng = \sum_{i\in\omega} m_i g_i$ for some integers n and m_i, $i \in \omega$. Since direct components are pure, $n | \sum_{i\in I} m_i g_i$ implies $n | m_i g_i$ for every $i \in \omega$, and $g = \sum_{i\in I} \frac{m_i}{n} g_i$. After reductions we have $g = \sum_{i\in I} \frac{m'_i}{n_i} g_i$, where $\frac{m'_i}{n_i}$ is irreducible. It suffices to show that $\frac{m'_i}{n_i} \in Q^{(P)}$.

Assume there is i such that $\frac{m'_i}{n_i} \notin Q^{(P)}$. Equivalently, for some $p_k \in \widehat{P}$, we have $m'_i \neq 0$ and $n_i = p_k n'_i$, where n'_i is an integer (recall that $\frac{m'_i}{n_i}$ is irreducible).

We have $h_k(\frac{m'_i}{n_i} g_i) = h_k(\frac{m'_i}{n'_i} \frac{g_i}{p_k}) \leq h_k(\frac{g_i}{p_k})$, since m'_i is not divisible by p_k. But $h_k(\frac{g_i}{p_k}) < h_k(g_i)$ (recall that $h_k(g_i)$ is finite). It is straightforward from the definitions of h_k that $h_k(g) = \min\{h_k(\frac{m'_i}{n_i}g_i) : i \in I, m_i \neq 0\}$, since each g_i belongs to a separate direct component of $\mathcal{G}$. Therefore $h_k(g) \leq h_k(\frac{m'_i}{n_i}g_i) < h_k(g_i)$. But $\chi(g_i) = \alpha$. Thus, $\chi(g) \ngeq \alpha$ and $g \notin \mathcal{G}[\alpha]$, and this contradicts our choice of g. Therefore, $\mathcal{G}[\alpha] \cong G_P$.

We show that if B is an excellent $\widehat{P}$-basis of $\mathcal{G}[\alpha]$, then $\mathcal{G} = (B)_{Q(\alpha)}$ (recall Notation 3.1.1).

For every $b \in B$ consider the minimal pure subgroup which contains b (recall that we denote this group by $[b]$. Consider $\langle B\rangle = \sum_{b\in B}[b] \leqq \mathcal{G}$. In fact $\langle B\rangle = \bigoplus_{b\in B}[b]$, because B is linearly independent within $\mathcal{G}[\alpha]$ and, therefore, within $\mathcal{G}$ as well.

By our choice, $b \in \mathcal{G}[\alpha]$. Thus, $\chi(b) \geq \alpha$ within $\mathcal{G}$. We show that in fact $\chi(b) = \alpha$. Assume $\chi(b) > \alpha$. We have $b = pa$ for some $a \in \mathcal{G}[\alpha]$ and $p \in \widehat{P}$. But B is $\widehat{P}$-independent. This contradicts the fact that $p|1 \cdot b$ and 1 is evidently not divisible by p. Therefore, we have

$$[b] = Q(\alpha)b.$$

It remains to prove that $\mathcal{G} \subseteq \langle B\rangle$. Pick any nonzero $g \in \mathcal{G}$. There exist integers m and n such that $(m, n) = 1$ and $\chi(\frac{m}{n}g) = \alpha$. To see this we use the fact that $\chi(g) \in \mathbf{f}$. It is enough to make only finitely many changes to $\chi(g)$ to make it equival to α.

Equivalently, $\frac{m}{n}g \in \mathcal{G}[\alpha]$. We have $\frac{m}{n}g = \sum\limits_{b\in B, r_b\in Q^{(P)}} r_b b$, by Lemma 3.3.1. By our assumption, $\chi(b) = \chi(\frac{m}{n}g) = \alpha$, for every $b \in B$. Obviously, $m|\frac{m}{n}g$ in $\mathcal{G}$. Therefore, by the definition of α and B, we have $m|b$ *in* $Q(\alpha)b$. Thus, there exist $x_b \in [b] = Q(\alpha)b$ such that $mx_b = b$. We can set $g = \sum_{b\in B} nr_b x_b$, where $nr_b x_b \in [b]$. This shows $\mathcal{G} = (B)_{Q(\alpha)}$. □

3.4 Effective content of S-independence.

Theorem 3.4.1. *Every computably presentable homogeneous completely decomposable torsion-free abelian group is Δ^0_3-categorical.*

The proof of the Theorem 3.4.1 is based on the lemma below. The proof of this lemma uses Theorem 3.3.1.

Lemma 3.4.1. Let $\mathcal{G} = \bigoplus_{i\in\omega} H$, where $H \leqq Q$, the type $\mathbf{t}(H)$ is $\mathbf{f}$, and $\alpha = (\alpha_0, \alpha_1, \ldots)$ is a characteristic of type $\mathbf{f}$. Let G_1 and G_2 be computable presentations of $\mathcal{G}$.

Suppose that both $G_1[\alpha]$ and $G_2[\alpha]$ have Σ^0_n excellent $\widehat{P}$-bases. Then there exists an Δ^0_n isomorphism from G_1 onto G_2.

We first prove Theorem 3.4.1, and then prove Lemma 3.4.1. We need to show that a given homogeneous completely decomposable group satisfies the hypothesis of Lemma 3.4.1 with $n = 3$.

Proof of Theorem 3.4.1. Let G be a computable presentation of $\mathcal{G} \cong \bigoplus_{i\in\omega} H$, where $H \leq Q$. Let α be a characteristic of type $\mathbf{t}(H)$ and $P = \{p_k : \alpha_k = \infty \text{ in } \alpha\}$. By Theorem 3.3.1 and Lemma 3.4.1, it suffices to construct a excellent $\widehat{P}$-basis of $G[\alpha]$ which is Σ^0_3.

We are building $C = \bigcup_n C_n$. Assume that we are given C_{n-1}. At step n of the procedure, we do the following:

1. Pick the n-th element g_n of $G[\alpha]$.

2. Find an extension C_n of C_{n-1} in $G[\alpha]$ such that (a) C_n is a finite $\widehat{P}$-independent set, and (b) g_n is linearly dependent of C_n.

Let $G = \bigoplus_{i\in I} Re_i$, where $\chi(e_i) = \alpha$ and $R \cong H$. Observe that at stage n of the procedure we have $g_n \cup C_{n-1} \subset (\{e_0, \ldots, e_k\})_{Q^{(P)}}$, for some k. By Proposition 3.3.1, the needed extension denoted by C_n can be found.

It suffices to check that the construction is effective relative to $0''$. We use computable infinitary formulas in the proofs of the claims below. See [3] for a background on computable infinitary formulas.

By Theorem 3.3.1, we have $G[\alpha] \cong G_P$, where $P = \{p : p^\infty | h\}$ is a Π^0_2 set of primes.

Claim 3.4.1. *The group $G[\alpha]$ is c.e. in $0''$.*

Proof. Pick any $h \in G$ with $\chi(h) = \alpha$. By its definition, for every $g \in G$, the property $\chi(g) \geq \alpha$ is equivalent to

$$\bigwedge_{p-\text{prime}} \bigwedge_{k\in\omega} ((\exists x) p^k x = h \to (\exists y) p^k y = g).$$

Therefore, the group $G[\alpha]$ is a Π^0_2-subgroup of G. □

Claim 3.4.2. *There is a $0''$-computable procedure which decides if a given finite set $B \subseteq G[\alpha]$ is $\widehat{P}$-independent, uniformly in the index of B.*

Proof. It suffices to show that the property "B is a $\widehat{P}$-independent set in $G[\alpha]$" can be expressed by a Π^0_2 infinitary computable formula in the signature of abelian groups with parameters elements from B.

Note that in general $P \in \Pi^0_2$. By Claim 6.3.1, the group $G[\alpha]$ is a Π^0_2-subgroup of G. Thus, the condition "B is a $\widehat{P}$-independent set in $G[\alpha]$" seems to be merely Π^0_3:

$$\bigwedge_{\overline{m}\in Z^{<\infty}} \bigwedge_{p-\text{prime}} ([p \notin P \wedge (\exists x)(x \in G[\alpha] \wedge px = \sum_{b\in B} m_b b)] \to \bigwedge_b p|m_b).$$

The idea is to substitute the Σ^0_3 formula $(\exists x)(x \in G[\alpha] \wedge px = \sum_{b\in B} m_b b)$ by an equivalent Σ^0_2 one, using a non-uniform parameter $c \in G$ such that $\chi(c) = \alpha$. More specifically, we are going to show that for every $p_v \notin P$, the formula

$$(\exists x)(x \in G[\alpha] \wedge p_v x = \sum_{b\in B} m_b b)$$

is equivalent to

$$(\exists k)(\exists y \in G)(\alpha_v < k \wedge p_v^k y = \sum_{b\in B} m_b b),$$

where α_v is the v-th component of α corresponding to p_v and

$$\alpha_v < k \Leftrightarrow \neg(\alpha_v \geq k) \Leftrightarrow \neg(\exists \xi)(p_v^k \xi = c).$$

Suppose there is $x \in G[\alpha]$ such that $p_v x = \sum_{b\in B} m_b b$. Since $h_v(x) \geq \alpha_v$, we have $p_v^{\alpha_v} y = x$ and $p_v^{\alpha_v+1} y = p_v x$, for some $y \in G$, so we can set $k = \alpha_v + 1$. For the converse, suppose there exist such k and y. Then $p_v x = p_v^k y$ for $x = p_v^{k-1} y$. We have $k > \alpha_v$, and therefore $(k-1) \geq \alpha_v$. But $h_v(x) \geq (k-1)$ because $x = p_v^{k-1} y$ is divisible by p_v^{k-1}, and thus $h_v(x) \geq \alpha_v$. The characteristic of x differs from the characteristic of y only at the position for the prime p_v. Thus, for every $w \neq v$,

$$h_w(x) = h_w(p_v^k y) = h_w(\sum_{b\in B} m_b b)) \geq \alpha_w,$$

since $\sum_{b\in B} m_b b \in G[\alpha]$. Therefore, $\chi(x) \geq \alpha$ and $x \in G[\alpha]$. □

By Claim 3.4.1 and Claim 3.4.2, the procedure is computable relative to $0''$. Assuming Lemma 3.4.1, this completes the proof of the theorem. □

Proof of Lemma 3.4.1. Recall that G_1 and G_2 are computable presentations of $\mathcal{G}$ such that both $G_1[\alpha]$ and $G_2[\alpha]$ have Σ^0_n excellent $\widehat{P}$-bases. We need to show that there exists an Δ^0_n isomorphism from G_1 onto G_2. Let B_1 and B_2 be excellent $\widehat{P}$-bases of G_1 and G_2, respectively.

Observe that the group $Q(\alpha)$ is isomorphic to a c.e. additive subgroup R of $(Q, +, \times)$. Furthermore, we may assume that $1 \in R$. To see this pick h with $\chi(h) = \alpha$ non-uniformly, and then apply Theorem 3.1.1 to the group $[h]$. By Theorem 3.3.1, we have

$$G_1 = \bigoplus_{b \in B_1} Rb \cong G_2 = \bigoplus_{b' \in B_2} Rb'.$$

To build a Δ^0_n isomorphism from G_1 to G_2 first define the map from B_1 onto B_2 using a standard back-and-forth argument. Then extend it to the whole G_1 using the fact that $r \cdot b$ can be found effectively and uniformly, for every $r \in R$ and $b \in B_1$. □

By Proposition 3.2.2 and Remark 3.2.2, "computable presentation of G_P" can be equivalently understood as "computable presentation of the group G_P" or "computable presentation of the $Q^{(P)}$-module G_P". Before we turn to a more detailed study of Δ^0_2-categorical completely decomposable groups, we prove a fact about excellent $\widehat{P}$-bases of the group G_P which is of an independent interest for us:

Theorem 3.4.2. *If a computable presentation of G_P has a Σ^0_2 basis which generates it as a free $Q^{(P)}$-module, then this presentation possesses a Π^0_1 basis which generates it as a free $Q^{(P)}$-module.*

Proof. Recall that, by Lemma 3.3.1, a basis generates G_P as a free $Q^{(P)}$-module if and only if this basis is an excellent $\widehat{P}$-basis. The proof of the theorem is based on Lemma 3.3.1 and the short technical lemma below.

Lemma 3.4.2. Suppose $\{e_i : i \in \omega\} \subset G_P$ is such that $G_P = \bigoplus_{i \in \omega} Q^{(P)} e_i$, and suppose $\{b_1, \dots, b_k\} \subset G_P \setminus \{0\}$. For any integer $m \neq 0$, the set $B = \{e_0, b_1, \dots, b_k\}$ is $\widehat{P}$-independent if and only if $B_m = \{e_0, b_1, \dots, b_{k-1}, b_k + me_0\}$ is $\widehat{P}$-independent. Furthermore, $(B)_{Q^{(P)}} = (B_m)_{Q^{(P)}}$, for every m.

Note that for the (obvious) second part of Lemma 3.4.2 *we do not assume that B is $\widehat{P}$-independent.*

Proof of Lemma 3.4.2. Suppose $B = \{e_0, b_1, \dots, b_k\}$ is $\widehat{P}$-independent. We show that $B_m = \{e_0, b_1, \dots, b_{k-1}, b_k + me_0\}$ is $\widehat{P}$-independent as well.

Pick an arbitrary $p \in \widehat{P}$. Suppose that p divides $g = n_0 e_0 + \sum_{1 \le i \le k-1} n_i b_i + n_k(b_k + me_0) = (n_0 + n_k m)e_0 + \sum_{1 \le i \le k} n_i b_i$. Recall that the set $B = \{e_0, b_1, \dots, b_k\}$ is $\widehat{P}$-independent. Therefore, $p | n_i$, for every $1 \le i \le k$. As a consequence, p divides $n_0 e_0 = g - n_k m e_0 - \sum_{1 \le i \le k} n_i b_i$. By our assumption on the element e_0, we have $p | n_0$. □

Suppose that $E = \{e_0, e_1, \ldots\}$ is a Σ^0_2 excellent $\widehat{P}$-basis of $G = \bigoplus_{i\in\omega} Q^{(P)} e_i = \{g_0 = 0, g_1, \ldots\}$ which is a computable group. We fix a computable relation R such that $x \in E$ if and only if $(\exists^{<\infty} y)R(x, y)$. We build a co-c.e set of elements B such that the following requirements are met:

R_0: $e_0 \in B$;

R_j: if $g_j = e_k$ for some k then B contains exactly one element of the form $(e_k + me_0)$.

We also require that the only elements that enter B are due to one of these requirements. There is no priority order on the requirements.

We first show that if all the requirements are met, then the set B is an excellent $\widehat{P}$-basis of G. Assume R_j is met, for every j. It follows that for every k there exists m such that $e_k + me_0 \in B$. Also, if B contains two elements of the form $e_k + me_0$ and $e_k + ne_0$, then necessarily $n = m$. We show that B is an excellent $\widehat{P}$-basis of G. Note that, if B is not $\widehat{P}$-independent, then there is a finite subset B_0 of B which is not $\widehat{P}$-independent. By (a multiple application of) Lemma 3.4.2, this contradicts the choice of $E = \{e_0, e_1, \ldots, \}$. It remains to apply the second part of Lemma 3.4.2 and see that the $Q^{(P)}$-spans of B and E coincide.

All strategies in the construction will share the same global restraint. More specifically, in the construction the strategies will put restraints onto certain elements of the group. The desired set B will consist of elements which eventually become forever restrained by the strategies.

Strategy for R_0: Permanently restrain e_0.

Strategy for R_j, $j > 0$: If R_j currently has no witness then pick a witness c_j which is equal to $g_j + me_0$, where m is the least such that $g_j + me_0$ is not restrained and is not yet enumerated into $\overline{B}$. Declare c_j restrained (thus, c_j is now our witness, and our current guess is $c_j \in B$). If c_j is the n^{th} element of the group, $c_j = g_n$, then enumerate each g_x with $x < n$ into $\overline{B}$ unless g_x is already in $\overline{B}$ or is restrained. If, at a later stage, a fresh y is found such that $R(g_j, y)$ holds, then enumerate $g_j + me_0$ into $\overline{B}$, and initialize R_j by making c_j undefined.

Construction.

Stage s. Let R_j, $j \leq s$, act according to their instructions.

End of construction.

The set B consists of elements which eventually become forever restrained by strategies. Also note that each element of the group can be restrained at most once. Thus, the set $\overline{B}$ is c.e.

To see why R_j is met note that the requirement eventually puts a permanent restraint on its witness $g_j + me_0$ if an only if $(\exists^{<\infty} y)R(g_j, y)$. This is the same as saying that $g_j = e_k$, for some k. □

3.5 Semi-low sets, and Δ^0_2-categoricity.

Recall that a set A is semi-low if the set $H_A = \{e : W_e \cap A \neq \emptyset\} = \{e : W_e \nsubseteq \overline{A}\}$ is computable in $\emptyset'$.

Theorem 3.5.1. *A computably presentable completely decomposable abelian group G is Δ^0_2-categorical if and only if G is isomorphic to G_P where $\widehat{P}$ is semi-low.*

The proof of this theorem is split into several parts. Each part corresponds to a different hypothesis on the isomorphism type of G. Different cases will need different techniques and strategies.

Proof. We need the following technical notion:

Definition 3.5.1. *Let $\alpha = (h_i)_{i\in\omega}$ be a c.e. characteristic (see Definition 3.2.1), and let $h_{i,s}$ be its non-decreasing uniform computable approximation: $h_i = \sup_s h_{i,s}$, for every i. We say that α has* a computable settling time *if there is a (total) computable function $\psi : \omega \to \omega$ such that*

$$h_i = \begin{cases} h_{i,\psi(i)}, & \text{if } h_i \text{ is finite,} \\ \infty, & \text{otherwise,} \end{cases}$$

for every i. We also say that ψ is a computable settling time for $(h_{i,s})_{i,s\in\omega}$.

This is the same as saying that, given i, there exists an effective (and uniform) way to compute a stage s after which the approximation of h_i either does not increase, or increases and tends to infinity. Note that this is the property of a characteristic, not the property of some specific computable approximation. Indeed, given an approximation of α having a computable settling time, we can define a computable settling time for any other computable approximation of α. Furthermore, as can be easily seen, this is a type-invariant property. Thus, we can also speak of types having computable settling times.

If a homogeneous completely decomposable group G of type $\mathbf{f}$ is computable, then $\mathbf{f}$ is c.e. (see Proposition 3.2.1). Suppose that G is a computable homogeneous completely decomposable group of type $\mathbf{f}$, and let $\alpha = (h_i)_{i\in\omega}$ be a characteristic of type $\mathbf{f}$. We consider the cases:

1. The type $\mathbf{f}$ of G has no computable settling time. In this case G is not Δ^0_2-categorical by Proposition 3.5.2. Observe that if $\mathbf{f}$ has no computable settling time then the set $Fin(\alpha) = \{i : 0 < h_i < \infty\}$ has to be infinite (see, e.g., Proposition 3.2.2). Thus, G can not be isomorphic to G_P, for a set of primes P.

2. The type $\mathbf{f}$ of G has a computable settling time, $Fin(\alpha) = \{i : 0 < h_i < \infty\}$ is empty (finite), and the set $\{i : h_i = 0\}$ is semi-low. In other words, the group G is isomorphic to G_P with $\widehat{P}$ semi-low. In this case G is Δ^0_2-categorical, by Proposition 3.5.1 below.

3. The type $\mathbf{f}$ of G has a computable settling time, the set $Fin(\alpha) = \{i : 0 < h_i < \infty\}$ is empty (finite), and the set $\{i : h_i = 0\}$ is not semi-low. Here G is again isomorphic to G_P, but in this case G is not Δ^0_2-categorical, by Proposition 3.5.3 below.

4. The type $\mathbf{f}$ of G has a computable settling time, and the set $Fin(\alpha) = \{i : 0 < h_i < \infty\}$ is infinite and not semi-low. As in the above case[1], G is not Δ^0_2-categorical, by Proposition 3.5.3.

5. The type $\mathbf{f}$ of G has a computable settling time, and the set $Fin(\alpha) = \{i : 0 < h_i < \infty\}$ is infinite and semi-low. The group is not Δ^0_2-categorical, by Proposition 3.5.4 below.

We first discuss why case (3) and case (4) above can be collapsed into one case. First, define $Inf(\alpha) = \{i : h_i = \infty\}$ and $V = \{i : 0 < h_{i,\psi(i)} < \infty\}$, where ψ is a computable settling time for α. Note that V is c.e. Evidently, $\overline{Inf(\alpha)} = Fin(\alpha) \cup \{i : h_i = 0\}$ and $Fin(\alpha) = \overline{Inf(\alpha)} \cap V$. We claim that "$Fin(\alpha)$ is not semi-low" implies "$\overline{Inf(\alpha)}$ is not semi-low". We assume that $\overline{Inf(\alpha)}$ is semi-low and observe that $\{e : W_e \cap Fin(\alpha) \neq \emptyset\} = \{e : W_e \cap V \cap \overline{Inf(\alpha)} \neq \emptyset\} = \{e : W_{s(e)} \cap \overline{Inf(\alpha)} \neq \emptyset\}$ for a computable function s. Therefore, $H_{Fin(\alpha)} \leq_m H_{\overline{Inf(\alpha)}} \leq_T \emptyset'$, as required.

Therefore, cases (3) and (4) are both collapsed into

(3′) If $\mathbf{f}$ has a computable settling time and $\overline{Inf(\alpha)}$ is not semi-low, then G is not Δ^0_2-categorical.

[1]We distinguish these two cases only because these cases correspond to (algebraically) different types of groups. We discuss a bit later why these cases are essentially not different.

Now we state and prove the propositions which cover all the cases above.

Recall that, by Proposition 3.2.2, the group G_P has a computable presentation as a group (module) if and only if P is c.e.

Proposition 3.5.1. If $\widehat{P}$ is semi-low (and co-c.e.) then G_P is Δ^0_2-categorical.

Proof. The proof may be viewed as a simpler version of the proof of Theorem 3.4.1. Let $G = \{g_0 = 0, g_1, \ldots\}$ be a computable copy of G_P. By Lemma 3.2.1, it is enough to build a Σ^0_2 excellent $\widehat{P}$-basis of G.

We are building $C = \bigcup_n C_n$. Assume that we are given C_{n-1}. At stage n of the construction, we do the following:

1. Pick the n-th element g_n of G.

2. Find an extension C_n of C_{n-1} in G such that (a) C_n is a finite $\widehat{P}$-independent set, and (b) $C_n \cup \{g_n\}$ is linearly dependent.

The algebraic part of the verification is the same as in Theorem 3.4.1 (and is actually simpler). Thus, it is enough to show that (a) in (2) above can be checked effectively and uniformly in $\emptyset'$. Given a finite set F of elements of G, define a c.e. set V consisting of primes which could potentially witness that F is $\widehat{P}$-dependent:

$$V = \Big\{p : \bigvee_{\overline{m} \in Z^{card(F)}} \Big[p | (\sum_{g \in F} m_g g) \wedge (\bigvee_{g \in F} p \nmid m_g)\Big]\Big\}.$$

The c.e. index of V can be obtained uniformly from the index of F. It can be easily seen from the definition of $\widehat{P}$-independence that

$$V \cap \widehat{P} = \emptyset \text{ if and only if } F \text{ is } \widehat{P}\text{-independent.}$$

By our assumption on $\widehat{P}$, this can be decided effectively in $\emptyset'$. □

Fix a computable listing $\{\Phi_e(x, y)\}_{e \in \omega}$ of all partial computable functions of two arguments. We say that $\lim_s \Phi_e(x, s)$ exists if $\Phi_e(x, s) \downarrow$ for every e and s and the sequence $(\Phi_e(x, s))_{s \in \omega}$ stabilizes. In the upcoming propositions we will use the following:

Notation 3.5.1. *Fix an effective listing $\{\Psi_e(x, s)\}_{e \in \omega}$ of total computable functions of two arguments satisfying the property:*

$$(\lim_s \Phi_e(x, s) \text{ exists}) \Rightarrow (\lim_s \Phi_e(x, s) = \lim_s \Psi_e(x, s)),$$

for every x and e. (We may assume that $\Psi_e(x, 0) = 0$, for every x and e.)

Proposition 3.5.2. Suppose that the type $\mathbf{f}$ of a computably presentable $G = \bigoplus_{i\in\omega} H$ has no computable settling time. Then G is not Δ^0_2-categorical.

Proof idea. Let $\alpha = (h_i)_{i\in\omega}$ be a characteristic of type $\mathbf{f}$. We build two computable groups, A and B, both isomorphic to G. The group A is a "nice" copy of G. The group B is a "bad" copy of G in which the e^{th} elementary direct component is used to defeat the e^{th} potential Δ^0_2-isomorphism from B onto A.

The first main idea of the strategy uses Baer's theory of types. We wait for the e^{th} potential isomorphism to converge on some specifically chosen element b_e from the e^{th} elementary direct component of B. We pick a fresh number j so large that, if the e^{th} potential isomorphism is indeed an isomorphism, the characteristic $\chi(b_e) = (d_i)_{i\in\omega}$ of b_e and the characteristic $\alpha = (h_i)_{i\in\omega}$ have to be equal starting from the j^{th} position. We may choose such a number j using that A is "nice" (to be explained in more detail). From this moment on, make sure $d_{k,s} = h_{k,s} - 1$ for $k \geq j$ least such that $h_{k,t} > 0$, where t is the current stage of the construction and $s \geq t$. By the choice of $\mathbf{f}$, such a position k can be found. Note that the e^{th} potential isomorphism is merely a (partial) Δ^0_2 function, and at a later stage it may output a new potential image of b_e. In this case we make $d_{k,s} = h_{k,s}$ and repeat the strategy.

The strategy would work if we had no symbols ∞ in $\mathbf{f}$. If we have ∞ on $\mathbf{f}$, then it may happen that

$$h_k = \lim_t h_{k,t} = \infty,$$

for the k we pick at the final iteration of the strategy (if the strategy iterates infinitely often then we win). In this case the strategy fails because both $h_{k,s}$ and $d_{k,s} = h_{k,s} - 1$ tend to infinity.

The second main idea is to pick a new fresh position k_1 for which $h_{k_1,s} > 0$ if we see $h_{k,s} > h_{k,t}$ at a later stage s. We may keep iterating this strategy defining k_2 when both h_k and h_{k_1} increase, etc. Nonetheless, this strategy is not sufficient if

$$h_{k_i} = \lim_t h_{k_i,t} = \infty$$

for every i.

The third main idea uses the notion of computable settling time. More specifically, each time we pick a new position k_i as described above, we additionally attempt to define a computable settling time ψ for α. If we have to make one more iteration as described in the previous paragraph, we set $\psi(k_i) = t$. We also define ψ on arguments between k_i and k_{i+1} (to be explained formally in the construction).

We keep introducing k_{i+1}, k_{i+2} etc. This process never terminates only if every position we pick corresponds to ∞ in α. Thus, we will succeed in defining a computable settling time for $\mathbf{f}$, contradicting the choice of $\mathbf{f}$ (to be explained in more detail). Therefore, we eventually pick a position k_j such that $h_{k_j} < \infty$. The groups A and B are both isomorphic to G by Theorem 3.1.1, because the characteristic of b_e belongs to $\mathbf{f}$. (Several minor technical details have not been mentioned in this sketch.)

Proof of Proposition 3.5.2. In the construction below we identify elements of A and B and the corresponding elements of ω. It suffices to build two computable presentations, A and B, of the group $G = \bigoplus_{i\in\omega} H$, and meet the requirements:

R_e : $\lim_t \Psi_e(b_e, t)$ exists $\Rightarrow$ $\lim_t \Psi_e(x, t)$ is not an isomorphism from B to A.

The nonzero element b_e is a witness for the R_e strategy below. More specifically, we enumerate $A = \bigoplus_{n\in\omega} Ha_n$ and $B = \bigoplus_{e\in\omega} C_e b_e$ in such a way that the sets $\{a_n : n \in \omega\}$ and $\{b_e : e \in \omega\}$ are computable. Let $(h_i)_{i\in\omega}$ be a characteristic of type $\mathbf{f}$. Fix a computable approximation $(h_{i,s})_{i,s\in\omega}$ of $(h_i)_{i\in\omega}$ such that (1) $h_{i,s} \leq h_{i,s+1}$, and (2) $h_i = \lim_s h_{i,s}$, for every i and s.

We make sure $\chi(a_n) = (h_i)_{i\in\omega}$, for every n, while the characteristic $\chi(b_e) = (d(e)_i)_{i\in\omega}$ of b_e will be merely equivalent to $(h_i)_{i\in\omega}$, for each e (thus, $C_e \cong H$, for each e).

The construction is injury-free, and we do not need any priority order on the strategies.

For every e, the strategy for R_e defines its own computable function ψ_e which[2] is an attempt to define a computable settling time for $(h_i)_{i\in\omega}$. To define ψ_e the strategy uses the sequence $(k_{e,i})_{i\in\omega}$ (to be defined in the construction).

Strategy for R_e: If at a stage s of the construction the parameter $k_{e,0}$ is undefined then:

1. Compute $\Psi_e(b_e, s)$. From this moment on, the strategy is always waiting for $t > s$ such that $\Psi_e(b_e, t) \neq \Psi_e(b_e, s)$. As soon as such a t is found, R_e initializes by making all its parameters undefined and also making $d(e)_{j,t} = h_{j,t}$ for every j we have seen so far.

2. Let $a \in A$ be such that $a = \Psi_e(b_e, s)$. Find integers c_n and c such that $ca = \sum_n c_n a_n$. Let j be a fresh large index such that (1) the prime p_j does not occur in the decompositions of the coefficients c and c_n, (2) $h_{j,s} > 0$, and (3) $d(e)_{j,s} < h_{j,s}$.

3. Once j is found[3], declare $\psi_e(j) = s$. From this moment on, make sure

[2]Since it will be clear from the construction at which stage ψ_e is defined (if ever), we omit the extra index t in $\psi_{e,t}$ and write simply ψ_e. We omit the index t for parameters $k_{e,i,t}$ as well.

[3]We may assume that at stage s such an index j can be found, otherwise we speed up the approximation $(h_{i,s})_{i,s\in\omega}$ during the construction.

$d(e)_{j,t} = h_{j,t} - 1$ for every $t \geq s$, unless the strategy initializes. Set $k_{e,0} = j$, and proceed.

Now assume that the parameters $k_{e,0}, \ldots k_{e,y}$ have already been defined by the strategyat $\leq s$. We also assume that $\psi_e(i)$ has already been defined for each i such that $k_{e,0} \leq i \leq max\{k_{e,x} : 0 \leq x \leq y\}$. Assume also that $k_{e,y}$ was first defined at stage $u < s$. Then do the following:

I. Wait for a stage $t \geq s$ (of the construction) such that either (a) $h_{i,t} > h_{i,s}$ for some i such that $k_{e,0} \leq i \leq max\{k_{e,x} : 0 \leq x \leq y\}$ and $i \notin \{k_{e,0}, \ldots k_{e,y}\}$, or (b) $h_{i,u} < h_{i,t}$ for each $i \in \{k_{e,0}, \ldots k_{e,y}\}$. While waiting, make $d(e)_{j,r} = h_{j,r}$ (r is the current stage of the construction), where $j \leq r$ and $j \notin \{k_{e,0}, \ldots k_{e,y}\}$.

II. If (a) holds for some i, then set $k_{e,(y+1)} = i$. If (b) holds, then let i be a fresh large index such that $h_{i,t} > 0$ (and $d(e)_{i,t} < h_{i,t}$), and set $k_{e,(y+1)} = i$. In this case also define $\psi_e(j)$ to be equal to the current stage for every j such that $max\{k_{e,x} : 0 \leq x \leq y\} < j \leq k_{e,(y+1)}$. Then proceed to III.

III. Set $d(e)_{i,v} = h_{i,v} - 1$ at every later stage v, where $i = k_{e,(y+1)}$, unless the strategy initializes.

End of strategy.

Construction. At stage 0, start enumerating A and B as free abelian groups over $\{a_n\}_{n\in\omega}$ and $\{b_e\}_{k\in\omega}$, respectively. Initialize R_e, for all e.

At stage s, let strategies R_e, $e \leq s$, act according to their instructions. If R_e acted at the previous stage, then return to its instructions at the position it was left at the previous stage.

Make $\chi(a_n) = (h_{i,s})_{i\in\omega}$ in A_s for every $n \leq s$, and $\chi(b_e) = (d(e)_{i,s})_{i\in\omega}$ in B_s for every $e \leq s$, by making a_n and b_e divisible by corresponding powers of primes.

End of construction.

Verification. For each e, the following cases are possible:

1. $\lim_s \Psi_e(b_e, s)$ does not exist. In this case the strategy initializes infinitely often. By the way the strategy is initialized, the characteristic of b_e is identical to α.

2. $\lim_s \Psi_{e,s}(b_e, s)$ exists and is equal to $\Psi_e(b_e, l)$. The domain of ψ_e should be finite. For if it were not, the it would be co-finite and then α would have a computable settling time. The computable settling time can be defined using

a (non-uniform) expansion of ψ_e to the finite set on which it is not defined. Therefore, the only possibility is that there is a parameter $k_{e,y}$ such that the $k_{e,y}^{th}$ position in α is finite. For otherwise we would be able to extend the definition of ψ_e again and again (see the construction). However, the strategy ensures $\lim_s \Phi_{e,s}(b_e, s)$ is not an isomorphism since the characteristic of b_e and α differ at $k_{e,y}^{th}$ position. We conclude that α differs from $\chi(b_e)$ in at most finitely many positions, and the differences are finitary.

In both cases $\chi(b_e)$ is equivalent to α. By Theorem 3.1.1, $A \cong B \cong G$. □

Recall that cases (3) and (4) were both reduced to:

Proposition 3.5.3. Let G be computable homogeneous completely decomposable abelian group of type $\mathbf{f}$, and suppose $\alpha = (\sup_s h_{i,s})_{i\in\omega}$ in $\mathbf{f}$ has computable settling time ψ. Furthermore, suppose $\overline{Inf(\alpha)}$ is not semi-low. Then G is not Δ_2^0-categorical.

Proof idea. We build two computable groups, A and B, both isomorphic to G. The group A is a "nice" copy of G. The group $B = \bigoplus_{e\in\omega} \bigoplus_{n\in\omega} C_{e,n} b_{e,n}$ is a "bad" copy of G in which the e^{th} direct component is used to defeat the e^{th} potential Δ_2^0-isomorphism from B onto A.

Recall that $Inf(\alpha)$ is a c.e. set. Given e, we attempt to define a functional $\Gamma(e, n, s)$ such that $H_{\overline{Inf(\alpha)}}(n) = \lim_s \Gamma(e, n, s)$. For every n, we pick an element $b_{e,n}$ in B and attempt to destroy the e^{th} potential Δ_2^0-isomorphism from B to A. We start by setting $\Gamma(e, n, 0) = 0$. We wait for j to appear in $W_{n,s} \setminus Inf(\alpha)_s$. If we never see such a j, then our attempt to define $\Gamma(e, n, s)$ is successful. If we find such a j, make $b_{e,n}$ divisible by a large power of p_j destroying the potential isomorphism (this power depends on our current guess on the isomorphic image of $b_{e,n}$ in A). We will set $\Gamma(e, n, t) = 1$ only if the e^{th} potential isomorphism changes on $b_{e,n}$ at a later stage t. We make $\Gamma(e, n, r) = 0$ as soon as j enters $Inf(\alpha)$, and then we start waiting for a new fresh number to show up in $W_n \setminus Inf(\alpha)$. If we see such a number then we repeat the above strategy with this number in place of j.

Our attempt to define $\Gamma(e, n, s)$ necessarily fails for at least one index n. Therefore, the e^{th} potential isomorphism will be defeated at the element $b_{e,n}$. Algebra is sorted out using Theorem 3.1.1.

Note that the algebraic strategy above differs from the one we used in Proposition 3.5.2. More specifically, we make elements divisible instead of keeping elements non-divisible. This strategy could not be used in Proposition 3.5.2, because it would not be consistent with the infinitary outcome (the case when the

e^{th} potential isomorphism changes infinitely often). We will see that this is not a problem here.

Proof of Proposition 3.5.3. We build two computable copies of G by stages. Recall that the first copy $A = \bigoplus_i Ha_i$ is a "nice" copy with $\chi(a_i) = \alpha$, for every i. The second ("bad") copy $B = \bigoplus_{e\in\omega} \bigoplus_{n\in\omega} C_{e,n}b_{e,n}$ is built in such a way that $\chi(b_{e,n})$ is equivalent to α, for every e and n.

Recall Notation 3.5.1. It suffices to meet the requirements:

$R_e : (\forall n) \lim_t \Psi_e(b_{e,n}, t)$ exists $\Rightarrow \lim_t \Psi_e(x, t)$ is not an isomorphism from B to A.

The strategy for R_e initially attempts to define a total $0'$-computable function Γ such that $\Gamma(n) = 0$ iff $W_n \subseteq Inf(\alpha)$. If we succeeded, this would imply

$$H_{\overline{Inf(\alpha)}} = \{n : W_n \cap \overline{Inf(\alpha)} \neq \emptyset\} = \{n : W_n \not\subseteq Inf(\alpha)\} \leq_T \emptyset',$$

contradicting the hypothesis. In the following, we write I in place of $Inf(\alpha)$. Also, we omit e in $\Gamma(e, n, s)$ and write simply $\Gamma(n, s)$. We also assume at most one number can be enumerated into W_n at every stage. We split R_e into substrategies $R_{e,n}$, $n \in \omega$:

Substrategy $R_{e,n}$. Permanently assign the element $b_{e,n}$ to $R_{e,n}$. Suppose that the strategy becomes active first time at stage s of the construction. Then:

1. Start by setting $\Gamma(n, s) = 0$ (we may suppose that $\Gamma(n, j) = 0$, for every $j < s$). At a later stage t, we define $\Gamma(n, t)$ to be equal to $\Gamma(n, t - 1)$, unless we have a specific instruction not to do so.

2. Wait for a stage $t > s$ and a number $j \in W_{n,t} \setminus I_t$. (Recall that we assume that at most one number can be enumerated into W_n at a stage.)

3. We let $p = p_j$ with $j \in W_{n,t} \setminus I_t$ at a later stage t. Find $a \in A_t$ such that $a = \Psi_e(b_e, t)$ (recall that the enumeration of A is controlled by us). Find integers c_n and c such that $ca = \sum_n c_n a_n$. Let k be a fresh large natural number such that (i) the prime $p = p_j$ has power at most $[k/2]$ in the decompositions of the coefficients c and c_n, and (ii) $h_{j,\psi(j)} < [k/2]$, where ψ is the computable settling time. Note that (*i*) and (*ii*) imply k is so large that p^k does not divide $a = \Psi_e(b_{e,n}, t)$ within A, unless $j \in I_t$. Make $b_{e,n}$ divisible by p^k within B.

   Wait for one of the two things to happen:

   I. (*I* changes first). We see $j \in I_u$ at a later stage $u > t$, and $\Psi_e(b_{e,n}, v) = \Psi_e(b_{e,n}, t)$ for each $v \in (t, u]$. We return to (2) with u in place of s.

II. (Ψ_e changes first). We see $\Psi_e(b_{e,n}, u) \neq \Psi_e(b_{e,n}, t)$ for $u > t$, and $j \in W_{n,v} \setminus I_v$ for each $v \in (t, u]$. Then set $\Gamma(n, u) = 1$ and start waiting for a stage $w > u$ such that $j \in I_w$. If such a stage w is found, then we set $\Gamma(n, w) = 0$ and go to (2) with w in place of s (and we do nothing, otherwise).

End of strategy.

Construction. At stage 0, start enumerating A and B as free abelian groups over $\{a_i\}_{i\in\omega}$ and $\{b_{e,n}\}_{e,n\in\omega}$.

At stage s, let strategies $R_{e,n}$, $e, n \leq s$, act according to their instructions. If $R_{e,n}$ acted at the previous stage, then return to its instruction at the position it was left at the previous stage.

Make $\chi(a_i) = \alpha = (h_j)_{j\in\omega}$ in A for every i. For every $e, n \in \omega$, make $\chi_j(b_{e,n}) = h_j$ in B for every j except at most one position, according to the instructions of $R_{e,n}$. We do so by making a_i and $b_{e,n}$ divisible by corresponding powers of primes.

End of construction.

Verification. By Theorem 3.1.1, $A \cong B \cong G$. Assume that $\lim_s \Psi_{e,s}(b_{e,n}, s)$ exists for every n (thus, II does not get visited infinitely often). Given n, consider the cases:

- $R_{e,n}$ eventually waits forever at substage (2). Then $\lim_s \Gamma(n, s) = 0$ and $W_n \subseteq I$. Thus, we have a correct guess about $H_{\overline{Inf(\alpha)}}$.

- $R_{e,n}$ visits I of (3) again and again from some point on (every time returning to (2)). Then $\lim_s \Gamma(n, s) = 0$ and $W_n \subseteq I$, and we again have a correct guess about $H_{\overline{Inf(\alpha)}}$.

- $R_{e,n}$ eventually waits forever at substage (3). Then $b_{e,n}$ witnesses that $\lim_s \Psi_e(b_{e,n}, s)$ is not an isomorphism.

There should be at least one n for which $\lim_s \Gamma(n, s) \neq H_{\overline{Inf(\alpha)}}(n)$. Therefore, for at least one n, the strategy $R_{e,n}$ eventually waits forever at substage (3). Thus, R_e is met. ◻

Proposition 3.5.4. If the type $\mathbf{f}$ of a computable homogeneous completely decomposable group G has a computable settling time, and $Fin(\alpha) = \{i : 0 < h_i < \infty\}$ is infinite and semi-low for $\alpha = (\alpha_i)_{i\in\omega}$ of type $\mathbf{f}$, then G is not Δ^0_2-categorical.

Proof idea. We combine the algebraic strategy from Proposition 3.5.2 and the guessing procedure based on the hypothesis $Fin(\alpha) = \{i : 0 < h_i < \infty\}$ is semi-low. As before, we are building two computable copies, A and B, of G.

If $Fin(\alpha)$ were an infinite computable set, then the algebraic strategy would be rather straightforward. To destroy the e^{th} potential Δ^0_2-isomorphism from B to A, we pick a large $j \in Fin(\alpha)$ and make the witness $b_e \in B$ *not divisible* by p_j. If the potential isomorphism changes at a later stage, we make $h_i(b_e) = \alpha_i$ and repeat the strategy for another fresh and large $i \in Fin(\alpha)$. We have already discussed a similar algebraic strategy in the idea of proof of Proposition 3.5.2 (the case with no ∞'s in α).

However, $Fin(\alpha)$ is merely semi-low. Recall that the type has a computable settling time. Therefore, we can produce a computable approximation $(h_{i,s})_{i,s\in\omega}$ of α such that, for every i, either $\alpha_i = h_{i,0}$ or $\alpha_i = \infty$. We focus on the computable set $N = \{i : h_{i,0} \neq 0\} = Inf(\alpha) \cup Fin(\alpha)$. Note that $Inf(\alpha) = \{i : \alpha_i = \infty\}$ is c.e.

Imagine the e^{th} potential Δ^0_2 isomorphism has settled on its witness $b_e \in B$ (if it never settles we win). To successfully run the algebraic strategy, we need to find at least one $i \in Fin(\alpha)$. We find a fresh large $i \in N$ and keep b_e not divisible by p_i. We can do so because i is so large that b_e has not been declared divisible by $p_i^{h_{i,0}}$ yet. At the same time we start enumerating a c.e. set first setting $W = \emptyset$, and ask if $W \cap Fin(\alpha) = \emptyset$ (recall that the guessing procedure is Δ^0_2). We do nothing and wait until we get the answer $W \cap Fin(\alpha) = \emptyset$. Note that we should eventually see this answer, otherwise we get a contradiction by keeping W empty. Then we enumerate i into W. We do not make b_e divisible by any further prime until we see:

(1.) i enters $Inf(\alpha)$. Then we pick next least $j \in N$, enumerate j into W, and repeat the strategy keeping b_e untouched.

(2.) The current guess becomes $W \cap Fin(\alpha) \neq \emptyset$. We allow the construction to continue building the elementary component corresponding to b_e but keep b_e not divisible by p_i. If i never enters $Inf(\alpha)$ we win. If at a later stage i enters $Inf(\alpha)$, we wait until our guess is $W \cap Fin(\alpha) = \emptyset$. Again, it should eventually happen, otherwise we get a contradiction by not changing W. Then we make b_e infinitely divisible by p_i, pick a large fresh $v \in N$, enumerate v into W, and repeat the whole strategy with v in place of i (again, keep b_e untouched etc).

Note that we eventually reach (2.) with some $j \in W$, and either j never enters $Inf(\alpha)$ or we change our guess on $W \cap Fin(\alpha)$. In the latter case will reach (2.) again with another number, and either win or change the guess once more. We can not change the guess infinitely often, because $Fin(\alpha)$ is semi-low. Thus, eventually the algebraic strategy succeeds.

In the formal construction each strategy defines its own *sequence* of c.e. sets. Every set from the sequence corresponds to a potential image of b_e, which can be changed at a later stage. If the image changes, we start enumerating the next set from the e^{th} sequence. Since the construction is effective and uniform, we may assume that the indexes of these c.e. sets are listed by a computable function, and the index of this function is given ahead of time. We give all details in the formal proof below.

Proof of Proposition 3.5.4. Let Γ be a computable function such that $\lim_s \Gamma(n,s)$ guesses $Fin(\alpha) \cap W_n = \emptyset$ correctly. As in the proof of Proposition 3.5.2, we are building two computable copies,

$$A = \bigoplus_{n\in\omega} Ha_n \text{ and } B = \bigoplus_{e\in\omega} C_e b_e,$$

of G. We make $\chi(a_n) = \alpha$ and $\chi(b_e) = (d(e))_{i\in\omega} \simeq \alpha$, for every n and e. Recall Notation 3.5.1. The requirements are:

R_e : If $\lim_t \Psi_e(b_e, t)$ exists, then $\lim_t \Psi_e(x,t)$ is not an isomorphism from B to A.

For every e, the strategy for R_e will enumerate its own sequence of c.e. sets. The indexes for the sets are listed by a computable function g of two arguments:

$$\{W_{g(e,s)}\}_{s\in\omega}.$$

Let $(h_{i,s})_{i,s\in\omega}$ be a computable approximation of α such that, for every i, either $\alpha_i = h_{i,0}$ or $\alpha_i = \infty$. Also, let $n(0), n(1)\ldots$ be an effective increasing enumeration of the infinite computable set $N = \{i : h_{i,0} \neq 0\}$.

The strategy for R_e: Suppose $s = 0$ or $\Psi_e(b_e, s) \neq \Psi_e(b_e, s-1)$. Do the following substeps:

1. Make $\chi(b_e) = (d(e))_{i\in\omega}$ and α equal at all positions seen so far.

2. Begin enumerating $W_{g(e,s)}$ by setting $W_{g(e,s)} = \emptyset$.

3. Wait for a stage u such that $\Gamma(g(e,s),u) = 0$.

4. Let $a \in A$ be such that $a = \Psi_e(b_e, s)$. If $a = 0$ do nothing. If $a \neq 0$, find integers c_m and c such that $ca = \sum_m c_m a_m$. Let $n(i) \in N$ be a fresh large number such that (1) the prime $p_{n(i)}$ does not occur in the decompositions of the coefficients c and c_m, (2) $h_{n(i),0} > 0$, and (3) $d(e)_{k,s} = 0$ for every $k \geq n(i)$.

5. Enumerate $n(i)$ into $W_{g(e,s)}$. Keep $d(e)_{n(i),l} = 0$ for $l \geq s$ (unless we have a specific instruction not to do so). *Restrain* the element b_e by not allowing the construction to make it divisible by any prime greater than $p_{n(i)}$.

6. Wait for one of the following three things to happen:

   I. $\Psi_e(b_e, s) \neq \Psi_e(b_e, t)$ at a later stage t. Then declare b_e not restrained and restart the strategy with t in place of s (go to (1); for instance, make b_e divisible by the corresponding power of $p_{n(i)}$).

   II. The number $n(i)$ enters the c.e. set $Inf(\alpha)$ at a stage $s > t$ (thus, $h_{n(i)} = \infty$). Make b_e infinitely divisible by $p_{n(i)}$ and return to (5) with $n(i+1)$ in place of $n(i)$ keeping b_e restrained.

   III. $\Gamma(g(e,s), t) = 1$ (thus, we believe $W_{g(e,s)} \cap Fin(\alpha) \neq \emptyset$ and $j \in Fin(\alpha)$). We remove the restraint from the element b_e allowing the construction to make b_e divisible by p_i with $i \notin W_{g(e,s)}$ if needed. We keep b_e not divisible by $p_{n(i)}$.

   If at a later stage r the number $n(i)$ enters $Inf(\alpha)_r$ (thus, $W_{g(e,s),r} \subseteq Inf(\alpha)_r$), then make b_e infinitely divisible by $p_n(i)$. In this case also wait for a stage $w \geq r$ such that $\Gamma(g(e,s), w) = 0$. Then return to (4) with a new fresh and large $n(j)$.

End of strategy.

Construction: At stage 0, start enumerating A and B as free abelian groups over $\{a_n\}_{n\in\omega}$ and $\{b_e\}_{k\in\omega}$, respectively.

At stage s, let strategies R_e, $e \leq s$, act according to their instructions. If R_e acted at the previous stage, then return to its instruction at the position it was left at the previous stage.

Make $\chi(a_n) = (h_{i,s})_{i\in\omega}$ in A_s for every $n \leq s$, and $(h_{i,s})_{i\in\omega} = (d(e)_{i,s})_{i\in\omega}$ in B_s for every $e \leq s$ which is not restrained, unless R_e keeps $d(e)_{i,s} = 0$.

End of construction.

Verification. If $\lim_t \Psi_e(b_e, t)$ does not exist, then we reach I of (6) infinitely often and, therefore, $\chi(b_e) = \alpha$. Assume that $\lim_t \Psi_e(b_e, t)$ exists. Let s be the stage after which $\Psi_e(b_e, t)$ never changes again and

$$\Psi_e(b_e, s) = \lim_t \Psi_e(b_e, t).$$

Let $u \geq s$ be a stage such that $\lim_t \Gamma(g(e,s),t) = \Gamma(g(e,s),u)$. The set $W_{g(e,s)}$ is designed to make $\lim_t \Gamma(g(e,s),t) = 1$. If $\Gamma(g(e,s),u) = 0$ was the case, then we would add more elements to $W_{g(e,s)}$ at a stage $v \geq u$ and eventually put some $n(j) \in Fin(\alpha)$ into $W_{g(e,s)}$, a contradiction.

By the definition of Γ, if $\lim_t \Gamma(g(e,s),t) = 1$, then there is at least one $j \in W_{g(e,s)} \cap Fin(\alpha)$. Furthermore, the strategy guarantees that there is exactly one such a j, namely the last witness $n(i)$ which visits III of the strategy at some stage and stays there from this stage on. As a consequence, the element b_e will eventually be unrestrained (see the construction).

The algebraic strategy guarantees b_e is not divisible by $p_{n(i)}$ while the image is (see the second paragraph of *proof idea*). Furthermore, b_e is declared not restrained as soon as we reach III with $n(i)$, meaning that the characteristic of b_e satisfies the property $d(e)_j = \alpha_j$ for each $j \neq n(i)$. It remains to apply Theorem 3.1.1. □

We note that in the proposition above the algebraic strategy from Proposition 3.5.3 would not succeed. Theorem 3.5.1 is proved. □

Corollary 3.5.1. *For a c.e. set P, the following are equivalent:*

1. *G_P has a Σ^0_2 excellent $\widehat{P}$-basis;*
2. *G_P has a Σ^0_2-basis as a free $Q^{(P)}$-module;*
3. *G_P has a Π^0_1-basis as a free $Q^{(P)}$-module;*
4. *G_P is Δ^0_2-categorical;*
5. *$\widehat{P}$ is semi-low.*

Proof. The proof is a combination of Theorem 3.5.1, Theorem 3.4.2, and Lemma 3.2.1. □

Corollary 3.5.2. *Each computable copy of the free abelian group of rank ω has a Π^0_1 set of free generators.*

Proof. The free abelian group can be viewed as the free Z-module. It remains to apply Theorem 3.4.2 and Theorem 3.5.1 with $\widehat{P}$ the set of all primes. □

Chapter 4

An effective transformation

This chapter studies properties of a certain computable functor (effective transformation) from computable trees to computable abelian groups. We begin with recalling some definitions and basic facts about rank-homogeneous trees (from [16]).

4.1 Rank-homogeneous trees

Definition 4.1.1 (tree rank)**.** *Let T be a subtree of $\omega^{<\omega}$. We define the* tree rank *of $x \in T$, denoted by $tr(x)$, by induction.*

1. *$tr(x) = 0$ if x has no successor,*
2. *for $\alpha > 0$, $tr(x) = \alpha$ if α is the least ordinal greater than $tr(y)$ for all successors y of x,*
3. *$tr(x) = \infty$ if x does not have ordinal tree rank.*

Tree rank is sometimes called *foundation rank*. Note that $tr(x) = \infty$ if and only if x extends to a path.

Definition 4.1.2 (rank-homogeneous tree)**.** *A tree $T \subseteq \omega^{<\omega}$ is* rank-homogeneous *provided that for all x at level n,*

1. *if $tr(x)$ is an ordinal, then for all y at level $n+1$ such that $tr(y) < tr(x)$, x has infinitely many successors z such that $tr(z) = tr(y)$,*
2. *if $tr(x) = \infty$, then for all y at level $n+1$, x has infinitely many successors z such that $tr(z) = tr(y)$.*

For a rank-homogeneous tree T, let $R(T)$ be the set of pairs (n, α) such that there is an element at level n of tree rank α (where α is an ordinal, not ∞). Note that the top node in T has rank ∞ just in case $R(T)$ has no pair of the form $(0, \alpha)$. Also note if T has a node of rank ∞, then the top node must have rank ∞, and if the top node has rank ∞, then there are nodes of rank ∞ at all levels. Thus, from the set of pairs $R(T)$ in which the second components are ordinals, we can deduce all of the information that would be given if we included pairs with second component ∞.

Proposition 4.1.1. *Suppose T, T' are rank-homogeneous trees. Then $T \cong T'$ iff $R(T) = R(T')$.*

Proof. Clearly, if $T \cong T'$, then $R(T) = R(T')$. Suppose $R(T) = R(T')$. To see that there is an isomorphism, we show that the set of finite partial rank-preserving isomorphisms between subtrees of T and T' has the back-and-forth property. The subtrees must be closed under predecessor in the large trees, and the finite partial isomorphisms must preserve all ranks, both ordinals and ∞. Given a finite subtree of one of the large trees, we can reach any further node by a finite sequence of steps in which the node being added is a successor of one already included. Therefore, it is enough to prove the following.

Claim: Let p be a rank-preserving isomorphism from the finite subtree τ of T onto the finite subtree τ' of T', and let $a \in T - \tau$ be a successor of $b \in \tau$. Suppose $b' = p(b)$. Then there exists a', a successor of b' in T', not already in $ran(p)$, such that a' and a have the same rank.

The rank of $p(b)$ is the same as that of b. If a has rank ∞, then b and b' also have rank ∞, and b' has infinitely many successors of rank ∞. If a has ordinal rank α, then b and b' have rank either ∞ or some $\beta > \alpha$. In either case, b' has infinitely many successors of rank α. We choose a' to be a successor of b', of the proper rank, not already in $ran(p)$.

□

The class of countable rank homogeneous trees is denoted by RHT.

4.2 The transformation

Hjorth [50] gave a transformation from trees to torsion-free Abelian groups which enabled him to show that the isomorphism relation on these groups is not Borel.

Downey and Montalbán [32] built on Hjorth's ideas to show that the isomorphism relation on these groups is analytic complete. The transformation

$$G : T \to G(T)$$

from [50] and [32] is described below.

We consider the elements of $\omega^{<\omega}$ as a basis for a Q-vector space V^*. Let T be a subtree of $\omega^{<\omega}$, and let V be the subspace of V^* with basis T. Let T_n be the set of elements at level n of T. If u is at level $n > 0$, let u^- be the predecessor of u. Let $(p_n)_{n\in\omega}$ be the standard computable list of primes, in increasing order. We let $G(T)$ be the subgroup of V generated by the vector space elements of the following forms:

1. $\dfrac{v}{(p_{2n})^k}$, where $v \in T_n$, and $k \in \omega$,

2. $\dfrac{v + v'}{(p_{2n+1})^k}$, where $v \in T_n$, v' is a successor of v, and $k \in \omega$.

If P is a finite set of prime numbers, we let $Q^{(P)}$ be the set of rationals of the form $\dfrac{k}{m}$, where $k \in Z$ and m is a product of powers of elements of P.

Elementary facts.

1. $Q^{(\emptyset)} = Z$

2. $Q^{(P)} \cap Q^{(R)} = Q^{(P\cap R)}$

3. $Q^{(P)} + Q^{(R)} = Q^{(P\cup R)}$

Recall that $\emptyset$ is the top node in the tree T. Note that each element of $G(T)$ can be expressed in the form

$$h = \Sigma_{v\in V} a_v v + \Sigma_{u\in U} b_u (u^- + u)$$

where

1. U, V are finite subsets of T, $\emptyset \notin U$,

2. if $v \in V \cap T_n$, then $a_v \in Q^{(\{p_{2n}\})}$,

3. if $u \in U \cap T_{n+1}$, then $b_u \in Q^{(\{p_{2n+1}\})}$.

The transformation described above takes the full class of trees to the class *TFA* of torsion-free Abelian groups. Our goal is to show that the restriction of the transformation to the class *RHT* of rank-homogeneous trees is $1-1$ on isomorphism types.

4.3 The injectivity on *RHT*

The main result of the section, and of the whole chapter, is:

Theorem 4.3.1. *For every two rank-homogeneous trees T and T', the groups $G(T)$ and $G(T')$ are isomorphic if, and only if, $T \cong T'$.*

Preliminary remarks. The idea of this technical result can not be described in two or three sentences. We, however, give some intuition which lies behind the proof below. Consider the transformation $T \to G(T)$. The first idea would be: given a group of the form $G(T)$, reconstruct vertices of T and understand which ones are adjacent. This is not possible: we will show that, in general, non-isomorphic trees may give rise to isomorphic groups. Thus, we have to use the special features of the class of rank-homogeneous trees. In particular, we know that the collection of ranks of vertices at different levels of a rank-homogeneous tree uniquely determines the isomorphism type of the tree. We do not distinguish between elements of the tree T and the corresponding elements of $G(T)$, which we call *vertex elements*. We will describe elements of $G(T)$ that resemble vertex elements. We call these elements vertex-like. We will also describe a relation on these elements that resembles the successor relation. From this, we obtain a notion of rank for vertex-like elements. We will use this new notion of rank to provide, for each $n \in \omega$ and each countable ordinal α, a sentence in $L_{\omega_1,\omega}$ that is true in $G(T)$ if and only if T has a node at level n of tree rank α. From this, it follows that rank-homogeneous trees that give rise to isomorphic groups must be isomorphic. These is all done by a careful analysis of infinite divisibility within the group $G(T)$. The proof uses the machinery from [50, 32] and is (essentially) purely algebraic.

Proof. The results in [32] use only a few simple facts, which they extract from the proofs in [50]. We begin with these same facts, but we shall need more. Recall that $\emptyset$ is the top node in the tree T. We write $p^\infty|h$ if h is divisible by all powers of p.

Lemma 4.3.1. Let $h \in G(T)$, say $h = \sum_{v \in V} q_v v$, where V is a finite set of vertex elements and $q_v \in Q - \{0\}$. If p is a prime and $p^\infty|h$, then there is some $g \in G(T)$ such that $g = \sum_{v \in V} r_v v$, where $p^\infty|g$, and for all $v \in V$, $r_v \in Q^{(\{p\})} - Z$.

Proof. We multiply h by an appropriate integer and then divide by a power of p. $\square$

The next two lemmas are given explicitly in [32].

Lemma 4.3.2. Let h be an element of $G(T)$, say $h = \sum_{v \in V} r_v v$, where V is a finite subset of T and $r_v \in Q - \{0\}$. If $(p_{2n})^\infty | h$, then for all $v \in V$, v has length n.

Proof. We take $g = \sum_{v \in V} r'_v v$ as in Lemma 4.3.1. For $v \in V$, the coefficient r'_v has the form $a_v + (\sum_{u \in U_v} b_u) + b_v$, where U_v consists of successors of v, if v has length m, then $a_v \in Q^{(\{p_{2m}\})}$, if $u \in U_v$, then $b_u \in Q^{(\{p_{2m+1}\})}$, and $b_v \in Q^{(\{p_{2m-1}\})}$. Since $r'_v \in Q^{(\{p_{2n}\})} - Z$, we must have $m = n$ and $a_v \neq 0$. Note that $(\sum_{u \in V_v} b_u) + b_v$ must be in Z. $\square$

Lemma 4.3.3. Let h be an element of $G(T)$, say $h = \sum_{v \in V} r_v v$, where V is a finite subset of T and $r_v \in Q - \{0\}$. If $(p_{2n+1})^\infty | h$, then for all v of length n in V, v has a successor $u \in U$.

See [32] for a proof. We will not give the proof because we will actually need more (see Lemma 4.3.4). It is useful to keep in mind the following example showing that the predecessor of v, even if it exists, may not be in U:

Example: Let $h = u - u' = (v + u) - (v + u')$, where $v \in T_n$ and u, u' are successors of v in T_{n+1}. Then $p_{2n+1}^\infty | h$, although in our expression for h, the coefficient of v is 0.

The following is taken from Hjorth [50] (Propositions 2.2 and 2.5).

Proposition 4.3.1. *Let φ be a homomorphism from $G(T)$ to Q such that $\varphi(v) = 1$ for $v \in T_n$ and $\varphi(v) = -1$ for $v \in T_{n+1}$. Let $h = \sum_{v \in V} c_v v + \sum_{u \in U} a_u u$, where $V \subseteq T_n$ and $U \subseteq T_{n+1}$. If $(p_{2n+1})^\infty | h$, then $\varphi(h) = 0$. Moreover, for each $v \in V$, if $h_v = c_v v + \sum_{u \in U_v} a_u u$, then $(p_{2n+1})^\infty | h_v$, and $\varphi(h_v) = 0$ (here $U_v \subseteq U$ contains all successors of v in U).*

Using Proposition 4.3.1, we obtain:

Lemma 4.3.4.

1. Suppose $h = a_\emptyset \emptyset + \sum_{u \in U} a_u u$, where $U \subseteq T_1$. If $(p_1)^\infty | h$, then $a_\emptyset = \sum_{u \in U} a_u$.

2. Suppose $h = \sum_{v \in V} a_v v + \sum_{u \in U} b_u u$, where $U \subseteq T_{n+1}$, and V is the set of predecessors of these elements. For $v \in V$, let U_v be the set of successors of v. If $(p_{2n+1})^\infty | h$, then for each $v \in V$, $a_v = \sum_{u \in U_v} b_u$.

Proof. For 1, we consider a homomorphism φ taking $\emptyset$ to 1 and taking elements at level 1 to -1. We have $\varphi(h) = 0 = a_\emptyset - \sum_{u \in U} a_u$. By Proposition 4.3.1, $a_\emptyset = \sum_{u \in U} a_u$. For 2, we consider a homomorphism φ taking all elements of V to 1 and all elements of U to -1. By Proposition 4.3.1, for each $v \in V$, $\varphi(a_v v + \sum_{u \in U_v} b_u) = 0 = a_v - \sum_{u \in U_v} b_u$. Therefore, $a_v = \sum_{u \in U_v} b_u$. □

Note that in Lemma 4.3.4, in Case 1, we may have $a_\emptyset = 0$ and $\sum_{u \in U} a_u = 0$, and in Case 2, we may have $a_v = 0$, and $\sum_{u \in U_v} b_u = 0$. We need a refinement of Lemma 4.3.2.

Lemma 4.3.5. Suppose $(p_{2n})^\infty | h$.

1. If $n > 0$, then h can be expressed in the form $\sum_{v \in V} r_v v$, where $V \subseteq T_n$ and r_v is in $Q^{(\{p_{2n}, p_{2n-1}\})}$.

2. If $n = 0$, then h has the form $r\emptyset$, where $r \in Q^{(\{p_0\})}$.

Proof. We consider the two cases separately.

Case 1: Suppose $n > 0$. By Lemma 4.3.2, h can be expressed in the form $\sum_{v \in V} r_v v$, where $V \subseteq T_n$, and $r_v \in Q$. Just because $h \in G(T)$, we have $h = \sum_{u \in U} a_u u + \sum_{u \in W} b_u (u + u^-)$, where if $u \in T_k$, then $a_u \in Q^{(\{p_{2k}\})}$, and $b_u \in Q^{(\{p_{2k-1}\})}$. For u at level $k \neq n$, the coefficient of u in the expression for h must be 0. This coefficient has the form $a_u + (\sum_{w^- = u} b_w) + b_u$.

Claim: For all $k > n$, for u at level k (appearing in our decomposition), a_u and b_u are integers.

Proof of Claim. We work our way back from the largest $k > n$ with some u at level k that appears. For the greatest k, if u is at level k, and u appears, then no successor of u appears. We have $0 = a_u + b_u$, where $a_u \in Q^{(\{p_{2k}\})}$ and $b_u \in Q^{(\{p_{2k-1}\})}$. Then both a_u and b_u must be integers. Supposing that the claim holds for $k' > k$, where $k > n$, let u be an element at level k that appears. We have $0 = a_u + (\sum_{w^- = u} b_w) + b_u$, where $a_u \in Q^{(\{p_{2k-1}\})}$, $b_u \in Q^{(\{p_{2k-1}\})}$, and $\sum_{w^- = u} b_w \in Z$. Again a_u and b_u must be integers. □

Using the Claim, we can complete the proof for Case 1. For v at level n, the coefficient is $r_v = a_v + (\sum_{w^- = v} b_w) + b_v$, where $\sum_{w^- = v} b_w \in Z$, $a_v \in Q^{(\{p_{2n}\})}$ and $b_v \in Q^{(\{p_{2n-1}\})}$. Therefore, $r_v \in Q^{(\{p_{2n}, p_{2n-1}\})}$.

Case 2: Suppose $n = 0$. Then the only possible v is $\emptyset$, so $h = r\emptyset$. Since there is no $\emptyset^-$, we have $r = a_\emptyset + \sum_{w^- = \emptyset} b_w$. By the argument above, $\sum_{w^- = \emptyset} b_w \in Z$. Since $a_\emptyset$ is in $Q^{(\{p_0\})}$, r is also. □

A node in T_n has the feature that there is a successor chain of length n leading from $\emptyset$ to it. We try to describe this in the group $G(T)$. We define first the *pseudo-vertex-like* elements at level n, and then the *vertex-like* elements at level n. We start with the definition of pseudo-successor:

Definition 4.3.1 (pseudo-successor)**.** *Suppose nonzero h and g are so that $p_{2n}^{\infty}|g$ and $p_{2n+2}^{\infty}|h$, for some $n \geq 0$. We say that h is a* pseudo-successor *of g if $(p_{2n+1})^{\infty}|(g+h)$.*

Lemma 4.3.6. There is a computable infinitary formula $\Theta(x)$ such that for all $T \in RHT$ with $T_1 \neq \emptyset$, $\Theta(x)$ is satisfied just by $\emptyset$ and $-\emptyset$.

Proof. We let $\Theta(x)$ say the following:

1. $(p_0)^{\infty}|x$,
2. for primes $q \neq p_0$, $q \not| x$,
3. x has a pseudo-successor,
4. $\dfrac{1}{p_0}x$ has no pseudo-successor.

It is not difficult to see that $\emptyset$ and $-\emptyset$ satisfy $\Theta(x)$. We must show that other elements do not. If x satisfies Condition 1, we can apply Part 2 of Lemma 4.3.5, to see that x has the form $r\emptyset$, where $r \in Q^{(\{p_0\})}$. Then r has the form $\dfrac{z}{(p_0)^m}$, where $z \in Z$. Condition 2 implies that z is not divisible by any primes other than p_0. Therefore, x has the form $\pm p^k\emptyset$. Condition 3 says that x has a successor. Using this, we show that $k \geq 0$. Take y such that $(p_2)^{\infty}|y$. By Part 1 of Lemma 4.3.5, $y = \sum_{v\in V} s_v v$, where $V \subseteq T_1$ and $s_v \in Q^{(\{p_2,p_1\})}$. If $(p_1)^{\infty}|(x+y)$, then by Lemma 4.3.4, $\pm(p_0)^k = \sum_{v\in V} s_v$. This implies that the right-hand side is an integer, and then the left-hand side is as well. Therefore, $x = \pm p_0^k$, where $k \geq 0$. Finally, we show that if x satisfies Condition 4, then k cannot be positive. If $k > 0$, then $\dfrac{1}{p_0}x = p_0^{k-1}\emptyset$. This satisfies Conditions 1 and 2. Moreover, if $v \in T_1$, then $p_0^{k-1}v$ is a successor of $\frac{1}{p_0}x$, contradicting Condition 4. Therefore, x must have the form $\pm\emptyset$. □

Definition 4.3.2 (pseudo-vertex-like)**.** *An element $h \in G(T)$ is* pseudo-vertex-like*, or* p.v.l.*, at level n, if one of the following holds:*

1. *$n = 0$ and $\Theta(x)$ holds, or*

2. $n > 0$ *and*

   (a) $p_{2n}^{\infty} | h$,

   (b) *there exists a sequence* $g_0, g_1, \ldots, g_n = h$, *such that* g_0 *satisfies the formula* $\Theta(x)$ *from Lemma 4.3.6, and for all* $i < n$, *we have* $p_{2i}^{\infty} | g_i$ *and* $p_{2i+1}^{\infty} | (g_i + g_{i+1})$.

It is easy to see that all vertex elements are pseudo-vertex-like.

Remark. For each n, we have a computable infinitary formula that defines the set of p.v.l. elements of $G(T)$. The formula is independent of T. For each n, we have a computable infinitary formula defining in $G(T)$ the set of pairs (g, h) such that g is p.v.l. at level n and h is a pseudo-successor of g. The formula is independent of T.

We define rank for p.v.l. elements by analogy with tree rank. We write $rk(h)$ for the rank of h in the group $G(T)$, and $tr(v)$ for the tree rank of v in the tree T.

Definition 4.3.3 (rank)**.** *Let h be p.v.l. at level n.*

1. $rk(h) = 0$ *if h has no pseudo-successors,*

2. *for* $\alpha > 0$, $rk(h) = \alpha$ *if all pseudo-successors of h have ordinal rank, and* α *is the least ordinal greater than these ranks,*

3. $rk(h) = \infty$ *if h does not have ordinal rank.*

We note that $rk(h) = \infty$ if and only if there is an infinite sequence $(g_i)_{i \in \omega}$ such that each g_i is p.v.l., $g_0 = h$ and g_{i+1} is a pseudo-successor of g_i.

Lemma 4.3.7. Suppose h is p.v.l at level n, expressed in the form $\sum_{v \in V} r_v v$, where V is a finite subset of T_n and $r_v \neq 0$. Then for all v, $tr(v) \geq rk(h)$.

Proof. We show by induction on α that if $rk(h) > \alpha$, then for all $v \in V$, $tr(v) \neq \alpha$. (We allow the possibility that $rk(h) = \infty$.) Let $rk(h) > 0$. Let g be a p.v.l. pseudo-successor for h. Then $(p_{2n+1})^{\infty} | (h + g)$. Say $g = \sum_{u \in U} s_u u$, where U is a set of vertex elements at level $n + 1$ and $s_u \neq 0$. By Lemma 4.3.3, for each $v \in V$, there is some $u \in U$ such that u is a successor of v. Therefore, $tr(v) \neq 0$.

Consider $\alpha > 0$, where the statement holds for $\beta < \alpha$. Suppose $rk(h) > \alpha$. Let g be a p.v.l. pseudo-successor of h such that $rk(g) \geq \alpha$. Say $g = \sum_{u \in U} s_u u$, where U is a set of vertex elements at level $n + 1$ and $s_u \neq 0$. By the Induction Hypothesis, $tr(u) \neq \beta$ for any $\beta < \alpha$, so $tr(u) \geq \alpha$. By Lemma 4.3.3, some $u \in U$ is a successor of v. Then $tr(v) \neq \alpha$. Finally, we show that if $rk(h) = \infty$, then for all $v \in V$, $tr(v) = \infty$.

There must be an infinite sequence of p.v.l. elements $(g_k)_{k\in\omega}$ such that $g_0 = h$ and g_{k+1} is a pseudo-successor of g_k. We have $g_k = \sum_{u\in U_k} s_u u$, where U_k is a set of vertex elements at level $n+k$, and $s_u \neq 0$. For each element of U_k, there is a successor in U_{k+1}. We obtain a chain of successors, starting with $v = v_0 \in U_0$, and choosing v_{k+1} a successor of v_k in U_{k+1}. Therefore, $tr(v) = \infty$. □

Remark. For each n and α, we have a formula of $L_{\omega_1,\omega}$ defining in $G(T)$ the set of p.v.l. elements at level n of rank α. The formula is independent of T. Moreover, it lies in the least admissible set containing the ordinal α.

It is again helpful to consider an example.

Example: Let $v \in T_1$ and let u and u' be successors of v in T_2. Suppose that both u and u' have successors in T_3. Let $g = \dfrac{1}{p_4}u + \dfrac{p_4 - 1}{p_4}u'$. Since $p_4^\infty | u, u'$, we have $p_4^\infty | g$. Since $p_4(v+g) = (v+u) + (p_4 - 1)(v+u')$, we see that $p_3^\infty | (v+g)$. Therefore, g is p.v.l. and it is a pseudo-successor of v. We can show that g has no pseudo-successor, even though we have expressed it in terms of u and u', both of which have successors in T_3. Suppose that h is its a pseudo-successor at level 3. Then $h = \sum_{w\in W} r_w w$, where $W \subseteq T_3$ and $r_w \in Q$. By Lemma 4.3.5, we must have $r_w \in Q^{(p_6,p_5)}$. We must have $p_5^\infty | (g+h)$. By Lemma 4.3.4, if $W_u, W_{u'}$ are, respectively, the sets of successors of u, u' in W, then $\sum_{w\in W_u} r_w = \dfrac{1}{p_4}$, and $\sum_{w\in W_{u'}} r_w = \dfrac{p_4 - 1}{p_4}$. This is a contradiction.

We strengthen the definition of p.v.l. element in order to rule out examples like the one above, in which g has no successor, but it has a decomposition in terms of elements all having successors.

Definition 4.3.4 (vertex-like)**.** *Let $g \in G(T)$. We say that g is* vertex-like*, or* v.l.*, if*

1. *g is p.v.l. at some level n, and*
2. *either*
   (a) *$rk(g) > 0$, or*
   (b) *$rk(g) = 0$ and for any decomposition $g = \sum_j r_j g_j$ such that all g_j are p.v.l. at level n, there exists j such that $rk(g_j) = 0$.*

Lemma 4.3.8. If v is a vertex element, then it is vertex-like.

Proof. We already noted that a vertex element is p.v.l. Suppose v is at level n, and $rk(v) = 0$. Then v has no successors. We must show that if $v = \sum_j r_j g_j$, where each g_j is p.v.l. at level n, then for some j, $rk(g_i) = 0$. Suppose that for all j, $rk(g_j) \neq 0$. Say h_j is a p.v.l. pseudo-successor of g_j at level $n+1$. By Lemma 4.3.2, each g_j has a decomposition in terms of tree elements at level n. Since $v = \sum_j r_j g_j$, v must appear with non-zero coefficient in the decomposition of some g_j. Then by Lemma 4.3.4, the corresponding h_j has a decomposition that involves successors of v with non-zero coefficients. This is a contradiction. □

We would like to show that if g is v.l. at level n, expressed in the form $\sum_{v \in V} r_v v$, where $V \subseteq T_n$ and $r_v \neq 0$, then $rk(g)$ is the minimum of $tr(v)$, for $v \in V$.

Lemma 4.3.9. Suppose g is v.l. at level n. Say $g = \sum_{v \in V} r_v v$, where $V \subseteq T_n$. Then $rk(g) = 0$ iff there exists $v \in V$ such that $tr(v) = 0$.

Proof. First, suppose there exists $v \in V$ such that $tr(v) = 0$. By Lemma 4.3.7, $tr(v) \geq rk(g)$, so $rk(g) = 0$. Next, suppose $rk(g) = 0$. The elements of V are p.v.l. and one of the decompositions of g is $\sum_{v \in V} r_v v$. By the definition of vertex-like, there is some v such that $rk(v) = 0$. Then v has no pseudo-successors, so v has no successors in T. Therefore, $tr(v) = 0$. □

Lemma 4.3.10. If g is v.l. at level n and $rk(g) > 0$, then g has a decomposition $\sum_{v \in V} m_v v$ where all coefficients m_v are integers.

Proof. By Lemma 4.3.5, g can be expressed in the form $\sum_{v \in V} r_v v$, where $V \subseteq T_n$ and $r_v \in Q^{(\{p_{2n}, p_{2n-1}\})}$. Since $rk(g) > 0$, we have a p.v.l. pseudo-successor g', expressed in the form $\sum_{u \in U} s_u u$, where $U \subseteq T_{n+1}$ and $s_u \in Q^{(\{p_{2n+2}, p_{2n+1}\})}$. Consider $h = g + g'$. Since $(p_{2n+1})^\infty | h$, we can apply Lemma 4.3.4. For each $v \in V$, let U_v be the set of successors of v in U. We have $r_v = \sum_{u \in U_v} s_u$. It follows that $\sum_{u \in U_v} s_u$ and r_v are integers. □

Suppose g is a v.l. element at level n. Recall that the definition of v.l. has two conditions, with the second split into two cases. If Condition 2 (a) holds for g, then Lemma 4.3.9 says that g can be expressed as a sum of vertex elements on level n with integer coefficients. If $rk(g) = 0$, then the decomposition of g involves some terminal vertex element.

Lemma 4.3.11. Let g be v.l. at level n, with a decomposition $\sum_{v \in V} r_v v$, where $V \subseteq T_n$ and all coefficients r_v are non-zero. Then $rk(g) = min_{v \in V} tr(v)$.

Proof. By Lemma 4.3.7, $tr(v) \geq rk(g)$ for all $v \in V$. We show by induction on α that if $tr(v) \geq \alpha$ for all $v \in V$, then $rk(g) \geq \alpha$. For $\alpha = 0$, the statement is trivially true. Suppose $\alpha > 0$, where the statement holds for all $\beta < \alpha$. If g satisfies Condition 2 (b) from the definition of v.l., then by Lemma 4.3.10, there is some $v \in V$ such that $tr(v) = 0$. Suppose $rk(g) = \beta$, where $0 < \beta < \alpha$. For all $v \in V$, $tr(v) > \beta$, so v has a successor u_v with $tr(u_v) \geq \beta$. By Lemma 4.3.10, we may suppose that all r_v are integers. We have a successor h of g, of the form $\sum_{v \in V} r_v u_v$. This h is vertex-like at level $n+1$, and by the Induction Hypothesis, $rk(h) \geq \beta$. Then $rk(g)$ cannot be β after all. Finally, suppose $tr(v) = \infty$ for all $v \in V$. For each v, there is an infinite successor chain, and we can use these to form an infinite chain of successors of g, so $rk(g) = \infty$. □

We are ready to finish the proof of the theorem. Recall that for a tree T, $R(T)$ is the set of pairs (n, α) such that there is some $v \in T$ at level n with $tr(v) = \alpha$. Proposition 4.1.1 says that for rank-homogeneous trees T, T', $T \cong T'$ if and only if $R(T) = R(T')$. Let $T, T' \in RHT$. We show that $T \cong T'$ iff $G(T) \cong G(T')$. We let $R(G(T))$ be the set of pairs (n, α) such that there is a v.l. $g \in G(T)$ at level n with $rk(g) = \alpha$. We can show that $R(T) = R(G(T))$. If $(n, \alpha) \in R(T)$, then there is a $v \in T$ and a corresponding vertex element v in $G(T)$ witnessing the rank. By lemma 4.3.11, the group rank of v is equal to the tree rank of v. Therefore, $(n, \alpha) \in R(G(T))$. On the other hand, if $(n, \alpha) \in R(G(T))$, witnessed by $g = \sum_{v \in V} r_v v$, then by Lemma 4.3.11, there is some vertex element $v \in V_n$ such that $tr(v) = \alpha$. Therefore, $(n, \alpha) \in R(T)$. This completes the proof that $G : T \to G(T)$ is $1-1$ on isomorphism types. □

4.4 The failure of injectivity

Consider the following isomorphism invariant of a given tree T which says how many nodes the tree T has at the n-*th* level: $L(T) = \langle |T_n| : n \leq height(T)\rangle$. If T has all leaves at one level, then $L(T)$ is a tuple of the form $\langle n_0, n_1, \ldots, n_h\rangle$, where $1 = n_0 \leq n_1 \leq \ldots \leq n_h$. The previous theorem can not be improved to the class of all trees:

Proposition 4.4.1. Suppose that finite trees T and U have all leaves at one level. Then $G(T) \cong G(U)$ if and only if $L(T) = L(U)$.

Proof. Consider the pure subgroup G_n of $G(T)$ generated by the image of T_n. Note that $g \in G_n$ if and only if $p_{2n}^{\infty}|g$. Observe that the number of nodes in T_n is equal to the rank of G_n. Note also that $n > height(T)$ implies that the rank of G_n is 0. Thus, the number of nodes in T_n depends on the isomorphism type of $G(T)$ only, for every $n \geq 0$. This gives the proof of the "only if" part.

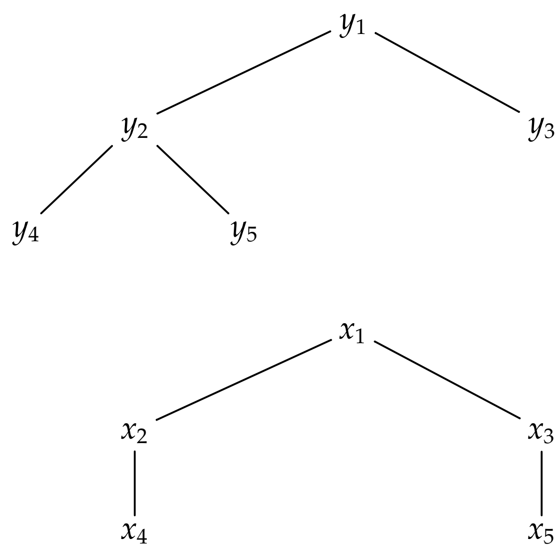
For the "if" part of the proposition, for each $n \leq height(T)$ consider the subgroups H_n and F_n of G_n generated by $\{(v - w) : v, w \in T_n$ have the same predecessor$\}$ and $\{v : v \in T_n\}$, respectively. Observe that $F_{n-1} \cong F_n/H_n$ via a homomorphism ϕ_n which takes each node (element, corresponding to it) to its predecessor. The groups F_n, F_{n-1} and H_n are free, therefore the subgroup H_n detaches as a summand of F_n (see, e.g., [39], Theorem 14.4).

Now assume we are given two trees, T and U, such that $L(T) = L(U)$. Let $G = G(T)$ and $G' = G(U)$. As above, define H_n, F_n and ϕ_n for G, and similarly define H'_n, F'_n and ϕ'_n for G'. Since $L(T) = L(U)$, we have $rk(H_n) = rk(H'_n)$ and $rk(F_n) = rk(F'_n)$, for each $n \leq h = height(T) = height(U)$. Furthermore, by the above observation, we have $F_n \cong F_0 \oplus H_1 \oplus \ldots \oplus H_h$ and $F'_n \cong F'_0 \oplus H'_1 \oplus \ldots \oplus H'_h$. Let $\varphi : F_0 \to F'_0$ be an isomorphism. It is clear how to extend this isomorphism to an isomorphism ψ of F_n onto F'_n so that $\psi(H_n) = H'_n$ for each $n \leq h$.

Using ψ we can define an isomorphism $\theta : \bigoplus_{0\leq n\leq h} F_n \to \bigoplus_{0\leq n\leq h} F'_n$ which maps each summand F_n to the corresponding F'_n and such that $\theta\phi_n = \phi'_n\theta$ on their domain and range, for each n. It remains to prove that the map θ can be extended to an isomorphism from $G(T)$ onto $G(U)$.

The map can be extended to a map $\theta : \bigoplus_{0\leq n\leq h} G_n \to \bigoplus_{0\leq n\leq h} G'_n$. It remains to show that the infinite divisibility can be preserved as well. Given $a \in F_n$, we have $\theta(a + \phi_n(a)) = \theta(a) + \theta(\phi_n(a)) = \theta(a) + \phi_n(\theta(a))$, and for $b \in F'_n$, we have $\theta^{-1}(a + \phi'_n(a)) = \theta^{-1}(a) + \theta^{-1}(\phi'_n(a)) = \theta^{-1}(a) + \phi_n(\theta^{-1}(a))$. This shows that θ can be correctly extended to the whole group preserving infinite divisibility. This completes the proof of the proposition. ◻

It is natural to ask if Proposition 4.4.1 can be extended to the case of finite trees having leaves not at the same level. Consider the trees (T (top) and U (bottom)):



Fact 4.4.1. *$L(T) = L(U)$ but $G(T) \not\cong G(U)$.*

Proof. The subgroup of $G(U)$ generated by elements at level 1 having successors and predecessors has rank 2, while the similarly defined subgroup of $G(T)$ has rank 1. It remains to note that the group generated by elements at level 1 having successors and predecessors can be defined by a first-order formula in the language of abelian groups augmented by predicates for infinite prime divisibility ($p_i^\infty|$ for $i \in \omega$).

◻

Chapter 5

Jump degrees of torsion-free abelian groups

The main result of the chapter is:

Theorem 5.0.1. *For every computable ordinal α and degree $\mathbf{a} > \mathbf{0}^{(\alpha)}$, there is a torsion-free abelian group having proper α^{th} jump degree $\mathbf{a}$.*

Proof idea. The basic idea is rather simple. We wish to have a way of coding a set S into a group $\mathcal{G}^\alpha_S$ so that S Σ^0_α if, and only if, $\mathcal{G}^\alpha_S$ has a computable copy. As we will discuss in more detail, providing such a coding is sufficient for proving the theorem (as it follows from [2], say). Although the basic idea is rather simple, the coding requires a lot of work. The main basic *technical* idea is to use infinite divisibility by various primes to effectively encode a potential Σ^0_α representation of S into the group (similarly to how it is done in the case of trees) so that the isomorphism type of the outcome depends only on the set S. Using the machinery introduced in the previous chapter, one can certainly obtain several codings of this kind.

The main problem is, however, not in coding but in undoing the set S from the group $\mathcal{G}^\alpha_S$. We have to come up with a computable infinitary formula which isolates the set within the group. This task is the main difficulty of the proof, because not every coding which one can come up with has this property (an evidence is the failure of injectivity, see Proposition 4.4.1). Thus, we have to adjust the coding to make the set definable within the group. This is done by a careful analysis of infinite divisibility within the group. We develop a new language and an algebraic machinery which enables us to define S within $\mathcal{G}^\alpha_S$.

The structure of the chapter. Section 5.1 provides the reader with further background and necessary notations and conventions. Section 5.2 describes the encoding of sets $S \subseteq \omega$ into groups $\mathcal{G}_S^\alpha$ (this encoding depends on α). Theorem 5.0.1 is demonstrated in Section 5.3.

5.1 Background, Notation, and Conventions

In this section, we review basic terminology and results relevant to torsion-free abelian groups. We also introduce some classical notation and adopt some conventions that will simplify the exposition.

The groups $\mathcal{G}$ constructed for Theorem 5.0.1 will have countably infinite rank. The key coding mechanism will be the existence or nonexistence of elements divisible by arbitrarily high powers of a prime. (Recall the definitions of infinite divisibility.)

Remark 5.1.1. Within any presentation of $\mathcal{G}$, the set $\{x \in G : p^\infty \,|\, x\}$ of elements infinitely divisible by p is $\Pi_2^c(\mathcal{G})$. Indeed, this set is a subgroup of $\mathcal{G}$ under the group operation (which we use without further mention).

Notation 5.1.1. *In the following, $D(\mathcal{G})$ stands for the divisible closure of $\mathcal{G}$.*

Note that the countable divisible torsion-free abelian groups are the groups Q^n (for $n \in \omega$) and Q^ω, and the divisible closure of Z is Q. Classically, the divisible closure $D(\mathcal{G})$ exists, is unique, and contains $\mathcal{G}$ as a subgroup. In terms of effective algebra, Smith (see [90]) proved that every computable torsion-free abelian group has a computable divisible closure and that there is a uniform procedure for passing from $\mathcal{G}$ to $D(\mathcal{G})$.[1] However, in general the divisible closure is not effectively unique (i.e., unique up to computable isomorphism) and the canonical image of $\mathcal{G}$ in $D(\mathcal{G})$ is computably enumerable but not necessarily computable (see [37] and [93] for a complete discussion of these issues). Therefore, when we consider a particular copy $\mathcal{G}$ of a torsion-free abelian group, we use $D(\mathcal{G})$ to denote the canonical divisible closure as in [90]. Thus, we have a uniform way to pass from any given copy of $\mathcal{G}$ to a copy of $D(\mathcal{G})$. In our construction, we will use a more limited notion of closure under divisibility by certain primes.

[1] One forms $D(\mathcal{G})$ from pairs $\langle g, n\rangle$ with $g \in \mathcal{G}$ and $n \geq 1$ modulo the computable equivalence relation $\langle g, n\rangle \sim \langle h, m\rangle$ if and only if $mg = nh$.

Definition 5.1.1. *If $p \in \omega$ is prime and $\mathcal{G}$ is a torsion-free abelian group, define the* p-closure of $\mathcal{G}$ *(denoted $[\mathcal{G}]_p$) to be the smallest subgroup $\mathcal{H}$ of $D(\mathcal{G})$ containing $\mathcal{G}$ having the property $(\forall g \in G)\,[p^\infty \,|\, g]$.*

More generally, if P is a set of prime numbers and $\mathcal{G}$ is a torsion-free abelian group, define the P-closure of $\mathcal{G}$ *(denoted $[\mathcal{G}]_P$) to be the smallest subgroup $\mathcal{H}$ of $D(\mathcal{G})$ containing $\mathcal{G}$ having the property $(\forall g \in G)(\forall p \in P)\,[p^\infty \,|\, g]$. We often write $[\mathcal{G}]_{p_0,p_1}$ for $[\mathcal{G}]_P$ with $P = \{p_0, p_1\}$, $[\mathcal{G}]_{P,q}$ for $[\mathcal{G}]_{P\cup\{q\}}$, and so on. If $\mathcal{G}$ is any torsion-free abelian group and P is any set of prime numbers, we say that $\mathcal{G}$ is* P-closed *if $\mathcal{G} \cong [\mathcal{G}]_P$.*

The following lemma says that if $\mathcal{G}$ is P-closed, then the result of closing $\mathcal{G}$ under additional primes will still be P-closed. In particular, we can view the prime closure $[\mathcal{G}]_P$ as the result of closing $\mathcal{G}$ under each of the individual primes in P in any order.

Lemma 5.1.1. If $\mathcal{G}$ is a torsion-free abelian group, P is a set of primes, and q is a prime not in P, then $[[\mathcal{G}]_P]_q \cong [\mathcal{G}]_{P,q}$.

Proof. Since $[[\mathcal{G}]_P]_q$ is clearly a subgroup of $[\mathcal{G}]_{P,q}$ and since every element of $[[\mathcal{G}]_P]_q$ is infinitely divisible by q, it suffices to show that each element of $[[\mathcal{G}]_P]_q$ is infinitely divisible by each prime $p \in P$. Fix $p \in P$ and $g \in [[\mathcal{G}]_P]_q$. We need to find $h \in [[\mathcal{G}]_P]_q$ such that $ph = g$. By the definition of $[[\mathcal{G}]_P]_q$, there is a $k \geq 0$ such that $q^k g \in [\mathcal{G}]_P$; let $\widehat{h}$ be this element. Let $\widehat{g} \in [\mathcal{G}]_P$ be such that $p\widehat{g} = \widehat{h}$ and let $h \in [[\mathcal{G}]_P]_q$ be such that $q^k h = \widehat{g}$. Then

$$q^k(ph) = p(q^k h) = p\widehat{g} = \widehat{h} = q^k g.$$

Since $\mathcal{G}$ is torsion-free, the equality $q^k(ph) = q^k g$ implies $ph = g$ as required. □

By an obvious variation of the construction in [90], there is an effective way to pass from $\mathcal{G}$ to a copy of $[\mathcal{G}]_P$ which is uniform in both $\mathcal{G}$ and P. As above, the closure operation sending $\mathcal{G}$ to $[\mathcal{G}]_P$ is not necessarily effectively unique so we fix this uniform procedure to define a particular copy of $[\mathcal{G}]_P$ given a particular copy of $\mathcal{G}$.

Convention 5.1.1. We will write statements such as $([Z]_{\rho_1,P} \setminus [Z]_P) \cap [Z]_{\rho_2,\rho_3,P} = \emptyset$. Such statements are intended to apply within a fixed (one-dimensional) copy of Q, where $Z \leqq Q$ is fixed as well. In particular, the indicated prime closures of Z should all be seen as being taken within a fixed copy of Q.

Often, we will write elements of the form $\dfrac{x+y}{p}$ as $\dfrac{x}{p} + \dfrac{y}{p}$ even though $\dfrac{x}{p}$ and $\dfrac{y}{p}$ may not exist within the group. We justify this by passing to the divisible closure of the group and considering the canonical image of the group within its divisible closure. Thus, $\dfrac{x+y}{p} = \dfrac{x}{p} + \dfrac{y}{p}$ in $D(\mathcal{G})$ even though $\dfrac{x}{p}$ and $\dfrac{y}{p}$ may not be in the image of $\mathcal{G}$.

Definition 5.1.2. *A* rooted torsion-free abelian group $\mathcal{G}$ *is a torsion-free abelian group with a distinguished element (termed the* root *of* $\mathcal{G}$*).*

We use rooted torsion-free abelian groups to help build our groups inductively. When we consider isomorphisms, we always consider group isomorphisms with no assumption that roots are preserved. That is, the root is only used as a tool in the inductive definitions and is not a formal part of the algebraic structure.

Definition 5.1.3. *Let* $\mathcal{G}$ *be a torsion-free abelian group and* $\{d_i\}_{i\in I} \subseteq D(\mathcal{G})$ *be a subset of its divisible closure. We define the* extension of $\mathcal{G}$ by $\{d_i\}_{i\in I}$, *denoted*

$$\langle \mathcal{G};\, d_i : i \in I \rangle\,,$$

to be the smallest subgroup of $D(\mathcal{G})$ *containing* $\mathcal{G}$ *and* d_i *for* $i \in I$.

Note that if $\mathcal{G}$ is computable and $\{d_i\}_{i\in I}$ is a computable set of elements of $D(\mathcal{G})$ (indeed, computably enumerable suffices), then the subgroup $\langle \mathcal{G}; d_i : i \in I\rangle$ is computably enumerable in $D(\mathcal{G})$. Since there is a uniform procedure to produce a computable copy of any computably enumerable subgroup of $D(\mathcal{G})$ (by letting n denote the n-th element enumerated into the subgroup and defining the group operations accordingly) we have a uniform procedure to pass from $\mathcal{G}$ to $\langle \mathcal{G}; d_i : i \in I\rangle$.

We continue by introducing some (important) conventions that will be used throughout the paper without further mention.

Convention 5.1.2. If β is any nonzero ordinal, when we write $\beta = \delta + i$ or $\beta = \delta + 2\ell + i$ for some $i \in \omega$, we require δ to be either zero or a limit ordinal (allowing zero only if $\beta < \omega$) and ℓ to be a nonnegative integer.

If i is even, we say the ordinal β is *even*; if i is odd, we say the ordinal β is *odd*.

When at limit ordinals, it will be necessary to approximate the ordinal effectively from below. We therefore fix a computable ordinal λ and increasing cofinal sequences for ordinals less than λ.

Definition 5.1.4. *Fix a computable ordinal λ.*

Fix a computable function $f : \lambda \times \omega \to \lambda$ such that $f(\alpha+1, n) = \alpha$ for all successor ordinals $\alpha + 1 \in \lambda$ and $n \in \omega$, and such that $\{f(\alpha, n)\}_{n\in\omega}$ is a sequence of increasing odd ordinals (greater than one) with $\alpha = \cup_{n\in\omega} f(\alpha, n)$ for all limit ordinals $\alpha \in \lambda$.

We denote $f(\alpha, n)$ by $f_\alpha(n)$.

5.2 The Group $\mathcal{G}_S^\alpha$ (For Successor Ordinals α)

Fixing a computable successor ordinal α below λ, the group $\mathcal{G}_S^\alpha$ will be a direct sum of rooted torsion-free abelian groups $\mathcal{G}_S^\alpha(n)$ coding whether n is or is not in S. It will be useful to have a plethora of disjoint sets of primes. We therefore partition the prime numbers into uniformly computable sets $P = \{p_\beta\}_{\beta\in\alpha+1}$, $Q = \{q_\beta\}_{\beta\in\alpha+1}$, $U = \{u_{\beta,k}\}_{\beta\in\alpha+1,k\in\omega}$, $V = \{v_{\beta,k}\}_{\beta\in\alpha+1,k\in\omega}$, $D = \{d_n\}_{n\in\omega}$, and $E = \{e_n\}_{n\in\omega}$,.

More specifically, the isomorphism type of $\mathcal{G}_S^\alpha(n)$ will be either $[\mathcal{G}(\Sigma_\alpha^0)]_{d_n}$ or $[\mathcal{G}(\Pi_\alpha^0)]_{d_n}$, or $[\mathcal{H}(\Sigma_\alpha^0)]_{d_n}$ or $[\mathcal{H}(\Pi_\alpha^0)]_{d_n}$ (all described later) depending on whether α is even or odd (deciding $\mathcal{G}$ versus $\mathcal{H}$) and whether n is in S (deciding Σ versus Π). The group $\mathcal{G}_S^\alpha(n)$ will be X-computable (uniformly in n) if $S \in \Sigma_\alpha^0(X)$. Conversely, there will be an effective enumeration $\{\Upsilon_n\}_{n\in\omega}$ of computable infinitary Σ_α^c sentences such that $\mathcal{G}_S^\alpha \models \Upsilon_n$ if and only if $n \in S$. Thus, the group $\mathcal{G}_S^\alpha$ will be X-computable if and only if $S \in \Sigma_\alpha^0(X)$.

The definition of the rooted torsion-free abelian groups $\mathcal{G}(\Sigma_\alpha^0)$, $\mathcal{G}(\Pi_\alpha^0)$, $\mathcal{H}(\Sigma_\alpha^0)$, and $\mathcal{H}(\Pi_\alpha^0)$ is done by recursion. Unfortunately, the recursion is not straightforward for technical reasons within the algebra (discussed in Remark 5.2.1). Indeed, we introduce additional rooted torsion-free abelian groups $\mathcal{G}(\Sigma_\alpha^0(m))$ for $m \in \omega$ if α is an even ordinal.

We define some of these groups pictorially in Section 5.2.1. The hope is these examples provide enough intuition to the reader so that the formal definition of $\mathcal{G}_S^\alpha$ (and all the auxiliary groups) is not (too) painful.

5.2.1 Defining $\mathcal{G}(\Sigma_\beta^0)$, $\mathcal{G}(\Sigma_\beta^0(m))$, $\mathcal{G}(\Pi_\beta^0)$, $\mathcal{H}(\Sigma_\beta^0)$, and $\mathcal{H}(\Pi_\beta^0)$ Pictorially

For each successor ordinal $\beta \geq 3$, we give a pictorial description of the groups $\mathcal{G}(\Sigma_\beta^0)$ (if β is odd), $\mathcal{G}(\Sigma_\beta^0(m))$ (if β is even), and $\mathcal{G}(\Pi_\beta^0)$. The recursion starts with $\mathcal{G}(\Sigma_2^0(m))$ as Z with root $r = p_1^m$ and $\mathcal{G}(\Pi_2^0)$ as $[Z]_{p_1}$ with root $r = 1$. The recursion continues as illustrated in Figure 5.1 and Figure 5.2.

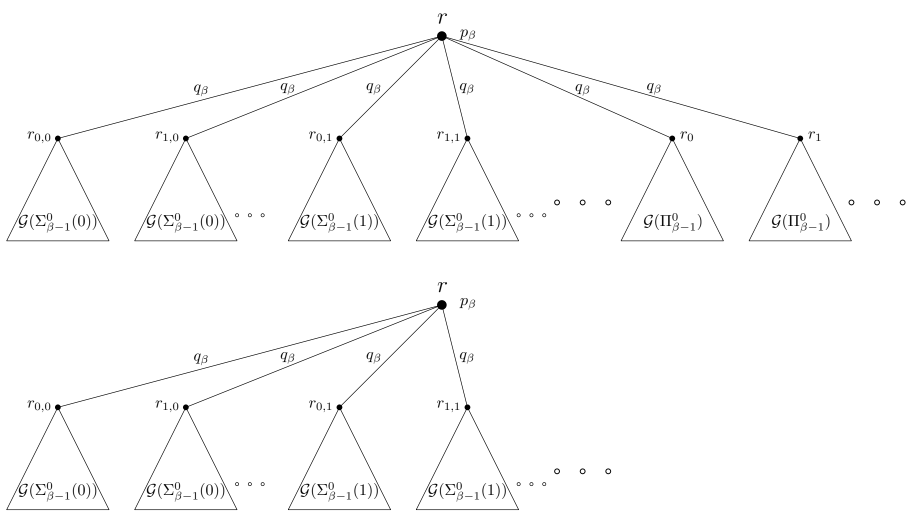


Figure 5.1: $\mathcal{G}(\Sigma^0_\beta)$ (Top) and $\mathcal{G}(\Pi^0_\beta)$ (Bottom) if $\beta = \delta + 2\ell + 1 > 1$

For each even ordinal $\beta = \delta + 2\ell + 2 > 2$, we give a pictorial description of the groups $\mathcal{H}(\Sigma^0_\beta)$ and $\mathcal{H}(\Pi^0_\beta)$. Though their definition relies on $\mathcal{G}(\Sigma^0_{\beta-1})$ and $\mathcal{G}(\Pi^0_{\beta-1})$ as illustrated in Figure 5.3, no further recursion is required.

Within these figures, the recursively defined rooted torsion-free abelian groups are denoted with triangles (the text inside specifies which recursively defined group), with the root denoted by a circle at the top. A line segment connecting two roots and with a label p denotes the sum of the roots is made infinitely divisible by p. Brackets around a recursively defined rooted group with a label p denotes the p-closure. A prime p next to a root r denotes r is made infinitely divisible by p.

5.2.2 Defining $\mathcal{G}^\alpha_S$ Formally

Having pictorially described some of the associated groups, we formalize the definition of $\mathcal{G}^\alpha_S$. Of course, doing so requires formalizing the definition of all the auxiliary groups.

Definition 5.2.1. *For each ordinal β with $1 < \beta \leq \alpha$, define rooted torsion-free abelian groups $\mathcal{G}(\Sigma^0_\beta)$ and $\mathcal{G}(\Pi^0_\beta)$ (if β is odd) or $\mathcal{G}(\Sigma^0_\beta(m))$ for $m \in \omega$ and $\mathcal{G}(\Pi^0_\beta)$ (if β is even) by recursion as follows.*

- *For $\beta = 2$, define $\mathcal{G}(\Sigma^0_\beta(m))$ to be the group $\mathbb{Z}$ with root $r = p_1^m$ and define $\mathcal{G}(\Pi^0_\beta)$ to be the group $[\mathbb{Z}]_{p_1}$ with root $r = 1$.*

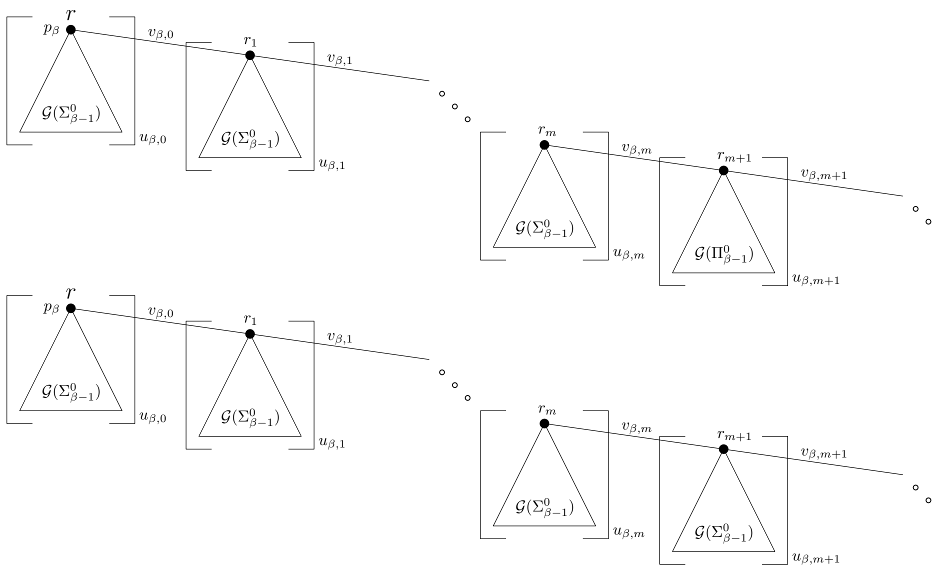


Figure 5.2: $\mathcal{G}(\Sigma^0_\beta(m))$ (Top) and $\mathcal{G}(\Pi^0_\beta)$ (Bottom) if $\beta = \delta + 2\ell + 2 > 2$

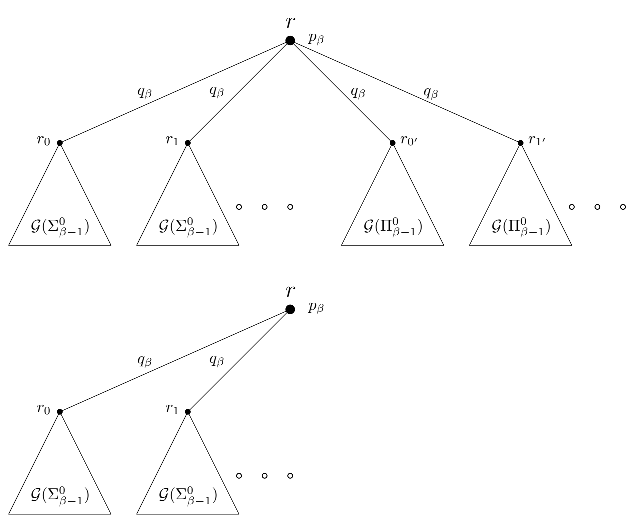


Figure 5.3: $\mathcal{H}(\Sigma^0_\beta)$ (Top) and $\mathcal{H}(\Pi^0_\beta)$ (Bottom) if $\beta = \delta + 2\ell + 2 > 2$

- *For odd $\beta = \delta + 2\ell + 1 \geq 3$, define $\mathcal{G}(\Sigma^0_\beta)$ to be the group*

$$\left\langle [\mathbb{Z}]_{p_\beta} \oplus \bigoplus_{k\in\omega} \bigoplus_{m\in\omega} \mathcal{G}(\Sigma^0_{\beta-1}(m)) \oplus \bigoplus_{k\in\omega} \mathcal{G}(\Pi^0_{\beta-1});\ q_\beta^{-t}(r + r_k), q_\beta^{-t}(r + r_{k,m}) : k, m, t \in \omega \right\rangle,$$

with root $r = 1$ in $[\mathbb{Z}]_{p_\beta}$, where r_k is the root of the kth copy of $\mathcal{G}(\Pi^0_{\beta-1})$ and $r_{k,m}$ is the root of kth copy of $\mathcal{G}(\Sigma^0_{\beta-1}(m))$.

For odd $\beta = \delta + 2\ell + 1 \geq 3$, define $\mathcal{G}(\Pi^0_\beta)$ to be the group

$$\left\langle [\mathbb{Z}]_{p_\beta} \oplus \bigoplus_{k\in\omega} \bigoplus_{m\in\omega} \mathcal{G}(\Sigma^0_{\beta-1}(m));\ q_\beta^{-t}(r + r_{k,m}) : k, m, t \in \omega \right\rangle,$$

with root $r = 1$ in $[\mathbb{Z}]_{p_\beta}$, where $r_{k,m}$ is the root of kth copy of $\mathcal{G}(\Sigma^0_{\beta-1}(m))$.

These are illustrated in Figure 5.1.

- *For even $\beta = \delta + 2\ell + 2 > 2$, define $\mathcal{G}(\Sigma^0_\beta(m))$ to be the group*

$$\left\langle \bigoplus_{0\leq k\leq m} \left[\mathcal{G}(\Sigma^0_{\beta-1})\right]_{u_{\beta,k}} \oplus \bigoplus_{k>m} \left[\mathcal{G}(\Pi^0_{\beta-1})\right]_{u_{\beta,k}};\ p_\beta^{-t} r_0, v_{\beta,k}^{-t}(r_k + r_{k+1}) : k, t \in \omega \right\rangle,$$

with root $r = r_0$, where r_k is the root of the kth copy of $\mathcal{G}(\Sigma^0_{\beta-1})$ or of $\mathcal{G}(\Pi^0_{\beta-1})$ depending on whether $k \leq m$ or $k > m$.

For even $\beta = \delta + 2\ell + 2 > 2$, define $\mathcal{G}(\Pi^0_\beta)$ to be the group

$$\left\langle \bigoplus_{k\in\omega} \left[\mathcal{G}(\Sigma^0_{\beta-1})\right]_{u_{\beta,k}};\ p_\beta^{-t} r_0, v_{\beta,k}^{-t}(r_k + r_{k+1}) : k, t \in \omega \right\rangle,$$

with root $r = r_0$, where r_k is the root of the kth copy of $\mathcal{G}(\Sigma^0_{\beta-1})$.

These are illustrated in Figure 5.2.

- *For limit $\beta = \delta > 0$, define the group $\mathcal{G}(\Sigma^0_\beta(m))$ to be*

$$\left\langle \bigoplus_{0\leq k\leq m} \left[\mathcal{G}(\Sigma^0_{f_\beta(k)})\right]_{u_{\beta,k}} \oplus \bigoplus_{k>m} \left[\mathcal{G}(\Pi^0_{f_\beta(k)})\right]_{u_{\beta,k}};\ p_\beta^{-t} r_0, v_{\beta,k}^{-t}(r_k + r_{k+1}) : k, t \in \omega \right\rangle,$$

with root $r = r_0$, where r_k is the root of the kth copy of $\mathcal{G}(\Sigma^0_{f_\beta(k)})$ or of $\mathcal{G}(\Pi^0_{f_\beta(k)})$

depending on whether $k \leq m$ or $k > m$.

Define $\mathcal{G}(\Pi^0_\beta)$ to be the group

$$\left\langle \bigoplus_{k\in\omega} \left[\mathcal{G}(\Sigma^0_{f_\beta(k)})\right]_{u_{\beta,k}};\ p_\beta^{-t} r_0, v_{\beta,k}^{-t}(r_k + r_{k+1}) : k, t \in \omega \right\rangle,$$

with root $r = r_0$, where r_k is the root of the kth copy of $\mathcal{G}(\Sigma^0_{f_\beta(k)})$.[2]

This completes the formal descriptions of these groups.

For odd $\beta \geq 3$, recall that the group $\mathcal{G}(\Sigma^0_\beta)$ has the form

$$\left\langle [Z]_{p_\beta} \oplus \bigoplus_{k\in\omega} \bigoplus_{m\in\omega} \mathcal{G}(\Sigma^0_{\beta-1}(m)) \oplus \bigoplus_{k\in\omega} \mathcal{G}(\Pi^0_{\beta-1});\ q_\beta^{-t}(r + r_k), q_\beta^{-t}(r + r_{k,m}) : k, m, t \in \omega \right\rangle.$$

We refer to the subgroups (indexed by k and m) of the form $\mathcal{G}(\Sigma^0_{\beta-1}(m))$ with roots $r_{k,m}$ as the $\mathcal{G}(\Sigma^0_{\beta-1}(m))$ *components* of $\mathcal{G}(\Sigma^0_\beta)$. Similarly, we refer to the $\mathcal{G}(\Pi^0_{\beta-1})$ subgroups (indexed by k) with root r_k as the $\mathcal{G}(\Pi^0_{\beta-1})$ *components* of $\mathcal{G}(\Sigma^0_\beta)$. We use similar language in the case of even β and limit β, as well as for other groups defined inductively within the chapter.

We emphasize the components of a group do not detach as direct summands (because of the divisibility introduced by the primes q_β and $v_{\beta,k}$). For clarity, we always refer to direct components (when the directness is an issue) without omitting the word "direct".

When we speak of components, we mean the components which are used in the inductive definition of these groups (or their prime closures), and we do not care if there are alternate ways to present the group. More formally, every such group will be considered as an image of one *canonical copy* given by the definition, and a subgroup is a component if and only if it is an image of a component which was used in the definition of this canonical copy. The isomorphism is chosen once and forever.

The important relationship between $\mathcal{G}(\Sigma^0_\beta)$ and $\mathcal{G}(\Pi^0_\beta)$ for odd β and $\mathcal{G}(\Sigma^0_\beta(m))$ and $\mathcal{G}(\Pi^0_\beta)$ for even β is whether each embeds within the other. For small β, one can see that the groups defined above satisfy the following embeddability relations:

[2]We emphasize that the definition of $\mathcal{G}(\Sigma^0_\beta(m))$ and $\mathcal{G}(\Pi^0_\beta)$ for $\beta = \delta + 2\ell + 2$ is identical to the case of $\beta = \delta$ as by definition $f_{\delta+2\ell+2}(k) = \delta + 2\ell + 1$ for all k. We separate them, here and in some later proofs, in hopes of not obfuscating the intuition.

if $\beta > 1$ is odd, then $\mathcal{G}(\Sigma^0_\beta) \nleqq \mathcal{G}(\Pi^0_\beta)$ and $\mathcal{G}(\Pi^0_\beta) \leqq \mathcal{G}(\Sigma^0_\beta)$; if $\beta > 0$ is even, then $\mathcal{G}(\Pi^0_\beta) \nleqq \mathcal{G}(\Sigma^0_\beta(m))$ and $\mathcal{G}(\Sigma^0_\beta(m)) \leqq \mathcal{G}(\Pi^0_\beta)$ for all $m \in \omega$. For larger ordinals β, the formal proof of these properties is less straightforward. Moreover, stronger properties of such groups are needed to run a successful induction. We avoid these formal difficulties by not using these embedability relations in later proofs, stating them only in order to aid intuition. Though they will not be formally used, the reader may find it useful to keep in mind which groups are "bigger".

The embeddability relations discussed reflect the utility of the coding. Informally, we will ask

Is there a large subgroup attached to x?

about an element x that is infinitely divisible by an appropriate prime. The answer will allow us to extract whether the Σ^0_β outcome or the Π^0_β outcome was the case.

We (informally) justify not using a simpler recursive scheme to define the groups $\mathcal{G}(\Sigma^0_\beta)$ and $\mathcal{G}(\Pi^0_\beta)$ in the following remark.

Remark 5.2.1. It would of course be simpler if Definition 5.2.1 used only the odd recursion schema (for all successor ordinals). Unfortunately, the embeddability relations would not be satisfied in this case, e.g., when $\beta = 4$ it would be the case that $\mathcal{G}(\Sigma^0_\beta) \leqq \mathcal{G}(\Pi^0_\beta)$ and $\mathcal{G}(\Pi^0_\beta) \leqq \mathcal{G}(\Sigma^0_\beta)$. The reason is $\mathcal{G}(\Sigma^0_4)$ would contain infinitely many copies of $\mathcal{G}(\Pi^0_3)$ and infinitely many copies of $\mathcal{G}(\Sigma^0_3)$ whereas $\mathcal{G}(\Pi^0_4)$ only would contain infinitely many copies of $\mathcal{G}(\Sigma^0_3)$. As $\mathcal{G}(\Pi^0_3) \leqq \mathcal{G}(\Sigma^0_3)$, it would follow that $\mathcal{G}(\Sigma^0_4) \leq \mathcal{G}(\Pi^0_4)$. Hence asking if there is a *large* subgroup would not distinguish between the Σ^0_β and the Π^0_β outcomes.

For even successor ordinals $\beta \geq 4$, we will need additional auxiliary groups $\mathcal{H}(\Sigma^0_\beta)$ and $\mathcal{H}(\Pi^0_\beta)$.

Definition 5.2.2. *For each even computable ordinal $\beta = \delta + 2\ell + 2 \geq 4$, define rooted torsion-free abelian groups $\mathcal{H}(\Sigma^0_\beta)$ and $\mathcal{H}(\Pi^0_\beta)$ as follows.*

Define $\mathcal{H}(\Sigma^0_\beta)$ to be the group

$$\left\langle [Z]_{p_\beta} \oplus \bigoplus_{k\in\omega} \mathcal{G}(\Sigma^0_{\beta-1}) \oplus \bigoplus_{k'\in\omega} \mathcal{G}(\Pi^0_{\beta-1});\ q_\beta^{-t}(r + r_k), q_\beta^{-t}(r + r_{k'}) : k, k', t \in \omega \right\rangle,$$

with root $r = 1$ in $[Z]_{p_\beta}$, where r_k is the root of the kth copy of $\mathcal{G}(\Sigma^0_{\beta-1})$ and $r_{k'}$ is the root of k'th copy of $\mathcal{G}(\Pi^0_{\beta-1})$.

Define $\mathcal{H}(\Pi^0_\beta)$ to be the group

$$\left\langle [\mathbb{Z}]_{p_\beta} \oplus \bigoplus_{k\in\omega} \mathcal{G}(\Sigma^0_{\beta-1});\ q_\beta^{-t}(r+r_k) : k,t\in\omega \right\rangle$$

with root $r = 1$ in $[\mathbb{Z}]_{p_\beta}$, where r_k is the root of the kth copy of $\mathcal{G}(\Sigma^0_{\beta-1})$.

These are illustrated in Figure 5.3.

It is now possible to define $\mathcal{G}^\alpha_S$ for $S \subseteq \omega$.

Definition 5.2.3. *For each successor ordinal $\alpha \geq 3$ and set $S \subseteq \omega$, define a torsion-free abelian group $\mathcal{G}^\alpha_S$ as follows.*

- *If $\alpha = \delta + 2\ell + 1 \geq 3$, define $\mathcal{G}^\alpha_S$ to be the group*

$$\mathcal{G}^\alpha_S := \bigoplus_{n\in S} \left[\mathcal{G}(\Sigma^0_\alpha)\right]_{d_n} \oplus \bigoplus_{n\notin S} \left[\mathcal{G}(\Pi^0_\alpha)\right]_{d_n}.$$

- *If $\alpha = \delta + 2\ell + 2$, define $\mathcal{G}^\alpha_S$ to be the group*

$$\mathcal{G}^\alpha_S := \bigoplus_{n\in S} \left[\mathcal{H}(\Sigma^0_\alpha)\right]_{d_n} \oplus \bigoplus_{n\notin S} \left[\mathcal{H}(\Pi^0_\alpha)\right]_{d_n}.$$

The following definition and associated observation will be exploited in later sections when we wish to express elements as sums of roots of subcomponents.

Definition 5.2.4. *If $\mathcal{G}$ is any group within this section, or any direct product of prime closures of such groups, we let $R_\mathcal{G}$ denote the set of roots of the recursively nested components of $\mathcal{G}$.*

Of course, some elements serve as the root of more than one component at different ordinal levels. For example, if β is odd, then the root of a $\mathcal{G}(\Sigma^0_{\beta-1}(m))$ component of $\mathcal{G}(\Pi^0_\beta)$ is also the root of a $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,0}}$ component. However, this root appears only once in $R_\mathcal{G}$. The fact below is a straightforward consequence of the definition of $\mathcal{G}$:

Fact 5.2.1. *The set $R_\mathcal{G}$ is a basis for both $\mathcal{G}$ and $D(\mathcal{G})$.*

5.3 Proof of Theorem 5.0.1

Having defined $\mathcal{G}_S^\alpha$ for each successor ordinal $\alpha \geq 3$, it of course remains to verify the desired properties. We state these explicitly.

Lemma 5.3.1. For each successor ordinal $\alpha \geq 3$, there is an effective enumeration $\{\Upsilon_n\}_{n\in\omega}$ of computable Σ_α^c sentences such that $\mathcal{G}_S^\alpha \models \Upsilon_n$ if and only if $n \in S$.

Lemma 5.3.2. For each successor ordinal $\alpha \geq 3$, if $S \in \Sigma_\alpha^0(X)$, then $\mathcal{G}_S^\alpha$ has an X-computable copy.

We also note that the lemmas hold with all possible uniformity. Assuming Lemma 5.3.1 and Lemma 5.3.2, we prove Theorem 5.0.1. Lemma 5.3.1 is demonstrated in Section 5.3.1 and Lemma 5.3.2 is demonstrated in Section 5.3.2. We note that the proof of Theorem 5.0.1 from Lemmas 5.3.1 and 5.3.2 is identical to the case of linear orders (see [2]).

Proof of Theorem 5.0.1. As we have mentioned before, the basic idea is standard. We, however, give a proof here. A reader familiar with such arguments may skip it. Only the (trivial) coding at the limit case perhaps deserves some attention.

Fix a computable ordinal α, a degree $\mathbf{a} > \mathbf{0}^{(\alpha)}$, and a set $A \in \mathbf{a}$. The cases $\alpha = 1, 2$ follow at once from [71], and we may assume that $\alpha \geq 3$. If α is a successor ordinal $\beta + 1$, we argue the torsion-free abelian group $\mathcal{G} := \mathcal{G}^\alpha_{A\oplus\overline{A}}$ has proper α^{th} jump degree $\mathbf{a}$. Recall that a relation on a structure $\mathcal{M}$ (having a finite language, say) is relatively intrinsically Σ_α^0 if it is $\Sigma_\alpha^0(\mathcal{B})$ for every $\mathcal{B} \cong \mathcal{M}$. A relation on $\mathcal{M}$ is relatively intrinsically c.e. if, and only if, it is definable by an infinitary computable Σ_α^c formula in the language of $\mathcal{M}$ (see [3] for a proof). By Lemma 5.3.1 and Lemma 5.3.2, we have

$$\begin{aligned}\mathrm{DegSp}(\mathcal{G}) &= \{X : A \oplus \overline{A} \in \Sigma_\alpha^0(X)\} \\ &= \{X : A \in \Delta_\alpha^0(X)\}\end{aligned}$$

(for α finite, these are $\Sigma^0_{\alpha+1}(X)$ and $\Delta^0_{\alpha+1}(X)$). It follows $\{X^{(\alpha)} : X \in \mathrm{DegSpec}(\mathcal{G})\}$ contains precisely those sets that compute A. Thus $\mathcal{G}$ has α^{th} jump degree $\mathbf{a}$. On the other hand, if $\beta < \alpha$, the set $\{X^{(\beta)} : X \in \mathrm{DegSpec}(\mathcal{G})\}$ has no element of least degree (see Lemma 1.3 of [2]). Thus $\mathcal{G}$ does not have β^{th} jump degree for any $\beta < \alpha$.

If α is a limit ordinal, fix an α-generic set B such that $B^{(\alpha)} \equiv_T B \oplus \emptyset^{(\alpha)} \equiv_T A$. Viewing B as a subset of $\omega \times \omega$ (see [2]), we write $B_n := \{k : (n,k) \in B\}$. We argue the torsion-free abelian group

$$\mathcal{G} := \bigoplus_{n \in \omega} [\mathcal{G}_{B_n}]_{e_n}$$

has proper α^{th} jump degree $\mathbf{a}$, where $\mathcal{G}_{B_n}$ is the group $\mathcal{G}_{B_n}^{f_\alpha(n)}$ associated with the set B_n and the ordinal $f_\alpha(n)$. Making use of the uniformity in both Lemma 5.3.1 and Lemma 5.3.2 and that the prime e_n distinguishes the subgroup $\mathcal{G}_{B_n}^{f_\alpha(n)}$ from $\mathcal{G}_{B_{n'}}^{f_\alpha(n')}$ for $n' \neq n$, we have

$$\mathrm{DegSp}(\mathcal{G}) \quad = \quad \{X : B_n \in \Sigma^0_{f_\alpha(n)}(X) \text{ uniformly in } n\}.$$

We explain the case when $\alpha = \omega$, the case of an arbitrary limit ordinal α is not much different. The reader can also find a proof for arbitrary limit α in the classical paper [2]. In fact, it does not matter they $\mathcal{G}$ is a group, ordering, or something else; it only matters that $\{X : B_n \in \Sigma^0_{f_\alpha(n)}(X)$ uniformly in $n\}$ serves as its degree spectrum, where $(f_\alpha(n))_{n\in\omega}$ is a uniformly computable sequence of notations in a fixed segment of $\mathcal{O}$ below α.

From now on, we follow the discussion after Lemma 3.1 of [2]. Lemma 1.1 of [2] allows us to produce an ω-generic set B such that $B^{(\omega)}$ has degree $\mathbf{d} = deg_T(A)$. The rest of the argument relies on the specific construction of B which will not be given here, but can be found in [2]. As we have mentioned before, $B \subseteq \omega \times \omega$. The basic facts on the (hyper)jumps and the uniformity of indices when performing (hyper)jumps (see [3]) implies that DegSp($\mathcal{G}$) contains B. Furthermore, if $D \in$ DegSp($\mathcal{G}$), then $B^{(\omega)} \leq_T D^{(\omega)}$. (See [2] for technical details.) Therefore, $\mathbf{d}$ is an ω-jump of $\mathcal{G}$. Towards a contradiction, suppose that $\mathcal{G}$ has an $n'th$ jump degree for some $n \in \omega$. This means that there is $Y \in$ DegSp($\mathcal{G}$) such that $Y^{(n)} \leq_T X^{(n)}$, for all $X \in$ DegSp($\mathcal{G}$). Thus, *in particular*, $Y^{(n)} \leq_T D^{(n)}$ for those D such that B_k is $\Delta^0_{f_\alpha(n)}(D)$ uniformly in n. The choice of B, however, implies that for every n, the collection $\{Y^{(n)} : Y \in \mathrm{DegSp}(\mathcal{G})\}$ has no least element under $\leq_T$. The proof of the latter can be found in [2], Lemma 1.4. □

5.3.1 Proof of Lemma 5.3.1

The definition of the Σ^c_α sentences $\{\Upsilon_n\}_{n\in\omega}$ is done recursively, mirroring the recursive nature of the definition of $\mathcal{G}^\alpha_S$. Before we start constructing formulas $\Phi_\beta(x)$ and $\Psi_\beta(x)$ connected semantically to $\mathcal{G}(\Sigma^0_\beta)$, $\mathcal{G}(\Sigma^0_\beta(m))$, and $\mathcal{G}(\Pi^0_\beta)$, we demonstrate two divisibility lemmas that isolate aspects of the odd and even inductive steps. The proofs of these are similar to proofs of lemmas by Downey and Montalbán (see Lemma 2.3 and Lemma 2.4 of [32]). For the proof of Lemma 5.3.3(1), we make explicit whether we are viewing elements of $\mathcal{B}$ as belonging to $\mathcal{B}$ or the divisible closure of $\mathcal{B}$. For later parts of Lemma 5.3.3 and Lemma 5.3.4, we do not make it explicit as which it should be is clear from context. Before stating these two divisibility lemmas, we note a number of simple number theoretic facts (without proof) that we will use repeatedly (without mention).

Fact 5.3.1. *The following facts hold of prime closures of Z.*

- *For any primes p_0 and p_1, $[Z]_{p_0} + [Z]_{p_1} = [Z]_{p_0,p_1}$, where the sum $[Z]_{p_0} + [Z]_{p_1}$ denotes the set of all $q \in Q$ such that $q = a + b$ for some $a \in [Z]_{p_0}$ and $b \in [Z]_{p_1}$.*
- *For all sets of primes P_0 and P_1, $[Z]_{P_0} \cap [Z]_{P_1} = [Z]_{P_0 \cap P_1}$.*
- *If P_0 and P_1 are disjoint sets of primes, then $([Z]_{P_0} \setminus Z) \cap [Z]_{P_1} = \emptyset$ and $0 \notin ([Z]_{P_0} \setminus Z) + [Z]_{P_1}$.*

Lemma 5.3.3. Fix pairwise disjoint sets of prime numbers F_1, F_2, and P and a prime number $\rho \notin F_1 \cup F_2 \cup P$. For each $i \in \omega$, fix a copy of $[Z]_{F_1}$ and let x_i denote the element 1 in this copy. For each $i, j \in \omega$, fix a copy of $[Z]_{F_2}$ and let $y_{i,j}$ denote the element 1 in this copy. Let $\mathcal{B}$ be the group

$$\mathcal{B} := \left[\left\langle \bigoplus_{i\in\omega}[Z]_{F_1} \oplus \bigoplus_{i,j\in\omega}[Z]_{F_2}; \frac{x_i + y_{i,j}}{\rho^k} : i, j, k \in \omega \right\rangle\right]_P .$$

Then $\mathcal{B}$ has the following properties:

1. For any $z \in \mathcal{B}$ and $\sigma_1 \in F_1$, we have $\sigma_1^\infty \mid z$ if and only if $z = \sum_i m_i x_i$ with $m_i \in [Z]_{F_1,P}$.
2. For any $y \in \mathcal{B}$ and $\sigma_2 \in F_2$, we have $\sigma_2^\infty \mid y$ if and only if $y = \sum_{i,j} m_{i,j} y_{i,j}$ with $m_{i,j} \in [Z]_{\rho,F_2,P}$.
3. Fixing an integer ℓ, if $\rho^\infty \mid \sum_j m_{\ell,j} y_{\ell,j}$, then $\sum_j m_{\ell,j} = 0$.
4. If z can be expressed as $z = \sum m_i x_i$ with $m_i \in [Z]_P$, then for each $\sigma_1 \in F_1$ and $\sigma_2 \in F_2$, the element z satisfies the formula

$$\sigma_1^\infty \,|\, z \ \wedge\ (\exists y \in \mathcal{B}) \left[\rho^\infty \,|\, (z + y) \ \wedge\ \sigma_2^\infty \,|\, y\right]. \qquad (\dagger)$$

5. If $z \in \mathcal{B}$ satisfies (†) with witness $y \in \mathcal{B}$, then $z = \sum_i m_i x_i$ with $m_i \in [Z]_P$ and $y = \sum_{i,j} m_{i,j} y_{i,j}$ with $m_{i,j} \in [Z]_{\rho,F_2,P}$ for all i, j and $m_i = \sum_j m_{i,j}$ for all i (noting that $m_i = 0$ is possible).

We also note that the lemma can be carried through in the case when $\mathcal{B}$ detaches as a summand of a larger group C and for every $\mathcal{B} \leq A \leq C$ and every prime z which occurs in the definition of B, we have $z^\infty | x$, $x \neq 0$ and $x \in A$ implies $x \in \mathcal{B}$. The lemma holds even if $\mathcal{B}$ is pure in a larger group C, with similar restrictions.

Proof. For (1), the backward direction is immediate. For the forward direction, we express z (in the divisible closure) as $z = \sum_i m_i x_i + \sum_{i,j} m_{i,j} y_{i,j}$ with $m_i, m_{i,j} \in Q$ (allowing the possibility of a coefficient being zero). If $\sigma_1^\infty \mid z$, as the summation is finite, there is a $[Z]_{\sigma_1}$-multiple $\hat{z}$ of z in $\mathcal{B}$ with

$$\hat{z} = \sum_i \frac{\hat{m}_i}{\sigma_1^{n_i}} x_i + \sum_{i,j} \frac{\hat{m}_{i,j}}{\sigma_1^{n_{i,j}}} y_{i,j}$$

(with the right hand side expressed in the divisible closure) where $\hat{m}_i, \hat{m}_{i,j} \in Z$, $\hat{m}_i \neq 0$ implies $\sigma_1 \nmid \hat{m}_i$, $\hat{m}_{i,j} \neq 0$ implies $\sigma_1 \nmid \hat{m}_{i,j}$, and $n_i, n_{i,j} > 0$. Since the coefficient of $y_{i,j}$ in any element of $\mathcal{B}$ (viewed in the divisible closure) is an element of $[Z]_{\rho,F_2,P}$[3] and $([Z]_{\sigma_1} \backslash Z) \cap ([Z]_{\rho,F_2,P}) = \emptyset$, it must be that $\hat{m}_{i,j} = 0$ for all i, j, and so $m_{i,j} = 0$ for all i, j. The coefficient of $y_{i,j}$ in any element of $\mathcal{B}$ (viewed in the divisible closure) is an element of $[Z]_{\rho,F_2,P}$: this is an immediate consequence of the fact that every element of $\mathcal{B}$ is a formal sum $\sum_i a_i x_i + \sum_{i,j} b_{i,j}(x_i + y_{i,j}) + \sum_{i,j} c_{i,j} y_{i,j}$ with $a_i \in [Z]_{F_1,P}$, $b_{i,j} \in [Z]_{\rho,P}$, and $c_{i,j} \in [Z]_{F_2,P}$. Thus, in the divisible closure, the coefficient of any fixed $y_{i,j}$ is an element of $[Z]_{\rho,P} + [Z]_{F_2,P} = [Z]_{\rho,F_2,P}$.

Thus if $\sigma_1^\infty \mid z$, then $z = \sum_i m_i x_i$ (in the divisible closure) with $m_i \in Q$. From the structure of elements of $\mathcal{B}$, we have $m_i \in [Z]_{\rho,F_1,P}$. Fix i. If $m_i \notin [Z]_{F_1,P}$, then there would be a non-$[Z]_P$-multiple of $x_i + y_{i,j}$ in z for some j, in particular a $[Z]_{\rho,P} \backslash [Z]_P$-multiple. Then the coefficient of this $y_{i,j}$ in z (in the divisible closure) would be in $[Z]_{\rho,P} \backslash [Z]_P + [Z]_{F_2,P}$. However $0 \notin [Z]_{\rho,P} \backslash [Z]_P + [Z]_{F_2,P}$, yielding a contradiction to the form $z = \sum_i m_i x_i$. Thus $m_i \in [Z]_{F_1,P}$ for all i, completing the proof of (1).

For (2), the argument is similar and we leave the minor change in details to the reader.

[3] Though we justify this here, we omit such arguments in the rest of the chapter as all are similar to the argument here.

For (3), as $\rho^\infty \mid \sum_j m_{\ell,j}y_{\ell,j}$, there is a $[\mathbb{Z}]_\rho$-multiple $\hat{z}$ of $\sum_j m_{\ell,j}y_{\ell,j}$ in $\mathcal{B}$ with

$$\hat{z} = \sum_j \frac{\hat{m}_{\ell,j}}{\rho^{n_{\ell,j}}} y_{\ell,j}$$

where $\hat{m}_{\ell,j} \in \mathbb{Z}^{\neq 0}$, $\rho \nmid \hat{m}_{\ell,j}$, and $n_{\ell,j} > 0$. Indeed, we may assume that $\sum_j \dfrac{\hat{m}_{\ell,j}}{\rho^{n_{\ell,j}}} \in [\mathbb{Z}]_\rho \backslash \mathbb{Z}$ (in particular, that it is not an element of $[\mathbb{Z}]_{P,F_2}$) if $\sum_j m_{\ell,j} \neq 0$. From the structure of elements of $\mathcal{B}$, we have

$$\hat{z} = a_\ell x_\ell + \sum_j b_{\ell,j}(x_\ell + y_{\ell,j}) + \sum_j c_{\ell,j} y_{\ell,j}$$

with $a_\ell \in [\mathbb{Z}]_{F_1,P}$, $b_{\ell,j} \in [\mathbb{Z}]_{\rho,P}$, and $c_{\ell,j} \in [\mathbb{Z}]_{F_2,P}$. As $\sum_j \dfrac{\hat{m}_{\ell,j}}{\rho^{n_{\ell,j}}} \notin [\mathbb{Z}]_{P,F_2}$, it must be the case that $\sum_j b_{\ell,j} \notin [\mathbb{Z}]_P$. However this would imply the coefficient of x_ℓ is nonzero as $0 \notin [\mathbb{Z}]_{F_1,P} + [\mathbb{Z}]_{\rho,P} \backslash [\mathbb{Z}]_P$. This would contradict the form of $\hat{z}$, showing (3).

For (4), we note if $z = \sum_i m_i x_i$ with $m_i \in [\mathbb{Z}]_P$, then $y = \sum_i m_i y_{i,0}$ is in $\mathcal{B}$. Moreover, by Parts (1) and (2), this y witnesses z satisfying (†), showing (4).

For (5), fix z and y with $\sigma_1^\infty \mid z$, $\rho^\infty \mid z + y$, and $\sigma_2^\infty \mid y$. By Part (1), we have $z = \sum_i m_i x_i$ with $m_i \in [\mathbb{Z}]_{F_1,P}$. By Part (2), we have $y = \sum_{i,j} m_{i,j} y_{i,j}$ with $m_{i,j} \in [\mathbb{Z}]_{\rho,F_2,P}$. As $\rho^\infty \mid z + y$, there is a $[\mathbb{Z}]_\rho$-multiple $\hat{z} + \hat{y}$ of $z + y$ in $\mathcal{B}$ with

$$\hat{z} + \hat{y} = \sum_i \frac{\hat{m}_i}{\rho^{n_i}} x_i + \sum_{i,j} \frac{\hat{m}_{i,j}}{\rho^{n_{i,j}}} y_{i,j}$$

where $\hat{m}_i, \hat{m}_{i,j} \in \mathbb{Z}$, $\hat{m}_i \neq 0$ implies $\rho \nmid \hat{m}_i$, $\hat{m}_{i,j} \neq 0$ implies $\rho \nmid \hat{m}_{i,j}$, and $n_i, n_{i,j} > 0$. As

$$\hat{w} := \sum_i \frac{\hat{m}_i}{\rho^{n_i}} x_i + \sum_i \frac{\hat{m}_i}{\rho^{n_i}} y_{i,0}$$

is in $\mathcal{B}$ (by virtue of it being a sum of $[\mathbb{Z}]_\rho$-multiples of $x_i + y_{i,0}$) and infinitely divisible by ρ, the element

$$\hat{z} + \hat{y} - \hat{w} = \sum_{i,j} \frac{\hat{m}_{i,j}}{\rho^{n_{i,j}}} y_{i,j} - \sum_i \frac{\hat{m}_i}{\rho^{n_i}} y_{i,0}$$

is in $\mathcal{B}$ and is infinitely divisible by ρ. By Part (3), this implies $\dfrac{\hat{m}_i}{\rho^{n_i}} = \sum_j \dfrac{\hat{m}_{i,j}}{\rho^{n_{i,j}}}$ for all i. This is equivalent to $m_i = \sum_j m_{i,j}$ for all i.

As $m_{i,j} \in [Z]_{\rho,F_2,P}$ for all i, j, fixing i, the sum $\sum_j m_{i,j}$ is in $[Z]_{\rho,F_2,P}$. As $[Z]_{F_1,P} \cap [Z]_{\rho,F_2,P} = [Z]_P$, it follows $m_i \in [Z]_P$ for all i. This shows (5). $\square$

Lemma 5.3.4. Fix pairwise disjoint sets of primes F_i, for $i \in \omega$, and P, and fix a sequence of distinct primes ρ_n, for $n \in \omega$, such that $\rho_n \notin (\cup_{i\in\omega} F_i) \cup P$ for each n. Let $\mathcal{B}$ be the group

$$\mathcal{B} := \left[\left\langle \mathcal{F}; \frac{x_{i,j}}{\sigma_i^k}, \frac{x_{i,j} + x_{i+1,j}}{\rho_i^k} : i, j, k \in \omega \text{ and all } \sigma_i \in F_i \right\rangle\right]_P$$

where $\mathcal{F}$ is the free abelian group on the elements $x_{i,j}$ for $i, j \in \omega$. Then $\mathcal{B}$ has the following properties:

1. Fixing an integer ℓ, a prime $\sigma_\ell \in F_\ell$, and an element $z \in \mathcal{B}$, if $\sigma_\ell^\infty \mid z$, then $z = \sum_j m_{\ell,j} x_{\ell,j}$ with $m_{\ell,j} \in [Z]_{F_\ell,P}$.

2. Fixing an integer ℓ, if $z = \sum_j m_{\ell,j} x_{\ell,j}$ is nonzero, then $\rho_i^\infty \nmid z$ for any i.

3. Fixing primes $\sigma_i \in F_i$ for $0 \le i \le k+1$, if $z_0, \ldots, z_{k+1} \in \mathcal{B}$ satisfy

$$\sigma_i^\infty \mid z_i \text{ for all } i \le k+1 \text{ and } \rho_i^\infty \mid (z_i + z_{i+1}) \text{ for all } i \le k$$

   then there are constants $m_j \in [Z]_P$ such that $z_i = \sum_j m_j x_{i,j}$ for all $0 \le i \le k+1$.

The same remark as in the previous lemma, on direct detachement and pureness, holds for this lemma.

Proof. For (1), we express z as $z = \sum_{i,j} m_{i,j} x_{i,j}$ with $m_{i,j} \in Q^{\neq 0}$. As $\sigma_\ell^\infty \mid z$ and the summation is finite, there is a $[Z]_{\sigma_\ell}$-multiple $\hat{z}$ of z in $\mathcal{B}$ with

$$\hat{z} = \sum_{i,j} \frac{\hat{m}_{i,j}}{\sigma_\ell^{n_{i,j}}} x_{i,j}$$

where $\hat{m}_{i,j} \in Z^{\neq 0}$, $\sigma_\ell \nmid \hat{m}_{i,j}$, and $n_{i,j} > 0$. Thus the coefficient of $x_{i,j}$ in $\hat{z}$ is an element of $[Z]_{\sigma_\ell} \backslash Z$. On the other hand, the coefficient of $x_{i,j}$ in any element of $\mathcal{B}$ is an element of $[Z]_{F_i,P} + [Z]_{\rho_i,P} + [Z]_{\rho_{i-1},P}$[4]. As $([Z]_{\sigma_\ell} \backslash Z) \cap [Z]_{F_i,\rho_i,\rho_{i-1},P} = \emptyset$ if $i \neq \ell$, it follows that z can be expressed as $z = \sum_j m_{\ell,j} x_{\ell,j}$ with $m_{\ell,j} \in Q^{\neq 0}$.

[4]We ignore the degenerate case of $i = 0$ as it is actually simpler.

We show $m_{\ell,j} \in [Z]_{F_\ell,P}$ for all j. Fixing j, if $m_{\ell,j}$ were not in $[Z]_{F_\ell,P}$, there would necessarily be a non-$[Z]_P$-multiple of either $x_{\ell,j}+x_{\ell+1,j}$ or $x_{\ell-1,j}+x_{\ell,j}$ in $\hat{z}$. This implies the coefficient of $x_{\ell+1,j}$ or $x_{\ell-1,j}$ is nonzero in $\hat{z}$ as $0 \notin [Z]_{\rho_\ell,P}\backslash[Z]_P + [Z]_{\rho_{\ell+1},P} + [Z]_{F_{\ell+1},P}$ and $0 \notin [Z]_{\rho_\ell,P}\backslash[Z]_P + [Z]_{\rho_{\ell-1},P} + [Z]_{F_{\ell-1},P}$. However this contradicts the form of $\hat{z}$, so it must be that $m_{\ell,j} \in [Z]_{F_\ell,P}$, showing (1).

For (2), fix an ℓ, a nonzero element $z = \sum_j m_{\ell,j}x_{\ell,j}$, and an integer i towards a contradiction. As we are assuming $\rho_i^\infty \mid z$ for a contradiction, there is a $[Z]_{\rho_i}$-multiple $\hat{z}$ of z in $\mathcal{B}$ with

$$\hat{z} = \sum_j \frac{\hat{m}_{\ell,j}}{\rho_i^{n_{\ell,j}}} x_{\ell,j}$$

where $\hat{m}_{\ell,j} \in Z^{\neq 0}$, $\rho_i \nmid \hat{m}_{\ell,j}$, and $n_{\ell,j} > 0$. Fix j and note that the coefficient of $x_{\ell,j}$ in $\hat{z}$ is an element of $[Z]_{\rho_i}\backslash Z$. On the other hand, the coefficient of $x_{\ell,j}$ in any element of $\mathcal{B}$ is an element of $[Z]_{F_\ell,P}+[Z]_{\rho_\ell,P}+[Z]_{\rho_{\ell-1},P}$. As $([Z]_{\rho_i}\backslash Z)\cap([Z]_{F_\ell,P}+[Z]_{\rho_\ell,P}+[Z]_{\rho_{\ell-1},P}) = \emptyset$ if $\ell \notin \{i, i+1\}$, we must have $\ell \in \{i, i+1\}$. We show that either yields a contradiction.

If $\ell = i$, as $([Z]_{\rho_\ell}\backslash Z) \cap ([Z]_{F_\ell,P} + [Z]_{\rho_{\ell-1},P}) = \emptyset$, the term $x_{\ell,j} + x_{\ell+1,j}$ must have a nonzero coefficient in the expression of $\hat{z}$ as an element of $\mathcal{B}$. Indeed, this coefficient must be an element of $[Z]_{\rho_\ell}\backslash Z$ as the coefficient of $x_{\ell,j}$ in $\hat{z}$ is in $[Z]_{\rho_\ell}\backslash Z$. This implies the coefficient of $x_{\ell+1,j}$ in $\hat{z}$ is nonzero as $0 \notin ([Z]_{\rho_\ell}\backslash Z)+[Z]_{\rho_{\ell+1},P}+[Z]_{F_{\ell+1},P}$, contradicting the form of z. If $\ell = i+1$, identical reasoning suffices to contradict the form of z. We have thus shown (2).

For (3), we induct on k. For $k = 0$, by Part (1), we have $z_i = \sum_j m_{i,j}x_{i,j}$ with $m_{i,j} \in [Z]_{F_i,P}$ for $i \in \{0,1\}$. As $\rho_0^\infty \mid (z_0 + z_1)$, there is a $[Z]_{\rho_0}$-multiple $\hat{z}$ of $z := z_0 + z_1$ in $\mathcal{B}$ with

$$\hat{z} = \sum_j \frac{\hat{m}_{0,j}}{\rho_0^{n_{0,j}}} x_{0,j} + \sum_j \frac{\hat{m}_{1,j}}{\rho_0^{n_{1,j}}} x_{1,j}$$

with $\rho_0 \nmid \hat{m}_{0,j}$, $\rho_0 \nmid \hat{m}_{1,j}$, $n_{0,j} > 0$, and $n_{1,j} > 0$. We rewrite $\hat{z}$ as

$$\hat{z} = \sum_j \frac{\hat{m}_{0,j}}{\rho_0^{n_{0,j}}}\left(x_{0,j} + x_{1,j}\right) + \sum_j \frac{\hat{m}_{1,j} - \rho_0^{n_{1,j}-n_{0,j}}\hat{m}_{0,j}}{\rho_0^{n_{1,j}}} x_{1,j}.$$

As the first summation and $\hat{z}$ are both in $\mathcal{B}$ and infinitely divisible by ρ_0, so is the second summation. By Part (2), the second summation must be zero. Thus $\hat{m}_{1,j}/\rho_0^{n_{1,j}} = \hat{m}_{0,j}/\rho_0^{n_{0,j}}$ for all j, and so $m_{0,j} = m_{1,j}$ for all j, with this value an element of $[Z]_{F_0,P} \cap [Z]_{F_1,P} = [Z]_P$, completing the base case.

Assuming Part (3) for k, we show it true for $k+1$. As in the base case, write $z_i = \sum_j m_{i,j} x_{i,j}$ with $m_{i,j} \in [Z]_{F_i,P}$ for $i \leq k+2$ by Part (1). By the induction hypothesis, for each fixed j, the values of $m_{i,j}$ for $0 \leq i \leq k+1$ are equal. Let m_j denote this common value. Since m_j is in $[Z]_{F_0,P} \cap \cdots \cap [Z]_{F_k,P}$, it must be in $[Z]_P$. As $z_{k+2} = \sum_j m_{k+2,j} x_{k+2,j}$ with $m_{k+2,j} \in [Z]_{F_{k+2},P}$, the same analysis as in the base case implies $m_{k+2,j} = m_{k+1,j} = m_j$. □

We continue by introducing various formulas that capture structural aspects of the groups. These formulas describe how group elements interact in terms of infinite divisibility by certain primes. When defining these formulas and verifying their properties, we often restrict quantification from ranging over all group elements to ranging only over those elements which are infinitely divisible by certain primes.

To make this notion precise, we define the (computable infinitary) language of infinite divisibility. The signature of this language is the same as the signature of the language of groups except that for each prime p, we add a relation symbol for the relation $p^\infty \mid x$. That is, we treat $p^\infty \mid t$ for each prime p and term t as an atomic statement. We build up formulas in this language in the standard computable infinitary manner.

Definition 5.3.1. *For any formula φ in the infinite divisibility language and any prime q, we define the relativized formula φ^q by recursion as follows:*

- *If φ is atomic, then $\varphi^q =_{def} \varphi$.*
- *If $\varphi := (\bigwedge_i \beta_i)$, then $\varphi^q =_{def} \bigwedge_i \beta_i^q$; similarly for $\bigvee$, $\neg$, and $\longrightarrow$.*
- *If $\varphi := (\exists x)\,\beta(x)$, then $\varphi^q =_{def} (\exists x)\,[q^\infty \mid x \wedge \beta^q(x)]$.*
- *If $\varphi := (\forall x)\,\beta(x)$, then $\varphi^q =_{def} (\forall x)\,[q^\infty \mid x \longrightarrow \beta^q(x)]$.*

Thus, a formula φ^q restricts all quantification to be over elements which are infinitely divisible by the prime q. The following lemma is a formal statement of this property.

Lemma 5.3.5. Let $\mathcal{G}$ be a torsion-free abelian group, let q be a prime, and let $\mathcal{G}_q$ be the subgroup consisting of the elements infinitely divisible by q. If $\mathcal{G}_q$ is a pure subgroup, then for any formula $\varphi(\overline{x})$ in the language of infinite divisibility and any parameters $\overline{a}$ from $\mathcal{G}_q$, we have

$$\mathcal{G} \models \varphi^q(\overline{a}) \qquad \text{if and only if} \qquad \mathcal{G}_q \models \varphi(\overline{a}). \tag{5.1}$$

In particular, if $\mathcal{G}$ is $\{q\}$-closed, then $\mathcal{G}_q = \mathcal{G}$ and hence

$$\mathcal{G} \models \varphi^q(\overline{a}) \qquad \text{if and only if} \qquad \mathcal{G} \models \varphi(\overline{a}).$$

Proof. Suppose $\mathcal{G}_q$ is a pure subgroup. We proceed by induction on $\varphi(\overline{x})$. If $\varphi(\overline{x})$ is atomic, then $\varphi^q(\overline{a})$ is the same as $\varphi(\overline{a})$. If $\varphi(\overline{a})$ has the form $t_0(\overline{a}) = t_1(\overline{a})$, then (5.1) follows because $\mathcal{G}_q$ is a subgroup. If $\varphi(\overline{a})$ has the form $p^\infty \mid t(\overline{a})$, then (5.1) follows because $\mathcal{G}_q$ is pure. The inductive cases for $\bigwedge$, $\bigvee$, $\longrightarrow$ and $\neg$ follow immediately by definition, leaving only the quantifier cases. It suffices to consider the case for $\exists$.

Suppose $\varphi^q(\overline{a})$ has the form $((\exists x)\beta(x,\overline{a}))^q$ and $\mathcal{G} \models (\exists x)[q^\infty \mid x \wedge \beta^q(x,\overline{a})]$ with a fixed witness x. Since x is infinitely divisible by q, we have $x \in \mathcal{G}_q$. By the inductive hypothesis $\mathcal{G}_q \models \beta(x,\overline{a})$ and hence $\mathcal{G}_q \models (\exists x)\beta(x,\overline{a})$ as required. Conversely, suppose $\mathcal{G}_q \models (\exists x)\beta(x,\overline{a})$ with fixed witness $x \in \mathcal{G}_q$. By the inductive hypothesis, $\mathcal{G} \models \beta^q(x,\overline{a})$, and since every element of $\mathcal{G}_q$ is infinitely divisible by q, we have $q^\infty \mid x$. Therefore, we have $\mathcal{G} \models (\exists x)[q^\infty \mid x \wedge \beta^q(x,\overline{a})]$ as required. □

Because the language of infinite divisibility is infinitary, we can express the relation $p^\infty \mid x$ using the standard formula $\varphi_p(x)$ given by

$$\varphi_p(x) := \bigwedge_{k \in \omega} (\exists y)\left[p^k y = x\right].$$

In any group, the atomic relation $p^\infty \mid x$ and the formula $\varphi_p(x)$ are equivalent in the sense that they are satisfied by the same elements. Thus, we can always translate formulas in the language of infinite divisibility into formulas in the (computable infinitary) language of group theory.[5] Notice, however, that some caution is required because the relativized formulas $(p^\infty \mid x)^q$ and $\varphi_p^q(x)$ are not (always) equivalent: the former is satisfied by those elements infinitely divisible by p, whereas the latter is satisfied by those elements infinitely divisible by p and q.

When we measure the quantifier complexity of a formula in the language of infinite divisibility, we will always mean its complexity as a formula in the language

[5] As all of our languages are computable infinitary languages, we drop explicit reference to this fact from now on.

of group theory. Given the remarks in the previous paragraph, we need to be careful how we translate relativized formulas in the language of infinite divisibility into formulas in the language of group theory for the purposes of measuring complexity. Thus, when we say a formula φ^q (in the language of infinite divisibility) is in Σ^c_β or Π^c_β, we mean that the following formula ψ (in the language of group theory) is in the complexity class.

- First, use the recursive definition of relativized quantifiers to write φ^q in an unrelativized form in the language of infinite divisibility.
- Second, replace each occurrence of an atomic formula $p^\infty \mid t$ in this unrelativized formula by the corresponding formula $\varphi_p(t)$ to obtain a formula ψ in the language of group theory.

By performing the translation in this order, we ensure that we do not add additional divisibility conditions on the witnesses for $p^\infty \mid t$ and thus each atomic fact $p^\infty \mid t$ remains Π^c_2 even if it is under the scope of a relativizing prime.

We need one further convention before giving our formulas. Note that this convention does not change the quantifier complexity of any formula.

Convention 5.3.1. When we quantify over group elements using $(\exists z)$ or $(\forall z)$, the quantification is restricted to nonzero group elements. Hence $(\exists z)\,[\psi(\overline{z})]$ is an abbreviation for $(\exists z)\,[z \neq 0 \wedge \psi(\overline{z})]$ and $(\forall z)\,[\psi(\overline{z})]$ is an abbreviation for $(\forall z)\,[z = 0 \vee \psi(\overline{z})]$.

In a similar manner, we regard each of the formulas $A_\beta(x)$, $\Phi_\beta(x)$, $\Psi_\beta(x)$, $B_\beta(x)$, and $\Theta_\beta(x)$ (all defined later) as having an additional conjunct $x \neq 0$. In most cases, we could show that such a conjunct is unnecessary, but it easier to add it and ignore the issue of the zero element. The point of this convention is merely to keep our formulas a reasonable size and to avoid repeatedly stating assumptions that elements are not the zero element.

The formulas $A_\beta(x)$ below capture when an element x is a sum of roots of $\mathcal{G}(\Sigma^0_\beta(m))$ components (for even β). The formulas $\Phi_\beta(x)$ and $\Psi_\beta(x)$ capture when an element x is a sum of roots of $\mathcal{G}(\Sigma^0_\beta)$ components and a sum of roots of $\mathcal{G}(\Pi^0_\beta)$ components, respectively.

Definition 5.3.2. *For each even ordinal β, we let $A_\beta(x)$ be the computable infinitary formula* $A_\beta(x) := p^\infty_\beta \mid x \wedge (\exists w)\left[u^\infty_{\beta,1} \mid w \wedge v^\infty_{\beta,0} \mid (x+w)\right]$.

Definition 5.3.3. *For each ordinal β with $\beta \geq 2$, we define computable infinitary formulas $\Phi_\beta(x)$ (for odd β) and $\Psi_\beta(x)$ (for even β) by recursion as follows.*

- *If $\beta = 2$, define $\Psi_\beta(x)$ to be the formula $\Psi_2(x) := p_1^\infty \mid x$.*
- *If $\beta = 3$, define $\Phi_\beta(x)$ to be the formula*

$$\Phi_3(x) := p_3^\infty \mid x \ \wedge \ (\exists y)\Big[q_3^\infty \mid (x+y) \ \wedge \ \Psi_2(y)\Big].$$

- *If $\beta = \delta + 2\ell > 2$, define $\Psi_\beta(x)$ to be the formula*

$$\Psi_\beta(x) := \bigwedge_{m\in\omega} (\exists x_0, \ldots, x_m)\Big[x_0 = x \wedge \bigwedge_{k\leq m} u_{\beta,k}^\infty \mid x_k \wedge \bigwedge_{k<m} v_{\beta,k}^\infty \mid (x_k + x_{k+1}) \wedge \Phi_{f_\beta(m)}^{u_{\beta,m}}(x_m)\Big].$$

  Note that when β is a successor ordinal, the last conjunct is $\Phi_{\beta-1}^{u_{\beta,m}}(x_m)$.

- *If $\beta = \delta + 2\ell + 1 > 3$, define $\Phi_\beta(x)$ to be the formula*

$$\Phi_\beta(x) := p_\beta^\infty \mid x \wedge (\exists y)\left[q_\beta^\infty \mid (x+y) \wedge A_{\beta-1}(y) \wedge \Psi_{\beta-1}(y)\right].$$

Lemma 5.3.6. The complexity of $A_\beta(x)$ is Σ_3^c (independent of β). If $\beta = \delta + 2\ell \geq 2$, then $\Psi_\beta \in \Pi_\beta^c$. If $\beta = \delta + 2\ell + 1 \geq 3$, then $\Phi_\beta \in \Sigma_\beta^c$. Furthermore, the relativization of these formulas to any prime does not change their complexity.

Proof. These statements follow immediately from $p^\infty \mid x$ being Π_2^c and induction. □

Fact 5.3.2. *Let ρ_0, ρ_1 and ρ_2 be distinct prime numbers and let $\psi(x)$ be the formula $\rho_0^\infty \mid x \wedge (\exists y)[\rho_1^\infty \mid y \wedge \rho_2^\infty \mid (x+y)]$. The following properties hold for any prime q.*

1. *If $\mathcal{G} \models \psi^q(x)$ for a fixed $x \in \mathcal{G}$ with witness y and $\mathcal{H}$ is a pure subgroup of $\mathcal{G}$ with $x, y \in \mathcal{H}$, then $\mathcal{H} \models \psi^q(x)$ with witness y.*
2. *If $\mathcal{H} \models \psi^q(x)$ for a fixed $x \in \mathcal{H}$ with witness y and $\mathcal{H}$ is a subgroup of $\mathcal{G}$, then $\mathcal{G} \models \psi^q(x)$ with the same witness.*

In particular, these properties hold for $A_\beta(x)$.

More generally, we have the following fact about our formulas as a consequence of them imposing only positive infinite divisibility conditions.

Fact 5.3.3. *Let $\varphi(x)$ be a formula of the form $A_\beta(x)$, $\Phi_\beta(x)$ or $\Psi_\beta(x)$. If $\mathcal{H} \models \varphi(x)$ for some fixed $x \in \mathcal{H}$ and if $\mathcal{H}$ is a subgroup of $\mathcal{G}$, then $\mathcal{G} \models \varphi(x)$.*

The next lemma gives the key properties needed to verify that our construction succeeds.

Lemma 5.3.7. Fix an odd ordinal $\beta \geq 3$ and a set of primes P disjoint from $\{p_\rho\}_{\rho\leq\beta} \cup \{q_\rho\}_{\rho\leq\beta} \cup \{u_{\rho,m}\}_{\rho\leq\beta,m\in\omega} \cup \{v_{\rho,m}\}_{\rho\leq\beta,m\in\omega}$. Let $\mathcal{G}$ be the group $[\oplus_{i\in\omega}C_i]_P$, where each C_i is either isomorphic to $\mathcal{G}(\Sigma^0_\beta)$ or $\mathcal{G}(\Pi^0_\beta)$. [6]

1. If $\beta = 3$, then $\mathcal{G} \models \Psi_2(y)$ if and only if y can be expressed as $y = \sum a_i y_i$ with each y_i a root of a $\mathcal{G}(\Pi^0_2)$ component and $a_i \in [Z]_{p_1,q_3,P}$.

2. For $\beta = \delta + 2\ell + 1 > 3$:

   (a) If $\mathcal{G} \models A_{\beta-1}(z) \wedge \Psi_{\beta-1}(z)$, then $z = \sum a_i z_i$ with z_i a root of a $\mathcal{G}(\Pi^0_{\beta-1})$ component, $a_i \in [Z]_{P,p_{\beta-2},q_\beta}$ (if $\beta - 1$ is not a limit) and $a_i \in [Z]_{P,q_\beta}$ (if $\beta - 1$ is a limit).

   (b) If $z = \sum a_i z_i$ with z_i a root of a $\mathcal{G}(\Pi^0_{\beta-1})$ component and $a_i \in [Z]_P$, then $\mathcal{G} \models A_{\beta-1}(z) \wedge \Psi_{\beta-1}(z)$.

3. For $\beta = \delta + 2\ell + 3 > 3$ and $k \geq 0$:

   (a) If $\mathcal{G} \models u^\infty_{\beta-1,k} \mid z \wedge \Phi^{u_{\beta-1,k}}_{\beta-2}(z)$, then $z = \sum a_i z_i$ with $a_i \in [Z]_{u_{\beta-1,k},q_\beta,P}$ and z_i a root of a $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ component.

   (b) If $z = \sum a_i z_i$ with $a_i \in [Z]_{u_{\beta-1,k},P}$ and z_i a root of a $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ component, then $\mathcal{G} \models u^\infty_{\beta-1,k} \mid z \wedge \Phi^{u_{\beta-1,k}}_{\beta-2}(z)$.

4. For $\beta = \delta + 1$ and $k \geq 0$:

   (a) If $\mathcal{G} \models u^\infty_{\beta-1,k} \mid z \wedge \Phi^{u_{\beta-1,k}}_{f_{\beta-1}(k)}(z)$, then $z = \sum a_i z_i$ with $a_i \in [Z]_{u_{\beta-1,k},q_\beta,P}$ and z_i a root of a $[\mathcal{G}(\Sigma^0_{f_{\beta-1}(k)})]_{u_{\beta-1,k}}$ component.

   (b) If $z = \sum a_i z_i$ with $a_i \in [Z]_{u_{\beta-1,k},P}$ and z_i a root of a $[\mathcal{G}(\Sigma^0_{f_{\beta-1}(k)})]_{u_{\beta-1,k}}$ component, then $\mathcal{G} \models u^\infty_{\beta-1,k} \mid z \wedge \Phi^{u_{\beta-1,k}}_{f_{\beta-1}(k)}(z)$.

Moreover, the same is true if $\mathcal{G}$ is a finite sum of such groups C_i.

Before proving Lemma 5.3.7, we establish some notation and some basic facts which will be useful in the proof. By Fact 5.2.1, we can write any element of $\mathcal{G}$, as an element of $D(\mathcal{G})$, in the form $\sum q_i x_i$ where each $q_i \in Q^{\neq 0}$ and $x_i \in R_{\mathcal{G}}$. We

[6]The astute reader will note the upcoming statements are almost, but definitely not, biconditionals as a consequence of differences of elements within distinct C_i. Though it is not too difficult to formulate (stating precisely is a bit more difficult) exact conditions for an element to satisfy the appropriate conjunction, we do not need them for our purposes.

will often use various divisibility conditions to narrow which roots x_i can occur in such a sum for particular elements and then use Lemma 5.3.3 and Lemma 5.3.4 to restrict the possible values for the coefficients q_i.

Definition 5.3.4. *If $X \subseteq R_{\mathcal{G}}$, denote by $\mathrm{Span}_{\mathcal{G}}(X)$ the set of all elements $g \in \mathcal{G}$ such that, in $D(\mathcal{G})$, $g = \sum q_i x_i$ where $q_i \in Q$ and $x_i \in X$ for all i.*

Lemma 5.3.8. For any set $X \subseteq \mathcal{G}$ (in particular any set $X \subseteq R_{\mathcal{G}}$), the set $\mathrm{Span}_{\mathcal{G}}(X)$ is a pure subgroup of $\mathcal{G}$.

Proof. The set $\mathrm{Span}_{\mathcal{G}}(X)$ is clearly a subgroup of $\mathcal{G}$. To see that it is pure, fix $g \in \mathrm{Span}_{\mathcal{G}}(X)$, $n > 0$, and $h \in \mathcal{G}$ such that $nh = g$ in $\mathcal{G}$. We need to show that $h \in \mathrm{Span}_{\mathcal{G}}(X)$. Write $g = \sum q_i x_i$ (in $D(\mathcal{G})$) with $q_i \in Q$ and $x_i \in X$. Because $\mathcal{G}$ is torsion-free, the element h is the unique element satisfying $nh = g$. Therefore, in $D(\mathcal{G})$, we have $h = \sum (q_i/n) x_i$ and hence $h \in \mathrm{Span}_{\mathcal{G}}(X)$. □

Note that in the context of torsion-free abelian groups, the subgroup $\mathrm{Span}_{\mathcal{G}}(X)$ need not separate as a direct summand of $\mathcal{G}$. Nevertheless, in the proof of Lemma 5.3.7, we will often be able to describe the isomorphism types of such subgroups. The next lemma pertains to any torsion-free abelian group.

Lemma 5.3.9. Let $\mathcal{H}$ be a torsion-free abelian group which is P-closed for a set P of primes. Let ρ be a prime and let $h \in \mathcal{H}$ be infinitely divisible by ρ. Then for any $q \in [Z]_P$, the element qh is infinitely divisible by ρ.

Proof. Let $g \in \mathcal{H}$ satisfy $\rho^k g = h$. Since $\mathcal{H}$ is $[Z]_P$-closed and $q \in [Z]_P$, we can multiply this equation by q in $\mathcal{H}$ to obtain $(q\rho^k)g = qh$. Thus, the element qg witnesses that qh is divisibly by ρ^k. □

We return to the proof of Lemma 5.3.7. We work both within $\mathcal{G}$ and $D(\mathcal{G})$ during this proof and often rely on context to indicate which group we are working in.

Proof of Lemma 5.3.7. Before establishing Lemma 5.3.7, we say a word about its proof. For $\beta = 3$, we demonstrate (1) directly. For $\beta > 3$, we demonstrate (2), (3), and (4) by simultaneous induction on β. The base case of the induction is the case $\beta = 5$ for (3). The induction cases proceed as follows. To prove (2) for β, we use that (3) and (4) hold for values less than or equal to β; to prove (3) for β, we use that (2) holds for values less than β; and to prove (4) for β, we use that (3) holds for values less than β. Because (3) includes our base case, we begin with the proof of (3) after showing (1).

(1) For $\beta = 3$, we show y can be so expressed if $\mathcal{G} \models \Psi_2(y)$, i.e., if $\mathcal{G} \models p_1^\infty \mid y$. Working in $D(\mathcal{G})$, we express y as $y = \sum a_i y_i$ where $a_i \in Q$ and y_i is the root of a $\mathcal{G}(\Sigma_2^0(m))$ component, a $\mathcal{G}(\Pi_2^0)$ component, or a $[Z]_{p_3}$ component. We note that it is impossible that any y_i is the root of a $\mathcal{G}(\Sigma_2^0(m))$ component. For if one were, with y_j the root of a $\mathcal{G}(\Sigma_2^0(m_j))$ component, there would be a $[Z]_{p_1}$-multiple $\hat{y}$ of y in $\mathcal{G}$ with

$$\hat{y} = \sum_i \frac{\hat{a}_i}{p_1^{n_i}} y_i$$

where $\hat{a}_i \in Z^{\neq 0}$, $p_1 \nmid \hat{a}_i$, and $n_j > m_j$. However, this is impossible as the coefficient of the root of any $\mathcal{G}(\Sigma_2^0(m_j))$ component in $\mathcal{G}$ has the form a/p_1^k where $a \in [Z]_{q_3,P}$ and $k \leq m_j$.

Thus, we have that $y = \sum a_i y_i$ where each y_i is the root of a $[Z]_{p_3}$ component or a $\mathcal{G}(\Pi_2^0)$ component. In other words, we have $y \in \mathcal{B}$ where $\mathcal{B} := \mathrm{Span}_{\mathcal{G}}(X)$ and X is the set of roots of $\mathcal{G}(\Pi_2^0)$ components and $[Z]_{p_3}$ components of $\mathcal{G}$. Hence $\mathcal{B}$ can be written as a direct sum of subgroups

$$\left[\left\langle [Z]_{p_3} \oplus \bigoplus_{k\in\omega} [Z]_{p_1}; q_3^{-t}(r + r_k) : k, t \in \omega \right\rangle\right]_P$$

since $\mathcal{G}(\Pi_2^0) \cong [Z]_{p_1}$.[7] Since $\mathcal{G} \models p_1^\infty \mid y$ and $\mathcal{B}$ is a pure subgroup of $\mathcal{G}$ (by Lemma 5.3.8), we have that $\mathcal{B} \models p_1^\infty \mid y$. Applying Lemma 5.3.3(2) to $\mathcal{B}$ (with $F_1 = \{p_3\}$, $F_2 = \{p_1\}$, $\rho = q_3$, and $P = P$) yields that each y_i is the root of a $\mathcal{G}(\Pi_2^0)$ component and each $a_i \in [Z]_{p_1,q_3,P}$.

Conversely, suppose $y = \sum a_i y_i$ with $a_i \in [Z]_{p_1,q_3,P}$ and y_i the root of a $\mathcal{G}(\Pi_2^0)$ component. Since y is the sum of roots of $\mathcal{G}(\Pi_2^0)$ components, we have $y \in \mathcal{B}$ (where $\mathcal{B}$ is as in the other direction). Since each $a_i \in [Z]_{p_1,q_3,P}$, Lemma 5.3.3(2) implies that $p_1^\infty \mid y$.

(3) For the base case when $\beta = 5$, we first show (3)(a). Fix $k \in \omega$ and suppose that $\mathcal{G} \models u_{4,k}^\infty \mid z \wedge \Phi_3^{u_{4,k}}(z)$, recalling $u_{4,k}^\infty \mid z \wedge \Phi_3^{u_{4,k}}(z)$ is

$$u_{4,k}^\infty \mid z \wedge p_3^\infty \mid z \wedge (\exists y)[u_{4,k}^\infty \mid y \wedge q_3^\infty \mid (z + y) \wedge p_1^\infty \mid y]. \qquad (\ddagger)$$

We need to show that $z = \sum a_i z_i$ with each $a_i \in [Z]_{u_{4,k},q_5,P}$ and each z_i a root of a $[\mathcal{G}(\Sigma_3^0)]_{u_{4,k}}$ component.

[7] Of course, here we mean r to be the root of the $[Z]_{p_3}$ component and r_k to be the root of the kth copy of $[Z]_{p_1}$. When obvious, we omit such explanation.

Since $u_{4,k}^{\infty} \mid z$, the element z must be a sum $z = \sum w_i$, where each w_i comes from a $[\mathcal{G}(\Sigma_3^0)]_{u_{4,k}}$ or $[\mathcal{G}(\Pi_3^0)]_{u_{4,k}}$ component (which we denote by $\mathcal{G}_i$). Indeed, since $p_3^{\infty} \mid z$ by hypothesis, each w_i is a multiple of the root of $\mathcal{G}_i$. Hence, the element z must be a sum $z = \sum a_i z_i$ where $a_i \in Q$ and each z_i is the root of $\mathcal{G}_i$. We endeavor to show that, in fact, each $a_i \in [Z]_{u_{4,k},q_5,P}$ and each $\mathcal{G}_i$ is a $[\mathcal{G}(\Sigma_3^0)]_{u_{4,k}}$ component.

Fix a witness y for (‡). Since $u_{4,k}^{\infty} \mid y$, the element y must also be contained within the $[\mathcal{G}(\Sigma_3^0)]_{u_{4,k}}$ and $[\mathcal{G}(\Pi_3^0)]_{u_{4,k}}$ components. Furthermore, since $p_1^{\infty} \mid y$, the element y must have the form $y = \sum b_j y_j$ where each $b_j \in Q$ and y_j is the root of a $\mathcal{G}(\Pi_2^0)$ component. Since the $[\mathcal{G}(\Pi_3^0)]_{u_{4,k}}$ components do not contain $\mathcal{G}(\Pi_2^0)$ components, each y_j is the root of a $\mathcal{G}(\Pi_2^0)$ subcomponent of a $[\mathcal{G}(\Sigma_3^0)]_{u_{4,k}}$ component. Thus $z, y \in \mathcal{B}$ where $\mathcal{B} := \mathrm{Span}_{\mathcal{G}}(X)$ and X contains the roots of the $[\mathcal{G}(\Pi_3^0)]_{u_{4,k}}$ components, the roots of the $[\mathcal{G}(\Sigma_3^0)]_{u_{4,k}}$ components, and the roots of the $\mathcal{G}(\Pi_2^0)$ components of the $[\mathcal{G}(\Sigma_3^0)]_{u_{4,k}}$ components. By Lemma 5.3.8, the group $\mathcal{B}$ is a pure subgroup of $\mathcal{G}$.

To describe the isomorphism type of $\mathcal{B}$, we need to analyze which primes infinitely divide the roots occurring in X. The point is that a particular element of X may be the root of components at more than one level and each level will introduce different infinite divisibilities. Because of these considerations, we split into cases depending on whether $k > 0$ or $k = 0$.

First, consider the case when $k > 0$ and let r be the root of a $[\mathcal{G}(\Pi_3^0)]_{u_{4,k}}$ component or a $[\mathcal{G}(\Sigma_3^0)]_{u_{4,k}}$ component. The root r is infinitely divisible by p_3 (since it is a root at level 3), by $u_{4,k}$ (by the prime closure of the component added at level 4) and by all the primes in P (by the prime closure of $\mathcal{G}$). Because $k > 0$, the element r is not the root of a component at level 4 and because the level 3 (at which r is a root) is odd, the element r is not the root at level 2. Similarly, if r is the root of a $\mathcal{G}(\Pi_2^0)$ subcomponent of a $[\mathcal{G}(\Sigma_3^0)]_{u_{4,k}}$ component, then r is infinitely divisible by p_1 (since it is the root of $\mathcal{G}(\Pi_2^0)$), by $u_{4,k}$ (by the prime closure) and by all the primes in P. Again, the element r is not the root at any other level. Thus, when $k > 0$, the group $\mathcal{B}$ is isomorphic to a direct sum of infinitely many copies of $[Z]_{p_3,u_{4,k},P}$ (coming from the roots of the $[\mathcal{G}(\Pi_3^0)]_{u_{4,k}}$ components) and infinitely many copies of

$$\left[\left\langle [Z]_{p_3} \oplus \bigoplus_{k\in\omega} [Z]_{p_1} ; q_3^{-t}(r + r_k) : k, t \in \omega \right\rangle\right]_{u_{4,k},P} \tag{5.2}$$

(coming from the roots of the $[\mathcal{G}(\Sigma_3^0)]_{u_{4,k}}$ components and the roots of their $\mathcal{G}(\Pi_2^0)$ subcomponents).

We show that each z_i in the sum $z = \sum a_i z_i$ is the root of a $[\mathcal{G}(\Sigma_3^0)]_{u_{4,k}}$ component. If not, then we can suppose without loss of generality that z_0 is the root of a $[\mathcal{G}(\Pi_3^0)]_{u_{4,k}}$

component, that is, the element z_0 is the element 1 in a direct summand of $\mathcal{B}$ of the form $[Z]_{p_3,u_{4,k},P}$. Since $q_3^\infty \mid (\sum a_i z_i + \sum b_j y_j)$, there is a $[Z]_{q_3}$-multiple $\hat{w}$ of $z + y$ in $\mathcal{B}$ such that

$$\hat{w} = \sum_i \frac{\hat{a}_i}{q_3^{k_i}} z_i + \sum_j \frac{\hat{b}_j}{q_3^{\ell_j}} y_j$$

where $\hat{a}_i, \hat{b}_j \in Z$, $q_3 \nmid \hat{a}_i$, $k_i > 0$, $q_3 \nmid \hat{b}_j$ and $\ell_j > 0$ (assuming $\hat{a}_i, \hat{b}_j \neq 0$). However, the coefficient of z_0 in any element of $\mathcal{B}$ must be from $[Z]_{p_3,u_{4,k},P}$. Hence, we have $\hat{a}_0 = 0$ and therefore $a_0 = 0$.

Having established that each z_i is the root of a $[\mathcal{G}(\Sigma_3^0)]_{u_{4,k}}$ component, it follows that $z, y \in \mathcal{B}'$ where $\mathcal{B}' := \mathrm{Span}_{\mathcal{G}}(X')$ with $X' \subseteq X$ containing only the roots of the $[\mathcal{G}(\Sigma_3^0)]_{u_{4,k}}$ components and the roots of their $\mathcal{G}(\Pi_2^0)$ subcomponents. That is, the group $\mathcal{B}'$ is the subgroup of $\mathcal{B}$ consisting of the direct sum of the infinitely many copies of the group in (5.2). Since $\mathcal{B}'$ is a pure subgroup of $\mathcal{G}$, we have by Fact 5.3.2(1)

$$\mathcal{B}' \models p_3^\infty \mid z \wedge (\exists y)[q_3^\infty \mid (z + y) \wedge p_1^\infty \mid y]$$

(with our fixed element $y \in \mathcal{B}'$ as witness). Therefore, we can apply Lemma 5.3.3(5) (with $F_1 := \{p_3\}$, $F_2 := \{p_1\}$, $\rho := q_3$ and $P := P \cup \{u_{4,k}\}$) to conclude that $z = \sum a_i z_i$ with $a_i \in [Z]_{u_{4,k},P}$.

Second, consider the case when $k = 0$. In this case, we have $z = \sum a_i z_i$ where each z_i is the root of a $[\mathcal{G}(\Sigma_3^0)]_{u_{4,0}}$ component since there are no $[\mathcal{G}(\Pi_3^0)]_{u_{4,0}}$ components. A root r of a $[\mathcal{G}(\Sigma_3^0)]_{u_{4,0}}$ component is infinitely divisible by p_3 (since it is a root at level 3), by $u_{4,k}$ (by the prime closure), by p_4 (since $k = 0$ and hence r is also the root of a $\mathcal{G}(\Pi_4^0)$ or $\mathcal{G}(\Sigma_4^0(m))$ component) and by all the primes in P. Furthermore, if r, r' are roots of (distinct) $\mathcal{G}(\Pi_4^0)$ or $\mathcal{G}(\Sigma_4^0(m))$ components within the same C_i, then $r - r'$ is infinitely divisible by q_5. (This divisibility does not add to the infinite divisibility of either r or r', but it does effect the isomorphism type of $\mathcal{B}$.) However, if r, r' are roots of such components in different C_i, then $r - r'$ is not divisible by q_5. To smooth out this difference in divisibility and to simplify the calculations, we work in $[\mathcal{B}]_{q_5}$.

The group $[\mathcal{B}]_{q_5}$ is isomorphic to the direct sum of infinite many copies of

$$\left[\left\langle [Z]_{p_3,p_4} \oplus \bigoplus_{k\in\omega} [Z]_{p_1}; q_3^{-t}(r + r_k) : k, t \in \omega \right\rangle\right]_{u_{4,0},q_5,P}$$

(coming from the roots of the $[\mathcal{G}(\Sigma^0_3)]_{u_{4,0}}$ components and their $\mathcal{G}(\Pi^0_2)$ subcomponents). Since $\mathcal{B}$ is a pure subgroup of $\mathcal{G}$ and $\mathcal{B}$ is a subgroup of $[\mathcal{B}]_{q_5}$, we have by Fact 5.3.2(1) and Fact 5.3.2(2)

$$[\mathcal{B}]_{q_5} \models p_3^\infty \,|\, z \wedge (\exists y)[q_3^\infty \,|\, (z+y) \wedge p_1^\infty \,|\, y]$$

with our fixed element $y \in \mathcal{B}$ as the witness. Applying Lemma 5.3.3(5) (with $F_1 := \{p_3, p_4\}$, $F_2 := \{p_1\}$, $\rho := q_3$ and $P := P \cup \{u_{4,0}, q_5\}$), we conclude that $a_i \in [Z]_{P,u_{4,0},q_5}$. This completes the proof of (3)(a) when $\beta = 5$.

To prove (3)(b) when $\beta = 5$, assume $z = \sum a_i z_i$, where $a_i \in [Z]_{u_{4,k},P}$ and each z_i is the root of a $[\mathcal{G}(\Sigma^0_3)]_{u_{4,k}}$ component. Let $\mathcal{G}_i$ denote the $[\mathcal{G}(\Sigma^0_3)]_{u_{4,k}}$ component containing z_i. We need to show that $\mathcal{G}$ satisfies

$$u_{4,k}^\infty \,|\, z \wedge p_3^\infty \,|\, z \wedge (\exists y)[u_{4,k}^\infty \,|\, y \wedge q_3^\infty \,|\, (z+y) \wedge p_1^\infty \,|\, y].$$

Since z_i is the root of $\mathcal{G}_i$, we have $p_3^\infty \,|\, z_i$. By Lemma 5.3.9 and the fact that $\mathcal{G}_i$ is $P \cup \{u_{4,k}\}$-closed, it follows that $p_3^\infty \,|\, a_i z_i$ and $u_{4,k}^\infty \,|\, a_i z_i$. Hence, we have $p_3^\infty \,|\, z$ and $u_{4,k}^\infty \,|\, z$.

Let y_i be the root of a $\mathcal{G}(\Pi^0_2)$ component inside $\mathcal{G}_i$ and let $y := \sum a_i y_i$. Since $\mathcal{G}(\Pi^0_2) \cong [Z]_{p_1}$, we have $p_1^\infty \,|\, y_i$. As $\mathcal{G}_i$ is $P \cup \{u_{4,k}\}$-closed, it follows that $p_1^\infty \,|\, a_i y_i$ (by Lemma 5.3.9) and that $u_{4,k}^\infty \,|\, a_i y_i$. Hence, both p_1 and $u_{4,k}$ infinitely divide y. By the definition of $\mathcal{G}(\Sigma^0_3)$, we have $q_3^\infty \,|\, (z_i + y_i)$ and applying Lemma 5.3.9 one more time, we obtain $q_3^\infty \,|\, (a_i z_i + a_i y_i)$. Therefore $\mathcal{G}$ satisfies $\Phi_3^{u_{4,k}}(z)$ with witness y.

This completes the base case of $\beta = 5$.

Next, we show (3) for $\beta > 5$ supposing (2) holds for $\beta - 2$. To prove (3)(a), we suppose $\mathcal{G} \models u_{\beta-1,k}^\infty \,|\, z \wedge \Phi_{\beta-2}^{u_{\beta-1,k}}(z)$, recalling $u_{\beta-1,k}^\infty \,|\, z \wedge \Phi_{\beta-2}^{u_{\beta-1,k}}(z)$ is

$$u_{\beta-1,k}^\infty \,|\, z \wedge p_{\beta-2}^\infty \,|\, z \wedge (\exists y)[u_{\beta-1,k}^\infty \,|\, y \wedge q_{\beta-2}^\infty \,|\, (z+y) \wedge A_{\beta-3}^{u_{\beta-1,k}}(y) \wedge \Psi_{\beta-3}^{u_{\beta-1,k}}(y)]. \qquad (\ddagger\ddagger)$$

We need to show that $z = \sum a_i z_i$ with each $a_i \in [Z]_{u_{\beta-1,k},q_\beta,P}$ and each z_i a root of a $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ component.

As in the $\beta = 5$ case, together $u_{\beta-1,k}^\infty \,|\, z$ and $p_{\beta-2}^\infty \,|\, z$ imply that $z = \sum a_i z_i$, where $a_i \in Q$ and each z_i is the root of a $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ or $[\mathcal{G}(\Pi^0_{\beta-2})]_{u_{\beta-1,k}}$ component. We endeavor to show that, in fact, each $a_i \in [Z]_{u_{\beta-1,k},q_\beta,P}$ and each z_i is the root of a $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ component.

Fix a witness y for ($\ddagger\ddagger$). Our first goal is to show that y is a sum of roots of $\mathcal{G}(\Pi^0_{\beta-3})$ components. Since $u_{\beta-1,k}^\infty \,|\, y$, the element y lies within the $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ and $[\mathcal{G}(\Pi^0_{\beta-2})]_{u_{\beta-1,k}}$ components. Thus, we have $y \in \mathcal{H} := \mathrm{Span}_{\mathcal{G}}(X)$ where X contains the roots of the $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ and $[\mathcal{G}(\Pi^0_{\beta-2})]_{u_{\beta-1,k}}$ components as well as the roots of all the components nested via the recursive construction inside these components. Note that $\mathcal{H}$ is the subgroup of $\mathcal{G}$ consisting of the elements infinitely divisible by $u_{\beta-1,k}$. Because $\mathcal{G}$ satisfies $A^{u_{\beta-1,k}}_{\beta-3}(y) \wedge \Psi^{u_{\beta-1,k}}_{\beta-3}(y)$, we have that $\mathcal{H}$ satisfies $A_{\beta-3}(y) \wedge \Psi_{\beta-3}(y)$ by Lemma 5.3.5.

We describe the isomorphism type of $\mathcal{H}$ in two cases: when $k > 0$ and when $k = 0$. If $k > 0$, then the roots of the $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ and $[\mathcal{G}(\Pi^0_{\beta-2})]_{u_{\beta-1,k}}$ components are not roots of components at any level other than $\beta - 2$. Thus, the group $\mathcal{H}$ is an infinite direct sum of $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k},P}$ and $[\mathcal{G}(\Pi^0_{\beta-2})]_{u_{\beta-1,k},P}$ groups.

If $k = 0$, then note that there are no $[\mathcal{G}(\Pi^0_{\beta-2})]_{u_{\beta-1,0}}$ components. Each root of a $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,0}}$ component is also the root of a $\mathcal{G}(\Sigma^0_{\beta-1}(m))$ or a $\mathcal{G}(\Pi^0_{\beta-1})$ component. Thus, each such root is infinitely divisible by $p_{\beta-1}$ (in addition to the divisibility imposed at level $\beta - 2$). Furthermore, if r, r' are roots of $\mathcal{G}(\Sigma^0_{\beta-1}(m))$ or $\mathcal{G}(\Pi^0_{\beta-1})$ components from the same C_i, then $q^\infty_\beta \mid (r - r')$. If they are roots from different C_i, then we have no such q_β divisibility. To incorporate the extra divisibility by $p_{\beta-1}$ and to smooth out this divisibility difference by q_β, we study $[\mathcal{H}]_{q_\beta,p_{\beta-1}}$. The group $[\mathcal{H}]_{q_\beta,p_{\beta-1}}$ is isomorphic to an infinite direct sum of $[\mathcal{G}(\Sigma^0_{\beta-2})]_{P'}$ groups where $P' := P \cup \{u_{\beta-1,0}, q_\beta, p_{\beta-1}\}$.

In each of the $k > 0$ and $k = 0$ cases, we can apply Part (2)(a) for $\mathcal{H}$ or $[\mathcal{H}]_{q_\beta,p_{\beta-1}}$ and $\beta - 2$ to conclude that $y = \sum b_j y_j$ is a sum of roots y_j of $\mathcal{G}(\Pi^0_{\beta-3})$ components in $\mathcal{G}$ (with appropriate coefficients, which depend on which case we are in). Thus, we have established our first goal.

Our second goal is to show that in the sum $z = \sum a_i z_i$, where each z_i is the root of a $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ component (as opposed to a $[\mathcal{G}(\Pi^0_{\beta-2})]_{u_{\beta-1,k}}$ component) and each coefficient a_i lies in $[Z]_{p_{\beta-1},q_\beta,P}$. We have $z, y \in \mathcal{B} := \mathrm{Span}_{\mathcal{G}}(X)$ where X contains the roots of the $[\mathcal{G}(\Pi^0_{\beta-2})]_{u_{\beta-1,k}}$ components, the roots of the $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ components, and the roots of their $\mathcal{G}(\Pi^0_{\beta-3})$ components. We split into cases depending on whether $k > 0$ or $k = 0$ and proceed with an analysis of the infinite divisibilities as in the $\beta = 5$ case.

First, suppose that $k > 0$. A root r of a $[\mathcal{G}(\Pi^0_{\beta-2})]_{u_{\beta-1,k}}$ or $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ component is infinitely divisible by $p_{\beta-2}$ (being a root at level $\beta-2$), by $u_{\beta-1,k}$ (by prime closure), and by the primes in P (by prime closures). Since $\beta - 2$ is odd, the element r is not a root at a lower level; since $k > 0$, the element r is not a root at a higher level.

A root r of a $\mathcal{G}(\Pi^0_{\beta-3})$ subcomponent of a $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ component is infinitely divisible by $p_{\beta-3}$ (being a root at level $\beta - 3$), by $u_{\beta-1,k}$ (by prime closure), and by the primes in P (by divisible closures). In addition, if $\beta - 3$ is not a limit ordinal, then r is also the root of a $[\mathcal{G}(\Sigma^0_{\beta-4})]_{u_{\beta-3,0}}$ component and hence is infinitely divisible by $p_{\beta-4}$ and $u_{\beta-3,0}$. Notice that the recursion stops at this point because $\beta - 4$ is an odd ordinal and hence r is not the root at any lower level. If $\beta - 3$ is a limit ordinal, then r is also the root of a $[\mathcal{G}(\Sigma^0_{f_{\beta-3}(0)})]_{u_{\beta-3,0}}$ component and hence is infinitely divisible by $p_{f_{\beta-3}(0)}$ and $u_{\beta-3,0}$. Again, the recursion stops at this point because $f_{\beta-3}(0)$ is an odd ordinal. Recall that if $\beta - 3$ is not a limit, then $f_{\beta-3}(0) = \beta - 4$. Thus, we can also describe the infinite divisibility by $p_{\beta-4}$ (in the case when $\beta - 3$ is not a limit) as infinite divisibility by $p_{f_{\beta-3}(0)}$. In future analyses, we will combine these cases in this manner.

From this analysis, when $k > 0$, the group $\mathcal{B}$ is isomorphic to the direct sum of infinitely many copies of $[Z]_{p_{\beta-2},u_{\beta-1,k},P}$ (from the roots of $[\mathcal{G}(\Pi^0_{\beta-2})]_{u_{\beta-1,k}}$ components) and infinitely many copies of

$$\left[\left\langle [Z]_{F_1} \oplus \bigoplus_{j\in\omega} [Z]_{F_2}; \frac{x + y_j}{\rho^k} : j,k \in \omega \right\rangle\right]_{P,u_{\beta-1,k}} \tag{5.3}$$

where $F_1 := \{p_{\beta-2}\}$, $F_2 := \{p_{\beta-3}, u_{\beta-3,0}, p_{f_{\beta-3}(0)}\}$ and $\rho := q_{\beta-2}$ (from the roots of $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ components and their $\mathcal{G}(\Pi^0_{\beta-3})$ subcomponents). A divisibility argument almost identical to the one used in the $\beta = 5$ case (using the fact that $q^\infty_{\beta-2} \mid (z + y)$) shows that none of the z_i elements can come from the $[Z]_{p_{\beta-2},u_{\beta-1,k},P}$ summands. Therefore, each z_i is the root of a $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ component.

Let $\mathcal{B}'$ be the subgroup of $\mathcal{B}$ consisting of the direct sum of infinitely many copies of the group in Equation (5.3). Since $\mathcal{B}'$ is a pure subgroup of $\mathcal{G}$ containing y and z and $\mathcal{G}$ satisfies

$$p^\infty_{\beta-2} \mid z \;\wedge\; q^\infty_{\beta-2} \mid (z + y) \;\wedge\; p^\infty_{\beta-3} \mid y,$$

we have that this formula is also satisfied in $\mathcal{B}'$ (by Fact 5.3.2(1)). Applying Lemma 5.3.3(5) to $\mathcal{B}'$ with the above values for F_1, F_2, and ρ yields that each $a_i \in [Z]_{P,u_{\beta-1,k}}$, completing the case when $k > 0$.

Second, suppose $k = 0$. The analysis of the isomorphism type of $\mathcal{B}$ is almost identical to the case when $k > 0$ except for three points. First, there are no components of the form $[\mathcal{G}(\Pi^0_{\beta-2})]_{u_{\beta-1,0}}$ and hence no argument is needed to conclude that each z_i is the root of a $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,0}}$ component. Second, the root of a $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,0}}$ component is also the root of a $\mathcal{G}(\Sigma^0_{\beta-1})$ or $\mathcal{G}(\Pi^0_{\beta-1})$ component and hence is infinitely divisible by $p_{\beta-1}$ in addition to the infinite divisibilities given above. Third, to smooth out the fact that q_β infinitely divides $r - r'$ when r, r' are roots of $\mathcal{G}(\Sigma^0_{\beta-1})$ or $\mathcal{G}(\Pi^0_{\beta-1})$ components from the same C_i, we work with $[\mathcal{B}]_{q_\beta}$. With these observations, the group $[\mathcal{B}]_{q_\beta}$ is isomorphic to the direct sum of infinitely many copies of

$$\left[\left\langle [Z]_{F_1} \oplus \bigoplus_{j\in\omega} [Z]_{F_2}; \frac{x + y_j}{\rho^k} : j, k \in \omega \right\rangle\right]_{P, u_{\beta-1,k}, q_\beta}$$

where $F_1 := \{p_{\beta-2}, p_{\beta-1}\}$, $F_2 := \{p_{\beta-3}, u_{\beta-3,0}, p_{f_{\beta-3}(0)}\}$ and $\rho := q_{\beta-2}$. Since $\mathcal{B}$ is a pure subgroup of $\mathcal{G}$ and $\mathcal{G}$ satisfies

$$p^\infty_{\beta-2} \,|\, z \,\wedge\, q^\infty_{\beta-2} \,|\, (z + y) \,\wedge\, p^\infty_{\beta-3} \,|\, y$$

this formula is also satisfied in $\mathcal{B}$ (by Fact 5.3.2(1)). Since $[\mathcal{B}]_{q_\beta}$ is an expansion of $\mathcal{B}$, it remains true in $\mathcal{B}'$ (by Fact 5.3.2(2)). We apply Lemma 5.3.3(5) to conclude that each $a_i \in [Z]_{P, u_{\beta-1,k}, q_\beta}$.

To prove (3)(b) when $\beta > 5$, fix an element $z = \sum a_i z_i$ with each $a_i \in [Z]_{u_{\beta-1,k}, P}$ and each z_i the root of a $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ component (which we denote by $\mathcal{G}_i$). We have $u^\infty_{\beta-1} \,|\, z_i$ and $p^\infty_{\beta-2}|z_i$ as a consequence of the structure of $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ components and z_i being the root. As $a_i \in [Z]_{u_{\beta-1,k}, P}$ and $\mathcal{G}_i$ is $\{u_{\beta-1,k}, P\}$-closed, it follows that $u^\infty_{\beta-1} \,|\, a_i z_i$ and $p^\infty_{\beta-2}|a_i z_i$ and hence that $u^\infty_{\beta-1} \,|\, z$ and $p^\infty_{\beta-2}|z$.

Let $y := \sum a_i y_i$, where y_i is the root of a $\mathcal{G}(\Pi^0_{\beta-3})$ subcomponent of $\mathcal{G}_i$. Since each $\mathcal{G}_i$ is a $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ component, it follows from the structure of these components that $u^\infty_{\beta-1,k} \,|\, y$ and $q^\infty_{\beta-2} \,|\, (z + y)$. It remains to show that $\mathcal{G}$ satisfies $A^{u_{\beta-1,k}}_{\beta-3}(y)$ and $\Psi^{u_{\beta-1,k}}_{\beta-3}(y)$.

Let $\mathcal{B} := \mathrm{Span}_{\mathcal{G}}(X)$ where X contains the roots of the $[\mathcal{G}(\Sigma^0_{\beta-2})]_{u_{\beta-1,k}}$ components and the roots of any component nested via the recursive construction inside such a component. Note that $y, z \in \mathcal{B}$ and that $\mathcal{B}$ is the subgroup of $\mathcal{G}$ consisting of the elements which are infinitely divisible by $u_{\beta-1,k}$. Applying Part (2)(b) to $\mathcal{B}$ with $\beta - 2$ and $P = P \cup \{u_{\beta-1,k}\}$, we get that $\mathcal{B}$ satisfies $A_{\beta-3}(y) \,\wedge\, \Psi_{\beta-3}(y)$. Since $\mathcal{B}$ consists of the elements of $\mathcal{G}$ which are infinitely divisible by $u_{\beta-1,k}$, we have that $\mathcal{G}$ satisfies $A^{u_{\beta-1,k}}_{\beta-3}(y) \,\wedge\, \Psi^{u_{\beta-1,k}}_{\beta-3}(y)$ by Lemma 5.3.5 as required.

(2) We show (2) for β supposing (3) and (4) hold for values less than or equal to β.

To show (2)(a), we suppose $\mathcal{G} \models A_{\beta-1}(z) \wedge \Psi_{\beta-1}(z)$, recalling $A_{\beta-1}(z)$ is

$$p_{\beta-1}^{\infty}|z \wedge (\exists w)\left[u_{\beta-1,1}|w \wedge v_{\beta-1,0}^{\infty}|(z+w)\right].$$

Since $p_{\beta-1}^{\infty} \,|\, z$, we can express z as $z = \sum a_i z_i$ where $a_i \in Q$ and z_i is the root of a $\mathcal{G}(\Sigma_{\beta-1}^0(m))$ or $\mathcal{G}(\Pi_{\beta-1}^0)$ component (which we denote by $\mathcal{G}_i$). Since z_i is also the root of the $[\mathcal{G}(\Sigma_{f_{\beta-1}(0)}^0)]_{u_{\beta-1,0}}$ component inside this $\mathcal{G}(\Sigma_{\beta-1}^0(m))$ or $\mathcal{G}(\Pi_{\beta-1}^0)$ component, we have that z_i is infinitely divisible by $p_{f_{\beta-1}(0)}$ and $u_{\beta-1,0}$. As above, the recursion stops here since $f_{\beta-1}(0)$ is an odd ordinal and hence z_i is the element 1 in a copy of $[Z]_{p_{f_{\beta-1}(0)}}$.

Fix an element w witnessing $\mathcal{G} \models A_{\beta-1}(z)$. The condition $u_{\beta-1,1}^{\infty} \,|\, w$ implies that w is a sum of elements from $[\mathcal{G}(\Sigma_{f_{\beta-1}(1)}^0)]_{u_{\beta-1,1}}$ and $[\mathcal{G}(\Pi_{f_{\beta-1}(1)}^0)]_{u_{\beta-1,1}}$ components. The condition $v_{\beta-1,0}^{\infty} \,|\, (z+w)$ implies (by divisibility arguments similar to those already given many times) that $w = \sum b_i w_i$, where each $b_i \in Q$ and each w_i is the root of a $[\mathcal{G}(\Sigma_{f_{\beta-1}(1)}^0)]_{u_{\beta-1,1}}$ or $[\mathcal{G}(\Pi_{f_{\beta-1}(1)}^0)]_{u_{\beta-1,1}}$ component. Since $f_{\beta-1}(1)$ is an odd ordinal, the root of such a component is the element 1 in a copy of $[Z]_{p_{f_{\beta-1}(1)}}$. Thus, the element w_i is infinitely divisible by $p_{f_{\beta-1}(1)}$ and $u_{\beta-1,1}$ but the recursion stops at this point. (Note that for the same reason, the roots of $[\mathcal{G}(\Sigma_{f_{\beta-1}(k)}^0)]_{u_{\beta-1,k}}$ and $[\mathcal{G}(\Pi_{f_{\beta-1}(k)}^0)]_{u_{\beta-1,k}}$ components for $k \geq 2$ are infinitely divisible only by $p_{f_{\beta-1}(k)}$ and $u_{\beta-1,k}$.)

To find the coefficients a_i in $z = \sum a_i z_i$, let $\mathcal{B} := \mathrm{Span}_{\mathcal{G}}(X)$ where X contains the roots of the $[\mathcal{G}(\Sigma_{f_{\beta-1}(k)}^0)]_{u_{\beta-1,k}}$ and $[\mathcal{G}(\Pi_{f_{\beta-1}(k)}^0)]_{u_{\beta-1,k}}$ components for $k \in \omega$. As in the proof of Part (3), we work in $[\mathcal{B}]_{q_\beta}$ since, for roots r, r' of $[\mathcal{G}(\Sigma_{f_{\beta-1}(0)}^0)]_{u_{\beta-1,0}}$ components (which are also roots of $\mathcal{G}(\Sigma_{\beta-1}^0(m))$ or $\mathcal{G}(\Pi_{\beta-1}^0)$ components), it is the case that q_β infinitely divides $r - r'$ if and only if these roots come from the same C_i summand of $\mathcal{G}$.

The group $[\mathcal{B}]_{q_\beta}$ is isomorphic to the $P' := P \cup \{q_\beta\}$ closure of

$$\left\langle \mathcal{F}; \frac{x_{k,j}}{\sigma_k^{\ell}}, \frac{x_{k,j} + x_{k+1,j}}{\rho_k^{\ell}} : j, k, \ell \in \omega \text{ and all } \sigma_k \in F_k \right\rangle$$

where $\mathcal{F}$ is the free abelian group on $x_{k,j}$ (for $k, j \in \omega$), $F_0 := \{p_{f_{\beta-1}(0)}, p_{\beta-1}, u_{\beta-1,0}\}$, $F_k := \{p_{f_{\beta-1}(k)}, u_{\beta-1,k}\}$ (for $k > 0$) and $\rho_k := v_{\beta-1,k}$. In this presentation, for each fixed j, the element $x_{k,j}$ is the root of a $[\mathcal{G}(\Sigma_{f_{\beta-1}(0)}^0)]_{u_{\beta-1,k}}$ component or a $[\mathcal{G}(\Pi_{f_{\beta-1}(0)}^0)]_{u_{\beta-1,k}}$ component within a fixed $\mathcal{G}(\Sigma_{\beta-1}^0(m))$ or $\mathcal{G}(\Pi_{\beta-1}^0)$ component of $\mathcal{G}$. As j varies, we range over all $\mathcal{G}(\Sigma_{\beta-1}^0(m))$ and $\mathcal{G}(\Pi_{\beta-1}^0)$ components of $\mathcal{G}$. If $\beta - 1$ is a limit ordinal, then $p_{f_{\beta-1}(k)} \neq p_{f_{\beta-1}(k')}$ for $k \neq k'$. If $\beta - 1$ is not a limit ordinal, then $p_{f_{\beta-1}(k)} = p_{\beta-2}$ for all k. In this case, we can remove the primes $p_{f_{\beta-1}(k)}$ from F_k and add $p_{\beta-2}$ to P' (since $[\mathcal{B}]_{q_\beta}$

is $p_{\beta-2}$ closed). This change has the effect of including infinite divisibility by $p_{\beta-2}$ for our coefficients a_i.

Since $\mathcal{B}$ is a pure subgroup of $\mathcal{G}$, the group $\mathcal{B}$ is a subgroup of $[\mathcal{B}]_{q_\beta}$, and $z, w \in \mathcal{B}$, we have (applying both Fact 5.3.2(1) and Fact 5.3.2(2)) that $[\mathcal{B}]_{q_\beta}$ satisfies $A_{\beta-1}(z)$ with our element w as witness. Therefore, by Lemma 5.3.4(3), we obtain that $z = \sum a_i z_i$ with $a_i \in [Z]_{P,p_{\beta-2},q_\beta}$ (if $\beta - 1$ is not a limit ordinal) or $a_i \in [Z]_{P,q_\beta}$ (if $\beta - 1$ is a limit ordinal).

Next, we use the fact that $\mathcal{G} \models \Psi_{\beta-1}(z)$ to show each $\mathcal{G}_i$ is a $\mathcal{G}(\Pi^0_{\beta-1})$ component. For if one were not, then some $\mathcal{G}_\ell$ would be a $\mathcal{G}(\Sigma^0_{\beta-1}(m_0))$ component for some $m_0 \in \omega$. With $m := m_0 + 1$, we fix a sequence $g_0, g_1, \ldots, g_m$ witnessing that $\mathcal{G}$ satisfies the m-th conjunct of $\Psi_{\beta-1}(z)$. Since $\mathcal{G}$ satisfies $u^\infty_{\beta-1,m} \,|\, g_m$ and $\Phi^{u_{\beta-1,m}}_{f_{\beta-1}(m)}(g_m)$, we have by Part (3) or Part (4) (depending on the form of $f_{\beta-1}(m)$) that $g_m = \sum c_j y_j$ where each y_j is the root of a $[\mathcal{G}(\Sigma^0_{f_{\beta-1}(m)})]_{u_{\beta-1,m}}$ component. Since $g_0 = z = \sum a_i z_i$ and $v^\infty_{\beta-1} \,|\, (g_k + g_{k+1})$ for $0 \leq k < m$, one of the y_j roots in the summand for g_m must lie in the component $\mathcal{G}_\ell$. However, the group $\mathcal{G}_\ell$ is a $\mathcal{G}(\Sigma^0_{\beta-1}(m_0))$ component with $m_0 < m$, so it does not contain a $[\mathcal{G}(\Sigma^0_{f_{\beta-1}(m)})]_{u_{\beta-1,m}}$ component, yielding the desired contradiction. This completes the proof of (2)(a).

To prove (2)(b), fix an element $z = \sum a_i z_i$ with $a_i \in [Z]_P$ and z_i a root of a $\mathcal{G}(\Pi^0_{\beta-1})$ component of $\mathcal{G}$ (which we denote $\mathcal{G}_i$). We need to show that $\mathcal{G} \models A_{\beta-1}(z)$ and $\mathcal{G} \models \Psi_{\beta-1}(z)$. For the former, we need to show that $\mathcal{G}$ satisfies

$$p^\infty_{\beta-1}|z \;\wedge\; (\exists w)[u^\infty_{\beta-1,1} \,|\, w \;\wedge\; v^\infty_{\beta-1,0} \,|\, (z + w)].$$

From the structure of $\mathcal{G}(\Pi^0_{\beta-1})$ components, we have each z_i is the root r_0 of the $\left[\mathcal{G}(\Sigma^0_{f_{\beta-1}(0)})\right]_{u_{\beta-1,0}}$ component of $\mathcal{G}_i$. By Lemma 5.3.9, the condition $p^\infty_{\beta-1}|z$ is satisfied since $a_i \in [Z]_P$ and $p^\infty_{\beta-1}|z_i$ for each z_i.

To generate the witness w, for each i, let w_i be the root r_1 of the $\left[\mathcal{G}(\Sigma^0_{f_{\beta-1}(1)})\right]_{u_{\beta-1,1}}$ component of $\mathcal{G}_i$. The conditions $u^\infty_{\beta-1,1} \,|\, w_i$ and $v^\infty_{\beta-1,0} \,|\, (z_i + w_i)$ are satisfied since $z_i = r_0$ and $w_i = r_1$ in $\mathcal{G}_i$. As each $a_i \in [Z]_P$, it follows from Lemma 5.3.9 that $\mathcal{G} \models A_{\beta-1}(z)$ with witness $w := \sum a_i w_i$.

To see that $\mathcal{G} \models \Psi_{\beta-1}(z)$, we reason as follows. Fix $m \in \omega$. We show how to pick the witnessing elements $g_0, \ldots, g_m$ for the m-th conjunct. For each z_i, pick a sequence of elements $g_{i,0}, g_{i,1}, \ldots, g_{i,m}$ in $\mathcal{G}_i$ by setting $g_{i,0} := z_i$ (which is the r_0 root in $\mathcal{G}_i$) and $g_{i,k} := r_k$ (the root of the $[\mathcal{G}(\Sigma^0_{f_{\beta-1}(k)})]_{u_{\beta-1,k}}$ component of $\mathcal{G}_i$) for $0 < k \leq m$.

Since $a_i \in [Z]_P$ and each $\mathcal{G}_i$ is P-closed, we have (from the structure of $\mathcal{G}(\Pi^0_{\beta-1})$) that $u^\infty_{\beta-1,k} \,|\, a_i g_{i,k}$ for $k \leq m$ and $v^\infty_{\beta-1,k} \,|\, (a_i g_{i,k} + a_i g_{i,k+1})$ for $k < m$.

For each $0 \leq k \leq m$, let $g_k := \sum_i a_i g_{i,k}$. By the divisibility conditions above, we have $u^\infty_{\beta-1,k} \,|\, g_k$ for $k \leq m$ and $v^\infty_{\beta-1,k} \,|\, (g_k + g_{k+1})$ for $k < m$. Furthermore, $g_0 = z$. Therefore, it only remains to show that $\Phi^{u_{\beta-1,m}}_{f_{\beta-1}(m)}(g_m)$. We already have $u^\infty_{\beta-1,m} \,|\, g_m$. Since $g_m = \sum_i a_i g_{i,m}$ where $a_i \in [Z]_P$ and $g_{i,m}$ is the root of a $[\mathcal{G}(\Sigma^0_{f_{\beta-1}(m)})]_{u_{\beta-1,m}}$ component, it follows from Part (3)(b) or Part (4)(b), depending on the form of $f_{\beta-1}(m)$, that $\mathcal{G}$ satisfies $\Phi^{u_{\beta-1,m}}_{f_{\beta-1}(m)}(g_m)$ and hence $\mathcal{G} \models \Psi_{\beta-1}(z)$.

(4) As $f_{\beta-1}(k)$ is an odd ordinal and $f_{\beta-1}(k) < \beta - 1$ for all $k \in \omega$, the proof of Part (4) is essentially the same as the proof of Part (3) with the appropriate notational changes to reflect that $\beta - 1$ is a limit ordinal.

□

Lemma 5.3.10. Let $\beta = \delta + 2\ell + 1 \geq 3$. Then for $\mathcal{G} = \bigoplus_{n\in\omega} \mathcal{G}_n$, where $\mathcal{G}_n$ is either $[\mathcal{G}(\Sigma^0_\beta)]_{d_n}$ or $[\mathcal{G}(\Pi^0_\beta)]_{d_n}$, the following holds:

$$\mathcal{G} \models [(\exists x)\Phi_\beta(x)]^{d_n} \qquad \text{if and only if} \qquad \mathcal{G}_n \cong [\mathcal{G}(\Sigma^0_\beta)]_{d_n}.$$

Proof. Since $\mathcal{G}_n$ is the subgroup of elements of $\mathcal{G}$ which are infinitely divisible by d_n, we have by Lemma 5.3.5 that

$$\mathcal{G} \models [(\exists x)\Phi_\beta(x)]^{d_n} \qquad \text{if and only if} \qquad \mathcal{G}_n \models (\exists x)\Phi_\beta(x).$$

Therefore, it suffices to show that $\mathcal{G}(\Sigma^0_\beta) \models (\exists x)\Phi_\beta(x)$ and $\mathcal{G}(\Pi^0_\beta) \not\models (\exists x)\Phi_\beta(x)$.

First, we show that $\mathcal{G}(\Sigma^0_\beta) \models \Phi_\beta(r)$ where r is the root of $\mathcal{G}(\Sigma^0_\beta)$. That is, we show that $\mathcal{G}(\Sigma^0_\beta)$ satisfies

$$p^\infty_\beta \,|\, r \ \wedge\ (\exists y)[q^\infty_\beta \,|\, (r + y) \ \wedge\ A_{\beta-1}(y) \ \wedge\ \Psi_{\beta-1}(y)].$$

Since r is the root of $\mathcal{G}(\Sigma^0_\beta)$, we immediately obtain $p^\infty_\beta \,|\, r$. We claim that the root r_k of a $\mathcal{G}(\Pi^0_{\beta-1})$ component works for the choice of y in Φ_β. From the definition of $\mathcal{G}(\Sigma^0_\beta)$, we have $q^\infty_\beta \,|\, (r + r_k)$ and by Lemma 5.3.7(2)(b), we have that $\mathcal{G}(\Sigma^0_\beta)$ satisfies both $A_{\beta-1}(r_k)$ and $\Psi_{\beta-1}(r_k)$ as required.

Second, assume for a contradiction that $\mathcal{G}(\Pi^0_\beta) \models (\exists x)\Phi_\beta(x)$ and fix the witness x. The condition $p^\infty_\beta \,|\, x$ implies that x is a multiple of the root of $\mathcal{G}(\Pi^0_\beta)$. Fix the witness y such that $q^\infty_\beta \,|\, (x + y) \ \wedge\ A_{\beta-1}(y) \ \wedge\ \Psi_{\beta-1}(y)$. By Lemma 5.3.7(2)(a), the condition $A_{\beta-1}(y) \wedge \Psi_{\beta-1}(y)$ implies that y is a sum of multiples of the roots of $\mathcal{G}(\Pi^0_{\beta-1})$ components. However, the group $\mathcal{G}(\Pi^0_\beta)$ has no such components, giving the desired contradiction. $\square$

We continue by constructing sentences connected semantically to $\mathcal{H}(\Sigma^0_\beta)$ and $\mathcal{H}(\Pi^0_\beta)$. We first give lemmas similar to Lemma 5.3.7 for the groups $\mathcal{H}(\Sigma^0_\beta)$ and $\mathcal{H}(\Pi^0_\beta)$.

Lemma 5.3.11. Let $\beta = \delta + 2l + 2 \geq 4$ and let $\mathcal{H} \cong \mathcal{H}(\Pi^0_\beta)$. Let $y \in \mathcal{H}$ be a sum $y = \sum b_j y_j$ where each $b_j \in Z$ and each y_j is the root of a (distinct) $\mathcal{G}(\Sigma^0_{\beta-1})$ component of $\mathcal{H}$. Then $\mathcal{H} \models \Phi_{\beta-1}(y)$.

Proof. We need to show that $\mathcal{H}$ satisfies

$$p^\infty_{\beta-1} \,|\, y \wedge (\exists w)[q^\infty_{\beta-1} \,|\, (y + w) \wedge A_{\beta-2}(w) \wedge \Psi_{\beta-2}(w)].$$

By the structure of $\mathcal{G}(\Sigma^0_{\beta-1})$, we have $p^\infty_{\beta-1} \,|\, y_j$ for all j. Since $b_j \in Z$, we have $p^\infty_{\beta-1} \,|\, b_j y_j$ and hence $p^\infty_{\beta-1} \,|\, y$. For each j, let w_j be the root of a $\mathcal{G}(\Pi^0_{\beta-2})$ component within the $\mathcal{G}(\Sigma^0_{\beta-1})$ component with root y_j and let $w := \sum b_j w_j$. It follows from the structure of $\mathcal{G}(\Sigma^0_{\beta-1})$ that $q^\infty_{\beta-1} \,|\, (y + w)$. Therefore, it remains to show that $\mathcal{H}$ satisfies $A_{\beta-2}(w) \wedge \Psi_{\beta-2}(w)$.

The group $\mathcal{H}$ is built by taking a direct sum of the groups $[Z]_{p_\beta}$ (with root r) and $\bigoplus_{k\in\omega} \mathcal{G}(\Sigma^0_{\beta-1})$ (with roots r_k) and then adding extra elements (from the divisible closure of this sum) to witness $q^\infty_\beta \,|\, (r + r_k)$. Since $w = \sum b_j w_j$ with each $b_j \in Z$, we can view w as an element of the group $\bigoplus_{k\in\omega} \mathcal{G}(\Sigma^0_{\beta-1})$ in this construction of $\mathcal{H}$. By Lemma 5.3.7(2)(b) applied to w as an element of $\bigoplus_{k\in\omega} \mathcal{G}(\Sigma^0_{\beta-1})$, we have that $\bigoplus_{k\in\omega} \mathcal{G}(\Sigma^0_{\beta-1})$ satisfies $A_{\beta-2}(w) \wedge \Psi_{\beta-2}(w)$. Since $\bigoplus_{k\in\omega} \mathcal{G}(\Sigma^0_{\beta-1})$ is a subgroup of $\mathcal{H}$, Fact 5.3.3 implies that $\mathcal{H}$ satisfies $A_{\beta-2}(w) \wedge \Psi_{\beta-2}(w)$ as required. $\square$

Lemma 5.3.12. Let $\beta = \delta + 2l + 2 \geq 4$ and let $\mathcal{H} \cong \mathcal{H}(\Sigma^0_\beta)$. Let r be the root of a $\mathcal{G}(\Pi^0_{\beta-1})$ component of $\mathcal{H}$. Then $\mathcal{H} \not\models \Phi_{\beta-1}(r)$.

Proof. We show that there is no $w \in \mathcal{H}$ such that $\mathcal{H}$ satisfies

$$q^\infty_{\beta-1} \,|\, (r + w) \wedge A_{\beta-2}(w) \wedge \Psi_{\beta-2}(w).$$

For a contradiction, fix such an element $w \in \mathcal{H}$. To simplify dealing with q_β divisibility in $\mathcal{H}$, we work in the prime closure $[\mathcal{H}]_{q_\beta}$ and note that if w satisfies this formula in $\mathcal{H}$, then by Fact 5.3.3, it also satisfies the formula in $[\mathcal{H}]_{q_\beta}$.

The group $[\mathcal{H}]_{q_\beta}$ decomposes as a direct sum

$$[Z]_{p_\beta,q_\beta} \oplus \bigoplus_{i\in\omega}[C_i]_{q_\beta}$$

where each C_i is isomorphic to $\mathcal{G}(\Sigma^0_{\beta-1})$ or $\mathcal{G}(\Pi^0_{\beta-1})$. The divisibility condition $p^\infty_{\beta-2} \mid w$ (from the fact that $[\mathcal{H}]_{q_\beta} \models A_{\beta-2}(w)$) implies that $w = \sum a_i w_i$ where each $a_i \in Q$ and each w_i is the root of a $\mathcal{G}(\Sigma^0_{\beta-2}(m))$ or $\mathcal{G}(\Pi^0_{\beta-2})$ component. Therefore, as an element of $[\mathcal{H}]_{q_\beta}$, we have $w \in \bigoplus_{i\in\omega}[C_i]_{q_\beta}$. In addition, by arguments similar to previous ones, the condition $q^\infty_{\beta-1} \mid (r+w)$ implies that at least one w_i is the root of a $\mathcal{G}(\Sigma^0_{\beta-2}(m))$ subcomponent of the $\mathcal{G}(\Pi^0_{\beta-1})$ component with root r (as this component has no $\mathcal{G}(\Pi^0_{\beta-2})$ subcomponents).

Assume for a moment that $\bigoplus_{i\in\omega}[C_i]_{q_\beta}$ satisfies $A_{\beta-2}(w) \wedge \Psi_{\beta-2}(w)$. Under this assumption, Lemma 5.3.7(2)(a) implies that w is a sum of roots of $\mathcal{G}(\Pi^0_{\beta-2})$ components, contradicting the fact that at least one w_i is the root of a $\mathcal{G}(\Sigma^0_{\beta-2}(m))$ component. Therefore, to complete our proof, it suffices to show that $\bigoplus_{i\in\omega}[C_i]_{q_\beta}$ satisfies $A_{\beta-2}(w) \wedge \Psi_{\beta-2}(w)$.

To show $\bigoplus_{i\in\omega}[C_i]_{q_\beta}$ satisfies $A_{\beta-2}(w) \wedge \Psi_{\beta-2}(w)$, we use the fact that $\bigoplus_{i\in\omega}[C_i]_{q_\beta}$ is a pure subgroup of $[\mathcal{H}]_{q_\beta}$ (since it is a direct summand) along with the following observation. Because $[\mathcal{H}]_{q_\beta}$ is a direct sum, any element $z \in [\mathcal{H}]_{q_\beta}$ can be written (uniquely) in the form $z = z_0 + z_1$ where $z_0 \in [Z]_{p_\beta,q_\beta}$ and $z_1 \in \bigoplus_{i\in\omega}[C_i]_{q_\beta}$. If ρ is a prime and $\rho^\infty \mid z$, then $\rho^\infty \mid z_0$ and $\rho^\infty \mid z_1$. Therefore, if $\rho^\infty \mid z$ and ρ is not p_β or q_β, we can conclude that $z \in \bigoplus_{i\in\omega}[C_i]_{q_\beta}$.

Using this observation, we show that the following implications hold for all γ with $2 \leq \gamma \leq \beta - 2$. Let $\varphi(x)$ be either $A_\gamma(x)$ or $\Psi_\gamma(x)$ (if γ is even) or $\Phi_\gamma(x)$ (if γ is odd), and let ρ be any prime number. For any $x \in \bigoplus_{i\in\omega}[C_i]_{q_\beta}$, we have

$$[\mathcal{H}]_{q_\beta} \models \varphi(x) \qquad \text{implies} \qquad \bigoplus_{i\in\omega}[C_i]_{q_\beta} \models \varphi(x) \text{ and}$$
$$[\mathcal{H}]_{q_\beta} \models \varphi^\rho(x) \qquad \text{implies} \qquad \bigoplus_{i\in\omega}[C_i]_{q_\beta} \models \varphi^\rho(x).$$

Notice that establishing this property finishes our proof as $w \in \bigoplus_{i\in\omega}[C_i]_{q_\beta}$ and $[\mathcal{H}]_{q_\beta} \models A_{\beta-2}(w) \wedge \Psi_{\beta-2}(w)$, so by the property $\bigoplus_{i\in\omega}[C_i]_{q_\beta} \models A_{\beta-2}(w) \wedge \Psi_{\beta-2}(w)$.

First, consider the case when $\varphi(x)$ is $A_\gamma(x)$ and assume $[\mathcal{H}]_{q_\beta} \models A_\gamma(x)$. In this case, the existential witness y in $A_\gamma(x)$ is infinitely divisible by $u_{\gamma,1}$. As $u_{\gamma,1} \notin \{p_\beta, q_\beta\}$, we have $y \in \bigoplus_{i\in\omega}[C_i]_{q_\beta}$. Since $\bigoplus_{i\in\omega}[C_i]_{q_\beta}$ is a pure subgroup containing x and y, the group $\bigoplus_{i\in\omega}[C_i]_{q_\beta}$ satisfies $A_\gamma(x)$. The same proof works for $A^\rho_\beta(x)$.

Second, consider the cases when $\varphi(x)$ is $\Phi_\gamma(x)$ (for odd γ), $\Psi_\gamma(x)$ (for even γ) or a prime relativization of one of these formulas. We proceed by induction on γ and note that in each case the proof for the relativized formula is identical to the proof for the unrelativized formula. In each case, we assume $x \in \bigoplus_{i\in\omega}[C_i]_{q_\beta}$ and $[\mathcal{H}]_{q_\beta} \models \varphi(x)$.

The first base case is when $\beta = 2$. Since $\Psi_2(x)$ is $p_1^\infty \,|\, x$, $x \in \bigoplus_{i\in\omega}[C_i]_{q_\beta}$, and $\bigoplus_{i\in\omega}[C_i]_{q_\beta}$ is a pure subgroup, we have $\bigoplus_{i\in\omega}[C_i]_{q_\beta} \models \Psi_2(x)$.

The second base case is $\beta = 3$. The existential witness y in the formula $\Phi_3(x)$ satisfies $p_1^\infty \,|\, y$ (from $\Psi_2(y)$). As $p_1 \notin \{p_\beta, q_\beta\}$, we have $y \in \bigoplus_{i\in\omega}[C_i]_{q_\beta}$. Since $\bigoplus_{i\in\omega}[C_i]_{q_\beta}$ is a pure subgroup containing x and y, we have $\bigoplus_{i\in\omega}[C_i]_{q_\beta} \models \Phi_3(x)$.

For the inductive cases, suppose γ is even and $4 \leq \gamma \leq \beta - 2$. Consider the m-th conjunct of $\Psi_\gamma(x)$. The witnesses $x_0, \ldots, x_m$ satisfy $u_{\gamma,k}^\infty \,|\, x_k$ and thus are in $\bigoplus_{i\in\omega}[C_i]_{q_\beta}$ as $u_{\gamma,k} \notin \{p_\beta, q_\beta\}$. Since $\bigoplus_{i\in\omega}[C_i]_{q_\beta}$ is a pure subgroup and it satisfies $\Phi^{u_{\gamma,m}}_{f_\gamma(m)}(x_m)$ by the inductive hypothsis, we have that $\bigoplus_{i\in\omega}[C_i]_{q_\beta} \models \Psi_\gamma(x)$.

If γ is odd and $4 < \gamma \leq \beta - 2$, then the existential witness y in $\Phi_\gamma(x)$ satisfies $p_{\gamma-1}^\infty \,|\, y$ from $A_{\gamma-1}(y)$. Thus, as $p_{\gamma-1} \notin \{p_\beta, q_\beta\}$, we have $y \in \bigoplus_{i\in\omega}[C_i]_{q_\beta}$. Since $\bigoplus_{i\in\omega}[C_i]_{q_\beta}$ is a pure subgroup and satisfies $A_{\gamma-1}(y)$ and $\Psi_{\gamma-1}(y)$ by the inductive hypothesis, we have $\bigoplus_{i\in\omega}[C_i]_{q_\beta} \models \Phi_\gamma(x)$. □

Definition 5.3.5. *If $\beta \geq 4$ is not a limit ordinal, define $B_\beta(x)$ to be the formula*

$$B_\beta(x) := p_\beta^\infty \,|\, x \;\wedge\; (\exists w)\left[p_{\beta-1}^\infty \,|\, w \wedge q_\beta^\infty \,|\, (x + w)\right].$$

Definition 5.3.6. *If $\beta = \delta + 2\ell + 2 \geq 4$, define Θ_β to be the formula*

$$\Theta_\beta(x) := (\forall y)\left[\left(B_\beta(x) \wedge B_{\beta-1}(y) \wedge q_\beta^\infty \,|\, (x + y)\right) \rightarrow \Phi_{\beta-1}(y)\right].$$

Lemma 5.3.13. The complexity of $B_\beta(x)$ is Σ^c_3 (independent of β).

If $\beta = \delta + 2\ell + 2 \geq 4$, then $\Theta_\beta(x) \in \Pi^c_\beta$.

Proof. These statements follow immediately from $p^\infty \,|\, x$ being Π^c_2 and Lemma 5.3.6. □

Lemma 5.3.14. Let $\beta = \delta + 2\ell + 2 \geq 4$. Let $x \in \mathcal{H}(\Pi^0_\beta)$ satisfy $B_\beta(x)$ with fixed witness w. Then $x = ar$ where $a \in Z$ and r is the root of $\mathcal{H}(\Pi^0_\beta)$ and $w = \sum b_j w_j$ where $b_j \in [Z]_{p_{\beta-1}, q_\beta}$ and each w_j is the root of a $\mathcal{G}(\Sigma^0_{\beta-1})$ component of $\mathcal{H}(\Pi^0_\beta)$.

Proof. Since $p_\beta^\infty \,|\, x$, the element x must have the form $x = ar$ where $a \in Q$ and r is the root of $\mathcal{H}(\Pi^0_\beta)$. Since $p^\infty_{\beta-1} \,|\, w$, the element w must have the form $w = \sum b_j w_j$ where $b_j \in Q$ and w_j is the root of a $\mathcal{G}(\Sigma^0_{\beta-1})$ component of $\mathcal{H}(\Pi^0_\beta)$.

Let $\mathcal{B} := \mathrm{Span}_{\mathcal{H}(\Pi^0_\beta)}(X)$ where X contains the root of $\mathcal{H}(\Pi^0_\beta)$ and the roots of the $\mathcal{G}(\Sigma^0_{\beta-1})$ components of $\mathcal{H}(\Pi^0_\beta)$. Then $x, w \in \mathcal{B}$, $\mathcal{B}$ is a pure subgroup of $\mathcal{H}(\Pi^0_\beta)$, and $\mathcal{B}$ is isomorphic to

$$\left\langle [Z]_{p_\beta} \oplus \bigoplus_{k\in\omega} [Z]_{p_{\beta-1}}; q_\beta^{-t}(r + r_k) : k, t \in \omega \right\rangle.$$

Since $\mathcal{B}$ satisfies $p^\infty_\beta \,|\, x$, $p^\infty_{\beta-1} \,|\, w$, and $q^\infty_\beta \,|\, (x + w)$, we can apply Lemma 5.3.3(5) (with $P = \emptyset$) to conclude that $a \in Z$ and each $b_j \in [Z]_{p_{\beta-1}, q_\beta}$. □

Lemma 5.3.15. Let $\beta = \delta + 2\ell + 2 \geq 4$. If $x, y \in \mathcal{H}(\Pi^0_\beta)$ satisfy

$$B_\beta(x) \wedge B_{\beta-1}(y) \wedge q^\infty_\beta \,|\, (x + y),$$

then $x = ar$ and $y = \sum b_j y_j$ where $a, b_j \in Z$, r is the root of $\mathcal{H}(\Pi^0_\beta)$, and y_j is the root of a $\mathcal{G}(\Sigma^0_{\beta-1})$ component of $\mathcal{H}(\Pi^0_\beta)$.

Proof. By Lemma 5.3.14, the fact that $B_\beta(x)$ holds implies $x = ar$ with $a \in Z$ and r the root of $\mathcal{H}(\Pi^0_\beta)$. Since $B_{\beta-1}(y)$ implies $p^\infty_{\beta-1} \,|\, y$ and since $q^\infty_\beta \,|\, (x + y)$, the element y works as a witness w in the formula $B_\beta(x)$ for our fixed element x. Therefore, by the previous lemma $y = \sum b_j y_j$ where $b_j \in [Z]_{p_{\beta-1}, q_\beta}$ and y_j is the root of a $\mathcal{G}(\Sigma^0_{\beta-1})$ component. It remains to show the stronger conclusion that $b_j \in Z$.

Fix a witness w for $B_{\beta-1}(y)$. Since $p^\infty_{\beta-2} \,|\, w$, the element w must have the form $w = \sum c_i w_i$ where $c_i \in Q$ and w_i is the root of a $\mathcal{G}(\Sigma^0_{\beta-2}(m))$ component inside a $\mathcal{G}(\Sigma^0_{\beta-1})$ component of $\mathcal{H}(\Pi^0_\beta)$. Therefore $y, w \in \mathcal{B}$ where $\mathcal{B} := \mathrm{Span}_{\mathcal{H}(\Pi^0_\beta)}(X)$ where X contains the roots of the $\mathcal{G}(\Sigma^0_{\beta-1})$ components and the roots of their $\mathcal{G}(\Sigma^0_{\beta-2}(m))$ subcomponents.

To determine the isomorphism type of $\mathcal{B}$, we consider which primes infinitely divide the roots of such components. The root of a $\mathcal{G}(\Sigma^0_{\beta-1})$ component is infinitely divisible by $p_{\beta-1}$. The roots of $\mathcal{G}(\Sigma^0_{\beta-2}(m))$ components are infinitely divisible by $p_{\beta-2}$ and $u_{\beta-2,0}$ from the definition of $\mathcal{G}(\Sigma^0_{\beta-2}(m))$. Each of these roots is also the root of a $\mathcal{G}(\Sigma^0_{f_{\beta-2}(0)})$ component (inside $\mathcal{G}(\Sigma^0_{\beta-2}(m))$) and hence is also infinitely divisible by $p_{f_{\beta-2}(0)}$. However, the recursion stops at this point since the root of $\mathcal{G}(\Sigma^0_{f_{\beta-2}(0)})$ is the element 1 in a copy of $[Z]_{p_{f_{\beta-2}(0)}}$. Therefore, the group $\mathcal{B}$ is isomorphic to

$$\left[\left\langle \bigoplus_{i\in\omega}[Z]_{F_1} \oplus \bigoplus_{i,j\in\omega}[Z]_{F_2}; \frac{s_i + t_{i,j}}{q^l_{\beta-2}} : i, j, l \in \omega\right\rangle\right]$$

where $F_1 = \{p_{\beta-1}\}$, $F_2 = \{p_{\beta-2}, u_{\beta-2,0}, p_{f_{\beta-2}(0)}\}$, the s_i elements generate the copies of $[Z]_{F_1}$ (representing the roots of the $\mathcal{G}(\Sigma^0_{\beta-1})$ components) and the $t_{i,j}$ elements generate the copies of $[Z]_{F_2}$ (representing the roots of the $\mathcal{G}(\Sigma^0_{\beta-2}(m))$ subcomponents of the $\mathcal{G}(\Sigma^0_{\beta-1})$ component with root s_i). Since $y, w \in \mathcal{B}$ and $\mathcal{B}$ is a pure subgroup of $\mathcal{H}(\Pi^0_\beta)$ satisfying $p^\infty_{\beta-1} \mid y$, $p^\infty_{\beta-2} \mid w$, and $q^\infty_{\beta-1} \mid (y + w)$, we can conclude from Lemma 5.3.3(5) (with $P = \emptyset$) that the coefficients in the sum $y = \sum b_j y_j$ come from Z. □

Lemma 5.3.16. Let $\beta = \delta + 2\ell + 2 \geq 4$. Then for $\mathcal{H} = \bigoplus_{n\in\omega} \mathcal{H}_n$, where $\mathcal{H}_n$ is either $[\mathcal{H}(\Sigma^0_\beta)]_{d_n}$ or $[\mathcal{H}(\Pi^0_\beta)]_{d_n}$, the following holds:

$$\mathcal{H} \models \Big[(\forall x)\Theta_\beta(x)\Big]^{d_n} \qquad \text{if and only if} \qquad \mathcal{H}_n \cong [\mathcal{H}(\Pi^0_\beta)]_{d_n}.$$

Proof. Since $\mathcal{H}_n$ is the subgroup of elements of $\mathcal{H}$ which are infinitely divisible by d_n, we have

$$\mathcal{H} \models [(\forall x)\Theta_\beta(x)]^{d_n} \;\Leftrightarrow\; \mathcal{H}_n \models (\forall x)\Theta_\beta(x)$$

Therefore, it suffices to show that $\mathcal{H}(\Pi^0_\beta) \models (\forall x)\Theta_\beta(x)$ and $\mathcal{H}(\Sigma^0_\beta) \not\models (\forall x)\Theta_\beta(x)$.

First, we show that $\mathcal{H}(\Pi^0_\beta) \models (\forall x)\Theta_\beta(x)$. Fix elements $x, y \in \mathcal{H}(\Pi^0_\beta)$ satisfying $B_\beta(x) \wedge B_{\beta-1}(y) \wedge q^\infty_\beta \mid (x + y)$. By Lemma 5.3.15, we can write $y = \sum b_j y_j$ where each $b_j \in Z$ and y_j is the root of a $\mathcal{G}(\Sigma^0_{\beta-1})$ component. By Lemma 5.3.11, the element y satisfies $\Phi_{\beta-1}(y)$ as required.

Second, we show that $\mathcal{H}(\Sigma^0_\beta) \not\models (\forall x)\Theta_\beta(x)$ by proving that $\mathcal{H}(\Sigma^0_\beta) \not\models \Theta_\beta(r)$ where r is the root of $\mathcal{H}(\Sigma^0_\beta)$. Let y be the root of a $\mathcal{G}(\Pi^0_{\beta-1})$ component of $\mathcal{H}(\Sigma^0_\beta)$. It is immediate that $\mathcal{H}(\Sigma^0_\beta) \models B_\beta(r) \wedge B_{\beta-1}(y) \wedge q^\infty_\beta \mid (r + y)$. However, by Lemma 5.3.12, the group $\mathcal{H}(\Sigma^0_\beta)$ does not satisfy $\Phi_{\beta-1}(y)$. □

Finally, we are in a position to define the sentences $\{\Upsilon_n\}_{n\in\omega}$ required for Lemma 5.3.1 and to demonstrate their correctness.

Definition 5.3.7. *Define sentences Υ_n for $n \in \omega$ as follows.*

- *If $\alpha = \delta + 2\ell + 1 \geq 3$, let $\Upsilon_n := [(\exists x)\Phi_\alpha(x)]^{d_n}$.*

- *If* $\alpha = \delta + 2\ell + 2 \geq 4$, *let* $\Upsilon_n := \neg\,[(\forall x)\Theta_\alpha(x)]^{d_n}$.

Proof of Lemma 5.3.1. By Lemma 5.3.10 and Lemma 5.3.16, the sentences Υ_n have the desired semantic properties. As a consequence of Lemma 5.3.6 and Lemma 5.3.13, the formulas Υ_n have the desired quantifier complexity. Moreover, all the (sub)formulas are computable with all possible uniformity, so Υ_n is uniformly computably Σ^c_α. □

5.3.2 Proof of Lemma 5.3.2

The construction of an X-computable copy of $\mathcal{G}^\alpha_S$ if $S \in \Sigma^0_\alpha(X)$ is also done by recursion. We treat only the case when $X = \emptyset$, the more general case following by relativization.

Lemma 5.3.17. For every even ordinal $\beta = \delta > 0$ or $\beta = \delta + 2\ell + 2 \geq 2$ and Σ^0_β set S, there is a uniformly computable sequence $\{\mathcal{G}_n\}_{n\in\omega}$ of rooted torsion-free abelian groups such that $\mathcal{G}_n \cong \mathcal{G}(\Sigma^0_\beta(m))$ for some $m \in \omega$ if $n \in S$ and $\mathcal{G}_n \cong \mathcal{G}(\Pi^0_\beta)$ if $n \notin S$.

For every odd ordinal $\beta = \delta + 2\ell + 1 \geq 3$ and Σ^0_β set S, there is a uniformly computable sequence $\{\mathcal{G}_n\}_{n\in\omega}$ of rooted torsion-free abelian groups such that $\mathcal{G}_n \cong \mathcal{G}(\Sigma^0_\beta)$ if $n \in S$ and $\mathcal{G}_n \cong \mathcal{G}(\Pi^0_\beta)$ if $n \notin S$.

Moreover the passage from an index for the set S to an index for the sequence is effective.

Proof. The proof is done by induction on β. We treat the cases $\beta = 2$, $\beta = \delta+2\ell+2 \geq 4$, $\beta = \delta+2\ell+1 \geq 3$, and $\beta = \delta > 0$ separately. In all cases, we fix a predicate $(\exists s)\,[R(n,s)]$ describing membership of n in S, where $R(n,s)$ is $\Pi^0_{f_\beta(k)}$ for some k. Without loss of generality, we suppose $R(n,s_0)$ implies $(\forall s \geq s_0)\,[R(n,s)]$. Indeed, we suppose this property of all existential subpredicates.

For $\beta = 2$, it suffices to start with the group Z with root $r_n = 1$ for $\mathcal{G}_n$. When we see $\neg R(n,s)$ for a new existential witness s, we introduce the element $1/p^s$ into the group. It is easy to see the sequence $\{\mathcal{G}_n\}_{n\in\omega}$ has the desired properties.

For $\beta = \delta + 2\ell + 1 \geq 3$, it suffices to start with the group $[Z]_{p_\beta}$ with root $r_n = 1$ for $\mathcal{G}_n$. For each integer s, we construct (via induction as $\neg R(n,s)$ is $\Sigma^0_{\delta+2\ell}$) a rooted torsion-free abelian group $\mathcal{G}_{n,s}$ with root $r_{n,s}$ and introduce elements $(r_n + r_{n,s})/q^t_\beta$ for all $t \in \omega$. For each integer m, we construct infinitely many copies of $\mathcal{G}(\Sigma^0_{\beta-1}(m))$ with root $r_{n,k,m}$ (where k is the copy number) and introduce elements $(r_n + r_{n,k,m})/q^t_\beta$ for all $t \in \omega$. Again, it is easy to see the sequence $\{\mathcal{G}_n\}_{n\in\omega}$ has the desired properties.

For $\beta = \delta + 2\ell + 2 \geq 4$, we construct (via induction as $\neg R(n,s)$ is $\Sigma^0_{\delta+2\ell+1}$[8]) rooted torsion-free abelian groups $\mathcal{G}_{n,s}$ with root $r_{n,s}$. Within $\mathcal{G}_{n,0}$, we introduce elements $r_{n,0}/p^t_\beta$ for all $t \in \omega$. Within each group $\mathcal{G}_{n,s}$, we introduce elements $x/u^t_{\beta,s}$ for all $t \in \omega$ and $x \in \mathcal{G}_{n,s}$. For each integer s, we introduce elements $(r_{n,s} + r_{n,s+1})/v^t_{\beta,s}$ for all $t \in \omega$. Again, it is easy to see the sequence $\{\mathcal{G}_n\}_{n\in\omega}$ has the desired properties.

For $\beta = \delta$, we construct (via induction) rooted torsion-free abelian groups $\mathcal{G}_{n,s}$ with root $r_{n,s}$, where $\mathcal{G}_{n,0} \cong \mathcal{G}(\Sigma^0_{f_\beta(0)})$ and where, for $s > 0$, $\mathcal{G}_{n,s} \cong \mathcal{G}(\Pi^0_{f_\beta(s)})$ if $\emptyset^{f_\beta(s)}$ suffices to witness $n \in S$ and $\mathcal{G}_{n,s} \cong \mathcal{G}(\Sigma^0_{f_\beta(s)})$ otherwise. Within $\mathcal{G}_{n,0}$, we introduce elements $r_{n,0}/p^t_\beta$ for all $t \in \omega$. Within each group $\mathcal{G}_{n,s}$, we introduce elements $x/u^t_{\beta,s}$ for all $t \in \omega$ and $x \in \mathcal{G}_{n,s}$. For each integer s, we introduce elements $(r_{n,s} + r_{n,s+1})/v^t_{\beta,s}$ for all $t \in \omega$. Again, it is easy to see the sequence $\{\mathcal{G}_n\}_{n\in\omega}$ has the desired properties. $\square$

Lemma 5.3.18. For every even ordinal $\beta = \delta + 2\ell + 2 \geq 4$ and Σ^0_β set S, there is a uniformly computable sequence of rooted torsion-free abelian groups $\{\mathcal{H}_n\}_{n\in\omega}$ such that $\mathcal{H}_n \cong \mathcal{H}(\Sigma^0_\beta)$ if $n \in S$ and $\mathcal{H}_n \cong \mathcal{H}(\Pi^0_\beta)$ if $n \notin S$.

Proof. We fix a predicate $(\exists s)\,[R(n,s)]$ describing membership of n in S, where $R(n,s)$ is $\Pi^0_{\beta-1}$. Without loss of generality, we again suppose $R(n,s_0)$ implies $(\forall s \geq s_0)\,[R(n,s)]$. Indeed, we suppose this property of all existential subpredicates.

It suffices to start with the group $[Z]_{p_\beta}$ with root $r_n = 1$ for $\mathcal{H}_n$. For each integer s, we (via Lemma 5.3.17) construct a rooted torsion-free abelian group $\mathcal{G}_{n,s}$ with root $r_{n,s}$ and introduce elements $(r_n + r_{n,s})/q^t_\beta$ for all $t \in \omega$. We also construct infinitely many copies of $\mathcal{G}(\Sigma^0_{\beta-1})$ with root $r_{n,k}$ (where k is the copy number) and introduce elements $(r_n + r_{n,k})/q^t_\beta$ for all $t \in \omega$. Again, it is easy to see the sequence $\{\mathcal{G}_n\}_{n\in\omega}$ has the desired properties. $\square$

Proof of Lemma 5.3.2. Fix a Σ^0_α set S. From Lemma 5.3.17 (if α is odd) or Lemma 5.3.18 (if α is even), there is a uniformly computable sequence $\{\mathcal{G}_n\}_{n\in\omega}$ of groups given by the Σ^0_α predicate. Since it is possible to pass from the group $\mathcal{G}_n$ to $[\mathcal{G}_n]_{d_n}$ uniformly in an index for the group $\mathcal{G}_n$ and d_n, the group $\mathcal{G}^\alpha_S$ is computable. $\square$

[8]More precisely, we use the $\Sigma^0_{\delta+2\ell+1}$ predicate $\neg R(n,s)$ to control the construction of $\mathcal{G}_{n,s+1}$ and build $\mathcal{G}_{n,0} \cong \mathcal{G}(\Sigma^0_{\beta-1})$. This index shift is necessary as $\mathcal{G}(\Sigma^0_\beta(m))$ has $m+1$ (rather than m) subcomponents of type $\mathcal{G}(\Sigma^0_{\beta-1})$.

Part II

Computable metric spaces

Chapter 6

Computably isometric Banach spaces

This chapter studies computable isometries between metric spaces associated to Banach spaces. First, we give formal definitions and prove several not difficult but rather useful facts on computable Banach spaces. Next, we show that Hilbert space is computably categorical as a metric space. Then we prove that $C[0,1]$ is not computably categorical as a metric space by constructing a computable structure in which 0 is computable but the operation $x \to (1/2)x$ is not.

6.1 Background

6.1.1 Definitions and conventions

Recall that special points of a computable metric space are points from the dense computable sequence in M which we call a *computable structure* on M. We usually identify a special point α_i with its number i and say "find a special point such that ..." in place of "find a number i such that α_i ...". Recall also that only points having Cauchy names are regarded as computable. A Cauchy name of x is a computable sequence of special points converging to x "quickly" (with the rate of 2^{-s}). Also, recall:

Definition 6.1.1. *Let $\mathcal{M}$ and $\mathcal{N}$ be computable metric spaces. A map $F\colon \mathcal{M} \to \mathcal{N}$ is* computable *if there is a Turing functional Φ such that, for each x in the domain of F and for every Cauchy name χ for x, the functional Φ enumerates a Cauchy name for $F(x)$ using χ as an oracle*[1].

[1] That is, $(\Phi^{\chi}(n))_{n\in\omega}$ is a Cauchy name for $F(x)$.

To emphasize which computable structures we consider, we say that a map F is computable with respect to $(\alpha_i)_{i\in\omega}$ and $(\beta_i)_{i\in\omega}$ (written w.r.t. $(\alpha_i)_{i\in\omega}$ and $(\beta_i)_{i\in\omega}$). In the special case of isometric (more generally, bi-Lipschitz) maps, Definition 6.1.1 is equivalent to saying that for every special point α_i in $\mathcal{M}$ the point $F(\alpha_i)$ is computable uniformly in i.

Definition 6.1.2. *Computable structures $(\alpha_i)_{i\in\omega}$ and $(\beta_i)_{i\in\omega}$ on a complete separable metric space (M, d) are* equivalent up to computable isometry*, or* computably isometric*, if there exists a surjective self-isometry ϕ of M and an effectively uniform algorithm which on input i outputs a Cauchy name for $\phi(\alpha_i)$ in $(\beta_i)_{i\in\omega}$.*

Definition 6.1.2 can be equivalently restated as follows:

Definition 6.1.3. *Computable structures $(\alpha_i)_{i\in\omega}$ and $(\beta_i)_{i\in\omega}$ on a Polish space (M, d) are said to be* equivalent up to a computable isometry *or* (computably) isometric*, if there exists a surjective self-isometry U computable w.r.t. $(\alpha_i)_{i\in\omega}$ and $(\beta_i)_{i\in\omega}$.*

Note that if U is a computable surjective isometry, then U^{-1} is computable as well. Therefore, equivalence up to a computable isometry is an equivalence relation on computable metric spaces. Pour-El and Richards [85] used a similar notion restricted to Banach spaces in a different terminology. Their approach is equivalent to the one discussed in the next section.

Definition 6.1.4. *A metric space (M, d) is* computably categorical *if every two computable structures on M are computably isometric.*

Computable structures $(\alpha_i)_{i\in\omega}$ and $(\beta_i)_{i\in\omega}$ on a Polish space (M, d) are equivalent exactly if the identity map

$$\mathrm{Id} : (\mathrm{M}, \mathrm{d}, (\alpha_\mathrm{i})_{\mathrm{i}\in\omega}) \to (\mathrm{M}, \mathrm{d}, (\beta_\mathrm{i})_{\mathrm{i}\in\omega})$$

is computable.

6.1.2 Computable spaces with operations

We view a computable Banach space as a computable metric space with distinguished computable operations.

An operation is a function which maps tuples of points to points (such as the addition in a Banach space), or tuples of points to reals (such as the inner product in a Hilbert space). Also, we view a distinguished point x as function $T_x : M \to \{x\}$

such that $T_x(y) = x$, for every y. Thus, distinguished points are operations of a special kind.

Before we define a computable operation, we need one more definition. In the following, we view a direct power M^k of (M, d) as a metric space with the metric $d_k = \sup_{i \leq k} d(\pi_i x, \pi_i y)$, where π_i is the projection on the i-th component. Let $(\alpha_i)_{i\in\omega}$ be a computable structure on (M, d). The computable structure $[(\alpha_i)_{i\in\omega}]^k$ on (M^k, d_k) is the effective listing of k-tuples of special points from $(\alpha_i)_{i\in\omega}$.

For convenience, if an operation $X : M^k \to M$ is computable w.r.t. $[(\alpha_i)_{i\in\omega}]^k$ and $(\alpha_i)_{i\in\omega}$, we simply say that X is computable w.r.t. $(\alpha_i)_{i\in\omega}$. Similarly, instead of saying that an operation $X : M^k \to \mathbb{R}$ is computable w.r.t. $[(\alpha_i)_{i\in\omega}]^k$ and $(q_i)_{i\in\omega}$, where $(q_i)_{i\in\omega}$ is the usual effective listing of rationals, we say that X is computable w.r.t. $(\alpha_i)_{i\in\omega}$.

Recall that every Turing functional Φ_e can be effectively identified with its computable index e. For instance, we may speak of the index for the distance function d (which depends on the given computable structure). We may also speak of uniformly computable families of maps between computable metric spaces meaning that we can get an index for the functional effectively from the place of the operation on the list.

Definition 6.1.5. *Let $(M, d, (X_j)_{j\in J})$ be a metric space with distinguished operations $(X_j)_{j\in J}$, where J is a computable set. We say that $(\alpha_i)_{i\in\omega}$ is a* computable structure *on $(M, d, (X_j)_{j\in J})$ if $(M, d, (\alpha_i)_{i\in\omega})$ is a computable metric space and the operations $(X_j)_{j\in J}$ are computable w.r.t. $(\alpha_i)_{i\in\omega}$ uniformly in their indices.*

We say that an isometry U *respects* an operation if it commutes with it: $X \circ U = U \circ X$.

Definition 6.1.6. *A space $(M, d, (X_j)_{j\in J})$ is* computably categorical *if every two computable structures $(\alpha_i)_{i\in\omega}$ and $(\beta_i)_{i\in\omega}$ on $(M, d, (X_j)_{j\in J})$ are computably isometric via an isometry which respects X_j for every $j \in J$.*

Definition 6.1.7. *We say that operations $(Y_i)_{i\in I}$* effectively determine *operations $(X)_{j\in J}$ on a metric space (M, d) if, for any given computable structure $(\alpha_i)_{i\in\omega}$ on (M, d), the uniform computability of $(Y_i)_{i\in I}$ w.r.t. $(\alpha_i)_{i\in\omega}$ implies the uniform computability of $(X)_{j\in J}$ w.r.t. $(\alpha_i)_{i\in\omega}$.*

Notice that in the definition above we implicitly have that every isometry of M which respects $(Y_i)_{i\in I}$ respects $(X_j)_{j\in J}$ as well. The following consequence of Definition 6.1.6 and Definition 6.1.7 is a useful tool.

Fact 6.1.1. Suppose $(M, d, (Y_i)_{i\in I}, (X_j)_{j\in J})$ is computably categorical, where the operations $(Y_i)_{i\in I}$ effectively determine the operations $(X_j)_{j\in J}$. Then $(M, d, (Y_i)_{i\in I})$ is computably categorical.

6.2 Banach spaces

6.2.1 Computable Banach spaces

We view a Banach space as a metric space with distinguished points, maps and operators. Formally, a Banach space $\mathbb{B}$ is a tuple $(B, d, 0, +, (r\cdot)_{r\in Q})$, where d is the metric induced by the norm, 0 is the distinguished point for zero, + is the vector summation, and $r\cdot$ is the operator of scalar multiplication by r, for $r \in Q$ (rational numbers). We shall avoid this complex formal notation if possible. As a special case of Definition 6.1.5, we have:

Definition 6.2.1. *A collection of points $(\alpha_i)_{i\in\omega}$ is a computable structure on a Banach space $\mathbb{B}$ if $(B, d, (\alpha_i)_{i\in\omega})$ is a computable metric space and 0, $+$, and $(r\cdot)_{r\in Q}$ are uniformly computable operations w.r.t. to $(\alpha_i)_{i\in\omega}$.*

It is not hard to see that our approach is equivalent to the approach of Brattka, Hertling, and Weihrauch [12, page 466]. It is also equivalent to the existence of an *effectively separable* structure in the sense of Pour-El and Richards [85].

As a special case of Definition 6.1.6, we have:

Definition 6.2.2. *A Banach space $\mathbb{B}$ is computably categorical if every two computable structures on (B, d), w.r.t. which the operations 0, $+$, and $(r\cdot)_{r\in Q}$ are uniformly computable, are computably isometric via an isometry which respects 0, $+$, and $(r\cdot)_{r\in Q}$. We also say that $\mathbb{B}$ is computably categorical as a Banach space.*

It is not difficult to see that $\mathbb{B}$ is computably categorical as a Banach space if, and only if, every two effectively separable structures on $\mathbb{B}$ are isometric, as defined in Pour-El and Richards [85, Question on page 146].

Remark 6.2.1. *Note that, for a computable structure on a Banach space, the uniform computability of $(r\cdot)_{r\in Q}$ implies the computability of 0. By Fact 6.1.1 we may eliminate 0 from the list of computable operations and obtain equivalent notions of computable Banach space and computably categorical Banach space. However, we may keep 0 for convenience.*

If, for a Banach space $\mathbb{B}$, the associated metric space (B, d) is computably categorical, then we say that $\mathbb{B}$ is *computably categorical as a metric space*.

6.2.2 Applications of the Mazur-Ulam theorem

The classical theorem of Mazur and Ulam states that every surjective isometry of Banach spaces is affine. In other words, if $U : \mathbb{B}_1 \to \mathbb{B}_2$ is a surjective isometry of Banach spaces $\mathbb{B}_1$ and $\mathbb{B}_2$, then there exists a linear map $L : \mathbb{B}_1 \to \mathbb{B}_2$ such that $U(x) = L(x) + U(0)$, for every $x \in \mathbb{B}_1$. We show:

Fact 6.2.1. If a Banach space $\mathbb{B}$ is computably categorical as a metric space, then it is computably categorical as a Banach space.

Proof. Let $(\alpha_i)_{i\in\omega}$ and $(\beta_i)_{i\in\omega}$ be computable structures on $\mathbb{B}$ w.r.t. which 0, $+$, and $(r\cdot)_{r\in Q}$ are computable. By the assumption, there is a surjective isometry U computable w.r.t. $(\alpha_i)_{i\in\omega}$ and $(\beta_i)_{i\in\omega}$. We need to find a computable surjective isometry which respects 0, $+$, and $(r\cdot)_{r\in Q}$.

By our assumption, the point $U(0)$ is computable w.r.t. $(\beta_i)_{i\in\omega}$. We have $x - y = x + (-1) \cdot y$, showing that the subtraction operation is computable w.r.t. $(\beta_i)_{i\in\omega}$. Thus, the isometry $W(x) = U(x) - U(0)$ is computable w.r.t. $(\alpha_i)_{i\in\omega}$ and $(\beta_i)_{i\in\omega}$. By the Mazur-Ulam theorem, W respects 0, $+$, and $(r\cdot)_{r\in Q}$. ◻

Pour-El and Richards [85, page 146] showed that the space l_1 with the usual norm is not computably categorical as a Banach space. As a consequence of their result and Fact 6.2.1, we have:

Corollary 6.2.1. *The space l_1 is not computably categorical as a metric space.*

In Theorem 6.4.1 we will construct a computable structure on the metric space $(C[0,1], \sup)$ such that 0 is computable w.r.t. this structure, but the operation $(1/2)\cdot$ is not. By Fact 6.2.2 below, this will imply that $(C[0,1], \sup)$ is not computably categorical.

Fact 6.2.2. Let $\mathbb{B}$ be a Banach space. Suppose $(\alpha_i)_{i\in\omega}$ is a computable structure on (B, d) w.r.t. which $+$ and $(r\cdot)_{r\in Q}$ are uniformly computable, and suppose $(\beta_i)_{i\in\omega}$ is another computable structure on (B, d) w.r.t. which 0 is computable. If $(\beta_i)_{i\in\omega}$ is computably isometric to $(\alpha_i)_{i\in\omega}$, then $+$ and $(r\cdot)_{r\in Q}$ are uniformly computable w.r.t. $(\beta_i)_{i\in\omega}$.

Proof. Let U be a surjective isometry computable w.r.t. $(\beta_i)_{i\in\omega}$ and $(\alpha_i)_{i\in\omega}$. Recall that U^{-1} is computable w.r.t. $(\alpha_i)_{i\in\omega}$ and $(\beta_i)_{i\in\omega}$. By the theorem of Mazur and Ulam, there exists a linear map $L : X \to Y$ such that $U(x) = L(x) + U(0)$, for every $x \in B$.

Given $r \in Q$, we show that the operator $r\cdot$ is computable w.r.t. $(\beta_i)_{i\in\omega}$ uniformly in r, as follows. The operations $+$, $r\cdot$ and $v - w = v + (-1 \cdot w)$ are uniformly computable w.r.t. $(\alpha_i)_{i\in\omega}$. Therefore, the map $x \to U^{-1}(r(U(x) - U(0)) + U(0))$ is computable w.r.t. $(\beta_i)_{i\in\omega}$. On the other hand,

$$U^{-1}(r(U(x) - U(0)) + U(0)) = U^{-1}(rL(x) + U(0)) = U^{-1}(L(rx) + U(0)) = rx,$$

showing that $r \cdot x$ is a computable w.r.t. $(\beta_i)_{i\in\omega}$ uniformly in r.

The computability of + can be established similarly:

$$U^{-1}(U(\beta) + U(\gamma) - U(0)) = U^{-1}(L(\beta + \gamma) + U(0)) = \beta + \gamma.$$

□

6.3 Hilbert spaces

6.3.1 Operations on a Hilbert space

We view a Hilbert space as a Banach space of a special kind. For instance, for $\mathbb{H}$ a Hilbert space, the associated metric space (H, d) is defined by $d(x, y) = ||x - y||$. Recall Definition 6.1.7. We show:

Lemma 6.3.1. In the metric space (H, d) associated with a Hilbert space $\mathbb{H}$, the point 0 effectively determines the operations + and $(r\cdot)_{r\in Q}$.

Proof. Suppose $(\alpha_i)_{i\in\omega}$ is a computable structure on (H, d) w.r.t. which 0 is computable. Recall that $d(x, y) = ||x - y||$. For instance, $||x|| = d(0, x)$ is computable for every computable point x. It is well-known that the parallelogram identity

$$||x + y||^2 + ||x - y||^2 = 2||x||^2 + 2||y||^2$$

characterizes Hilbert spaces within the class of Banach spaces. We show:

Claim 6.3.1. *The opearation + is computable w.r.t.* $(\alpha_i)_{i\in\omega}$.

Proof. Given a positive rational $\epsilon < 1$ and Cauchy names for points x and y, find a special point z such that:

1. $|\,||z||^2 + ||x - y||^2 - 2||x||^2 - 2||y||^2\,| < \delta$,
2. $|\,||y - z|| - ||x||\,| < \delta$,
3. $|\,||x - z|| - ||y||\,| < \delta$,

where $\delta = \epsilon/(2||x||+2||y||+3)$. We may assume that $\delta < 1$. Applying the parallelogram identity, we obtain

$$|\,||z||^2 - ||x + y||^2\,| < \delta.$$

Using the well-known formula for inner product, we get

$$\|x + y - z\|^2 = \|x + y\|^2 + \|z\|^2 - 2\langle x, z\rangle - 2\langle y, z\rangle.$$

Applying this formula again, we obtain

$$\|y - z\|^2 = \|y\|^2 + \|z\|^2 - 2\langle y, z\rangle$$

and

$$\|x - z\|^2 = \|x\|^2 + \|z\|^2 - 2\langle x, z\rangle.$$

We combine the three equations above:

$$\|x + y - z\|^2 = (\|x + y\|^2 - \|z\|^2) + (\|x - z\|^2 - \|y\|^2) + (\|y - z\|^2 - \|x\|^2).$$

Taking into account $\delta < 1$, observe that

$$\begin{aligned} |\, \|x - z\|^2 - \|y\|^2 \,| &= |\, \|x - z\| - \|y\| \,| \cdot (\|x - z\| + \|y\|) \\ &< \delta(\|y\| + \delta + \|y\|) \\ &< \delta(2\|y\| + 1), \end{aligned}$$

and similarly $|\, \|y - z\|^2 - \|x\|^2 \,| < \delta(2\|x\| + 1)$.

Thus,

$$\begin{aligned} \|x + y - z\|^2 &\leq |\, \|x + y\|^2 - \|z\|^2 \,| + |\, \|x - z\|^2 - \|y\|^2 \,| + |\, \|y - z\|^2 - \|x\|^2 \,| \\ &< \delta + \delta(2\|y\| + 1) + \delta(2\|x\| + 1) \\ &= \delta(2\|x\| + 2\|y\| + 3) \\ &= \epsilon. \end{aligned}$$

Thus, we can produce a Cauchy name for $x + y$ uniformly in Cauchy names for x and y, proving the claim.

□

Claim 6.3.2. *The operations $(r\cdot)_{r\in Q}$ are uniformly computable w.r.t. $(\alpha_i)_{i\in\omega}$.*

Proof. By Claim 6.3.1, for every $n \in \omega$ the operation $n\cdot$ is computable uniformly in n. The point 0 is a computable by our assumption, therefore the operation $0\cdot = 0$ is computable. We show that $(-1)\cdot$ is computable. Given a rational $\eta > 0$ and a Cauchy name for x, find a special point α such that $d(0, x + \alpha) < \eta$. Note that

$\|(-x) - \alpha\| = \|x + \alpha\| = d(0, x + \alpha) < \eta$. Therefore, we can produce a Cauchy name for $-x$, showing $(-1)\cdot$ is a computable operation.

For every $q = \dfrac{n}{m} \in Q$, we have

$$q \cdot x = n \cdot (1/m)x.$$

Note also that the operation $m\cdot$ is bi-Lipschitz with constant m. Therefore, its inverse $(1/m)\cdot$ is computable, uniformly in m. We conclude that $q\cdot$ is computable, uniformly in r.

□

The theorem follows immediately from Claim 6.3.1 and Claim 6.3.2.

□

Remark 6.3.1. The proof of Lemma 6.3.1 actually shows that the indices for $+$ and $(r\cdot)_{r\in Q}$ can be obtained effectively from the computable structure and the computable indices for d and 0.

Remark 6.3.2. The inner product $\langle \cdot, \cdot \rangle$ is effectively determined by 0 in a Hilbert space. We have $\|x\| = d(0, x)$, $v - w = v + (-1 \cdot w)$, and

$$\langle u, v \rangle = \frac{1}{4}(\|u + v\|^2 - \|u - v\|^2).$$

6.3.2 Hilbert spaces are computably categorical

Using a different terminology, Pour-El and Richards [85] showed:

Theorem 6.3.1. *Every separable Hilbert space is computably categorical as a Banach space.*

Proof. The main idea is to use the Gram–Schmidt process. See the discussion on page 146 of Pour-El and Richards [85]. □

The following consequence of Lemma 6.3.1 strengthens their result:

Theorem 6.3.2. *Every separable Hilbert space is computably categorical as a metric space.*

Proof idea. Note that, if 0 was computable w.r.t. every computable structure of a metric space (H, d) associated to a Hilbert space $\mathbb{H}$, then Fact 6.1.1 (with 0 in place of $(Y_i)_{i\in\omega}$), Lemma 6.3.1 and Theorem 6.3.1 would imply (H, d) is computably categorical. Unfortunately, 0 does not have to be computable w.r.t. every computable structure on (H, d). On the other hand, we are given a computable structure on (H, d), not $\mathbb{H}$ itself, and zero and the operations are not specified in (H, d). We may pick any special point and declare it to be "zero". We have to define new vector space operations (to make this element a true zero), and then apply Fact 6.1.1, Lemma 6.3.1 and Theorem 6.3.1.

Proof. Suppose $\mathbb{H}$ is a Hilbert space, and suppose $(\alpha)_{i\in\omega}$ and $(\beta)_{i\in\omega}$ are computable structures on the associated metric space (H, d).

Declare $z_1 = \alpha_0$. Consider the isometry $V(x) = x + z_1$. Clearly, $V(0) = z_1$. Let

$$||x||_1 = d(z_1, x).$$

The operation $||x||_1$ satisfies the norm axioms with respect to the new vector space operations

$$x +_1 y = x + y - z_1 \text{ and } r \cdot_1 x = r(x - z_1) + z_1,$$

where z_1 plays the role of zero. Furthermore, the norm $|| \cdot ||_1$ satisfies the parallelogram equality, H is complete with respect to $|| \cdot ||_1$, and

$$||x +_1 (-1) \cdot_1 y||_1 = d(x, y),$$

for every $x, y \in H$. Thus,

$$\mathbb{H}_1 = (H, d, z_1, +_1, (r\cdot_1)_{r\in Q})$$

is a Hilbert space.

Similarly, we define a Hilbert space

$$\mathbb{H}_2 = (H, d, z_2, +_2, (r\cdot_2)_{r\in Q}),$$

where $z_2 = \beta_0$. Note that $\mathbb{H}_2 \cong \mathbb{H}_1 \cong \mathbb{H}$.

By Lemma 6.3.1, $+_1$ and $(r\cdot_1)_{r\in Q}$ are uniformly computable w.r.t. $(\alpha)_{i\in\omega}$. Similarly, $+_2$ and $(r\cdot_2)_{r\in Q}$ are uniformly computable w.r.t. $(\beta)_{i\in\omega}$. Recall that $\mathbb{H}$ is computably categorical as a Banach space (Theorem 6.3.1). It remains to apply Fact 6.1.1. □

We emphasize that the proof of Theorem 6.3.2 works in the case of any finite dimension:

Corollary 6.3.1 ([51]). *For every $n \in \mathbb{N}^+$, the metric space $\mathbb{R}^n$ with the usual Euclidean metric is computably categorical.*

6.4 The space $C[0, 1]$

Let $(l_i)_{i\in\omega}$ be the effective list of all continuous piecewise linear functions (written p.l.) on $C[0, 1]$ which have (finitely many) breakpoints, each breakpoint having rational coordinates in $[0, 1] \times \mathbb{R}$. In the following, we call these functions rational p.l. functions.

Notation 6.4.1. *In this section d stands for the pointwise supremum metric on $C[0, 1]$:*

$$d(f, g) = \sup_{x\in[0,1]} \{|f(x) - g(x)|\}$$

The sequence $(l_i)_{i\in\omega}$ of rational p.l. functions is a computable structure on $(C[0, 1], d)$. Furthermore, the operators $+$ and $(r\cdot)_{r\in Q}$ are uniformly computable w.r.t. $(l_i)_{i\in\omega}$. Thus, $(l_i)_{i\in\omega}$ makes $C[0, 1]$ a computable Banach space, not merely a computable metric space. Unlike Hilbert spaces, zero does not effectively determine vector space operations on $C[0, 1]$:

Theorem 6.4.1. *There is a computable structure on $(C[0, 1], d)$ in which 0 is a computable point but the operation $(1/2)\cdot$ is not computable.*

Proof idea. We build a computable structure $(f_i)_{i\in\omega}$ on $(C[0, 1], d)$ which consists of points Δ^0_2 with respect to $(l_i)_{i\in\omega}$. That is, the points are of the form $f_i = \lim_s f_{i,s}$, where $f_{i,s}$ is a computable double sequence of rational p.l. functions, but the computable sequence $(f_{i,s})_{s\in\omega}$ may not have an effective rate of convergence.

We diagonalize against the e'th Turing functional Ψ_e potentially witnessing the computability of $(1/2)\cdot$, as follows. We choose an interval I_e (which is disjoint from I_j for each $j \neq e$) and a witness the special point f_p not equal to 0 on I_e. As soon as $\Psi_{e,s}$ on f_p becomes close to our current guess on $f_p/2$ (if ever), we change the approximation of f_p by setting $f_{p,s+1}$ to be far enough from $f_{p,s}$ on I_e. This will make $\Psi_{e,s}$ too far from $(1/2) \cdot f_{p,s+1}$.

Although not every special point from $(f_i)_{i\in\omega}$ will be computable w.r.t. $(l_i)_{i\in\omega}$, we guarantee $\lim_s f_{0,s} = 0$. This makes 0 a special point in the new computable structure we are building. We also make sure $d(f_i, f_j)$ is computable uniformly in i, j by maintaining the equality $d(f_{v,s}, f_{u,s}) = d(f_{v,s+1}, f_{u,s+1})$ at each stage s and for every

u and v. This seems to conflict with our attempt to change f_p as described above. However, we are able to make the construction injury-free. Also, some extra work is needed to make $(f_i)_{i\in\omega}$ a dense sequence in $C[0,1]$.

Proof. We build a computable double sequence of rational p.l. functions $(f_{i,s})_{i,s\in\omega}$ such that $f_{0,s} = 0$ for all s (thus, 0 is a special point in the new strucutre). At each stage s of the construction and for every $u, v \in \omega$, we maintain the equality

$$d(f_{v,s}, f_{u,s}) = d(f_{v,s+1}, f_{u,s+1}). \tag{6.1}$$

To make sure the equality (6.1) holds for every u, v and s, we will introduce the notion of (J, δ) -variation (Definition 6.4.1).

At stage s of the construction we will have a finite collection $f_{0,s}, \ldots, f_{n(s),s}$ of rational p.l. functions, where $n(s)$ is nondecreasing in s. At the end of stage s we will have another collection of rational p.l. functions $f_{0,s+1}, \ldots, f_{n(s+1),s+1}$ so that the equality (6.1) above holds for every $u, v \leq n(s)$. Note that we will not necessarily have $f_{i,s+1} = f_{i,s}$ for every $i \leq n(s)$. For every i we need to meet the requirement:

R_i : $\lim_s f_{i,s}$ exists.

To meet the requirements $(R_i)_{i\in\omega}$ we will make sure that:

$$(\forall n)\,(\exists s)\,(\forall t, z > s)\,(\forall i)\,[\,d(f_{i,t}, f_{i,z}) < 2^{-n}\,]. \tag{6.2}$$

This will imply R_i is met, for every i. The condition (6.2) will be satisfied in the construction (to be shown in Claim 6.4.1). Since the R-requirements will have no conflicts with other requirements (will be clear form the proofs of Claim 6.4.1 and Claim 6.4.3), we may assume for notational convenience that $\lim_s f_{i,s}$ exists for every i. We denote $\lim_s f_{i,s}$ by f_i.

For every $j \in \omega$, we need to meet the following requirements:

P_j : l_j belongs to the closure of $(f_i)_{i\in\omega}$.

Strategy for P_j. If s is a stage such that $s = 2\langle k, j\rangle$ for some integer k, and the function l_j is not among $f_{0,s}, \ldots, f_{n(s),s}$, then set $f_{n(s)+1,t} = l_j$ for every $t \leq (s+1)$.
End of strategy.

Taking into account (6.2) one can see that the strategy guarantees there is a sequence of elements in $(f_i)_{i\in\omega}$ converging to l_j (to be shown in Claim 6.4.2). The most important requirements are:

N_e : Ψ_e does not represent $(1/2)\cdot$ in $(f_i)_{i\in\omega}$.

The *strategy for N_e* is less straightforward and requires some extra work.

Preliminary work towards N_e. First, we simplify the N_e requirements. Note that $(p, p, \ldots)$ is a Cauchy name for the special point f_p. There is a primitive recursive function s such that $\Psi_e^{(p,p,\ldots)}(n) = \Phi_{s(e)}(p, n)$, where $(\Phi_e)_{e\in\omega}$ is the effective listing of all partially computable functions of two arguments (without an oracle). Thus, it is sufficient to meet the requirements:

N'_e : $(\exists p)[\,(\Phi_e(p, n))_{n\in\omega}$ is a Cauchy name $\Rightarrow$ $\lim_n \Phi_e(p, n) \neq f_p/2\,]$.

(Note that if $\Phi_e(p, n)\uparrow$ for some n, then N'_e is met trivially.)

The special element f_p will be the witness for N'_e chosen by the strategy. We need a technical definition which will allow us to make changes to approximations of special points without conflicting the R- and P-requirements:

Definition 6.4.1. *Suppose J is a subinterval of $[0, 1]$, and suppose $h_0, \ldots, h_k$ are rational p.l. functions on $[0, 1]$. We say that a finite collection $(g_0, \ldots, g_k)$ of rational p.l. functions is a (J, δ)-variation of the collection $(h_0, \ldots, h_k)$ if:*

(a) $h_0 = g_0$ and $d(h_i, h_j) = d(g_i, g_j)$, for all $i, j \leq k$,

(b) $h_i = g_i$ on $[0, 1] \setminus J$ and $d(h_i, g_i) \leq \delta$, for every $i \leq k$.

The strategy for N'_e will work within its own interval I_e and have a rational p.l. function w_e with support I_e. The function w_e may eventually become the witness for N'_e. More specifically, fix a computable listing of computable disjoint subintervals $(I_e)_{e\in\omega}$ of $[0, 1]$, where $I_e = [a_e, b_e]$ and $a_e, b_e \in Q$ for every e. We define a rational p.l. function w_e as follows:

$$w_e(x) = \begin{cases} 0, \text{ if } x \notin (a_e, b_e), \\ 2^{-e}, \text{ if } x \in [a_e + (1/4)(b_e - a_e), a_e + (3/4)(b_e - a_e)] \\ \text{linear, otherwise.} \end{cases}$$

Strategy for N'_e.

(i) At stage $t = 2\cdot e + 1$, if w_e is not already among $f_{0,t}, \ldots, f_{n(t),t}$, then set $f_{n(t)+1,r} = w_e$ for every $r \leq t$. In the following, we assume that $p \leq n(t) + 1$ is such that $f_{p,t} = w_e$.

(ii) At stage $s > t$ wait for a computation $\Phi_{e,s}(p, -\log \xi_e)\downarrow = h$, where ξ_e is much smaller than 2^{-e-1} (choose $\xi_e = 2^{-2^e-10}$). We have the following possibilities:

Case 1. The function $f_{h,s}$ has not been defined so far. Then, for each $v \leq s$, set $f_{h,v} = g$ to be a rational p.l. function which is not among $f_{0,s}, \ldots, f_{n,s}$ and which satisfies $\sup_{I_e} |g - (1/2) f_{p,s}| > 10$. Stop the strategy.

Case 2. The function $f_{h,s}$ has already been defined, and $\sup_{I_e} |f_{h,s} - (1/2) f_{p,s}| > \xi_e$, In this case do nothing and stop the strategy.

Case 3. The function $f_{h,s}$ has already been introduced, and $\sup_{I_e} |f_{h,s} - (1/2) f_{p,s}| \leq \xi_e$. Find a sub-interval J of I_e with rational end-points and a $(J, 2^{-e})$-variation $(g_0, \ldots, g_{n(s)})$ of the collection $(f_{0,s}, \ldots, f_{n(s),s})$ such that, for some $y \in J$, we have $f_{p,s}(y) = 2^{-e}$ and $g_p(y) = g_h(y) = f_{h,s}(y)$. We will show in Claim 6.4.3 that at least one such a $(J, 2^{-e})$-variation exists and, therefore, can be found effectively. See Figure 1 below for better idea. Set $f_{i,s+1} = g_i$, for all $i \leq n(s)$. Stop the strategy.

End of strategy.

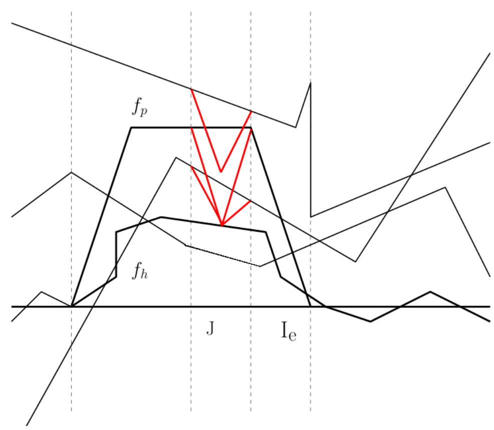


Figure 1. The figure illustrates a $(J, 2^{-e})$-variation. Within the interval J all the functions we change are linear. The colored lines show the variation.

Comments on the strategy for N'_e. Note that if Φ_e represents $(1/2)\cdot$ in $(f_i)_{i\in\omega}$ then $d(f_h, f_p/2) \leq \xi_e$. In the construction only the N'-strategies will possibly change the approximations of the special points $(f_i)_{i\in\omega}$, and each N'_e-strategy makes changes within its own sub-internal I_e of $[0, 1]$ disjoint from I_j, for $j \neq e$ (see the definitions of I_e and $(J, 2^{-e-1})$-variation). Thus, in Case 1 and Case 2 we guarantee $d(f_h, f_p/2) > \xi_e$, and Φ_e can not approximate $(1/2) \cdot f_p$ in $(f_i)_{i\in\omega}$.

Observe that if the strategy stops at Case 3, then

$$f_{h,s+1}(y) = f_{h,s}(y) = f_{p,s+1}(y) = f_p(y) \geq 2^{-e-1} - \xi_e.$$

By the choice of y, we have $f_{p,s}(y) = 2^{-e}$. By our assumption, $|f_{h,s}(y) - (1/2)f_{p,s}| \leq \xi_e$. Therefore, by the choice of ξ_e, we obtain

$$\begin{aligned} d(f_h, f_p/2) &\geq ||f_{h,s+1}(y) + \xi_e| - (1/2)f_{p,s+1}(y)| \\ &\geq |(2^{-e-1} + \xi_e) - (1/2)(2^{-e-1} - \xi_e)| \\ &> \xi_e, \end{aligned}$$

and N'_e is met. We put all the strategies together:

Construction.

At stage 0 of the construction set $f_{0,0} = 0$.

At stage $s > 0$ of the construction let the strategies act according to their instructions. For every $i \leq n(s)$, if $f_{i,s+1}$ have not been defined by the strategies, then set $f_{i,s+1} = f_{i,s}$.

End of construction.

The *verification* is split into several claims.

Claim 6.4.1. *The requirement R_i is met, for every i.*

Proof. Only the N'_e-requirements may change the approximation $(f_{i,s})_{s\in\omega}$ of a special point f_i, and each N'_e works within its own subinterval I_e. Furthermore, this change (if it is ever done by N'_e) is bounded by 2^{-e}. Therefore, if N'_e never reaches its Case 3, then the condition (6.2) is satisfied for $n = e$ and $s = 0$. If N'_e reaches its Case 3 at stage s', then (6.2) holds for $n = e$ and $s = s'$. Therefore, the condition (6.2) holds for every n. This implies R_i is met, for every i. □

Claim 6.4.2. *The requirement P_j is met, for every j.*

Proof. Let $s(k) = 2\langle k, j\rangle$. The strategy for P_j guarantees that the collection

$$f_{0,s(k)+1}, \ldots, f_{n(s(k)+1),s(k)+1}$$

contains a function $f_{m(k),s(k)}$ equal to l_j. Suppose also that that $s(k)$ is so large that for every $j \leq e$ the strategy for N_j never reaches its Case 3 after stage $s(k)$. Then $d(f_{m(k),s(k)}, l_j) \leq 2^{-e}$. Therefore, $(f_{m(k),s(k)})_{k\in\omega}$ converges to l_j. □

Claim 6.4.3. *The requirement N'_e is met, for every e.*

Proof. For functions f and h and a set $X \subseteq [0,1]$, write $f \leq_X g$ if $f(x) \leq g(x)$ for every $x \in X$. Define $<_X$ and $=_X$ similarly.

From now on we use notations from the strategy for N'_e. Assume we are at Case 3 of the strategy. We need to find an interval $J \subseteq I_e$ and a $(J, 2^{-e})$-variation $(g_0, \dots, g_{n(s)})$ of $(f_{0,s}, \dots, f_{n(s),s})$ such that, for some $y \in J$, we have $f_{p,s}(y) = 2^{-e}$ and $g_p(y) = g_h(y) = f_{h,s}(y)$. Find a rational point y and subinterval $J = [c, d] \subseteq I_e$ containing y, where $c, d \in Q$, such that:

1. $f_{p,s}(y) = 2^{-e}$ (notice that this implies $f_{p,s}(y) \neq f_{h,s}(y)$, by the choice of $f_{p,s}$ and $\xi_e << 2^{-e}$);
2. $f_{v,s}$ is linear on J, for each $v \leq n(s)$
3. for every $v, m \leq n(s)$ either $f_{m,s} <_J f_{v,s}$, or $f_{v,s} <_J f_{m,s}$ or $f_{v,s} =_J f_{m,s}$.

Recall that $f_{p,s}$ is equal to w_e on I_e, and there is a subinterval of I_e such that $f_{p,s}$ is equal to 2^{-e} when restricted to this subinterval. Note that the functions $f_{0,s}, \dots, f_{n(s),s}$ have finitely many breakpoints, and so do the functions $\{f_{k,s} - f_{m,s}\}_{k,m \leq n(s)}$. It is sufficient to choose $J = [c, d]$ so that $f_{p,s} =_J 2^{-e}$ and J does not contain any of these points. Let y be any rational point from J.

Denote $f_{i,s}$ restricted to $J = [c, d]$ by F_i. For every i, the function F_i is linear. Without loss of generality, we may assume

$$F_0 <_J \dots <_J F_h < \dots <_J F_p <_J \dots <_J F_{n(s)}.$$

Note that, by the choice of J, we must have $p < h$ in this list. (Note that we possibly have to change indexing and identify functions equal under $=_J$. We, however, slightly abuse our notation and assume that p and h remain untouched.) Let $k = p - h$:

$$F_0 <_J \dots F_h <_J \dots <_J F_{h+k} \dots <_J F_{n(s)}.$$

Given $i \in \{1, \dots, k\}$, define $\delta_i = |F_{h+i}(y) - F_h(y)|$. For each $j \leq n(s)$, define a new p.l. function G_j to be equal to F_j on the end-points of $j = [c, d]$, set

$$G_j(y) = \begin{cases} F_j(y) - \delta_k, \text{ if } j \geq h + k, \\ F_j(y) - \delta_{j-h}, \text{ if } h < j < h + k, \\ F_j(y), \text{ otherwise,} \end{cases}$$

and make it linear on $x \in J \setminus \{y\}$ (see Figure 1).

Recall that F_h is ξ_e-close to $F_p/2$, where ξ_e is much smaller that 2^{-e}. Recall also that, by the choice of y, we have $F_p(y) = 2^{-e}$. Therefore,

$$|F_p(y) - F_h(y)| < 2^{-e} \text{ and } \delta_i \leq 2^{-e},$$

for every $i \in \{1, \dots, k\}$. This implies that, for every $j \leq n(s)$,

$$\sup_J |G_j - F_j| \leq 2^{-e}.$$

Note that $G_i(c) = F_i(c)$ and $G_i(d) = F_i(d)$. Also,

$$|G_i(x) - G_j(x)| \leq |F_i(x) - F_j(x)|$$

for every $x \in (c, d)$, by the definition of G_i and G_j. Also, the functions $(F_i)_{i \leq n(s)}$ are linear, and $\sup_J |F_i - F_j| = \sup_{[c,d]} |F_i - F_j|$ realizes on c or d. We conclude that

$$\sup_J |G_i - G_j| = \sup_J |F_i - F_j|,$$

for every $i, j \leq n(s)$.

For every $j \leq n(s)$, define

$$g_j(x) = \begin{cases} G_j(x), & \text{if } x \in [c, d], \\ f_{j,s}(x), & \text{otherwise.} \end{cases}$$

We have $f_{p,s} \geq_{I_e} 0$ and $f_h \geq_{I_e} 0$. Thus, the definition ensures $g_0 = f_{0,s}$. It follows that $(g_0, \dots, g_n)$ is a $(J, 2^{-e})$-variation of $(f_{0,s}, \dots, f_{n(s),s})$. By its definition, we have $f_{p,s}(y) = 2^{-e}$ and $g_p(y) = g_h(y) = f_{h,s}(y)$. We proved the claim in the case when all the inequalities are strict.

The general case is done by a simple inductive argument. Suppose, say, $F_1 = F_2$, and suppose there is a $(J, 2^{-e})$-variation $(g_0, g_2 \dots, g_n)$ of $(f_{0,s}, f_{2,s} \dots, f_{n(s),s})$ with the needed properties. Define

$$g_1(x) = \begin{cases} f_1(x), & \text{if } x \notin J, \\ g_2(x), & \text{if } x \in J. \end{cases}$$

The collection $(g_0, g_1 \dots, g_n)$ is the needed $(J, 2^{-e})$-variation of $(f_{0,s}, f_{1,s} \dots, f_{n(s),s})$. It is important that $f_{p,s} \geq_{I_e} 0$ and $f_h \geq_{I_e} 0$. This concludes the proof of Claim 6.4.3. $\square$

It remains to observe that all the stages are effective, because at each stage we have a collection of rational p.l. functions, and all the questions we ask about these collections are effectively decidable. $\square$

Theorem 6.4.2. *The space $C[0,1]$ is not computably categorical.*

Proof. This follows from Theorem 6.4.1 and Fact 6.2.2. $\square$

Chapter 7

Computably categorical metric spaces

This chapter continues the study of isometris on computable metric spaces. In this chapter most of the metric spaces are not associated to Banach spaces. We show that the Urysohn space and the Cantor space are computably categorical, and give a necessary and sufficient condition for a subspace of $\mathbb{R}^n$ to be computably categorical.

7.1 Cantor space

Recall that Cantor space is the set of infinite strings of 0's and 1's. We show:

Theorem 7.1.1. *Cantor space* $\{0,1\}^\omega$ *with the metric* $d(\xi, \phi) = \max\{2^{-n} : \xi(n) \neq \phi(n)\}$ *is computably categorical.*

Proof idea: Let $(\alpha_i)_{i\in\omega}$ and $(\beta_i)_{i\in\omega}$ be computable structures on Cantor space. The computable structures are rational-valued metric spaces. Furthermore, every surjective isometry of these rational-valued subspaces is uniquely expandable to a surjective isometry of their closures. Therefore, it is sufficient to build a computable bijection $f : \omega \to \omega$ such that $d(\alpha_i, \alpha_j) = d(\beta_{f(i)}, \beta_{f(j)})$, for every $i, j \in \omega$.

Without loss of generality, we may assume $(\alpha_i)_{i\in\omega}$ is the usual computable structure on Cantor space given by the collection of infinite strings that are eventually 0. To define $f(i)$, we find the least j such that $\{\alpha_k\}_{k\leq i}$ and $\{\beta_{f(k)}\}_{k<j} \cup \beta_j$ are isometric via $\alpha_k \to \beta_k$ and $\alpha_i \to \beta_j$, and set $f(i) = j$. Since the distances are rational-valued, the definition of f is effective.

Cantor space is ultra-homogeneous. That is, any partial isometry between finite subsets of it can be extended to a surjective self-isometry of the whole space. Note that we are essentially using the (ultra)homogeneity of Cantor pace and that the collection of distances is a desecrate subset of computable reals. The same proof would work if we replace the standard distance on 2^ω by $\max\{2^{-n}\gamma\colon \xi(n) \neq \phi(n)\}$, where γ is a computable real.

All details are given in the formal proof below.

Proof. Let $(\alpha_i)_{i\in\omega}$ be the usual computable structure on Cantor space given by the infinite strings that are eventually 0, and let $(\beta_i)_{i\in\omega}$ be another computable structure on Cantor space. If we think of the Cantor space as of a binary tree, the special points in $(\alpha_i)_{i\in\omega}$ are enumerated level-by-level, excluding repetitions:

$$\emptyset,\ 1,\ 01,\ 11,\ 001\ldots$$

We define f in the following procedure:

Construction.

At *stage* 0, set $f(0) = 0$.

At *stage* $i > 0$, we assume that $f(k)$ has already been defined for every $i < k$. Say that j is *good for extension* if the isometry $\alpha_k \to \beta_{f(k)}$ of $\{\alpha_k\}_{k<i}$ onto $\{\beta_{f(k)}\}_{k<j}$ can be extended to an isometry of $\{\alpha_k\}_{k\leq i}$ onto $\{\beta_{f(k)}\}_{k<j} \cup \{\beta_j\}$. Find least j good for extension and set $f(i) = j$.

End of construction.

Verification. By the density of $(\beta_i)_{_{i\in\omega}}$, at every stage of the construction there exists at least one j good for extension. In fact, every special point β_j in a certain open ball in 2^ω is good for extension. The formal argument is not difficult and can be left to the reader. Thus, f is total. We show:

Claim 7.1.1. *f is computable.*

Proof. Recall that, for each v and w in Cantor space, the distance $d(v, w)$ is of the form 2^{-n}, for some n. Suppose we have a computable structure on Cantor space. Given unequal computable points v and w, we can effectively find n such that $d(v, w) = 2^{-n}$. We conclude that the construction is effective and, consequently, the function f is computable. □

Claim 7.1.2. *f is a bijection.*

Proof. The usual visualization of Cantor set by the complete binary tree may help in understanding of the proof below. Recall that special points from $(\alpha_i)_{_{i\in\omega}}$ are enumerated "level-by-level". The structure $(\beta_j)_{j\in\omega}$ can be visualized as a set of infinite paths through the complete binary tree. If at stage s we have to choose an extension of f to another point of $(\alpha_i)_{i\in\omega}$, we can pick any point in $(\beta_j)_{j\in\omega}$ within an open ball uniquely determined by the collection of distances between points in $\mathrm{dom}(f_s)$ and $\mathrm{range}(f_s)$. To see that the ball is uniquely determined by the distances, observe that the finite collection of distances are completely determined by finite initial segments of the special points $\mathrm{dom}_s(f)$ and $\mathrm{range}_s(f)$. An easy inductive argument shows that the way we list elements of $(\alpha_i)_{i\in\omega}$ ensures that every open ball of the form $\sigma 2^\omega$ will be among these "determinerd" balls. Furthermore, the balls 02^ω and 12^ω will correspond (in some order) to stages 0 and 1, the balls $\sigma 2^\omega$, where $lgth(\sigma) = n$, will appear (in some order) at stages $\sum_{k<n} 2^n \leq s < \sum_{k\leq n} 2^k$.

We pick $v \in \omega$ and show that β_v is in the range of f. Suppose f has already been defined for all $u < v$, and s is the least stage when that happened. Let n be largest having the property $\sum_{k<n} 2^n \leq s$. There must be a stage $t > \sum_{k\leq n} 2^k$ such that the open ball $\tau 2^\omega$, where $\tau \subset \beta_v$, is the ball determined by the distances in $\mathrm{dom}(f_t)$ and $\mathrm{range}(f_t)$. The construction ensures that the least special point from the ball must be put into the range of f. If it has not already happened to β_v, then β_v must be good for extension at stage t. □

Define a surjective self-isometry U of Cantor space:

$$\text{if } (\alpha_{g(i)})_{i\in\omega} \text{ is a Cauchy name then } U(\lim_i \alpha_{g(i)}) = \lim_i f(\alpha_{g(i)}).$$

By Claims 7.1.2 and 7.1.1, the isometry U witnesses computable categoricity of Cantor space. □

Remark 7.1.1. *Note that (the index for) U witnessing computable categoricity of* $\{0,1\}^\omega$ *may be obtained uniformly from (the indices for) the given computable structures on* $\{0,1\}^\omega$.

7.2 The Urysohn space

The rational Urysohn space $\mathbb{QU}$ is the Fraisse limit of finite rational-valued metric spaces. The Urysohn space $\mathbb{U}$ is the closure of $\mathbb{QU}$. We can effectively list all finite rational-valued metric spaces. Therefore, the points in $\mathbb{QU}$ form a computable structure of $\mathbb{U}$. We will need the following definitions which can be found in [57] and [69].

Definition 7.2.1. *Let X be a metric space. A map $f : X \to \mathbb{R}^+$ is a* Katetov map *if for $z \notin X$ setting $d(x,z) = f(x)$ defines a metric space on $X \cup \{z\}$ which extends X.*

A map f is Katetov if, and only if, $(\forall x, y \in X)|f(x) - f(y)| \leq d(x,y) \leq f(x) + f(y)$. The collection $E(X)$ of all Katetov maps together with the metric $\sup_{x \in X} |f(x) - g(x)|$ is a (complete) metric space. Informally, Katetov maps reflect all one-point metric extensions of a given metric space.

Definition 7.2.2. *A space X has the* approximate extension property *if for every finite subset A of X, for every $f \in E(X)$, and every $\epsilon > 0$ there exists a point $z \in X$ such that*

$$(\forall a \in A)|d(z,a) - f(a)| \leq \epsilon.$$

It is known that a Polish metric space has the approximate extension property if, and only if, it is isometric to the Urysohn space ([69]). The approximate extension property is equivalent to the extension property which is the approximate extension property with $\epsilon = 0$. The approximate extension property is central to the proof of the theorem below.

Theorem 7.2.1. *The Urysohn space is computably categorical.*

Proof idea. The proof is essentially an effective version of the usual back-and-forth argument. The problem is that we can not define the image of a point in one step. The approximate extension property allows us to run the argument on the special points with an "arbitrarily good precision". This property allows us to search for a point in a given computable structure $(\alpha_i)_{i \in \omega}$ which is "approximately" an image of a special point r form $\mathbb{QU}$. As was mentioned above, a Polish metric space has the approximate extension property if, and only if, it is isometric to the Urysohn space. The proof of this fact (see, e.g., [69]) guarantees that it is possible to define a sequence of "approximate" images of r so that it is a Cauchy sequence of points from $(\alpha_i)_{i \in \omega}$. As is described in detail in the formal proof below, one can use the approximate extension property to define an injection of $\mathbb{QU}$ into the closure of $(\alpha_i)_{i \in \omega}$.

However, it does not guarantee the map is surjective. We should force every special point from $(\alpha_i)_{i \in \omega}$ to be in the closure of the image of the embedding of $\mathbb{QU}$. Clearly, it can not be implemented as it was done for Cantor space, because we don't know the precise distances between special points in $(\alpha_i)_{i \in \omega}$ at a stage. We need to make sure that, for every ϵ and n, the special element α_n is in the ϵ-neighbourhood of the image of $\mathbb{QU}$.

At a stage s, the element α_n may be within the 2^{-m}–neighbourhood of the image of (a finite part of) $\mathbb{QU}$. At a later stage we discover it is outside the 2^{-m-3}–neighbourhood of the current image. In this case we need to put a new special point r from $\mathbb{QU}$ into the domain of our map so that α_n belongs to the 2^{-2m}–neighbourhood of the image of r.

For technical reasons, we use movable markers on the (numbers of) special elements from $\mathbb{QU}$ to implement this idea. The markers allow us to label elements for which the map has already been defined. The formal details can be found in the proof below.

Proof. Let $(r_i)_{i\in\omega}$ be the computable structure on $\mathbb{U}$ given by an effective listing of the points in $\mathbb{QU}$, and let $(\alpha_i)_{i\in\omega}$ be another computable structure on $\mathbb{U}$.

First, we define a computable double sequence $(f_{j,s})_{s,j\in\omega}$ of special points in $(\alpha_i)_{i\in\omega}$ such that $\lim_s f_{j,s}$ exists with the rate of convergence computable uniformly in j, and $d(r_i, r_j) = d(\lim_s f_{i,s}, \lim_s f_{j,s})$ for each $i, j \in \omega$. In the construction for every k we will have a strategy L_k which defines the computable sequence $(f_{k,s})_{s\in\omega}$ of special points in $(\alpha_i)_{i\in\omega}$.

Strategy L_0. Set $f_{0,s} = \alpha_0$ for every s.

Given $k > 0$, suppose that for every $i < k$ the (strategies L_i enumerating) computable sequences $(f_{i,s})_{s\in\omega}$ have already been defined, and for every $i < k$ the point $g_i = \lim_s f_{i,s}$ is computable uniformly in i. We have:

Strategy L_k. Suppose we need to define $f_{k,s}$, and either $s = 0$ or $f_{k,s-1}$ has already been defined. Find a special point α in $\mathbb{QU}$ such that for every $i < k$

$$|d(r_i, r_k) - d(g_i, \alpha)| < 2^{-s},$$

and $d(f_{k,s-1}, \alpha) < 2^{2-s}$ if $s > 0$. Set $f_{k,s} = \alpha$.

Clearly, if we can show that for every k, s at least one point α with the needed properties exists (to be shown), then L_k eventually finds α (or any other point satisfying these properties would work). If so, then $g_k = \lim_s f_{k,s}$ exists and is a computable point. Furthermore, the computable index for the Cauchy name of g_k can be obtained uniformly form the computable indices for the Cauchy names $(f_{i,s})_{s\in\omega}$ of $(g_i)_{i<k}$. Thus, the map $\mathrm{F} : r_j \to g_j$ is an isometry which is uniquely expandable to a computable isometric embedding of $\mathbb{U}$ into itself. To define a surjective embedding we need extra requirements (to be introduced later) and movable markers.

Movable markers. Recall that we identify special elements and their numbers. At each stage of the construction we will have markers $(\mathrm{m}_k)_{k\in\omega}$ on the special elements form $\mathbb{QU}$. If a special point r at stage s carries the marker m_k, then we write $\boxed{\mathrm{r}}_k^s$ or $\mathrm{m}_{k,s} = r$. As usual for movable markers arguments, we will describe how to effectively move all the markers at once instead of dealing with finitely many markers at every stage of the construction. At the end of every stage s, for every k there will be exactly one r such that $\mathrm{m}_{k,s} = r$, and every r will carry a marker. We will show that for every k there exists a stage t such that $\mathrm{m}_{k,s} = \mathrm{m}_{k,t}$ for every $s \geq t$, and for every r there exists k such that $r = \lim_s \mathrm{m}_{k,s}$.

We need the following important modifications of L_k:

- Replace every instance of r_k and r_i in the strategy by $\mathrm{m}_{k,t}$ and $\mathrm{m}_{i,t}$ respectively, where t is the stage of the construction at which L_k defines $f_{k,s}$.

- If L_k defines $f_{k,0}$ at stage t, declare $\mathrm{m}_{k,t} = \lim_s \mathrm{m}_{k,s} = \mathrm{m}_k$ and say that m_k has *settled*.

The isometric map will be defined by the rule $\lim_s \mathrm{m}_{k,s} \to g_k$ for every k.

The requirements for surjectivity. It is sufficient to meet, for every $v, n \in \omega$, the requirements:

$P_{v,n} : (\exists i)\, d(\alpha_v, g_i) < 2^{-n}$.

The strategy for $P_{v,n}$. Suppose we are at stage t of the construction, and $\mathrm{m}_{0,t}, \ldots, \mathrm{m}_{j,t}$ are all the markers which have already settled. Equivalently, $(j+1)$ is least such that $f_{j+1,0}$ has not been defined yet. Wait until one of the two possibilities is effectively recognized:

1. $d(\alpha_v, g_k) < 2^{-n}$ holds, for some $k \leq j$. In this case $P_{v,n}$ is met. Stop the strategy.

2. $d(\alpha_v, g_k) > 2^{-n-1}$ holds for every $k \leq j$. Find $\mathrm{m}_{h,t}$ with $h > j$ such that for every $k \leq j$
$$|d(\mathrm{m}_{k,t}, \mathrm{m}_{h,t}) - d(g_k, \alpha_v)| < 2^{-n-2}.$$
Define $\mathrm{m}_{j+1,t+1} = \mathrm{m}_{h,t}$. Declare m_{j+1} settled. (Note that temporarily one special point carries no marker.)

   For every $i > (j+1)$, in increasing order, find the least x such that r_x does not carry a marker m_l with $l < i$ and set $\mathrm{m}_{i,t+1} = r_x$. (In other words, move all the non-settled markers one step left, avoiding the settled markers.)

Then set $f_{j+1,s} = \alpha_v$, for every $s \leq n+2$. Stop the strategy. Note that the modified strategy L_{j+1} guarantees $d(g_{j+1}, \alpha_v) = d(g_{j+1}, f_{j+1,n+2}) < 2^{2-(n+2)} = 2^{-n}$, and $P_{v,n}$ is met.

Construction.

At *stage* t, let k be least such that $\mathrm{m}_{k,t}$ has not settled yet. For every $j \leq k$ let the modified L_j define $f_{j,s}$ for at least every $s \leq t$. Then let $P_{v,n}$ with $\langle v, n\rangle = t$ act according to its instructions, where $\langle \cdot, \cdot \rangle$ is the usual computable bijection of ω^2 onto ω.

Verification. We split the verification into claims.

Claim 7.2.1. *For every k, the strategy L_k defines a computable infinite sequence $(f_{k,s})_{s\in\omega}$ of special points.*

Proof. The statement is clear for $k = 0$. Suppose for every $i \leq k$ the strategy L_k defines a computable infinite sequence $(f_{i,s})_{s\in\omega}$ of special points, and either $s = 0$ or $f_{k,s-1}$ has already been defined. Note that, by inductive assumption, we may assume that we have indices for the Cauchy names of $(g_i)_{i<k}$ (observe that the values of $(f_{i,s})_{s\in\omega}$ depend on the values of $(f_{j,s})_{s\in\omega, j<i}$ only, and we can speed-up the enumerations $(f_{j,s})_{s\in\omega}$ and postpone the definition of $f_{i,s}$ if needed). Therefore, if a special point α satisfies the conditions $|d(r_i, r_k) - d(g_i, \alpha)| < 2^{-s}$ and $d(f_{k,s-1}, \alpha) < 2^{2-s}$, then we will eventually see that it indeed satisfies these conditions. We need only to show that at least one such α exists.

Recall that we need to find a special point α in $\mathbb{QU}$ such that for every $i < k$ $|d(r_i, r_k) - d(g_i, \alpha)| < 2^{-s}$, and $d(f_{k,s-1}, \alpha) < 2^{2-s}$ if $s > 0$.

Recall that a Polish metric space has the approximate extension property if, and only if, it has the extension property. The proof of this fact goes as follows. We suppose that a space has the approximate extension property. We take finitely many points and a Katetov map h on these points (equivalently, a potential 1-point extension of the finite metric space on these points) and wish to find a sequence which converges to some point in the space which realizes this map. We take $\epsilon = 1/2$ and take x_1 which realizes the Katetov map with precision ϵ. We can set $\epsilon = 1/4$ and define x_2 similarly, but there is no guarantee that $d(x_1, x_2)$ is small. A more careful analysis of the situation which makes use of the distance between Katetov maps shows that it can be done (see, e.g., [69], Exercise 3 and the proof of Theorem 3.4). It is not important for us how exactly it is done.

Now we take the Katetov map which is given by $d(g_i, U(r_k))$, where U is *some* self-isometry of the Urysohn space, and let $f_{k,s}$ play the role of x_s in the explanation above. The ultra homogeneity of the Urysohn space implies that the choice of U does not matter, and the density of of $(\alpha_i)_{i\in\omega}$ in $\mathbb{U}$ implies that we can set $f_{k,s} = \alpha$ for some special point α in $(\alpha_i)_{i\in\omega}$. □

Claim 7.2.2. *The requirement $P_{v,n}$ is met for every $v, n \in \omega$.*

Proof. Note that at least one of the two possibilities in the strategy for $P_{v,n}$ will be eventually effectively recognized. The effectiveness follows from the fact that the indices for $(f_{i,s})_{s\in\omega}$ are given ahead of time, as it is explained in the proof of Claim 7.2.1. The rest follows form the density of $(r_i)_{i\in\omega}$ and $(\alpha_i)_{i\in\omega}$ and the strategy for L_k. □

Claim 7.2.3. *For every k, the movable marker* m_k *settles, and every special point eventually carries a settled marker.*

Proof. At some stage t, the marker $m_{k,t}$ will be either used by an L- or P-strategy to define $f_{k,0}$. In both cases it will be declared settled. The construction is organized so that at the beginning of stage t the least j such that r_j does not carry a settled marker will be occupied by $\mathrm{m}_{k,t}$, where k is least such that $\mathrm{m}_{k,t}$ has not settled yet. The marker will be declared settled by L_k at stage t. □

Denote the element which eventually carries the marker m_k by c_k. Define F by setting $\mathrm{F}(c_k) = \lim_s f_{k,s}$, for every s, and then extending F to the whole $\mathbb{U}$. The map F is a surjective self-isometry computable w.r.t. $(r_i)_{i\in\omega}$ and $(\alpha_i)_{i\in\omega}$.

□

7.3 Subspaces of $\mathbb{R}^n$, and a generalization.

In this section we characterize computably categorical subspaces of $\mathbb{R}^n$, and also suggest a sufficient condition for an arbitrary metric space to be computably categorical. In Theorem 7.3.4 we will show that this general condition is also necessary for closed subspaces of $\mathbb{R}^n$.

7.3.1 Subspaces of $\mathbb{R}^n$.

In this subsection d stands for the Euclidean metric. We denote the linear span of $M \subseteq \mathbb{R}^n$ over $\mathbb{R}$ by $\langle M \rangle_{\mathbb{R}}$.

If $M = \mathbb{R}^n$, then one may use the Gram–Schmidt process to show $\mathbb{R}^n$ is computably categorical (Corollary 6.3.1). In the general case $M \subseteq \mathbb{R}^n$, there are two difficulties. First, M may not be a Banach space. Second, even if we isometrically and computably embed M into $\mathbb{R}^n$, we may not be able to run Gram - Schmidt within M. The definition below is central to this section:

Definition 7.3.1. *Let $M \subseteq (\mathbb{R}^n, d)$ be closed such that $\langle M \rangle_{\mathbb{R}}$ has dimension $m \leq n$, and assume (M, d) possesses a computable structure. We say that points $x_0, \dots, x_m \in M$ form an* intrinsically computable base *of (M, d) if:*

1. the vectors $x_0 - x_1, \dots, x_0 - x_m$ are linearly independent in $\mathbb{R}^n$,

2. for every computable structure $(\alpha_i)_{i\in\omega}$ on (M, d) there is a surjective self-isometry W of (M, d) such that $W(x_0), \dots, W(x_m)$ are computable in $(\alpha_i)_{i\in\omega}$.

We call m from the definition above *the dimension of M*. We show:

Theorem 7.3.1. *Let M be a closed subspace of $(\mathbb{R}^n, d)$ having dimension $m \leq n$ which possesses a computable structure. The following are equivalent:*

1. *(M, d) is computably categorical;*

2. *M has an intrinsically computable base $x_0, \dots, x_m$.*

Proof idea. The proof of (1) $\Rightarrow$ (2) is rather straightforward, we briefly outline (2) $\Rightarrow$ (1). Suppose we are given two computable structures on M. Using an intrinsically computable base, we embed M into $\mathbb{R}^n$ and define new computable structures on $\mathbb{R}^n$. Then we observe that these new structures are computably isometric via a surjective isometry which maps M onto itself. Then we show that the restriction of this self-isometry to M is computable w.r.t. the given computable structures on M, and conclude that M is computably categorical.

Proof. In the proof below, we consider the linear span of M within $\mathbb{R}^n$ and, without loss of generality, set $m = n$.

We prove (1) $\Rightarrow$ (2). Suppose $(\alpha_i)_{i\in\omega}$ is a computable structure on (M, d). By the density of $(\alpha_i)_{i\in\omega}$ and the choice of M and n, we can choose special points $x_0, \dots, x_n$

such that $x_0 - x_1, \ldots, x_0 - x_n$ are linearly independent. By our assumption, for every computable structure $(\beta_i)_{i\in\omega}$ on (M, d) there exists a computable isometry

$$U : (M, d, (\alpha_i)_{i\in\omega}) \to (M, d, (\beta_i)_{i\in\omega}).$$

The points $U(x_0), \ldots, U(x_n)$ are computable in $(\beta_i)_{i\in\omega}$.

We show (2) $\Rightarrow$ (1). We need the following fact. Although the fact is intuitively clear, we give a proof of it which uses elementary affine geometry.

Fact 7.3.1. *Every surjective self-isometry W of M can be uniquely extended to a surjective self-isometry $\overline{W}$ of $\mathbb{R}^n$. Furthermore, both W and its extension are completely determined by the images of $x_0, \ldots, x_n$.*

Proof. Note that the points $x_0, \ldots, x_n$ are affine independent which means that the smallest convex set containing the points has non-zero volume in $\mathbb{R}^n$ (recall that here $m = n$). The volume is determined by the value of the Cayley-Maneger matrix which involves only $d(x_i, x_j)$, for $i, j \leq n$. The isometry W preserves the values of $d(x_i, x_j)$ and, consequently, the images of $x_0, \ldots, x_n$ are also affine independent. Thus, $W(x_0) - W(x_1), \ldots, W(x_0) - W(x_n)$ are also linearly independent. The vectors $x_0 - x_i$ are linearly independent. Thus, every point $z \in M \subseteq \mathbb{R}^n$ is uniquely determined by $d(z, x_i)$, $i \leq n$. On the other hand, every point y from $\mathbb{R}^n$, and from M in particular, is uniquely determined by the distances $d(y, W(x_i))$, $i \leq n$. Notice that these distances are preserved under W. On the other hand, since $x_0 - x_1, \ldots, x_0 - x_n$ are linearly independent, every element z of $\mathbb{R}^n$ can be uniquely written as $z = \sum_{0<i\leq n} f_i(x_0 - x_i)$, where f_i are reals. Now it is clear that the map

$$\overline{W} : \sum_{0<i\leq n} f_i(x_0 - x_i) \to \sum_{0<i\leq n} f_i(W(x_0) - W(x_i))$$

is the needed extension. $\square$

By Fact 7.3.1, we may assume that $W_1(M)$ is a subset of an isometric copy of $\mathbb{R}^n$. Note that the proof of Fact 7.3.1 implies that $W_1(x_0) - W_1(x_1), \ldots, W_1(x_0) - W_1(x_n)$ are linearly independent within this copy.

Claim 7.3.1. *Let $v_1, \ldots, v_n$ be linear independent vectors in $\mathbb{R}^n$ with $||v_i||$ and $||v_i - v_j||$ computable. Then $(\sum_{1\leq i\leq n} r_i v_i)_{(r_1,\ldots,r_n)\in Q^n}$ is a computable structure on $(\mathbb{R}^n, d)$, where d is the Euclidean metric.*

Proof. The collection of points $(\sum_{1\leq i\leq n} r_i v_i)_{(r_1,\ldots,r_n)\in Q^n}$ is clearly dense in $\mathbb{R}^n$. We have

$$\begin{aligned} d_E^2(r_1v_1+\ldots+r_nv_n, q_1v_1+\ldots+q_nv_n) &= \|(r_1-q_1)v_1+\ldots+(r_n-q_n)v_n\|^2 \\ &= \sum_{0<i\leq m}(r_i-q_i)^2\|v_i\|^2 + \sum_{0<i<j\leq m} 2(r_i-q_i)(r_j-q_j)\langle v_i, v_j\rangle. \end{aligned}$$

BY the assumption, for every $i,j\in\{1,\ldots,n\}$ the reals $\|v_i-v_j\|$ and $\|v_i\|$. Thus, the real $\langle v_i, v_j\rangle = \frac{1}{2}(\|v_i\|^2+\|v_j\|^2-\|v_i-v_j\|^2)$ is computable, for every $i,j\in\{1,\ldots,n\}$. Therefore, the distances between the special points in $(r_1v_1+\ldots+r_nv_n)_{(r_1,\ldots,r_n)\in Q^n}$ are uniformly computable. □

Let $(\alpha_i)_{i\in\omega}$ and $(\beta_i)_{i\in\omega}$ be computable structures on M. Let W_1 and W_2 be self-isometries of M such that $W_1(x_0),\ldots,W_1(x_n)$ are computable in $(\alpha_i)_{i\in\omega}$ and $W_2(x_0),\ldots,W_2(x_n)$ are computable in $(\beta_i)_{i\in\omega}$, respectively. Let $v_i = W_1(x_0)-W_1(x_i)$ and $w_i = W_2(x_0)-W_2(x_i)$, for $i\in\{1,\ldots,n\}$. By the choice of $x_0,\ldots,x_n$, these points as well as their W-images are computable. Thus, the norms $\|v_i-v_j\|$, $\|w_i-w_j\|$, $\|w_i\|$ and $\|v_i\|$ are computable reals, for every $i,j\in\{1,\ldots,n\}$. Fixing some effective enumeration of all rational n-tuples, denote $(\sum_{1\leq i\leq n} r_iv_i)_{(r_1,\ldots,r_n)\in Q^n}$ by $(\gamma_i)_{i\in\omega}$, and $(\sum_{1\leq i\leq n} r_iw_i)_{(r_1,\ldots,r_n)\in Q^n}$ by $(\theta_i)_{i\in\omega}$.

Claim 7.3.2. *The point α_k is computable w.r.t. $(\gamma_i)_{i\in\omega}$ uniformly in k, for every $k\in\omega$.*

Proof. We have $\alpha_k = \sum_{0<i\leq n} f_iv_i$ for reals $f_1,\ldots,f_n$. Given $\delta>0$ and a tuple of rationals $(q_1,\ldots,q_n)$ such that $|f_i-q_i|<\delta$ for every $i\in\{1,\ldots,n\}$, we obtain

$$\begin{aligned} \|\sum_{0<i\leq n}(f_i-q_i)v_i\|^2 &= \sum_{0<i\leq n}(f_i-q_i)^2\|v_i\|^2 + \sum_{0<i<j\leq n} 2(f_i-q_i)(f_j-q_j)\langle v_i, v_j\rangle \\ &\leq \delta^2(\sum_{0<i\leq n}\|v_i\|^2 + \sum_{0<i<j\leq n} 2|\langle v_i, v_j\rangle|). \end{aligned}$$

The reals $\|v_i\|$ and $|\langle v_i, v_j\rangle|$ are computable, for every $i,j\in\{1,\ldots,n\}$. Therefore, the positive real

$$(\sum_{0<i\leq n}\|v_i\|^2 + \sum_{0<i<j\leq n} 2|\langle v_i, v_j\rangle|)$$

is computable. Thus, it is sufficient to show that f_i is computable, for every $i\in\{1,\ldots,n\}$.

By the choice of W_1, the real $d(\alpha_k, W_1(x_i))$ is computable, for every $i \leq n$. Therefore, the real

$$B_0 = d_E^2(\alpha_k, W_1(x_0)) = \| \sum_{0<i\leq n} f_i v_i \|^2$$

is computable, and so is

$$B_k = d_E^2(\alpha_k, W_1(x_i)) = \|(\sum_{0<i\leq n} f_i v_i) - v_k\|^2,$$

for every $i \in \{1, \ldots, n\}$.

We express B_i using the inner products $\langle v_i, v_j \rangle$ and the norms $\|v_i\|^2$, for $i, j \leq n$. After a simplification, we get the system of equations

$$\frac{B_0 - B_k}{2\|v_k\|^2} = f_k - \sum_{j\neq k} \frac{\langle v_k, v_j \rangle}{\|v_k\|^2} f_j.$$

The set $\{v_1, \ldots, v_n\}$ is a basis of $\mathbb{R}^n$, which implies that the matrix corresponding to this system is invertible. Furthermore, it has computable coefficients. Thus, for every $i \in \{1, \ldots, n\}$ the real f_i is computable. □

Claim 7.3.3. *The point β_k is computable w.r.t. $(\theta_i)_{i\in\omega}$ uniformly in k, for every $k \in \omega$.*

Proof. Similar to the proof of Claim 7.3.2. □

By Claim 7.3.1, $(\gamma_i)_{i\in\omega}$ and $(\theta_i)_{i\in\omega}$ are computable structures on $(\mathbb{R}^n, d)$. Taking into account Fact 7.3.1, observe that these structures are computably isometric via $W(r_1 v_1 + \ldots + r_n v_n) = r_1 w_1 + \ldots + r_n w_n$. By Claim 7.3.2, the embedding W_1 of M into $\mathbb{R}^n$ is computable w.r.t. $(\alpha_i)_{i\in\omega}$ and $(\gamma_i)_{i\in\omega}$. By Claim 7.3.3, the embedding W_2 of M into $\mathbb{R}^n$ is computable w.r.t. $(\beta_i)_{i\in\omega}$ and $(\theta_i)_{i\in\omega}$. Note that M is closed, and the inverse W_1^{-1} is computable on its domain w.r.t. $(\gamma_i)_{i\in\omega}$ and $(\beta_i)_{i\in\omega}$. Similarly, W_2^{-1} is computable w.r.t. $(\theta_i)_{i\in\omega}$ and $(\alpha_i)_{i\in\omega}$. The following diagram commutes, where U is W restricted to M:

$$\begin{tikzcd}
(M, d, (\alpha_i)_{i\in\omega}) \arrow[r, "U"'] \arrow[d, "W_1"] & (M, d, (\beta_i)_{i\in\omega}) \arrow[d, "W_2"'] \\
(\mathbb{R}^n, d, (\gamma_i)_{i\in\omega}) \arrow[r, "W"] & (\mathbb{R}^n, d, (\theta_i)_{i\in\omega})
\end{tikzcd}$$

This shows that U is computable w.r.t. $(\alpha_i)_{i\in\omega}$ and $(\beta_i)_{i\in\omega}$, proving the theorem. □

The usual computable structure on $\mathbb{R}^n$ is given by the tuples of rationals which are coordinates in an orthonormal base. The proof of Theorem 7.3.1 leads to a fact which resembles similar results on computable closures in computable algebra:

Fact 7.3.2. Let $M \subseteq (\mathbb{R}^n, d)$ be closed such that $\langle M \rangle_{\mathbb{R}} = \mathbb{R}^n$. For every computable structure $(\alpha_i)_{i\in\omega}$ on $\mathcal{M} = (M, d)$ there exists an isometric embedding $U : \mathcal{M} \to (\mathbb{R}^n, d)$ which is computable with respect to $(\alpha_i)_{i\in\omega}$ and the usual computable structure on $\mathbb{R}^n$. Furthermore, its inverse is computable on its domain.

Proof. By the choice of n and the density of $(\alpha_i)_{i\in\omega}$ in $\mathcal{M}$, there exist special points $\gamma_0, \ldots, \gamma_n$ in $(\alpha_i)_{i\in\omega}$ such that $\{\gamma_0 - \gamma_1, \ldots, \gamma_0 - \gamma_n\}$ is linearly independent in $\mathbb{R}^n$. Denote $\gamma_0 - \gamma_k$ by v_k, for every $k \in \{1, \ldots, n\}$. By Claim 7.3.1, the collection of points $(\sum_{1\leq i\leq n} r_i v_i)_{(r_1,\ldots,r_n)\in Q^n}$ is a computable structure on $\mathbb{R}^n$. By Claim 7.3.2, the embedding I of $\mathcal{M}$ into $\mathbb{R}^n$ is computable with respect to $(\alpha_i)_{i\in\omega}$ and $(\sum_{1\leq i\leq n} r_i v_i)_{(r_1,\ldots,r_n)\in Q^n}$. By Theorem 6.3.2, the computable structure $(\sum_{1\leq i\leq n} r_i v_i)_{(r_1,\ldots,r_n)\in Q^n}$ is computably isometric to the usual structure on $\mathbb{R}^n$ via a computable isometry U. The composition $\mathrm{I} \circ \mathrm{U} : \mathcal{M} \to \mathbb{R}^\backslash$ is the needed isometry. $\square$

As a consequence of Theorem 7.3.1, many common computable compact subsets of $(\mathbb{R}^n, d)$ with the inherited metric are computably categorical. In the following, $B_n(r)$ denotes the n-dimensional ball of radius r, and $\mathrm{Cube_n(r)}$ stands for the n-dimensional cube (with its inside) with edge of length r. In particular, $\mathrm{Cube_1(r)}$ is isometric to the interval $[0, r]$. The metrics on $B_n(r)$ and $\mathrm{Cube_n(r)}$ are Euclidean. The following fact is rather straightforward:

Fact 7.3.3. *Let $M_n(r)$ be either the ball $B_n(r)$ or the cube* $\mathrm{Cube_n(r)}$. *The space $M_n(r)$ possesses a computable structure if, and only if, r is left-c.e.*

Proof. Suppose r is left-c.e. and $r = \sup_s q_s$ for a computable sequence of positive rationals $(q_s)_{s\in\omega}$. If $M_n(r) = B_n(r)$ is a ball, then define a computable structure starting from the geometrical center of $M_n(r)$ and expanding the structure on the stages s such that $q_s > q_{s-1}$. More formally, define a computable sequence of finite rational-valued metric subspaces X_n such that for every m we have $X_{\langle k,m\rangle} \subseteq B_n(q_m)$ and, furthermore, $\bigcup_k X_{\langle k,m\rangle}$ is a computable structure on $B_n(q_m)$. The sequence can be organized so that $d(x, y)$ is computable for every $x, y \in \bigcup_{k,m\in\omega} X_{\langle k,m\rangle}$. The desired structure is $\bigcup_{k,m\in\omega} X_{\langle k,m\rangle}$. The case when $M_n(r) = \mathrm{Cube_n(r)}$ can be done similarly.

Suppose $(\alpha_i)_{i\in\omega}$ is a computable structure on $M_n(r)$. The real $\mu = \sup\{d(\alpha_i, \alpha_j) : i, j \in \omega\}$ is clearly left-c.e. Note that $\mu = 2r$ if $M_n(r) = B_n(r)$, and $\mu = r\sqrt{n}$ if $M_n(r) = \mathrm{Cube_n(r)}$. $\square$

We have:

Corollary 7.3.1 ([51]). *For every $n \in \omega$ and every computable real r, the cube* $\mathrm{Cube_n(r)}$ *(the ball $B_n(r)$) is computably categorical.*

Proof. We prove the corollary for $\mathrm{Cube_n(r)}$ with $n = r = 1$, the general case is not significantly different from the case $n = r = 1$. We show that $\{0, 1\}$ is an intrinsically computable base of $[0, 1]$.

Let $(\alpha_i)_{i\in\omega}$ be a computable structure on $\mathrm{Cube_1}(1) = [0, 1]$. Define two uniformly computable sequences, $(\theta_k)_{k\in\omega}$ and $(i_k)_{k\in\omega}$ of special points of $(\alpha_i)_{i\in\omega}$ by recursion. The sequence $(\theta_k)_{k\in\omega}$ will be a Cauchy name for 0, and the sequence $(i_k)_{k\in\omega}$ will be a Cauchy name for 1. Let v and w be special points such that $d(v, w) > 3/4$ and $d(v, 0) < d(w, 0)$. Set $\theta_0 = v$ and $i_0 = w$. We use these points as non-uniform parameters. For $k > 1$, search in $(\alpha_i)_{i\in\omega}$ for special points x and y such that $d(x, y) > 1-2^{-k-2}$ and $d(x, \theta_{k-1}) < d(x, i_{k-1})$. Set $\theta_k = x$ and $i_k = y$. This completes the definition of $(\theta_k)_{k\in\omega}$ and $(i_k)_{k\in\omega}$. Clearly, $d(\theta_k, 0) \leq 2^{-k}$ and $d(i_k, 1) \leq 2^{-k}$, for every $k \geq 0$. □

Remark 7.3.1. If in Definition 7.3.1 we replace "there exists a surjective self-isometry W" by "for every surjective self-isometry W" then the corresponding analog of the preceding theorem will state that every isometry from a space having such a base is computable w.r.t. to any given computable structures. The proof needs only minor adjustments.

As a consequence, we obtain:

Fact 7.3.4. *Every self-isometry of* $[0, 1]$ *with the usual metric is computable w.r.t.* $(\alpha_i)_{i\in\omega}$ *and* $(\beta_j)_{j\in\omega}$*, for each computable structures* $(\alpha_i)_{i\in\omega}$ *and* $(\beta_j)_{j\in\omega}$ *on* $[0, 1]$.

Proof. A straightforward modification of the proof of the preceding corollary shows that $\{0, 1\}$ is an intrinsically computable base of $[0, 1]$ having the stronger property from Remark 7.3.1. □

As a consequence, every two computable structures on $[0, 1]$ are equivalent (that is, the identity map is computable w.r.t. to these structures). It is natural to ask what is the number of non-isometric computable structures on an interval of length r, where r is left-c.e. We show:

Theorem 7.3.2. *Let n be a positive natural and r a left-c.e. real. The following are equivalent:*

1. *r is computable,*

2. *the space* $\mathrm{Cube_n(r)}$ *is computably categorical,*

3. *there exists only finitely many computable structures on* $\mathrm{Cube_n(r)}$ *which are pairwise not computably isometric.*

Proof. The implication (1) $\to$ (2) is given by Corollary 7.3.1, and the implication (2) $\to$ (3) is trivial. We show $\neg(1) \to \neg(3)$.

Suppose r is a non-computable left-c.e. real. Consider the simplest case $\mathrm{Cube_1(r)} = [0, \mathrm{r}]$. We define infinitely many structures on $[0, r]$, as follows. For a natural $m > 0$, let $\alpha(m)_i$ and $\beta(m)_i$ be the m'th rational in any fixed enumeration of positive rationals from the left cuts of $2^{-m}r$ and $(1-2^{-m})r$, respectively. Clearly, $(\alpha(m)_i)_{i\in\omega}$ is a computable structure on $[0, 2^{-m}r]$, and $\beta(m)_i$ is a computable structure on $[0, (1-2^{-m})r]$. Note that 0 does not belong to either $(\alpha(m)_i)_{i\in\omega}$ or $(\beta(m)_i)_{i\in\omega}$.

Define a computable structure $(h(m)_i)_{i\in\omega}$ on $[0, r]$, as follows. Let $h(m)_0 = 2^{-m}r$. Given $i > 0$, let

$$h(m)_i = \begin{cases} 2^{-m}r + \beta(m)_k, \text{if } i = 2k, \\ 2^{-m}r - \alpha(m)_k, \text{if } i = 2k-1. \end{cases}$$

Observe that the distance between any two points from $(h(m)_i)_{i\in\omega}$ is uniformly computable, and the sequence $(h(m)_i)_{i\in\omega}$ in dense in $[0, r]$.

We show that $(h(m)_i)_{i\in\omega}$ is not computably isometric to $(h(n)_i)_{i\in\omega}$ if $m > n$. Pick a surjective isometry U of $[0, r]$ and assume (towards a contradiction) that U is computable with respect to $(h(m)_i)_{i\in\omega})$ and $(h(n)_i)_{i\in\omega})$. The point $U(h(m)_0)$ has to be computable in $(h(n)_i)_{i\in\omega}$. Note that $U(h(m)_0)$ is either $2^{-m}r$ or $(1-2^{-m})r$. In either case, $d(U(h(m)_0), h(n)_0)$ is not computable as it is a rational multiple of r, contradicting the choice of U. This finishes the proof for the case $\mathrm{Cube_1(r)} = [0, \mathrm{r}]$.

Suppose $n > 1$ and r is a non-computable left-c.e. real. For each $m \in \omega$ define $(\overline{h(m)}_i)_{i\in\omega}$ as folows. The sequence $(\overline{h(m)}_i)_{i\in\omega}$ consists of n-tuples in which the first component is taken from $(h(m)_i)_{i\in\omega}$, and the other components are rationals $< r$. It is not difficult to see that $(\overline{h(m)}_i)_{i\in\omega}$ is a computable structure on $\mathrm{Cube_n(r)}$. An argument similar to the case $n = 1$ shows that $(\overline{h(m)}_i)_{i\in\omega}$ is not computably isometric to $(\overline{h(k)}_i)_{i\in\omega}$, if $m \neq k$. In the case $n > 1$ we have more than two isometries, but still only finitely many. The distance between $(h(n)_0, 0, \dots, 0)$ and $(h(n)_0, 0, \dots, 0)$ can be expressed as $r \cdot v$, where v is a computable real which depends on the isometry. This finishes the proof. $\square$

7.3.2 A sufficient condition

We say that a collection B of points of a metric space $\mathcal{M} = (M, d)$ is *an automorphism base* of $\mathcal{M}$ if and only if any nontrivial surjective self-isometry of $\mathcal{M}$ necessarily moves at least one of its elements – or, equivalently, the global action of any such surjective self-isometry is completely determined by that on B. We need an effective version of this notion:

Definition 7.3.2. *We say that a finite automorphism base $B = \{b_1, \dots, b_k\}$ of a Polish space $\mathcal{M} = (M, d)$ is an* effective automorphism base *of $\mathcal{M}$ if:*

1. For every computable structure $(\alpha_i)_{i\in\omega}$ on $\mathcal{M}$ there is a surjective self-isometry U of $\mathcal{M}$ such that $U(b_1), \dots, U(b_k)$ are computable in $(\alpha_i)_{i\in\omega}$,

2. For every rational $\epsilon > 0$ we can compute a rational $\delta \in (0, 1)$ such that for every $x, y \in \mathcal{M}$ the inequality $\sup_j |d(b_j, x) - d(b_j, y)| < \delta/C_{x,y}$ implies $d(x, y) < \epsilon$, where $C_{x,y} = 1 + \sup_i d(b_i, x) + \sup_i d(b_i, y)$.

Thus, points in $\mathcal{M}$ are effectively determined by their distances to points in an effective automorphism base. Clearly, if the diameter of $\mathcal{M}$ is finite we can eliminate $(1 + \sup_i d(b_i, x) + \sup_i d(b_i, y))$ from (2.) in Definition 7.3.2. If the diameter of $\mathcal{M}$ is infinite, then $C_{x,y}$ is needed to make Theorem 7.3.3 work for subspaces of $\mathbb{R}^n$. We show:

Theorem 7.3.3. *Suppose a Polish space $\mathcal{M}$ possesses a computable structure. If $\mathcal{M}$ has an effective automorphism base then $\mathcal{M}$ is computably categorical.*

Proof. Suppose $\{b_1, \dots b_k\}$ is an effective automorphism base for $\mathcal{M}$. Let $(\alpha_s)_{s\in\omega}$ and $(\beta_s)_{s\in\omega}$ be two computable structures on $\mathcal{M}$. Let W_1 and W_2 be surjective self-isometries such that $W_1(b_1), \dots W_1(b_k)$ are computable w.r.t. $(\alpha_s)_{s\in\omega}$ and $W_2(b_1), \dots W_2(b_k)$ are computable w.r.t. $(\beta_s)_{s\in\omega}$. Let $U = W_1^{-1} \circ W_2$. We show that U is computable with respect to $(\alpha_s)_{s\in\omega}$ and $(\beta_s)_{s\in\omega}$.

Note that $W_1(b_1), \dots, W_1(b_k)$ is an effective automorphism base. Therefore, without loss of generality, we may assume $W_1 = \mathrm{Id}$. Thus, we assume the points $U(b_0), \dots, U(b_k)$ are computable w.r.t. $(\beta_s)_{s\in\omega}$, and the points $b_0, \dots, b_k$ are computable w.r.t. $(\alpha_s)_{s\in\omega}$.

Given $\epsilon > 0$ and a special point α_s, compute the rational $\delta < 1$ corresponding to ϵ. Find a special point β_v such that

$$\sup_j |d(b_j, \alpha_s) - d(U(b_j), \beta_v)| < \frac{\delta}{2 + 2\sup_j d(b_j, \alpha_s)}.$$

By the density of $(\beta_s)_{s\in\omega}$, such a special point can be effectively found. We have $d(b_j, \alpha_s) = d(U(b_j), U(\alpha_s))$ and $d(b_j, U^{-1}(\beta_v)) = d(U(b_j), \beta_v)$. Note that $\delta < 1$ implies $\sup_j |d(U(b_j), U(\alpha_s)) - d(U(b_j), \beta_v)| < 1$. Consequently,

$$2 + 2\sup_j d(b_j, \alpha_s) > 1 + \sup_j d(U(b_j), \beta_v) + \sup_j d(U(b_j), U(\alpha_s))$$

and

$$|d(U(b_j), U(\alpha_s)) - d(U(b_j), \beta_v)| < \frac{\delta}{1 + \sup_j d(U(b_j), \beta_v) + \sup_j d(U(b_j), U(\alpha_s))}.$$

By the choice of δ, we obtain $d(U(\alpha_s), \beta_v) < \epsilon$, showing that $U(\alpha_s)$ is computable w.r.t. $(\beta_s)_{s\in\omega}$. Thus, U is a computable map. □

Corollary 7.3.2. *The unit circle S_1 with the distance given by the shortest arc between points is computably categorical.*

Proof. Let (α_i) be a computable structure on S_1. This structure contains special points α_0, α_1 such that $d(\alpha_0, \alpha_1) < 1/2$. It is straightforward to check that α_0, α_1 form an effective automorphism base of S_1. □

Effective automorphism bases yield an alternative characterization of computable categoricity of subspaces of $\mathbb{R}^n$:

Theorem 7.3.4. *A closed subspace $\mathcal{M}$ of $\mathbb{R}^n$ which possesses a computable structure is computably categorical if, and only if, $\mathcal{M}$ has an effective automorphism base.*

Proof. By Theorem 7.3.3, it is sufficient to show that $\mathcal{M}$ contains an effective automorphism base provided that $\mathcal{M}$ is computably categorical. Let $(\alpha_i)_{i\in\omega}$ be a computable structure on $\mathcal{M}$. Let n be least such that $\mathcal{M}$ isometrically embeds into $\mathbb{R}^n$. By Theorem 7.3.3, the space $\mathcal{M}$ contains an intrinsically computable base $b_0, \dots, b_n$. Without loss of generality, we may assume that $b_0, \dots, b_n$ are computable w.r.t. $(\alpha_i)_{i\in\omega}$. By Fact 7.3.2, we may assume that $\mathcal{M}$ is a subspace of $\mathbb{R}^n$ such that each α_i is computable in the usual structure on $\mathbb{R}^n$ uniformly in i. As a consequence, the points $b_0, \dots, b_n$ are computable in the usual computable structure on $\mathbb{R}^n$. By the definition of intrinsically computable base, the elements $b_0, \dots, b_n$ satisfy (1.) of Definition 7.3.2. It remains to check (2.) of Definition 7.3.2 for $b_0, \dots, b_n$.

Let $\epsilon < 1$, and suppose $x, y \in \mathcal{M}$ are such that

$$|d(b_j, x) - d(b_j, y)| < \delta/(1 + \sup_i d(b_i, x))$$

for every $j \in \{1, \ldots, k\}$.

Let $v_j = b_j - b_0$ for $j \in \{1, \ldots, n\}$. There are uniquely defined tuples of reals $\overline{f} = (f_1, \ldots, f_n)$ and $\overline{h} = (h_1, \ldots, h_n)$ such that

$$y = \sum_{1 \le j \le n} f_j v_j \text{ and } x = \sum_{1 \le j \le n} h_j v_j.$$

Define

$$\overline{u} = (d^2(b_0, y) - d^2(b_1, y), \ldots, d^2(b_0, y) - d^2(b_n, y));$$
$$\overline{w} = (d^2(b_0, x) - d^2(b_1, x), \ldots, d^2(b_0, x) - d^2(b_n, x)).$$

As we have seen in the proof of Claim 7.3.2 in Theorem 7.3.1, the computability of $b_0, \ldots, b_n$ implies that there is a computable matrix B with a computable inverse such that $\overline{f} = B \cdot \overline{u}$ and $\overline{h} = B \cdot \overline{w}$. Let D be the matrix which corresponds to the Gram–Schmidt orthogonalization of $v_1, \ldots, v_n$. By the choice of $v_1, \ldots, v_n$, the matrix D and its inverse are computable. Let $\|\cdot\|$ stand for the usual norm in the space $\mathbb{R}^n$ of n-tuples of reals. We have

$$d^2(y, x) = \|DB(\overline{u}) - DB(\overline{w})\|^2 \le \|DB\|^2 \|\overline{u} - \overline{w}\|^2,$$

where $\|DB\| > 0$ is a computable real. Let $\delta = \min\{\epsilon \dfrac{1}{4\|DB\| \sqrt{n}}, \epsilon\}$ and

$$\delta_1 = \frac{\delta}{1 + \sup_j d(b_j, x) + \sup_j d(b_j, y)}.$$

We obtain

$$\begin{aligned}
\|\overline{u} - \overline{w}\|^2 &= \sum_{1 \le j \le n} (d^2(b_0, y) - d^2(b_j, y) - d^2(b_0, x) + d^2(b_j, x))^2 \\
&< \sum_{1 \le j \le n} (2\delta_1 \sup_i (d(b_i, y) + d(b_i, x)))^2 \\
&\le 4n(\delta_1(1 + \sup_j d(b_j, x) + \sup_j d(b_j, y)))^2 \\
&= 16n\delta^2.
\end{aligned}$$

Therefore, $d(y, x) < \|DB\| \cdot 4\sqrt{n}\delta \le \epsilon$. This finishes the proof. $\square$

Thus, Theorem 7.3.3 is a generalization of (2) $\to$ (1) part of Theorem 7.3.2 to metric spaces which are not subspaces of $\mathbb{R}^n$. We obtain:

Corollary 7.3.3. *For a closed subspace $\mathcal{M}$ of $\mathbb{R}^n$, the following are equivalent:*

1. *$\mathcal{M}$ is computably categorical;*
2. *$\mathcal{M}$ has an intrinsically computable base;*
3. *$\mathcal{M}$ has an effective automorphism base.*

Proof. By Theorem 7.3.3 and Theorem 7.3.4. □

Chapter 8

K-triviality in metric spaces

In this chapter we generalize the notion of K-triviality to the more general setting of a computable metric space.

8.1 Preliminaries

We use the notation in [80]. In particular, we write $s \leq^+ t$ to denote that $s \leq t + O(1)$. Throughout the chapter we will use the usual Cantor pairing function $\langle a, b\rangle = a + \frac{1}{2}(a+b)(a+b+1)$ to encode pairs of natural numbers by a single number. We provide some formal definitions of concepts particular to this chapter, most of which have already been discussed above.

8.1.1 K-triviality for functions

Fix some effective encoding of tuples x over ω by binary strings, so that $K(x)$ is defined for any such tuple. The following extends the definition of K-triviality for subsets of ω, which can be identified with $\{0,1\}$-valued functions.

Definition 8.1.1. *We say that a function $f\colon \omega \to \omega$ is K-*trivial *if*

$$\exists b\, \forall n\, K(f\upharpoonright_n) \leq K(n) + b.$$

By the following, K-triviality for functions may be reduced to K-triviality for sets.

Proposition 8.1.1. *A function $f\colon \omega \to \omega$ is K-trivial if and only if its graph $\Gamma = \{\langle n, f(n)\rangle\colon n \in \omega\}$ is K-trivial in the usual sense of sets.*

Proof. First suppose that f is K-trivial. We have

$$K(\Gamma\restriction_{n(n-1)/2}) \leq^+ K(f\restriction_n) \leq^+ K(n),$$

since to describe $\Gamma\restriction_{n(n-1)/2}$ it is enough to know the values of f up to n. For a set A, if there is a computable increasing sequence $\{q_n\}_{n\in\omega}$ such that $K(A\restriction_{q_n}) \leq^+ K(n)$ then A is K-trivial [80, Ex. 5.2.9]. Therefore the set Γ is K-trivial.

Now suppose that Γ is K-trivial. Recall that a set X is called low for K if $K(y) \leq^+ K^X(y)$ for each string y. By [79], K-triviality implies lowness for K (also see [80, Section 5.4]). Thus, $K(f\restriction_n) \leq^+ K^\Gamma(f\restriction_n)$. Clearly $K^\Gamma(f\restriction_n) \leq^+ K(n)$. Thus $K(f\restriction_n) \leq^+ K(n)$, as required. ◻

Note that the result does not depend on the particular choice of a pairing function: if we use an alternative pairing function, then the graph of f in terms of that pairing function is m-equivalent to Γ. Hence it is K-trivial iff Γ is. We could also formulate a similar theorem for other encodings of function by sets; for instance, a function f is K-trivial iff S is K-trivial, where S is given by the rule that $1 + f(n)$ is the $n+1$-th element of S minus the n-th element of f.

Note that the implication from right to left in Proposition 8.1.1 relies on the hard result of [79], which has a non-uniform proof. It is not known whether Proposition 8.1.1 is uniform, that is, whether a constant for the K-triviality of f can be computed from an index for Γ (say, as an ω-c.e. set) and its K-triviality constant. We conjecture that this is not the case.

Let $f \oplus g$ be the function u such that $u(2n) = f(n)$ and $u(2n+1) = g(n)$. The following is a consequence of the corresponding fact for sets; see [80, 5.2.17].

Corollary 8.1.1. *If f, g are K-trivial as functions, then so is $f \oplus g$.*

Proof. Use that fact that K-trivials are downward closed under Turing reducibility (see [80]). ◻

8.1.2 Solovay functions

Recall that a computable function $h\colon \omega \to \omega$ is called a *Solovay function* [9] if the $\liminf_r[h(r) - K(r)]$ exists (and is a finite integer). Solovay [94] constructed an example of such a function. The following simpler, recent example is due to Merkle (unpublished). We include the short verification for completeness' sake.

Fact 8.1.1. *There is a Solovay function h.*

Proof. Let $\mathbb{U}$ denote the optimal prefix-free machine. Given $r = \langle \sigma, n, t \rangle$, if t is least such that $\mathbb{U}_t(\sigma) = n$, define $h(r) = |\sigma|$. Otherwise let $h(r) = r$.

We have $K(r) \leq^+ h(r)$ because there is a prefix-free machine M which on input σ outputs $r = \langle \sigma, \mathbb{U}(\sigma), t \rangle$ if t is least such that $\mathbb{U}_t(\sigma)$ halts. If σ is also a shortest string such that $\mathbb{U}(\sigma) = n$, then we have $h(r) = |\sigma| = K(n) \leq^+ K(r)$. □

The following generalizes a known fact for sets, also due to Merkle, to the setting of functions.

Fact 8.1.2. *Let h be as in the proof of Fact 8.1.1. Let $f\colon \omega \to \omega$ be a function such that $\forall r\, K(f\upharpoonright_r) \leq h(r) + b$. Then f is K-trivial via a constant $b + O(1)$.*

Proof. Given n, let σ be a shortest $\mathbb{U}$-description of n, and let t be least such that $\mathbb{U}_t(\sigma) = n$. Let $r = \langle \sigma, n, t \rangle$. Then

$K(f\upharpoonright_n) \leq^+ K(f\upharpoonright_r) \leq h(r) + b = |\sigma| + b = K(n) + b.$ □

8.2 Computable points and K-trivial points

In the following we fix a computable metric space $\mathcal{M} = (M, d, (\alpha_i)_{i\in\omega})$. We will use letters p, q etc. for special points in M. They will be identified with natural numbers via the listing above. Recall that a point x is computable if one can effectively determine arbitrarily good approximations of x that are special points.

8.2.1 K-trivial points: definition and examples

A positive rational δ is viewed as a fraction $\frac{n}{v}$ where $\gcd(n, v) = 1$, which we effectively encode by a single natural number i_δ. For $i = i_\delta$ and $p \in \omega$ we let $K(\delta) := K(i)$ and $K(p, \delta) := K(p, i)$. The following is the main definition of the chapter.

Definition 8.2.1. Let $b \in \omega$. We say that a point $x \in M$ is *K-trivial via b*, or K-trivial(b) for short, if for each positive rational δ there is a special point p such that

$$d(x, p) \leq \delta \;\wedge\; K(p, \delta) \leq K(\delta) + b. \tag{8.1}$$

A point $x \in M$ is called K-trivial if it is K-trivial via some b.

Choosing a different effective encoding of the rationals will merely lead to a different constant b in (8.1), without affecting the set of K-trivial points. Thus we might as well use the canonical encoding of a positive rational δ: If $\delta = \frac{n}{v}$ where $\gcd(n, v) = 1$ let $i_\delta = \langle n, v\rangle$. (This yields the listing $1, 1/2, 2, 1/3, 3, 1/4, 2/3, \ldots$ of the positive rationals.)

Clearly each computable point is K-trivial. There may be no others:

Example 8.2.1. *There is an infinite compact computable metric space $\mathcal{M}$ with an incomputable point such that each K-trivial point is computable.*

Proof. Let ω denote Chaitin's halting probability. We have $\omega = \lim_s \omega_s$ where $\omega_s = \sum\{2^{-|\sigma|} \colon \mathbb{U}_s(\sigma)\downarrow\}$. Let $\mathcal{M}$ be the computable metric space with domain $\{\omega_s \colon s \in \omega\} \cup \{\omega\}$, the metric inherited from the unit interval and with the computable structure given by $\alpha_s = \omega_s$.

Assume for a contradiction that ω is a K-trivial point. Given n pick $p = \omega_s$ as in (8.1) for $\delta = 2^{-n}$. We can compute $\omega_s \upharpoonright_n$ (the first n bits of the binary expansion of ω_s) from p and n. Since $d(\omega, \omega_s) \leq 2^{-n}$, this shows that $K(\omega\upharpoonright_n) \leq K(p, n) + O(1) \leq K(n) + O(1)$, which contradicts $K(\omega\upharpoonright_n) \geq n - O(1)$ for sufficiently large n. □

In Theorem 8.4.1 we will show that each computable complete metric space $\mathcal{M}$ without isolated points contains an incomputable K-trivial point. Instead of giving a direct construction, we will derive this from the corresponding fact in Cantor space, using two facts: Proposition 8.4.2 below that K-triviality is preserved by a computable map from one metric space to another, and a result of Brattka and Gherardi that there is a 1-1 computable map from Cantor space into $\mathcal{M}$, which hence also preserves incomputability of points.

8.2.2 Number and distribution of K-trivial points for a constant b

Firstly, we note that few numbers p can satisfy the second inequality in (8.1):

Fact 8.2.1. *For $\delta \in \mathbb{Q}^+$, we have $|\{p \in \omega \colon K(p, \delta) \leq K(\delta) + b\}| = O(2^b)$.*

Proof. Chaitin [18] proved that there are only $O(2^b)$ strings v of length n such that $K(v) \leq K(n) + b$. To prove the fact at hand, one adapts, for instance, the proof in [80, 2.2.26] of his result, with the change that one lets $M(\sigma) = n$ if $\mathbb{U}(\sigma) = \langle i, n\rangle$, where $n \in \omega$ encodes δ. □

We now show that a similar bound holds for the number of points that are K-trivial(b). Furthermore, such a point x is determined by a highly compressible special point p close by.

Proposition 8.2.1. Let $b \in \omega$. *(i).* At most $O(2^b)$ many $x \in M$ are K-trivial(b). *(ii).* There is a rational $\delta > 0$ as follows. If a point x is K-trivial via b, then there is a special point p with $K(p, \delta) \le K(\delta) + b$ such that x is the only K-trivial(b) point with $d(x, p) \le \delta$.

Proof. Suppose that distinct points $x_1, \ldots, x_k \in M$ are K-trivial(b). Pick a rational $\delta > 0$ such that 2δ is less than $d(x_i, x_j)$ for any $i \ne j$, and choose p_i for x_i, δ according to (8.1). Then all the p_i are distinct. By Fact 8.2.1, this implies that $k = O(2^b)$, hence (i) holds. If k is chosen maximal then x_i is the only K-trivial(b) point x such that $d(x, p_i) \le \delta$, which establishes (ii). ◻

8.2.3 Dyadic K-triviality

We mostly work with the apparently weaker form of Definition 8.2.1 of K-triviality in metric spaces where δ only ranges over rationals of the form 2^{-n}.

Definition 8.2.2. A point $x \in M$ is *dyadically K-trivial via b* if for each $n \in \omega$ there is a special point p such that

$$d(x, p) \le 2^{-n} \ \wedge \ K(p, n) \le K(n) + b. \tag{8.2}$$

Clearly, a point that is K-trivial via v is dyadically K-trivial via $v + O(1)$. We will show in our main result Theorem 8.3.1 that being dyadically K-trivial via b already implies having a Cauchy name that is K-trivial as a function via $2b + O(1)$, which in turn easily implies being K-trivial via $2b + O(1)$. Thus, up to a computable change in constants, dyadic K-triviality is the same as K-triviality.

From this point on, we use the usual "level-by-level" enumeration of special points $\sigma(0^\omega)$ in Cantor space: $0(0^\omega), 1(0^\omega), 00(0^\omega), 01(0^\omega), \cdots$, and we use the similar coding for Baire space. Suppose a point f in Cantor or Baire space is K-trivial in the usual sense. Then it is clearly dyadically K-trivial: given n, let p be the special point determined by $f \upharpoonright_n$, namely $p(i) = f(i)$ for $i < n$ and $p(i) = 0$ for $i \ge n$. The p is a witness in (8.2). Conversely, each dyadically K-trivial point f is K-trivial in the usual sense, because given n, if p is a witness in (8.2), we have $K(f \upharpoonright_n) \le K(p, n) + O(1)$.

Recall that the unit interval $[0, 1]$ is a computable metric space with the usual distance function and the computable structure given by an effective listing without repetition of the rationals in $[0, 1]$. Suppose a point x has binary expansion $0.A$ where A is an infinite bit sequence (set). If A is K-trivial then as a witness p in (8.2) we may use the dyadic rational $0.(A\upharpoonright_n)$. Conversely, suppose $x \in [0,1]$ is dyadically K-trival. Given n let p be a witness in (8.2), and let σ be the first n bits in the binary expansion of p. Then $K(\sigma) \le K(p,n) + O(1)$; furthermore, $0.\sigma - 0.(A\upharpoonright_n) = c2^{-n}$ for some $c \in \{-1,0,1\}$. Therefore $K(A\upharpoonright_n) \le K(n) + O(1)$.

8.2.4 We cannot replace the term $K(p,\delta)$ in Definition 8.2.1 by $K(p)$

We provide the example showing that, even in Cantor space,the suggested replacement is not an adequate generalization of K-triviality. The "dyadic" version of this generalization is that

$$\forall n\, \exists p\, [d(x,p) \le 2^{-n} \,\wedge\, K(p) \le K(n) + O(1)]. \tag{8.3}$$

This is in fact equivalent to

$$d(x,\alpha_p) \le \delta \text{ and } K(p) \le K(\delta) + O(1), \tag{8.4}$$

because from $\delta > 0$ one can compute the least n such that $2^{-n} \le \delta$. Then a witness p for n in (8.3) is also a witness for δ in the new definition. Recall from [27] that a K-trivial is not Turing complete.

Proposition 8.2.1. There is a Turing complete Π^0_1 set $A \in \{0,1\}^\omega$ satisfying condition (8.3).

Proof. For a string α over $\{0,1\}$, let $g(\alpha)$ be the longest prefix of α that ends in 1, and $g(\alpha) = \emptyset$ if there is no such prefix. We say that a set A is *weakly K-trivial* if

$$\forall n\, [K(g(A\upharpoonright_n)) \le^+ K(n)].$$

This implies (8.3): given n let $p = g(A\upharpoonright_n)0^\infty$, then $d(p,A) \le 2^{-n}$. (As an aside, we note that every K-trivial set is weakly K-trivial. Every weakly K-trivial set with an infinite computable subset is already K-trivial by [80, Ex. 5.2.9].)

We now build a Turing complete Π^0_1 set A that is weakly K-trivial. We maintain the condition that

$$\forall i\, \forall w\, [\gamma_i < w \to K(w) > i], \tag{8.5}$$

where γ_i is the i-th element of A. This implies that A is Turing complete, as follows. We build a prefix-free machine N. When i enters $\emptyset'$ at stage s, we declare that $N(0^i1) = s$. This implies $K(s) \le i + d$ for some fixed coding constant d. Now $i \in \emptyset' \leftrightarrow i \in \emptyset'_{\gamma_{i+d}}$, which implies $\emptyset' \le_T A$.

We let $A = \bigcap A_s$, where A_s is a cofinite set effectively computed from s, $A_0 = \omega$, $[s, \infty) \subseteq A_s$, and $A_{s+1} \subseteq A_s$ for each s. We view γ_i as a movable marker; γ_i^s denotes its position at stage s, which is the i-th element of A_s.

Construction of A and a prefix-free machine M.
Stage 0. Let $A_0 = \omega$.

Stage $s > 0$. Suppose that there is w such that $i := K_s(w) < K_{s-1}(w)$. By convention, we may assume that w is unique and $w < s$. Thus, there is a new computation $\mathbb{U}_s(\sigma) = w$ with $|\sigma| = i$ at stage s.

If $w \leq \gamma_i^{s-1}$ then let $A_s = A_{s-1}$. If $w > \gamma_i^{s-1}$ then, to maintain (8.5) at stage s, we *move* the marker γ_i: we let $A_s = A_{s-1} - [\gamma_i^{s-1}, s)$, which results in $\gamma_{i+k}^s = s + k$ for $k \geq 0$, while $\gamma_j^s = \gamma_j^{s-1}$ for $j < i$.

In any case, declare $M(\sigma) = g(A_s \upharpoonright_w)$.

Verification. Clearly, each marker γ_i moves at most 2^{i+1} times, so $A = \bigcap_s A_s$ is an infinite co-c.e. set. Furthermore, condition (8.5) holds because it is maintained at each stage of the construction.

We show by induction on s that

$$\forall n \, [K(g(A_s \upharpoonright_n)) \leq^+ K_s(n)]. \tag{8.6}$$

For $s = 0$ the condition is vacuous. Now suppose $s > 0$ and (8.6) holds for $s - 1$. We may suppose that w as in stage s of the construction exists, otherwise (8.6) holds at stage s by inductive hypothesis.

As in the construction let $i = K_s(w)$, and let σ be the string of length i such that $\mathbb{U}_s(\sigma) = w$. If $w \leq \gamma_i^s$ then $A_s = A_{s-1}$, so setting $M(\sigma) = g(A_s \upharpoonright_w)$ maintains (8.6).

Now suppose that $w > \gamma_i^s$. Let $n < s$. We verify (8.6) at stage s for n.

If $n \leq \gamma_i^{s-1}$ then $A_s \upharpoonright_n = A_{s-1} \upharpoonright_n$ and $K_s(n) = K_{s-1}(n)$, so the condition holds at stage s for n by inductive hypothesis. Now suppose that $n > \gamma_i^{s-1}$. By (8.5) at stage $s - 1$ we have $K_{s-1}(n) > i$, and hence $K_s(n) \geq i$ (equality holds if $n = w$). Because $n, w > \gamma_i^{s-1}$ and we move the marker γ_i at stage s, we have $g(A_s \upharpoonright_n) = g(A_s \upharpoonright_w)$. Thus, setting $M(\sigma) = g(A_s \upharpoonright_w)$ ensures that the condition (8.6) holds at stage s for n. □

8.3 A point is K-trivial iff it has a K-trivial Cauchy name

As before we fix a computable metric space $\mathcal{M} = (M, d, (\alpha_i)_{i \in \omega})$. Recall that a sequence $(p_s)_{s \in \omega}$ of special points is called a Cauchy name if $d(p_s, p_t) \leq 2^{-s}$ for each

$s, t \in \omega$, $s < t$. If $x = \lim_s p_s$, we say that $(p_s)_{s\in\omega}$ is a Cauchy name for x. Note that $d(x, p_s) \le 2^{-s}$ for each s. Via the underlying listing of special points $(\alpha_i)_{i\in\omega}$ we may view a Cauchy name as a function $\omega \to \omega$. Our main result is the corresponding fact for K-triviality.

Theorem 8.3.1. *Let x be a point in a computable metric space $\mathcal{M}$.*

(i) If x has a Cauchy name that is K-trivial as a function via $u \in \omega$, then x is a K-trivial point via $u + O(1)$.

(ii) If x is a K-trivial point via $v \in \omega$, then x has a Cauchy name that is K-trivial as a function via $2v + O(1)$.

We begin with proving the easier part (i). Recall that before Definition 8.2.1 we fixed an effective encoding of positive rationals δ by natural numbers i_δ. It is easy to verify that $\delta \ge 2^{-i_\delta}$ for each δ.

Suppose now the function f is K-trivial via u, and a Cauchy name for x. Given $\delta > 0$ let $i = i_\delta$. If n is least such that $\delta \ge 2^{-n}$, then we can take $p = f(n)$ as a witness for K-triviality in the sense of (8.1). We have $i \ge n$ ($f(n)$ appears among $f(0), \ldots, f(i)$), and hence

$$K(f(n), i) \le K(f\restriction_{i+1}) + O(1) \le K(i) + u + O(1).$$

Since $K(i)$ is the same as $K(\delta)$ and $K(p, i)$ is the same as $K(p, \delta)$, this shows (i).

In 8.2.2 we defined the auxiliary concept of dyadic K-triviality, and noted that being K-trivial via v implies being dyadically K-trivial via $v + O(1)$. The following lemma will close the circle by establishing (ii).

Lemma 8.3.1. Suppose that the point $x \in M$ is dyadically K-trivial via b. Then x has a Cauchy name f that is K-trivial as a function via $2b + O(1)$.

Proof. After adding a natural number to the Solovay function h from Fact 8.1.1, we may suppose that $\forall r\, K(r) \le h(r)$.

The c.e. tree T. By a *tree* we mean a set $T \subseteq \omega^{<\omega}$ that is closed under taking prefixes. As usual, $[T]$ denotes the set of (infinite) paths of a tree T. We define a c.e. tree T such that each $f \in [T]$ is a Cauchy name after leaving out $f(0)$ and $f(1)$. Let $T = \bigcup_s T_s$, where we define the trees

$$T_s = \{(p_1, \ldots, p_v) \;:\; \forall i \le v\, [K_s(p_i, i) \le h(i) + b] \;\wedge\; \forall i < v\, [d(p_i, p_{i+1}) \le 2^{-i+1}]\}. \tag{8.7}$$

A thin c.e. subtree G of T. In the setting of Example **??**, T is a full 2-branching tree, consisting of all the tuples of the form $(\langle r_0, 0\rangle, \langle r_1, 1\rangle, \ldots, \langle r_v, v\rangle)$ where $r_i \in \{0, 1\}$. This shows that T may contain lots of Cauchy names consisting of witnesses for dyadic K-triviality; we cannot expect that each such Cauchy name is K-trivial as a function. Therefore we will prune T to a c.e. subtree G that is so thin that all strings τ in it are *compressible* in the sense that $K(\tau) \le h(|\tau|) + b + O(1)$; hence each infinite path is dyadically K-trivial by Fact 8.1.2.

We say that a special point $p \in \omega$ is *present at level n* of a tree $B \subseteq \omega^{<\omega}$ if there is $\eta \in B$ such that η has length n and ends in p. While G is only a thin subtree of T, we will ensure that each special point present at a level n of T is also present at level n of G. This will show that $[G]$ still contains a function that (after leaving out the first two values) is a Cauchy name for x. (In Example **??** there are only two labels at each level of T, so for $G \subseteq T$ we can simply take the tuples where each r_i is 0, except possibly the last. Then the only path of G is a computable Cauchy name of the limit point x.)

We will build a computable enumeration $(G_s)_{s\in\omega}$ of the tree G where G_s is a tree contained in T_s for each s.

Why can each string in G be compressed? Suppose at a certain stage, a new leaf labelled p appears at level n of T, but is not yet present at level n of G. Suppose also that p is a successor on T of a node labelled q. Inductively, q is already present at level $n-1$ of G; that is, there already is a node $\overline{\eta}$ of length $n-1$ on G that ends in q. Since p is present at level n of T, there is a $\mathbb{U}$-*description* showing that $K(p, n) \le h(n)+b$ (that is, there is a string w with $|w| \le h(n) + b$ such that $\mathbb{U}(w) = \langle p, n\rangle$). Since p is not present at level n of G, this $\mathbb{U}$-description is "unused". Hence we can use it as a description of a new node $\eta = \overline{\eta}\frown p$ in G. This ensures that $K(\eta) \le h(n) + b + O(1)$.

Note that we make use of the fact that once a string η in G is compressible at a stage, it remains so at all later stages. This is the reason we need the Solovay function h. If we tried to satisfy the condition $K(\eta) \le K(n) + b + O(1)$, we might fail, because $K(n)$ on the right side could decrease later on. We also needed the Solovay function to ensure that T is c.e.

In the formal construction, we build a prefix-free machine L (see [80, Chapter 2]) to give short descriptions of these nodes. The argument above is implemented via maintaining the conditions (8.9) and (8.10) below.

A slower computable enumeration $(\widetilde{T}_s)_{s\in\omega}$ of T. We define a computable enumeration $(\widetilde{T}_s)_{s\in\omega}$ of T that grows "one leaf at a time". The $\widetilde{T}_s$ are subtrees of the T_s. Let $\widetilde{T}_0$ consist only of the empty string. If $s > 0$ and $\widetilde{T}_{s-1}$ has been defined, see whether there is $\tau \in T_s - \widetilde{T}_{s-1}$. If so, choose τ least in some effective numbering of $\omega^{<\omega}$. Pick v maximal such that $\tau \upharpoonright_v \in \widetilde{T}_{s-1}$, and put $\tau \upharpoonright_{v+1}$ into $\widetilde{T}_s$. Clearly we have $\widetilde{T}_s \subseteq T_s$ and $T = \bigcup_s \widetilde{T}_s$.

Three conditions that need to be maintained at each stage. For strings $\tau, \eta \in \omega^{<\omega}$ we write $\tau \sim \eta$ if they have the same length and end in the same element. Recall that each label present at a level n of T needs to be also present at level n of G. Actually, in the construction we ensure that for each stage s, each label p that is present at a level n of $\widetilde{T}_s$ is also present at level n of G_s:

$$\forall \tau \in \widetilde{T}_s \, \exists \eta \in G_s \, [\tau \sim \eta]. \tag{8.8}$$

To make sure that each $\eta \in G$ satisfies $K(\eta) \leq h(|\eta|) + b + O(1)$, we construct, along with $(G_s)_{s\in\omega}$, a computable enumeration $(L_s)_{s\in\omega}$ of (the graph of) a prefix-free machine L. Let m, n range over natural numbers, and v, w over strings. We maintain at each stage s the conditions

$$\forall \eta \in G_s \, \forall m \Big[0 < m \leq |\eta| \to \exists v \, [\, |v| \leq h(m) + b \, \wedge \, L_s(v) = \eta \upharpoonright_m] \Big]; \tag{8.9}$$

$$\text{if } \mathbb{U}_s(w) = \langle p, n \rangle, \text{ then} \Big[w \in \operatorname{dom}(L_s) \ \to \ p \text{ is at level } n \text{ of } G_s \Big]. \tag{8.10}$$

Construction of $(G_s)_{s\in\omega}$ *and* $(L_s)_{s\in\omega}$.

Stage 0. Let G_0 contain only the empty string. Let $L_0 = \emptyset$. Clearly the conditions (8.8,8.9,8.10) hold for $s = 0$.

Stage $s > 0$. Inductively we assume that (8.8,8.9,8.10) hold for $s - 1$.

If $\widetilde{T}_s - \widetilde{T}_{s-1}$ is empty go to the next stage. Otherwise there is a unique $\tau \in \widetilde{T}_s - \widetilde{T}_{s-1}$. Let $n = |\tau|$. By the definition of the computable enumeration $(\widetilde{T}_r)_{r\in\omega}$, we have $\tau = \overline{\tau}\,\widehat{}\,p$ for some $\overline{\tau} \in \widetilde{T}_{s-1}$. Since $\tau \in T_s$, by the definition in (8.7) we have $\mathbb{U}_s(w) = \langle p, n \rangle$ for some $\mathbb{U}$-description w such that $|w| \leq h(n) + b$.

If p is already present at level n of G_{s-1}, then go to the next stage. Otherwise, by (8.10) for $s - 1$, we have $w \notin \operatorname{dom}(L_{s-1})$, i.e., we have not yet used w as an L-description.

By (8.8) for $s - 1$, there is $\overline{\eta} \in G_{s-1}$ such that $\overline{\tau} \sim \overline{\eta}$. Now let $\eta = \overline{\eta}\,\widehat{}\,p$. (Note that $\eta \in T_s$ because $\overline{\eta} \in G_{s-1} \subseteq T_s$ and $K_s(p, n) \leq h(n) + b$.) Put η into G_s. Set $L_s(w) = \eta$. Then conditions (8.8,8.9,8.10) hold at stage s. Go to the next stage.

Verification. Given a function $f \colon \omega \to \omega$, let $\widehat{f}$ denote the function given by $\widehat{f}(n) =$

$f(n+2)$. If $f \in [T]$ then $\widehat{f}$ is a Cauchy name.

Note that the prefix-free machine $L = L_b$ is obtained uniformly in b, so we can build a prefix-free machine M such that $M(0^b 1\sigma) = L_b(\sigma)$ for each binary string σ. Hence, if $f \in [G]$ then by condition (8.9) and Fact 8.1.2, $\widehat{f}$ is K-trivial as a function via $2b + O(1)$. The following now concludes the proof of the lemma.

Claim 8.3.1. *There is $f \in [G]$ such that $\widehat{f}$ is a Cauchy name of x.*

Since x is dyadically K-trivial, for each n we can choose p_n as in (8.2). Then $(p_1, \dots, p_n) \in T$. So by (8.8) we can choose a string $\eta_n \in G$ of length n that ends in p_n.

For each n there are only finitely many p such that $K(p,n) \le h(n) + b$. So each level of T is finite. Thus, by König's Lemma, there is an infinite path f on the subtree of G consisting of the strings that are a prefix of some η_n. For each $r > 0$ there is n such that $f\upharpoonright_r \preceq \eta_n$. So $n \ge r$. Hence

$$d(f(r-1), x) \le d(f(r-1), p_n) + d(p_n, x) \le 2^{-r+2} + 2^{-n} \le 2^{-r+3}.$$

This shows that $\widehat{f}$ is a Cauchy name for x.

This concludes Lemma 8.3.1, and thereby establishes Theorem 8.3.1. $\square$

8.3.1 An analog of Theorem 8.3.1 for plain Kolmogorov complexity C.

We adapt the foregoing proof. We say that a function $f\colon \omega \to \omega$ is C-trivial via u if $C(f\upharpoonright_n) \le C(n) + u$ for each n.

A point $x \in M$ is called *C-trivial via v* if for each positive rational δ there is a special point p such that

$$d(x,p) \le \delta \ \wedge \ C(p,\delta) \le C(\delta) + v. \tag{8.11}$$

Clearly (i) of Theorem 8.3.1 holds in the setting of plain complexity C, via the same proof. To prove an analog of (ii) for C, we replace the Solovay function h by a logarithm function. Let $\log_2 n$ denote the largest integer k such that $2^k \le n$. Clearly, with a shift in constants, condition (8.11) implies its dyadic variant: for each n there is a special point p such that $d(x,p) \le 2^{-n} \ \wedge \ C(p,n) \le C(n) + v + O(1)$. Since $C(n) \le^+ \log_2 n$, this in turn implies the condition

$$\forall n\, \exists p\, [d(x,p) \le 2^{-n} \ \wedge \ C(p,n) \le \log_2 n + b]. \tag{8.12}$$

(where $b = v + O(1)$).

We now obtain an analog of Lemma 8.3.1.

Proposition 8.3.1. *Suppose that (8.12) holds for a point x. Then x has a Cauchy name f such that $C(f \upharpoonright_n) \leq \log_2 n + 2b + O(1)$.*

Proof. Let $h(n) = \log_2 n + d$ where the constant d is chosen so that $h(n) \geq C(n)$ for each n. Now we follow the proof of Theorem 8.3.1(ii) with this definition of h. The machines L_b and M are defined as before but based on a plain universal machine. So they now become plain machines. □

This leads to a characterization of computable points via algorithmic information theory.

Corollary 8.3.1. *A point x is C-trivial iff it is computable.*

Proof. The implication from right to left is clear. For the converse, adapting the proof of Chaitin's result in [80, Thm. 5.2.20(ii)] to functions shows that each function f as in the foregoing proposition is computable. So x has a computable Cauchy name. □

8.4 Preservation results and existence of K-trivial points

We apply our main result Theorem 8.3.1 in order to obtain information on K-trivial points that is not at all obvious from the Definition 8.2.1. First we provide some more background on computable metric spaces.

8.4.1 Computable maps, and equivalent computable structures

Definition 8.4.1. Let $\mathcal{M}, \mathcal{N}$ be computable metric spaces. A map $F\colon \subseteq M \to N$ is called *computable* if there is a Turing functional Φ such that for each x in the domain of F and for every Cauchy name α for x, Φ^α is a Cauchy name for $F(x)$.

Recall that a computable map F is called *effectively uniformly continuous* if there is a computable function $h\colon \omega \to \omega$ such that for $x, y \in M$, $d(x, y) \leq 2^{-h(n)}$ implies $d(F(x), F(y)) \leq 2^{-n}$. (For instance, if F is Lipschitz with constant 2^c, then F is effectively uniformly continuous via $h(n) = n + c$.) We now show that for such F, the Turing functional Φ above can be chosen to be a weak truth-table reduction; that is, the use is computably bounded.

Proposition 8.4.1. *Suppose a computable map F is effectively uniformly continuous via h. Then there is a Turing functional Φ as in Definition 8.4.1 such that for each α, n the use of $\Phi^\alpha(n)$ is bounded by $h(n+2)+1$.*

Proof. Suppose F is computable via the Turing functional $\widetilde{\Phi}$. To define the functional Φ, given a Cauchy name α for a point x, let $\alpha[n]$ denote the Cauchy name γ that follows α up to $h(n+2)$ and then repeats, that is, $\gamma(i) = \alpha(i)$ for $i < h(n+2)$ and $\gamma(i) = \alpha(h(n+2))$ for $i \geq h(n+2)$. Let

$$\Phi^\alpha(n) = \widetilde{\Phi}^{\alpha[n]}(n+2).$$

Then $h(n+2)+1$ is a use bound as required. To show that $\beta = \Phi^\alpha$ is a Cauchy name for $y = F(x)$, note that $d(x, \alpha_{h(n+2)}) \leq 2^{-h(n+2)}$, and hence $d(y, F(\alpha_{h(n+2)})) \leq 2^{-(n+2)}$. Furthermore, since $\alpha[n]$ is a Cauchy name for $\alpha(h(n+2))$, we have $d(F(\alpha_{h(n+2)}), \beta_n) \leq 2^{-(n+2)}$. Therefore

$$d(y, \beta_n) \leq d(y, F(\alpha_{h(n+2)})) + d(F(\alpha_{h(n+2)}), \beta_n) \leq 2^{-(n+1)}.$$

Therefore $d(\beta_s, \beta_t) \leq 2^{-s}$ for $t \geq s$, and β converges to y. □

Recall the following:

Definition 8.4.2. We say that computable structures $(q_i)_{i\in\omega}$ and $(r_i)_{i\in\omega}$ on a metric space (M, d) are *equivalent* if the identity and its inverse are computable when the identity is viewed as a map $\mathcal{M} \to \mathcal{N}$, where $\mathcal{M} = (M, d, (q_i)_{i\in\omega})$ and $\mathcal{N} = (M, d, (r_i)_{i\in\omega})$.

For instance, if we apply a computable permutation to the special points, we obtain an equivalent structure.

8.4.2 Preservation of K-triviality

The following generalizes the fact [79] that the class of K-trivial sets is closed downward under Turing reducibility.

Proposition 8.4.2. *Let $\mathcal{M}, \mathcal{N}$ be computable metric spaces, and let the map $F\colon M \to N$ be computable. If x is K-trivial in $\mathcal{M}$, then $F(x)$ is K-trivial in $\mathcal{N}$.*

Proof. Let α be a K-trivial Cauchy name for x. Since F is computable, there is a Cauchy name $\beta \leq_T \alpha$ for $F(x)$. Then β is K-trivial by Proposition 8.1.1 and the result of [79] that K-triviality for sets is closed downward under $\leq_T$. □

If F is effectively uniformly continuous, then one can also give a direct proof which avoids the hard result from [79]. Moreover, from a K-triviality constant for x one can effectively obtain a K-triviality constant for $F(x)$, which is not true in the general case.

To see this, note that, by Proposition 8.4.1, we have $\beta = \Phi^\alpha$ for a Turing functional Φ with use bounded by some computable function g. Then

$$K(\beta \upharpoonright_n) \leq^+ K(\alpha \upharpoonright_{g(n)}) \leq^+ K(g(n)) \leq^+ K(n).$$

The increase in constants is fixed, because it only depends on Φ and g.

Since the identity map is Lipschitz, we obtain that K-triviality in a computable metric space is invariant under changing of the computable structure to an equivalent one:

Corollary 8.4.1. *Suppose a point x is K-trivial via b with respect to a computable structure on a metric space. Then for any equivalent structure on the same space, x is K-trivial via $b + O(1)$.*

In the previous chapter we showed that all computable structures on the unit interval are equivalent (Fact 7.3.4). This means that K-triviality is intrinsic to the unit interval as a metric space.

8.4.3 Existence of non-computable K-trivial points

A Polish (i.e., complete separable metric) space is said to be *perfect* if it has no isolated points. In the following we take Cantor space $\{0,1\}^\omega$ as the computable metric space with the usual computable structure.

Proposition 8.4.1. [11, Proposition 6.2] Suppose the computable Polish space $\mathcal{M}$ is perfect. Then there is a computable injective map $F\colon \{0,1\}^\omega \to \mathcal{M}$ which is Lipschitz with constant 1.

Theorem 8.4.1. *Let $\mathcal{M}$ be a perfect computable Polish space. Then for every special point $p \in M$ and every $\delta > 0$ there exists a non-computable K-trivial point x such that $d(x,p) \leq \delta$.*

Proof. We may assume $\delta \in Q$. Note that $\{x\colon d(x,p) \leq \delta\}$ with the inherited computable structure is again a perfect computable metric space $\mathcal{N}$. By the result of Brattka and Gherardi there is a computable injective Lipschitz map $F\colon \{0,1\}^\omega \to \mathcal{N}$.

Let A be a non-computable K-trivial point in Cantor space. Then $x = F(A)$ is K-trivial in $\mathcal{N}$, and hence in $\mathcal{M}$, by Proposition 8.4.2; actually only the easier case of Lipschitz functions discussed after 8.4.2 is needed.

As Brattka and Gherardi point out before Proposition 6.2, the inverse of F is computable (on its domain). Thus, if x is computable then so is A, which is not the case. $\square$

8.4.4 K-trivial compact sets

Given a Polish space M, let $\mathbb{K}(M)$ denote the Polish space of compact subsets of M with the Hausdorff distance (the maximum distance that a point in one set can have from the other set). If M is a computable metric space then $\mathbb{K}(M)$ carries a natural computable structure where the special points are the finite sets of special points in M. Thus we have a notion of K-trivial compact sets.

If M is Cantor space $\{0,1\}^\omega$, then the computable structure on $\mathbb{K}(M)$ is easily seen to be equivalent in the sense of Definition 8.4.2 to the one where the special points are the clopen sets. Barmpalias et al. [8] studied a notion of K-triviality for compact subsets V in Cantor space. For instance, they built a non-empty K-trivial Π^0_1 class V without computable paths. We will show that their notion coincides with ours.

They noted that the definition they gave originally is equivalent to requiring that the subtree $T_V = \{\sigma\colon [\sigma] \cap V \neq \emptyset\}$ of $\{0,1\}^{<\omega}$ is K-trivial. For the latter condition it suffices to ask that $K(T_V[n]) \leq^+ K(n)$ for each n, where $T_V[n]$ is (a suitable string encoding of) the finite set $\{\sigma \in T_V\colon |\sigma| = n\}$.

Fact 8.4.1. *Let $V \subseteq \{0,1\}^\omega$ be closed. Then*
V is a K-trivial point in $\mathbb{K}(\{0,1\}^\omega) \Leftrightarrow T_V$ is K-trivial as a set.

Proof. $\Leftarrow$: Let $P_n = \bigcup\{[\sigma]\colon \sigma \in T_V[n]\}$. Then the P_n are witnesses for dyadic K-triviality of V according to Definition 8.2.2.
$\Rightarrow$: For each n let P_n be a clopen set such that $d(P_n, V) < 2^{-n}$ and $K(P_n, n) \leq^+ K(n)$. Then for each string σ of length n we have $[\sigma] \cap V \neq \emptyset \leftrightarrow [\sigma] \subseteq P_n$. This shows that T_V is K-trivial. $\square$

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