Computable Ordered Abelian Groups and Fields

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Abstract. We present transformations of linearly ordered sets into ordered abelian groups and ordered fields. We study effective properties of the transformations. In particular, we show that a linear order L has a Δ_2^0 copy if and only if the corresponding ordered group (ordered field) has a computable copy. We apply these codings to study the effective categoricity of linear ordered groups and fields.
Key words: computable algebra, effective categoricity.

We study complexity of isomorphisms between computable copies of ordered abelian groups and fields[1]. Recall that an ordered abelian group is one in which the order is compatible with the additive group operation. Ordered fields are defined in a similar manner. We say that an ordered abelian group $\mathcal{A} = (A; +, \leq)$ is computable if its domain A, the operation $+$, and the relation $\leq$ are computable. Similarly, a field is computable if its domain and its basic operations are computable. If $\mathcal{A}$ is computable and isomorphic to $\mathcal{B}$, we say that $\mathcal{A}$ is a computable copy (or equivalently, computable presentation) of $\mathcal{B}$. We mention that Malcev started a systematic study of computable abelian groups [18], and Rabin initiated a systematic development of the theory of computable fields [22].

One of the main themes in the study of abstract mathematical structures such as groups and fields is to find their isomorphism invariants. For instance, in the case of countable abelian p-groups, Ulm invariants are well-studied (see [26]). However, in the study of computable structures the isomorphism invariants do not always reflect computability-theoretic properties of the underlying structures. For instance, Malcev in [18] constructed computable abelian groups G_1 and G_2 that are isomorphic but not computably isomorphic. In fact, in G_1 the dependency relation is decidable, while in G_2 it is not. Another example is Khisamiev's criterion obtained for computable abelian p-groups of small Ulm length [13]. It is not clear how one can extend this result of Khisamaev to arbitrary Ulm length.

In the study of properties of computable structures, one can use different types of transformation methods between the structures. The idea consists in transforming a given class of computable structures into another class of computable structures in such a way that certain desirable properties of structures

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in the first class are preserved under the transformation to the second class. Here are some examples of this type. Hirschfeldt, Khoussainov, Shore and Slinko in [6] interpret directed graphs in the classes of groups, integral domains, rings, and partial orders. These transformations preserve degree spectra of relations and computable isomorphisms types. Goncharov and Knight [12] provide a method of transforming trees into p-groups to show that the isomorphism problem for these groups is Σ^1_1-complete. Downey and Montalban [5] code trees into torsion-free abelian groups to obtain the analogous result for the torsion-free case. For more examples see [11]. In this paper we present transformations of linearly ordered sets into ordered abelian groups and ordered fields. We study effective properties of these transformations. In particular, we prove the following result:

Theorem 1. *There are transformations Φ and Ψ of the class of linear orders into the classes of ordered abelian groups and ordered fields, respectively, such that both Φ and Ψ preserve the isomorphism types. Moreover, a linear order L has a Δ^0_2 copy if and only if the corresponding ordered group $\Phi(L)$ (ordered field $\Psi(L)$) has a computable copy.*

As a consequence of this theorem and the results from [12], *the isomorphism problems for ordered abelian groups and ordered fields are Σ^1_1-complete.*

The transformations Φ and Ψ are applied to investigate the complexity of isomorphisms between computable copies of ordered abelian groups and fields. We recall known concepts. A structure (e.g. group or field) is *computably categorical* if there is a computable isomorphism between any two computable copies of the structure. Computably categorical structures have been studied extensively. There are results on computably categorical Boolean algebras [8] [16], linearly ordered sets [23] [10], abelian p-groups [24] [9], torsion-free abelian groups [21], trees [17], ordered abelian groups [7] and linearly ordered sets with function symbol [4]. If a structure $\mathcal{A}$ is not computably categorical, then a natural question is concerned with measuring the complexity of the isomorphisms between computable copies of $\mathcal{A}$. Here one uses computable ordinals. Let α be a computable ordinal. A structure $\mathcal{A}$ is Δ^0_α-categorical, if any two computable copies of $\mathcal{A}$ are isomorphic via a Δ^0_α function. In [19] McCoy studies Δ^0_2-categorical linear orders and Boolean algebras. In [1] Ash characterizes hyperarithmetical categoricity of ordinals. In [17], for any given $n > 0$, a tree is constructed that is Δ^0_{n+1}-categorical but not Δ^0_n-categorical. Similar examples are built in the class of abelian p-groups [3]. There are also examples of Δ^0_n-categorical torsion-free abelian groups for small n [20]. In this paper we apply the transformations Φ and Ψ to construct Δ^0_α-categorical ordered abelian groups and fields. We prove the following theorem:

Theorem 2. *Suppose $\alpha = \delta + 2n + 1$ is a computable ordinal, where δ is either 0 or a limit ordinal, and $n \in \omega$. Then there is there is an ordered abelian group (ordered field) which is Δ^0_α-categorical but not Δ^0_β-categorical, for any $\beta < \alpha$.*

Here is a brief summary of the rest of the paper. Section 1 provides some necessary background and notation for ordered abelian groups. Sections 2 and 3 prove Theorem 1 and Theorem 2 for the case of ordered abelian groups. Section 4 outlines both theorems for the case of ordered fields.

1 Basic concepts

We briefly introduce basic concepts of ordered abelian groups and computability. Standard references are [25] for computability, [14] for the theory of ordered groups, and [26] for abelian groups. We use $+$ to denote the group operation, 0 for the neutral element, and $-a$ for the inverse of a. We use notation na for $\underbrace{a+\ldots+a}_{n \text{ times}}$, where $n \in \omega$ and $a \in A$. Note if $n = 0$ then $na = 0$. Therefore, every abelian group is a Z-module. So one can define the notion of linear independence (over Z). For more background on general linear algebra see [15].

Definition 3. *Let X be a set. For $x \in X$, set $Zx = \langle \{nx : n \in Z\}, +\rangle$, where $nx + mx = (n+m)x$. The free abelian group over X is the group $\bigoplus_{x \in X} Zx$.*

Thus, the free abelian group over X is isomorphic to the direct sum of $card(X)$ copies of Z. In this paper, when we write $\sum_{b \in B} m_b b$, where B is an infinite subset of an abelian group and all m_b's are integers, we mean that the set $B_0 = \{b \in B : m_b \neq 0\}$ is finite. Thus, $\sum_{b \in B} m_b b$ is just our notation for $\sum_{b \in B_0} m_b b$.

Recall that an ordered abelian group is a triple $\langle G, +, \leq\rangle$, where $(G, +)$ is an abelian group and $\leq$ is a linear order on G such that for all $a, b, c \in G$ the condition $a \leq b$ implies $a + c \leq b + c$.

If $\varphi : A \to B$ is a homomorphism of structures A and B, then we write $a\varphi$ to denote the φ-image of $a \in A$ in B, and we write $A\varphi$ for the range of φ. Recall that $\varphi : A \to B$ is a homomorphism between ordered abelian groups A and B if (1) $(a + b)\varphi = a\varphi + b\varphi$, and (2) $a \leq b$ implies $a\varphi \leq b\varphi$ for all $a, b \in A$.

If $G_1 = (A; +, \leq)$ is a computable ordered abelian group isomorphic to G then we say that G_1 is a computable copy of G. As our groups are infinite and countable, the domain A of G_1 can always be assumed to be ω.

Recall that the arithmetical hierarchy [25] can be extended to the hyperarithmetical hierarchy [2] by iterating the Turing jump up to any computable ordinal. We follow [2] in our notations for the hyperarithmetical hierarchy.

Definition 4. *Let α be a computable ordinal. A structure $\mathcal{A}$ is Δ^0_α-categorical if there exists a Δ^0_α-computable isomorphism between any two computable copies of $\mathcal{A}$. The structure is $\Delta^0_\alpha(X)$-categorical if any two X-computable copies of $\mathcal{A}$ have a $\Delta^0_\alpha(X)$ isomorphism between them.*

2 Proof of Theorem 1 for ordered abelian groups

We define a transformation Φ of the class of linear orders into the class of ordered abelian groups as follows.

Definition 5 (Definition of $\Phi(L)$ and $G(L)$). *Let $L = \langle\{l_i : i \in I\}, \leq\rangle$ be a linear order. Let $G(L)$ be the free abelian group defined over L and ordered lexicographically with respect to L. More formally,*

1. *The domain of $G(L)$ consists of formal finite sums $m_{i_1}l_{i_1} + m_{i_2}l_{i_2} + \ldots + m_{i_k}l_{i_k}$, where $l_{i_j} \in L$, $0 \neq m_{i_j} \in Z$ and $l_{i_1} > l_{i_2} > \ldots > l_{i_k}$ in L.*
2. *The operation $+$ is defined coordinate-wise to make $\langle G(L), +\rangle$ free over L.*
3. *The order is defined by the positive cone as follows: $m_{i_1}l_{i_1} + m_{i_2}l_{i_2} + \ldots + m_{i_k}l_{i_k} > 0$, where $L \models l_{i_1} > l_{i_2} > \ldots > l_{i_k}$, iff $m_{i_1} > 0$.*

It is easy to check that the structure $G(L)$ is an ordered abelian group. Now define $L^\star = L \cup \{\star\}$, where the new element $\star$ is the least element of order $L^\star$. We set $\Phi(L) = G(L^\star)$.

The linear order L can clearly be identified with a subset of $G(L)$ under the mapping $l \to 1 \cdot l$. If $i : L \to Li$ is an isomorphism of linear orders then we say that $i : L \to Li \subset G(Li)$ is an *L-embedding* of L into $G(Li)$.

Definition 6. *For $g \in G$, where G is an ordered abelian group, the absolute value of $|g|$ is g if $g \geq 0$, and $-g$ otherwise. Two elements $g, h \in G$ are Archimedean equivalent, written $g \sim h$, if there is an integer $m > 0$ such that $|mg| > |h|$ and $|mh| > |g|$. The equivalence classes of $\sim$ are called the Archimedean classes of G.*

The following lemma describes the Archimedean classes of $G(L)$.

Lemma 7. *Two nonzero elements $g, h \in G(L)$ are Archimedean equivalent if and only if they can be written as follows:*

$$g = m_j l_j + \textstyle\sum_{l_i < l_j} m_i l_i,\ h = n_j l_j + \sum_{l_i < l_j} n_i l_i,$$

where $m_j \neq 0$ and $n_j \neq 0$.

Proof. If there are such decompositions then g and h are Archimedean equivalent. Now suppose $g \sim h$. By the definition of $G(L)$, every nonzero element a of $G(L)$ can be presented uniquely as $a = m_{i_1}l_{i_1} + m_{i_2}l_{i_2} + \ldots + m_{i_k}l_{i_k}$, where $l_{i_1} > l_{i_2} > \ldots > l_{i_k}$ in L. Thus $g = m_j l_j + \sum_{l_i < l_j} m_i l_i$ and $h = n_k l_k + \sum_{l_t < l_k} n_t l_t$. Suppose $l_j \neq l_k$. Without loss of generality we may assume $l_j < l_k$. Then $|mg| < |h|$, for all $m \in Z^+$. This is a contradiction.

Definition 8. *Let $[a]_\sim$ be the Archimedean equivalence class of $a \in G = G(L)$. Let $\mathcal{L}(G) = \langle\{[a]_\sim : a \in G, a \neq 0\}, \preceq\rangle$, where $[a]_\sim \preceq [b]_\sim$ if $G \models |a| < |b|$ or $[a]_\sim = [b]_\sim$.*

Proposition 9. *The linear orders L_0 and L_1 are isomorphic if and only if the ordered abelian groups $G_0 = G(L_0)$ and $G_1 = G(L_1)$ are isomorphic.*

Proof. If $L_0 \cong L_1$ then clearly $G_0 \cong G_1$. Suppose $L_0 = \langle\{\nu_i : i \in I\}, \leq\rangle$. By *Lemma 7*, $\mathcal{L}(G_0) = \langle\{[\nu_i]_\sim : i \in I\}, \preceq\rangle \cong \langle\{\nu_i : i \in I\}, \leq\rangle = L_0 \cong \mathcal{L}(G_1) \cong L_1$. Thus $G_0 \cong G_1$ implies $L_0 \cong L_1$.

As a consequence of *Proposition 9*, *Theorem 9* and ([12], *Theorem 4.4 (d)*), we have the following:

Corollary 10. *The isomorphism problem for ordered abelian groups is Σ^1_1 -complete.*

Proposition 11. *Let L be a $0'$-computable linear order with least element. Then there is a computable presentation H of $G(L)$ and a $0'$-computable L-embedding $i : L \to H$.*

Proof. Without loss of generality we assume that the domain of L is ω, and the least element of L has ω-number 0. The diagram $D_0(L)$ of L is $0'$-computable. Thus its characteristic function is the limit of a computable function. At *Step t* we define a finite approximation of $D_0(L)$ enumerating this computable function, and we denote this approximation by $D^t_0(L)$. We require that $D^t_0(L)$ is consistent with the axioms of finite linear order on elements $\{0, 1, \ldots, t\}$. We denote this linear order by L_t. At *Step t*, the procedure enumerates a finite part $D^t_0(H)$ of $D_0(H)$, the finite ordered semigroup H_t which is described by $D^t_0(H)$, and a finite map $i_t : L_t \to H_t$.

The idea of the construction can be illustrated by the following example. Suppose at *Step* $t-1$ we have $L_{t-1} = \ldots l_k < l_{k+1} < \ldots$ and $i_{t-1} : L_{t-1} \to H_{t-1}$. At *Step t* we may have $l_{k+1} < l_k$. In this case we declare $i_{t-1}l_{k+1}$ to be equal to $M_t l_k$, where M_t is a big natural number. We will require the i_t-images of l_j to be positive in the group for every t. Therefore $i_{t-1}l_{k+1} = M_t l_k i_{t-1}$ will preserve the order, but will glue the Archimedean classes of these elements. Then we may add new free generators to our group and redefine i. The existence of least element makes the construction simpler.

Step 0. Set $H_0 = \{0, a_0\}$, $L_0 = \{0\}$, $D^0_0(H) = \{a_0 > 0, 0 + 0 = 0\}$, $D^0_0(L) = \emptyset$ and $0i_0 = a_0$.

Suppose that $D^{t-1}_0(L)$, L_{t-1}, $D^{t-1}_0(G)$, H_{t-1} and i_{t-1} have been defined by the previous stages. Then *Step t* proceeds as follows:

Step t. Perform at most t stages in approximation of $D_0(L)$ on $\{0, 1, \ldots, t\} \subset \omega$ starting with $D^{t-1}_0(L)$, until getting an approximation $D^t_0(L)$ that respects the axioms of a linear order with the least element 0.

We say that $n \in \omega$, $n \leq t-1$, *is at the wrong place* if there is $k \in \omega$ such that (1.) $\omega \models k < n$, and (2.) $D^{t-1}_0(L) \vdash n < k$ but $D^t_0(L) \vdash k < n$, or $D^{t-1}_0(L) \vdash k < n$ but $D^t_0(L) \vdash n < k$.

Let R^t be the set of all numbers that are declared to be at the wrong place at *Step t*. There is a disjoint partition $R^t = R^t_0 \cup R^t_1 \cup \ldots$ such that $R^t_j = [r^t_{j,1}, \ldots, r^t_{j,h(t)}]$ is an interval (with end-points, i.e. closed) in L_{t-1}, for every j. By our hypothesis, L has least element 0 (evidently $0 \notin R^t$ for all t). Therefore the partition $R^t = R^t_0 \cup R^t_1 \cup \ldots$ can be chosen in such a way that for every j there is a number $k^t_j \in \omega$ such that $k^t_j \notin R^t$ and $D^{t-1}_0(L)$ says that the successor of k^t_j is in R^t_j.

Let M_t be a natural number greater then any number we have ever used in our procedure so far. For every $R^t_j = [r^t_{j,1}, \ldots, r^t_{j,h(t)}]$ add to $D^{t-1}_0(H)$ formulas corresponding to the following new equations:

$r^t_{j,1} i_{t-1} = M_t \cdot k^t_j i_{t-1}$, $r^t_{j,2} i_{t-1} = M^2_t \cdot k^t_j i_{t-1}, \ldots,$ $r^t_{j,h(t)} i_{t-1} = M^{h(t)}_t \cdot k^t_j i_{t-1}$.
Let $D(t)$ be $D^{t-1}_0(H)$ extended by this set of formulas.

For every $k \in R^t$ and for $k = t$ add a new element $x_k > 0$ to H_{t-1}, and for every $k \leq t$ set

$$ki_t = \begin{cases} ki_{t-1}, \text{ if } n \notin R^t \text{ and } k \neq t, \\ x_k, \text{ otherwise.} \end{cases}$$

Extend the order on H_{t-1} to $(R^t \cup \{t\})i_t$. This new extended order must meet the following requirements: (1) $L_t i_t \cong L_t$ (as linear orders) and (2) for all $k, s \notin R^t$, $D^{t-1}_0(G) \vdash ki_{t-1} < si_{t-1}$ iff $D(t) \vdash ki_t < si_t$. These are the only restrictions on the extended order on new elements $\{x_k\}_{k \in R^t}$. Therefore we can make these new elements look Archimedean independent of the all previously defined elements. Thus, we can always find an extended order satisfying (1) and (2). We can put x_k between the previously defined Archimedean classes to make sure that $L_t i_t \cong L_t$. Requirement (2) will be satisfied automatically.

Set $H_t = \{\sum_{0 \leq k \leq t} n_k \cdot (ki_t) : |n_k| \leq M^{t+1}_t\}$ and define $+$ on this set using the restriction: $a, b \in H_t$ implies $a + b \in H_t$, for all a, b. Then extend the linear order on $L_t i_t$ to the whole of G_t lexicographically. Set $D^t_0(G) = D_0(H_t)$.

Lemma 12. *The map* $\lim_t i_t = i : L \longrightarrow Li$ *is a* $0'$*-computable isomorphism of linear orders.*

Proof. For every $k \in \omega$ and every t_0, $ki_{t_0} = \lim_t ki_t$ if and only if $k \notin R_t$, for all $t \geq t_0$. Indeed, $ki_t \neq ki_{t-1}$ if and only if k is declared to be at the wrong place at *Step t*. But the diagram of L is $0'$-computable, and the definition of R_t uses the natural order of ω. By a simple inductive argument, $\lim_t ki_t$ exists for all k. Thus i is $0'$-computable. By the construction, $L_t i_t \cong L_t$. By our assumption, for all $k, s \in L$ there is t_0 such that $k, s \notin R_t$, for all $t \geq t_0$. Thus $D_0(L) \vdash s < k \Leftrightarrow D^t_0(L) \vdash s < k \Leftrightarrow D^t_0(H) \vdash si_t < ki_t \Leftrightarrow D_0(H) \vdash si < ki$.

The output $D_0 = \cup_t D^t_0(H)$ of the procedure is a computable diagram of ordered abelian group $H = \cup_t H_t$. Finally, by our construction and by *Lemma 12*, $H \cong G(Li)$. This finishes the proof.

Now *Theorem 1* follows from *Propositions 9* and *11*. Indeed, $\Phi(L) = G(L^\star)$, $L^\star$ has least element $\star$, L is X-computable if and only if $L^\star$ is, and $L^\star \cong L^\star_0$ if and only if $L \cong L_0$.

3 Proof of Theorem 2 for ordered abelian groups

In this section all groups are free lexicographically ordered abelian groups. So let L be a linear order, and let $i : L \to G = G(Li)$ be an L-embedding. Consider the *projection* map $\pi : G \setminus \{0\} \to L$ defined by the following rule. If $g \in G$, $g \neq 0$ and $g = m_j l_j + \sum_{l_i < l_j} m_i l_i$, then $g\pi = (m_j l_j + \sum_{l_i < l_j} m_i l_i)\pi = l_j \in L$. Note that in this particular representation of g we list the corresponding free generators in the decreasing order. This map has a number of useful properties.

(1) $li\pi = l$, for all $l \in L$.

(2) π induces a homomorphism of the linear order $(G^+, \leq)$ onto L, where G^+ is the cone of (strictly) positive elements.

(3) For all nonzero $a, b \in G$, $a \sim b$ if and only if $a\pi = b\pi$ (see *Lemma 7*).

(4) Let $\tau : G \setminus \{0\} \to \mathcal{L}(G)$ be the canonical map with respect to $\sim$, i.e. $a\tau = [a]_\sim$ for all $a \in G$, $a \neq 0$. If $\gamma : \mathcal{L}(G) \to L$ is an isomorphism such that $i\tau = \gamma^{-1}$, then $\tau\gamma = \pi$ (by (1) $-$ (3) and the proof of *Proposition 9*).

Lemma 13. *Let L and L_0 be linear orders and let $i : L \to G$ and $i_0 : L_0 \to H$ be L- and L_0-embeddings respectively. Also let $\pi_0 : H \to L_0$ be the projection. If $\varphi : G \to H$ is an isomorphism (of ordered abelian groups) then $i\varphi\pi_0 : L \to L_0$ is an isomorphism (of linear orders).*

Proof. It is not hard to see that $i\varphi\pi_0$ is a homomorphism of linear orders. The isomorphism φ preserves Archimedean equivalence, and every Archimedean class in $\mathcal{L}(G)$ has exactly one representative xi for some $x \in L$. Thus every element of $\mathcal{L}(H)$ has exactly one representative $xi\varphi$ for some $x \in L$. Therefore $i\varphi\tau_0 : L \to \mathcal{L}(H)$ is an isomorphism, where $\tau_0 : G \setminus \{0\} \to \mathcal{L}(H)$ is the canonical homomorphism, see (4) above. Thus, $i\varphi\tau_0\gamma_0$ is an isomorphism of L onto L_0, where $\gamma_0 : \mathcal{L}(H) \to L_0$ is an isomorphism such that $i_0\tau_0 = \gamma_0^{-1}$. By property (4) of projection maps applied to π_0, $i\varphi\tau_0\gamma_0 = i\varphi\pi_0$.

Now we turn to computable properties of the embedding described above.

Let $G = G(L)$ be computable. By the proof of *Proposition 9*, L is isomorphic to $\langle G \setminus \{0\}, \leq\rangle / \sim$, where $\sim$ is the Archimedean equivalence. The relation $\sim$ is Σ^0_1, thus L has a Σ^0_1-presentation, and it is $0'$-computable. There is a $0'$-computable set of representatives of $\mathcal{L}(G)$ in G. Recall that we identify L with the set of free generators of $G(L)$.

Proposition 14. *Suppose $G = G(L)$ is computable. Let $S \subset G$ be a $0'$-computable set of representatives of $\mathcal{L}(G)$. There is a set of representatives $C \subset G$ of $\mathcal{L}(G)$ and a $0''$-computable bijective map $\beta : S \to C$ such that:*

(1) $s \sim s\beta$, for all $s \in S$;

(2) there is an automorphism ψ of G such that $L\psi = C$, and $l\psi \sim l$ for all $l \in L$.

Proof. Consider the following $0''$-computable procedure, which builds the set C. We will show later that this set satisfies the needed requirements. We will define β later as well.

At *Step 0* we set $C_0 = \{l_0\}$, for some $l_0 \in L$. Suppose that C_{t-1} has been defined at *Step (t-1)*. At *Step t* search for a finite family of elements $C(t)$ such that:

(1) $c > 0$, for all $c \in C(t)$;

(2) $c \nsim c'$, for all $c \neq c'$ such that $c, c' \in C_{t-1} \cup C(t)$;

(3) for every $h \in G$, every $c \in C(t)$ and every integer k s.t. $|k| > 1$, $h \sim c$ implies $kh + c \sim c$;

(4) if $g \in G$ is the element with ω-number t then $g = \sum_{c \in C_{t-1} \cup C(t)} m_c c$ (where m_c is an integer, for every c).

Set $C_t = C_{t-1} \cup C(t)$ and proceed to *Step t+1*.

Lemma 15. *For every t, Step t halts.*

Proof. If $C(t)$ satisfies (3) and $c \in C(t)$, then $c = l_j + \sum_{l_i < l_j} m_i l_i$, for some $l_j \in L$. Indeed, if $c = l_j + \sum_{l_i < l_j} m_i l_i$, then (3) holds, by *Lemma 7*. If $c = m_j l_j + \sum_{l_i < l_j} m_i l_i$ and $|m_j| > 1$, then (3) fails for $h = l_j$ and $k = -m_j$.

Thus, if C_{t-1} has been defined then $C_{t-1} = \{l_j + \sum_{i \in K_j} m_i l_i : j \in J_{t-1}\}$, where J_{t-1} and all the K_j's are finite. The element g with ω-number t can be written as $g = \sum_{k \in I(t)} n_k l_k$, for some finite set $I(t)$. If we set $C(t) = \{l_k : k \in [I(t) \cup \bigcup_{j \in J_{t-1}} K_j] \setminus J_{t-1}\}$ then the requirements $(1) - (4)$ will be satisfied. Thus, there is at least one extension of C_{t-1} which satisfies $(1) - (4)$.

Let $C = \bigcup_t C_t$. By *Lemma 15* and requirement (4) of *Step t*, every element of G can be written as $\sum_{c \in C} m_c c$. This decomposition is unique because C is linearly independent (by requirement (2)). Therefore, G (without order) is isomorphic to the free abelian group over C.

Now suppose $s \in S$, $s = \sum_{c \in C} m_c c$. By requirement (2) of the procedure and *Lemma 7*, there is the unique $c_s \in C$ such that $c_s \sim s$. Let $s\beta = c_s$. This map is $0''$-computable.

For every $l_j \in L$ let $l_j \psi = c$, where $c \in C$ and $c \sim l_j$ (this element is unique, as above). Then close ψ under $+$ and $-$ by setting $(g - h)\psi = g\psi - h\psi$. It is not hard to see that ψ is bijective and preserves $+$ and $-$. On the other hand, $\sum_{j \in J} n_j l_j = n_{j_0} l_{j_0} + \sum_{l_j < l_{j_0}} n_j l_j > 0$ iff $n_{j_0} > 0$. But $(\sum_{j \in J} n_j l_j)\psi = \sum_{j \in J} n_j (l_j + \sum_{l_i < l_j} m_i l_i) > 0$ iff $n_{j_0} > 0$. Thus ψ is an automorphism of G.

Proposition 16. *Let L be a (countable) linear order with the least element and let $\alpha > 2$. Then L is $\Delta^0_\alpha(0')$-categorical if and only if $G(L)$ is Δ^0_α-categorical.*

Proof. $(\Rightarrow)$. Suppose $G \cong G_0 \cong G(L)$. Consider the $0'$-computable sets $\mathcal{L}$ and $\mathcal{L}_0$ of representatives of $\mathcal{L}(G)$ and $\mathcal{L}(G_0)$ respectively. Sets $\mathcal{L}$ and $\mathcal{L}_0$ with corresponding induced orders are isomorphic to L (see *Proposition 9*). By our assumption, there is a Δ^0_α isomorphism $\varphi : \mathcal{L} \to \mathcal{L}_0$. By *Proposition 14* there are $0''$-computable bijective maps $\beta : \mathcal{L} \to G(L)$ and $\beta_0 : \mathcal{L}_0 \to G(L_0)$ such that $G = G(\mathcal{L}\beta)$ and $G_0 = G(\mathcal{L}_0\beta_0)$. Thus $\beta^{-1}\varphi\beta_0 : L \to L_0$ can be extended to a Δ^0_α isomorphism of G onto G_0.

$(\Leftarrow)$. Suppose that $G(L)$ is Δ^0_α-categorical. Let L_0 and L_1 be $0'$-computable copies of L. By *Proposition 11*, there are computable ordered abelian groups $G_0 \cong G_1 \cong G(L)$ and $0'$-computable L_0- and L_1-embeddings $i_0 : L_0 \to G_0$ and $i_1 : L_1 \to G_1$. By our assumption there is a Δ^0_α isomorphism $\varphi : G_0 \to G_1$. Let $\pi_1 : G_1 \to L_1$ be a $0'$-computable projection such that $i_1\pi_1 = id_{L_1}$. By *Lemma 13*, the map $i_0\varphi\pi_1 : G_0 \to G_1$ is a Δ^0_α isomorphism.

The following result has already been mentioned:

Theorem 17 (Ash, [1]). *Let α be a computable ordinal. Suppose $\omega^{\delta+n} \leq \alpha < \omega^{\delta+n+1}$, where δ is either 0 or a limit ordinal, and $n \in \omega$. Then α is $\Delta^0_{\delta+2n}$-categorical and is not Δ^0_β-categorical, for $\beta < \delta + 2n$.*

By Proposition 16 and relativized Theorem 17, $G(\omega^{\delta+n})$ is $\Delta^0_{\delta+2n+1}$-categorical and is not Δ^0_β-categorical, for $\beta < \delta + 2n + 1$. This proves *Theorem 2*.

4 Ordered fields

Let $L = \langle \{x_i : i \in I\}, \leq \rangle$ be a (nonempty) linear order. Consider the ring of polynomials $Z[L]$ and the field of rational fractions $Q(L)$ with the set L used as the set of indeterminates. Informally, we will define an order on $Q(L)$ such that x_i is infinitely large relative to x_j if $L \models x_i > x_j$.

First, we order the set of monomials $\prod_i x_i^{\epsilon_i}$ *lexicographically* relative to L. We illustrate this definition by the following examples. Suppose $L \models x_3 < x_2 < x_1$. Then $x_1x_3 > x_2^4x_3^3$ since the first monomial has indeterminate x_1 which is L-greater then any indeterminate in the second monomial. We have $x_1x_2 > x_1x_3^3$ since $L \models x_2 > x_3$. Finally, $x_1x_2^3 > x_1x_2^2x_3$ because the L-greatest indeterminate x_2 in which these two monomials differ has power 3 in the first monomial, and 2 in the second. For any given $p \in Z[L]$ we can find the largest monomial $B(p)$ of p relative to the order defined above. We denote by $\mathcal{C}(p)$ the integer coefficient of $B(p)$ and let $p > 0$ if $\mathcal{C}(p) > 0$. Finally, $\frac{p}{q} > 0$ $(q > 0)$ if and only if $p > 0$. One can see that $Q(L)$ with this order respects all the axioms of ordered fields (see [15]).

Definition 18. *Suppose L is a linear order. Recall $L^\star = L \cup \{\star\}$ with least element $\star$. Set $\Psi(L) = Q(L^\star)$.*

Definition 19. *Let F be an ordered field. For any $g \in F$ we define its absolute value $|g|$ to be g if $g \geq 0$, and $-g$ if $g < 0$. Two elements a and b of F are Archimedean equivalent, written $a \sim b$, if there is integer $m > 0$ such that $|a^m| > |b|$ and $|b^m| > |a|$.*

Suppose $a \in Q(L)$. Let $\pi(a) = x_i$ where x_i is the L-greatest element of L that appears in the numerator of a. Say, $\pi(\frac{x_1^3x_2+x_1^2x_3}{x_2x_3-x_3}) = x_1$, if $L \models x_1 > x_2 > x_3$. Note that for $a, b > 1$ and $a, b \nsim 1$, $a \sim b$ if and only if $\pi(a) = \pi(b)$. Denote by $Q_1(L)$ the set of all elements of $Q(L)$ which are greater than 1 and that are not Archimedean equivalent to 1. Then $(Q_1(L)/\sim) \cong L$. Therefore, we have established the following

Proposition 20. *Let L and L_0 be linear orders. Then $Q(L) \cong Q(L_0)$ if and only if $L \cong L_0$.*

The proofs of *Theorems 1* and *2* for fields are similar to the proofs for ordered groups. In the proof of *Theorem 1* we declare $x_i = x_j^M$ (for a large number M) to make $x_i \sim x_j$. In the proof of *Theorem 2* we need the same technical propositions. In the proof of *Proposition 14* for fields we use slightly modified requirements to the extensions which respect the new definition of Archimedean equivalence.

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