DECOMPOSABILITY AND COMPUTABILITY

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ABSTRACT. We present a new construction of indecomposable type **0** abelian groups of rank 2. We apply this new construction to study degree spectra of such groups. As a corollary, we obtain a new computability-theoretic proof showing that there exist continuum non-isomorphic type **0** indecomposable abelian groups of rank 2.

1. INTRODUCTION

In this note, using basic concepts of computability theory, we present a new method of constructing indecomposable abelian groups. Recall that a group is indecomposable if it cannot be split into the direct sum of its two non-trivial subgroups. In the theory of abelian groups, constructing indecomposable groups has always been an important topic. The earliest examples of indecomposable abelian groups of rank 2 (to be defined) are due to Prüfer [Pru23], Levi [Lev19], and Pontryagin [Pon34]. For more examples see, e.g., Kurosh [Kur37], Baer [Bae37], De Croot [dG57], and Fuchs [Fuc57]; we also cite two very recent papers [AMMS14, MS18].

Henceforth, all our groups are torsion-free, countable, and abelian. It is convenient to view such groups as modules over the integers; then the (Prüfer) rank of a group is the cardinality of its maximal linearly independent subset over Z. We will usually also assume that the rank of an indecomposable group is at least 2 since the case of rank 1 is trivial.

There is one special subclass of indecomposable groups which drew considerable attention. These are groups which are "homogeneous of type **0**"; to avoid a lengthy introduction to Baer type theory, we just say that these are exactly the groups such that all their rank 1 subgroups are infinite and cyclic. In what follows next we usually suppress "homogeneous" and write simply "type **0**". The earliest and easiest examples of indecomposable groups used infinite divisibility by primes, and therefore were not type **0**. For instance, it is not hard to show that the additive subgroup of Q^2 generated by $\{(\frac{1}{p^n}, 0), (0, \frac{1}{q^n}), (\frac{1}{w^n}, \frac{1}{w^n}) : n \in \omega\}$, where p, q, w are distinct primes, is indecomposable. See the exercises 1-3 on page 127 of [Fuc73] for more early examples of indecomposable groups using infinite divisibility, and for the relevant bibliographic references.

More effort is needed to construct an indecomposable group (of rank > 1) while keeping its type equal to **0**. Examples of type **0** indecomposable groups are usually built using somewhat indirect methods. For one such construction that uses algebraically independent p-adics over Q, see §88 of Chapter XIII in [Fuc73]. Further examples with additional algebraic properties can be found in, e.g., [FL71]. A much more direct proof is contained in [Pon34].

One naturally seeks to analyse the computability-theoretic content of such constructions and estimate the complexity of the constructed groups. One of the standard invariants that reflect the computational content of a countable group (more generally, a countable algebraic structure) $\mathcal{A}$ is its degree spectrum [Ric81]. We recall that a *computable presentation* (a computable copy, a constructivization) of an infinite countable group G is an isomorphic copy G' of G such that the domain of G' is the set ω of all natural numbers and the group operation $+'$ of G' is a Turing computable function [Mal62, Rab60]. For a Turing degree $\mathbf{a}$, define $\mathbf{a}$-computable presentation similarly. The degree spectrum of $\mathcal{A}$ is the set

$$DSp(\mathcal{A}) = \{\mathbf{a} : \mathcal{A} \text{ has an } \mathbf{a}\text{-computable presentation}\}.$$

Much work has been done on degree spectra of groups and other structures; for spectra of abelian groups see [Mel09, KKM13, Mel17, Dow97, CDS00]. If a class is very restricted from the computability-theoretic perspective, then this is reflected in the complexity of degree spectra that can be found in the class. The classical example is that every low_4 Boolean algebra is isomorphic to a computable one [KS00].

The class of type $\mathbf{0}$ indecomposable groups of rank 2 is seems very narrow indeed. Degree spectra of finite rank torsion-free abelian groups have been completely described [CDS00, Mel09, CHS07]. But there are many more restrictions on the class on top of having rank 2, so one might expect these restrictions to be reflected in the possible degree spectra. It is easy to build computable type $\mathbf{0}$ rank 2 indecomposable groups[1]. Can we say more? The main result shows that, perhaps unexpectedly, from the computability-theoretic perspective these groups are not any simpler than just arbitrary finite rank torsion-free abelian groups.

Theorem 1. *For every torsion-free abelian group G of finite rank there is a type $\mathbf{0}$ indecomposable group A of rank 2 such that $DSp(G) = DSp(A)$.*

We prove the theorem by effectively coding an infinite set into a group[2]. The result can potentially be proven in many different ways, but we believe that the construction that we give in our proof is new. It contains several fresh ideas that are potentially more valuable than the result itself. One idea is that if the constructed group was decomposable then it would be computable, but it is easy to see that it is not. Perhaps, this approach can find applications in some situations when algebraic arguments are less helpful. Also, to prove that the constructed group is homogeneous we use the asymptotic speed of growth of the function $k \to p_k$ (the kth prime number); as far as we know, this idea is new as well.

The result has several immediate corollaries. For example, for any non-computable Turing degree $\mathbf{a}$ there is a indecomposable type $\mathbf{0}$ group G of degree $\mathbf{a}$:

$$\{\mathbf{b} : \mathbf{b} \text{ computes a copy of } G\} = \{\mathbf{b} : \mathbf{b} \geq \mathbf{a}\}.$$

This is Corollary 1. Noting that our groups are not automorphically trivial [Kni86], the upper cone is *equal* to the set of degrees of copies of the group. We also conclude

[1]For example, the group constructed on pages 384-385 of [Pon34] is evidently computable. Since there exist computable (indeed, even automatic [BY17]) transcendental p-adic numbers, the aforementioned construction in [Fuc73] can be performed computably as well.

[2]An expert in degree spectra will quickly recognize that another way to state the result is that for every set S there is a group A in the class having spectrum $\{\mathbf{a} : S \text{ is c.e. in } \mathbf{a}\}$. We know that every torsion-free abelian group of finite rank has an enumeration degree in this sense [Mel09, CHS07] (and thus the least jump degree [CDS00]; we omit the definition). Thus, it is sufficient to code an arbitrary set of natural numbers in the $\exists$-diagram of the group A.

that there is a indecomposable rank 2 group of type **0** without Turing degree. Since there are continuum Turing degrees, we get a new computably-theoretic proof of the following classical:

Theorem 2 (Theorem 88.4 of [Fuc73]). *There exist continuum non-isomorphic indecomposable type* **0** *groups of rank* 2*.*

See Corollary 2 for more details.

Nothing beyond the basic definitions in algebra and computability theory (which will be presented when necessary) is needed to understand the proof of Theorem 1. A recursion theorist will perhaps appreciate our detailed proof of the theorem, but of course, it could potentially be compressed. The methods and results of this paper can surely be generalised to any finite rank. We keep this note short and leave the generalisation to the reader.

2. Proof of Theorem 1

Proof. As discussed in the introduction, given any set S it is sufficient to produce a group G such that G has an **a**-computable copy iff S is c.e. in **a**, and we can also assume $S \subseteq Z^+$ is not c.e. Fix an unbounded computable function $\varphi : Z^+ \to Z^+$. Let $p_1, p_2, \dots$ be the listing of all primes in the usual increasing order. Assume additionally that φ satisfies

$$(1) \qquad (\exists k_0)\,(\forall k > k_0)\;\; 0 < |t_2 - \varphi(k)t_1| < p_k,$$

where $t_1, t_2 \in Z$ are arbitrary such that at least one of them is not zero[3].

Define the group $G_{S,\varphi} = (e_1, x_i : i \in S) \leqq Q^2 = Qe_1 \oplus Qe_2$, where e_1 and e_2 are the standard unit vectors, and $x_i = \frac{e_1+\varphi(i)e_2}{p_i}$. Every element g of $G_{S,\varphi}$ can be written as $g = m_1e_1 + \sum_{j\in J} d_jx_j$, but this expression is not unique in general. It is not difficult to see that the rank of this group is equal to 2 since $e_1 + \varphi(i)e_2$ and $e_1 + \varphi(j)e_2$ are independent whenever $\varphi(i) \neq \varphi(j)$.

Lemma 1. *$G_{S,\varphi}$ has an* **a***-computable presentation if and only if S is c.e. in* **a***.*

Proof. If S is c.e. in **a** then the group clearly has an **a**-computable presentation since φ is a computable function. For the other implication, note that the group $G_{S,\varphi}$ is contained in the group $[Z]_{P_S}e_1 \oplus [Z]_{P_S}e_2$, where $P_S = \{p_i : i \in S\}$ and $[Z]_X$ stands for the localisation of Z by X. In particular, when $i \notin S$ and $p_i \not| GCD(m, n)$, we have $\frac{me_1+ne_2}{p_i} \notin [Z]_{P_S}e_1 \oplus [Z]_{P_S}e_2$ and, thus, $\frac{me_i+ne_2}{p_i} \notin G_{S,\varphi}$. On the other hand, for any $i \in S$ there exist m, n (more specifically, $m = 1$ and $n = \varphi(i)$) such that $\frac{me_i+ne_2}{p_i} \in G_{S,\varphi}$. Now let H be a computable presentation of the group, and let c_1, c_2 be isomorphic images of e_1, e_2, respectively. It follows that

$$S = \{i : (\exists m, n \in Z)(\exists y \in H)\,[p_iy = mc_1 + nc_2 \,\&\, p_i \not| GCD(m, n)]\},$$

which shows that S is computably enumerable in **a**. □

Write $\mathbf{t}(g)$ for the type of the least pure subgroup containing g. We will only use the cases $\mathbf{t}(g) \neq 0$ and $\mathbf{t}(g) = 0$, the former being the negation of the latter (which was defined in the introduction). Our next goal is to show that the type of every element in $G_{S,\varphi}$ is **0**. We first need two simple lemmas. For an integer m and a group-element x, we write $m|x$ to express that m divides x in the group, that is,

[3]We could take φ to be, e.g., logarithmic or linear since the asymptotic speed of growth of $k \to p_k$ is $nlog(n)$.

$my = x$ for some group element y. Recall that a subgroup A of G is pure [Fuc70] if, for every $a \in A$ and every integer m, $m|a$ in G implies $m|a$ in A.

Lemma 2. *The subgroup generated by e_2 is pure in $G_{S,\varphi}$.*

Proof. Suppose we have $ng = ce_2$, for some $n, c \in Z$ and $g = m_1e_1 + \sum_{j\in J} d_jx_j \in G_{S,\varphi}$. Then $n(m_1 + \sum_{j\in J} \frac{d_j}{p_j}) = 0$, because $\{e_1, e_2\}$ is a basis of $G_{S,\varphi}$. Since all these primes are distinct, this implies $p_j|d_j$, for all $j \in J$. On the other hand, $n\sum_{j\in J} \frac{d_j\varphi(j)}{p_j} = c$ and thus $c = n\sum_{j\in J} d'_j\varphi(j)$ for some $d'_j \in Z$ witnessing $n|ce_2$. □

Lemma 3. *If $\mathbf{t}(g) \neq 0$ for a nonzero $g \in G_{S,\varphi}$ then there is an infinite set of primes $P \subseteq P_S = \{p_i : i \in S\}$ such that $p|g$, for every $p \in P$.*

Proof. First, we claim that $G_{S,\varphi}$ has no non-zero elements of infinite p-height, for each fixed prime p. Recall that the p-height of an element x is the largest power h of p such that $p^h|x$. Every element $g \in G_{S,\varphi}$ can be viewed as an element of the group Q^2. If there is a non-zero element of an infinite p-height, then we would have $\frac{g}{p^n} = \frac{r_1e_1+r_2e_2}{p^n} = \frac{r_1}{p^n}e_1 + \frac{r_2}{p^n}e_2 \in G_{S,\varphi}$, for all n. In particular, after finitely many reductions we would have $\frac{r'_1e_1+r'_2e_2}{p^2} \in G_{S,\varphi}$, where the greatest common divisor of the numerators of r'_1 and r'_2 is not divisible by p. (If both numerators of r_1 and r_2 were divisible by a power of p, say p^3, then take $n = 5$.) But $G_{S,\varphi}$ is generated by fractions that do not have p^2 in their denominators, for any choice of p. Therefore, if the set $R = \{p : p \text{ is prime and } p|g\}$ was finite then $\mathbf{t}(g) = 0$ would hold.

Now assume $P = R \cap P_S$ is finite, and therefore $R \setminus P_S$ is infinite. As above, consider $g = r_1e_1 + r_2e_2$, and let U be the finite set of primes that divide both numerators of r_1 and r_2. Since $D = R \setminus P_S$ is infinite, there must be a prime $q \in D \setminus U$. For this prime we have,

$$\frac{r_1e_1 + r_2e_2}{q} \notin [Z]_{P_S}e_1 \oplus [Z]_{P_S}e_2,$$

contradicting $G_{S,\varphi} \subset [Z]_{P_S}e_1 \oplus [Z]_{P_S}e_2$ which holds by construction. The contradiction shows that $P = R \cap P_S$ must be infinite. □

Proposition 1. *$G_{S,\varphi}$ is a homogeneous group of type 0.*

Proof. Recall that φ satisfies (1). Suppose $G_{S,\varphi}$ is not homogeneous. Then, by Lemma 3, for some non-zero $g \in G_{S,\varphi}$ there must exist arbitrarily large primes p_k such that $p_k|g$ and $k \in S$. Suppose this element is $g = m_1e_1 + \sum_{j\in J} d_jx_j \neq 0$. Let $t_1, t_2 \in Z$ be such that $\prod_{j\in J} p_jg = t_1e_1 + t_2e_2$. Clearly, $g \neq 0$ implies that at least one of t_1, t_2 is not zero. For any $p_k \notin J$, $p_k|g$ if, and only if, $p_k|t_1e_1 + t_2e_2$. (Note that t_1, t_2 do not depend on k here.) If additionally $k \in S$ then $p_k|(t_1e_1 + t_2e_2)$ implies $p_k|t_1e_1 + t_2e_2 - p_kt_1x_k$. Recall that $x_k = \frac{e_1+\varphi(k)e_2}{p_k}$, thus

$$t_1e_1 + t_2e_2 - p_kt_1x_k = (t_2 - \varphi(k)t_1)e_2$$

is divisible by p_k in the group. By Lemma 2 we must have

$$t_2 - \varphi(k)t_1 = 0 \bmod p_k.$$

By the choice of g, the index $k \in S$ in the equation above can be chosen arbitrarily large. But (1) says that there exists k_0 such that $t_2 - \varphi(k)t_1 \neq 0 \bmod p_k$, for all $k > k_0$. This is a contradiction. □

Finally, to finish the proof of the theorem, recall that S is not computably enumerable. If $G_S = G_{\varphi,S}$ was non-trivially decomposable, $G_S = G_1 \oplus G_2$, then each summand G_i would have to be a type $\mathbf{0}$ rank 1 group and, thus, would be isomorphic to Z. However, G does not have a computable presentation by Lemma 1, but the group $Z \oplus Z$ is clearly computable[4]. $\square$

Corollary 1. *For any Turing degree $\boldsymbol{a} > \boldsymbol{0}$ there is an indecomposable type* $\mathbf{0}$ *group G of Q^2 having degree $\boldsymbol{a}$.*

Proof. Let $A \in \mathbf{a}$. Apply Theorem 2 above with $S = A \oplus \overline{A}$. $\square$

Corollary 2. *There exist continuum many non-isomorphic indecomposable type* $\mathbf{0}$ *groups of rank 2.*

Proof. Different choices of Turing degrees in the previous corollary will clearly give non-isomorphic groups since their spectra are different. $\square$

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[4]There are other ways to finish the proof, some purely algebraic, but none are quite as elementary. For example, with a bit more work, one could use the well-known criterion of Baer for complete decomposability of a finite rank homogeneous group; see §98 of [Fuc73]. (Note that our group is decomposable iff it is completely decomposable.) Another possibility is to argue directly: assume the group is free abelian and derive a contradiction using divisibility by primes, type $\mathbf{0}$ homogeneity of the group, and the choice of φ.

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