ON A QUESTION OF KALIMULLIN

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Abstract. We prove that for every computable limit ordinal α there exists a computable structure $\mathcal{A}$ which is Δ^0_α-categorical and α is smallest such, but nonetheless for every isomorphic computable copy $\mathcal{B}$ of $\mathcal{A}$ there exists a $\beta < \alpha$ such that $\mathcal{A} \cong_{\Delta^0_\beta} \mathcal{B}$. This answers a question raised by Kalimullin in personal communication with the third author.

1. Introduction

Much of classical mathematics is concerned with classification of mathematical structures by their isomorphism types. Two mathematical structures are usually identified if they are isomorphic. However, such a classification blurs fine-grained distinctions related to the algorithmic nature of the structures. For example, it is easy to construct two algorithmically presented versions (to be clarified) of a simple structure like $(\mathbb{N}, \leq)$ with wildly differing computability-theoretic properties, such as decidability (or non-decidability) of the adjacency relation. On the other hand, sometimes computable isomorphism type coincides with classical isomorphism type, which is the case for a dense countable linear ordering or a finitely presented group.

Computable structure theory [AK00, EG00] has grown to understand the computability-theoretic properties of *computably presented* structures. Recall that a countably infinite algebraic structure is *computable* if its domain is the natural numbers $\mathbb{N}$ and its operations are Turing computable functions. If an algebraic structure $\mathcal{A}$ is isomorphic to a computable structure $\mathcal{B}$, then we say that $\mathcal{B}$ is a computable copy, a computable presentation, or a constructivization of $\mathcal{A}$. As we noted above, an infinitely generated algebraic structure may have computable copies with wildly differing computability-theoretic properties. Based on this observation, Maltsev [Mal61] suggested that computable structures should be studied under computable isomorphism. In particular, we say that a countably infinite structure is *computably categorical* if it has a unique computable copy up to computable isomorphism. Although computably categorial structures are "unclassifiable" in general (see [DKL$^+$15]), computable categoricity tends to be very well-behaved within many standard algebraic classes. For instance, a Boolean alegrba is computably categorical iff it has only finitely many atoms [Gon97, LaR77], and a torsion-free abelian group is computably categorcial iff its rank is finite [Nur74]. For most of these "nice" algebraic classes, computable categoricity is equivalent to the stronger notion of relative computable categoricity; we omit the definition of relative computable categoricity, see [AK00]. In contrast to computable categoricity in general, relative computable categoricity admits a syntactical description, a so-called computably enumerable Scott family [AK00]. There has been many successful applications of various syntactical techniques to the study of relative computable categoricity and related notions, see e.g. the recent work of Montalban [Mon13, Mon15] that relate computable structure theory with descriptive set theory and model theory. On the other hand, the study of the more "wild" general computable categoricity enjoys applications of advanced recursion-theoretic techniques; it also often leads to novel methods and new results which are not necessarily related to categoricity questions (see e.g. [DM13, Gon81]). One of the most remarkable theorems of this kind says that there is a structure with exactly two computable copies up to computable isomorphism; see Goncharov [Gon80], and see Hirschfeldt [Hir01] for further applications of the technique of Goncharov.

As we see, there are two main strands within modern computable structure theory. The first seeks to relate definability with effectivity [AK00], and the other strand tends to be concerned with properties which revolve around the specifics of the computation level and the structures concerned, see book [EG00]. In this paper, we will be working on the interface of the two strands. More specifically, we answer the following question of Kalimullin. A few years ago, Kalimullin asked whether a computable structure could be arithmetically categorical (or Δ^0_ω-categorical) in the following unbounded way. Can there be a computable structure $\mathcal{A}$, such that for any computable structure $\mathcal{B}$ isomorphic to $\mathcal{A}$, there is an n such that $\mathcal{A}$ is Δ^0_n-isomorphic to $\mathcal{B}$, but for each $m < \omega$ there is a computable $\mathcal{C}_m$ with $\mathcal{C}_n$ ismorphic to $\mathcal{A}$ but *not* by a Δ^0_m-isomorphism? In this paper we answer this question affirmatively. In fact, we prove more.

Theorem 1.1. *for every computable ordinal α there exists a computable structure $\mathcal{A}_\alpha$ such that:*

- *For every computable structure $\mathcal{M}$, there exists a $\beta < \alpha$ such that $\mathcal{M} \cong_{\Delta^0_\beta} \mathcal{A}_\alpha$.*

- *For every $\beta < \alpha$, there exists a computable structure $\mathcal{B} \cong \mathcal{A}_\alpha$ such that $\mathcal{B} \not\cong_{\Delta^0_\beta} \mathcal{A}_\alpha$.*

We note that the structure $\mathcal{A}_\alpha$ witnessing Theorem 1.1 will be built up from structures that are themselves relatively Δ^0_β-categorcial. Although the isomorphism types of these substructures will depend on the construction, their nice uniform properties will allow us to exploit techniques borrowed from the "syntactical" strand, more specifically the result of Ash [Ash86]. The construction will be an iterated priority argument, along the lines of a "worker" argument. We believe that the easiest presentation is to give a direct proof rather than to try to use any of the existing metatheorems ([AK00, Mon14, LL97]), aside from using the above mentioned result of Ash[1].

2. Outline

The remainder of this paper will be focused on proving Theorem 1.1. Let α be a fixed computable limit ordinal. For notational convenience, when we wish to discuss Δ^0_β constructions as oracle constructions, we will use $0_{(\beta)}$ as the name of the oracle which is equal to $0^{(\beta-1)}$ when $1 \leq \beta < \omega$, and it equal to the β'th hyperjump (using specifically chosen Turing degree representatives that can uniformly resolve Δ^0_β questions) when $\omega \leq \beta < \alpha$.

Let $\langle \alpha_n : n \in \omega \rangle$ be a computable increasing sequence of ordinals whose limit is α, with $\alpha_0 > 0$. For each n, $\beta_n = 2 \cdot \alpha_n + 1$, and note that $\langle \beta_n : n \in \omega \rangle$ is also a computable increasing sequence of ordinals whose limit is α, but with $\beta_0 > 2$. Let $\langle \mathcal{M}_n : n \in \omega \rangle$ be a listing of all the computable structures in our signature, which will be specified later.

We will have isomorphism requirements for each $n \in \omega$, and diagonalization requirements for each $\beta < \alpha$ of the form $\lambda + 2n$ where λ is either 3 or a limit ordinal and $n \in \omega$.:

$$\mathcal{I}_n : \mathcal{M}_n \cong \mathcal{A} \to \mathcal{M}_n \cong_{\Delta^0_{\beta_n+1}} \mathcal{A}$$

$$\mathcal{D}_n : \exists \mathcal{B}_n (\mathcal{B}_n \text{ is computable } \& \ \mathcal{B}_n \cong \mathcal{A} \ \& \ \mathcal{B}_n \not\cong_{\Delta^0_{\beta_n}} \mathcal{A})$$

Note that if we meet all of these requirements, then we will have proved the theorem because we will have ensured that every isomorphic computable structure is $\cong_{\Delta^0_\beta} A$ for some $\beta < \alpha$, but also that there is no fixed $\beta < \alpha$ that suffices for all of these structures.

[1]Thanks to Noam Greenberg, we are aware that Kach and Montalban have (independently) announced the case $\alpha = \omega$. As far as we know, their proof has not yet been formally written or published.

The signature of our structure will have an edge relation, an ordering relation, and a collection of unary predicates $P_{n,i,j}$, denoting disjoint portions of the structure which we will use to meet requirement $\mathcal{D}_\beta$ for the ith potential Δ^0_β isomorphism. The j parameter will be used for injury purposes, to allow the requirement to start over with a blank slate any time it needs to. The ordering relation and edge relation will never hold between elements of different $P_{\beta,i,j}$ predicates.

In each $P_{n,i,j}$, we will discuss two different constructions. The "attempted construction" will be a computable construction that, if it is completed, will make progress towards meeting a $\mathcal{D}_n$ requirement. The "actual construction" will be the construction that is actually carried out in $P_{\beta,i,j}$, which might be different from the attempted construction due to interactions between requirements..

We first describe the properties that the attempted construction will have. We then describe the attempted construction, and verify that it has the required properties. Then we explain how to modify the attempted constructions to create the actual construction, and finally we verify that the actual construction proves the theorem.

3. Properties of the attempted construction

In each $P_{n,i,j}$, the attempted construction will have the following properties.

Note that in the attempted construction for $P_{n,i,j}$, $\mathcal{A} \upharpoonright P_{n,i,j}$ and $\mathcal{B}_n \upharpoonright P_{n,i,j}$ are constructed simultaneously, making them isomorphic on that component while diagonalizing against a $\Delta^0_{\beta_n}$ isomorphism between the two. If $\gamma \neq \beta$ then $\mathcal{B}_\gamma$ will simply copy $\mathcal{A}$ in each $P_{\beta,i,j}$.

(1) The attempted construction on $P_{n,i,j}$ is computable uniformly in n, i, j.

(2) If the attempted construction on $P_{n,i,j}$ is completed, then $\mathcal{A}$ is not isomorphic to $\mathcal{B}_n$ via the ith $\Delta^0_{\beta_n}$ map.

(3) If the attempted construction on $P_{n,i,j}$ is completed, then $\mathcal{A}$ is uniformly $\Delta^0_{\beta_n+1}$ isomorphic to $\mathcal{B}_n$ on $P_{n,i,j}$.

(4) The attempted construction on $P_{\beta,i,j}$ in A must be uniformly constructed by finite extensions with the "local rigidity property," defined below.

Definition 3.1. *A structure S is constructed by finite extensions with the* local rigidity property. *If there is a computable sequence $\langle S_n : n \in \omega\rangle$ of finite structures such that $S_n \subseteq S_{n+1}$, $\bigcup_n S_n = S$, and for each n, the inclusion map is the unique embedding $S_n \to S_{n+1}$.*

We call the S_n the stages *of the construction of S.*

4. Attempted construction

In this section, we describe the attempted construction and verify that it has the properties of the previous section. This section can be read before or after the next two sections, as the only relevant facts from this section are those listed in the properties of the attempted construction.

Our primary tool for the attempted construction will be the following result of Ash (Theorem 18.15 of [AK00]).

Theorem 4.1 (Ash)**.** *Let γ be a computable ordinal, and suppose L is a $\Delta^0_{2\gamma+1}$ linear ordering. Then $\omega^\gamma \cdot L$ has a computable copy $F(L)$. Moreover, there is a $\Delta^0_{2\gamma+1}$ function f taking $a \in L$ to the first element of the corresponding copy of ω^γ in $F(L)$.*

We remark that the proof of Theorem 4.1 is uniform as long as L does not have a first element.

Using Theorem 4.1 we now define our attempted construction on $P_{n,i,j}$.

For our attempted construction on $P_{n,i,j}$, we begin by constructing two linear orders in $\mathcal{A}$ and $\mathcal{B}_n$, that we will ensure are isomorphic, but not via the ith $\Delta^0_{\beta_n}$ isomorphism. This will be done by constructing linear orderings L_0 and L_1, computably in $0_{(\beta_n)}$, both of order type ω^*, and then constructing $F(L_0)$ as our linear ordering in $P_{\beta,i,j}$ in $\mathcal{A}$ and $F(L_1)$ as our linear ordering in $P_{\beta,i,j}$ in $\mathcal{B}_\beta$. Here, we are using Theorem 4.1 with $\gamma = \alpha_n$, and hence $2\gamma + 1 = \beta_n$.

The construction of L_0 and L_1 is as follows. Note that this construction is computable in $0_{(\beta)}$. Construct ω^* in both L_0 and L_1, letting x_0 and x_1 be the apparent last elements of the two orders. At any stage, if the ith $\Delta^0_{\beta_n}$ partial function converges, sending $f(x_0)$ to $f(x_1)$ then $0_{(\beta)}$ adds one more element to the end of L_1.

This ensures that $F(L_0)$ and $F(L_1)$ are isomorphic, but not by the ith Δ^0_β map.

To ensure that the construction in $P_{n,i,j}$ in $\mathcal{A}$ can be constructed locally rigidly in stages, we use our edge relation. From each element of our linear orders $F(L_0)$ and $F(L_1)$, we will have an extra element that is attached by the edge relation. The edge relation will not be used for any other purpose in the attempted construction. The additional elements will not be related to any elements via the ordering relation.

Note that the elements of the ordering can be identified as the elements that are related to any other element by the ordering relation, and the extra elements can be identified as the elements

We verify that this can be constructed locally rigidly in Lemma 4.5 This completes the construction.

Lemma 4.2. *The attempted construction on $P_{n,i,j}$ is computable uniformly in n, i, j.*

Proof. The uniformity follows from the fact that the sequence $\langle \alpha_n \rangle$ is computable, as well as the fact that the proof of Theorem 4.1 is uniform. □

Lemma 4.3. *If the attempted construction on $P_{n,i,j}$ is completed, then $\mathcal{A}$ is not isomorphic to $\mathcal{B}_n$ via the ith $\Delta^0_{\beta_n}$ map.*

Proof. Any isomorphism between $\mathcal{A}$ and $\mathcal{B}_n$ must also be an isomorphism when restricted to $P_{n,i,j}$, and must therefore also be an isomorphism when restricted to the linear order part of $P_{n,i,j}$. In our construction, we ensured that $F(L_0)$ and $F(L_1)$ were not isomorphic by the ith Δ^0_β map. □

Lemma 4.4. *If the attempted construction on $P_{n,i,j}$ is completed, then $\mathcal{A}$ is uniformly $\Delta^0_{\beta_n+1}$ isomorphic to $\mathcal{B}_n$ on $P_{n,i,j}$.*

Proof. First note that $0_{(\beta)}$ can compute L_0 and L_1, both of order type ω^*. Therefore $0'_{(\beta)} = 0_{(\beta+1)}$ can compute the isomorphism between L_0 and L_1, by repeatedly finding the last element of each and matching those elements together. While doing this, it also computes the successor relations on L_0 and L_1.

Also $0_{(\beta)}$ can compute the functions f_0 and f_1 taking each $a \in L_0, L_1$ to the first elements of the corresponding copies of ω^β in $F(L_0), F(L_1)$. Using f_0 and f_1 together with the successor relations on L_0 and L_1, $0_{(\beta+1)}$ can decompose $F(L_0)$ and $F(L_1)$ into individual copies of ω^β. So to compute the isomorphism between $F(L_0)$ and $F(L_1)$, it suffices to be able to compute the isomorphism on each individual copy of ω^β.

This can be done uniformly by $0_{(\beta)}$, and hence also by $0_{(\beta+1)}$. (Folklore, proved implicitly in the proof of Theorem 4.1.) □

Lemma 4.5. *The attempted construction on $P_{\beta,i,j}$ in A can uniformly be constructed by finite extensions with the local rigidity property.*

Proof. The way that we construct this structure in $\mathcal{A}$ in stages is as follows.

During the first stage, we put the first two elements into $F(L_0)$, and we attach an element to each of them.

After this, each time we wish to add a new element to $F(L_0)$, we will do so over the course of two stages, a and b. At the beginning of stage a, there will be a finite number of elements in $F(L_0)$, each with an element attached by an edge. In stage a, we add the new element to $F(L_0)$, and in stage b, we create a new element, which we attach by an edge to the element we added in stage a.

This construction is locally rigid, because members of the linearly ordered part must always map to members of the linearly ordered part, and they must do so in an order preserving manner. At the end of stage a, in any embedding, nothing can map to the new element because it does not have an extra element attached by an edge. Therefore all the linearly ordered elements must map to themselves, and so the elements attached to those elements must also map to themselves. At the end of stage b, the linear order hasn't been changed, so again, all linearly ordered elements must map to themselves, and so all of the elements must map to themselves. $\square$

5. Modifying attempted construction to actual construction

Recall that we have diagonalization requirements, $\mathcal{D}_n$, which say that there exist computable structures that are isomorphic, but for which the isomorphism is difficult to compute, and isomorphism requirements, $\mathcal{I}_n$, which say that for every computable structure, if it is isomorphic, then the isomorphism is easy to compute.

Item 2 from the properties of the attempted construction ensures that if the attempted construction were to succeed everywhere, then all of the $\mathcal{D}_n$ requirements would be met. In fact, if for every n and i there exists a j such that $P_{n,i,j}$ completes its attempted construction, then all of the $\mathcal{D}_n$ requirements will be met.

The purpose of this section is to describe how to modify a construction with the above properties to produce the actual construction, which will meet the $\mathcal{I}_n$ requirements while still meeting the $\mathcal{D}_n$ requirements. In the attempted construction, $\mathcal{A}$ and $\mathcal{B}_n$ are constructed simultaneously. In the actual construction, they will be, in some sense, constructed simultaneously, but a computable modification to $\mathcal{B}_n$, depending nonuniformly on n, will be required at the very end.

There will be a certain level of finesse required to ensure that the modification to $\mathcal{B}_n$ is a computable one. This is because a single $\mathcal{D}_n$ requirement can potentially cause the attempted construction to fail to be completed on countably many $P_{n,i,j}$, but the construction for $\mathcal{B}_n$ is only uniform on the $P_{n,i,j}$ that complete their constructions. The specific subtleties of the "wishes to act" definition in the construction are to ensure that at the end of the construction the required modification to $\mathcal{B}_n$ is computable.

Besides this requirement, the construction can be phrased as a tree construction, but we will instead describe the construction in explicit detail in order to ensure that the modifications to the $\mathcal{B}_n$ can be seen at the end to be computable.

We describe now, for a fixed value of n, how the actual construction on the $P_{n,i,j}$ will be implemented while simultaneously building an isomorphism between $\mathcal{A}$ and the $\mathcal{M}_m$ with $m < n$. Temporarily, let us assume that $n = 1$, and so we only need to construct an isomorphism between $\mathcal{A}$ and $\mathcal{M}_0$ while we implement our construction of $P_{1,i,j}$.

In the $P_{1,i,0}$, we will implement the attempted construction on $\mathcal{A}$ and $\mathcal{B}_1$ in stages, as in item 4 of the properties of the attempted construction. Each time we complete a stage, we will wait for $\mathcal{M}_0$ to also create the exact same object. We will then extend the isomorphism we have defined so far, and proceed to the next stage. Note that for this to function, is important that at each stage, there is only one injection from the structure at the previous stage to the structure at the next stage. Else, $\mathcal{M}_0$ could potentially produce an isomorphic object by a different embedding, preventing us from extending our isomorphism.

For $j > 0$, in the $P_{1,i,j}$, we will implement the attempted construction without any stages or waiting. However, each time $\mathcal{M}_0$ catches up to the construction on $P_{1,i,0}$, we will take the next j and initialize the construction in $P_{1,i,j}$ by expanding the construction to be the computable Fraisse limit of structures in the signature with a binary edge relation and a binary ordering relation. Note that when we do this, we no longer continue to insist that the edge relation only holds between special pairs, and we also no longer continue to insist that the ordering relation continues to be an ordering. So the structure is a sort of Rado graph, with two kinds of edges, one directed and one undirected.

When we do this, we also begin to define our isomorphism between $\mathcal{A}$ and $\mathcal{M}_0$ on the initialized $P_{1,i,j}$ using the standard algorithm for producing an isomorphism between Fraisse limits in the same signature. If $\mathcal{M}_0$ is actually isomorphic to $\mathcal{A}$, then in particular it will have have the same Fraisse limit in the initialized $P_{1,i,j}$, so this algorithm will produce a computable isomorphism between $\mathcal{M}_0$ and $\mathcal{A}$.

For a given value of i, if after a certain number of stages, $\mathcal{M}_0$ never catches up again, then we will only injure $P_{1,i,j}$ for finitely many values of j, so in particular, there will be a j such that we complete our attempted construction on $P_{\beta,i,j}$. This will also complete the requirement of $\mathcal{I}_1$, which will be vacuously true because $\mathcal{M}_1$ will not be isomorphic to $\mathcal{A}$, because $\mathcal{A}$ restricted to $P_{1,i,0}$ is frozen as a finite structure that is different from what $\mathcal{M}_1$ has.

If $\mathcal{M}_1$ catches up infinitely many times on every $P_{1,i,0}$, then we will complete our attempted construction on all of the $P_{1,i,0}$. Furthermore, we will complete the requirement of $\mathcal{I}_1$ on the $P_{\beta,i,j}$, because, if it is the case that $\mathcal{M}_1 \cong \mathcal{A}$, then our construction will have defined an isomorphism between $\mathcal{M}_1$ and $\mathcal{A}$. (The "waiting and extending in stages" strategy defines an isomorphism on the $P_{1,i,0}$, and for $j > 0$, we will have initialized every $P_{1,i,j}$ at some finite stage. Our Fraisse limit strategy will then produce a computable isomorphism between $\mathcal{A}$ and $\mathcal{M}_1$ as long as $\mathcal{M}_1$ also builds the same Fraisse limit that $\mathcal{A}$ builds.)

We now drop the assumption that $n = 1$. We continue to be working for a single fixed value of n. Fix a computable map sending ω to $\{X \subseteq n\}$ such that there are infinitely many elements mapped to each $X \subseteq n$. Let X_j be the image of $j \in \omega$ under this map.

For $m < n$, if $m \in X_j$, then $P_{n,i,j}$ will operate under the assumption that $\mathcal{M}_m$ will catch up to the construction infinitely many times. Otherwise $P_{n,i,j}$ will operate under the assumption that $\mathcal{M}_m$ will not catch up to the construction infinitely many times. The precise behavior is as follows.

The construction will be organized into steps, with the intention that in each step, some finite number of $P_{n,i,j}$ will complete one stage of their attempted constructions. At each step of the construction, we will have a finite number of "active" $P_{n,i,j}$, and we will begin by activating $P_{n,i,j}$ for the first $\langle i, j\rangle$ such that $P_{n,i,j}$ is not yet active.

After this, for every active $P_{n,i,j}$, we ask "Does there exist a pair $\langle \hat{i}, \hat{j}\rangle > \langle i, j\rangle$ such that $X_{\hat{j}} \supsetneq X_j$, and $P_{n,\hat{i},\hat{j}}$ has acted at least once?"

If the answer is "yes," then we permanently injure $P_{n,i,j}$, ensuring that $\mathcal{A}$ and $\mathcal{B}_n$ both construct Fraisse limits in $P_{n,i,j}$.

After we do this, we determine which of the $P_{n,i,j}$ wish to act at this step. We say that $P_{n,i,j}$ wishes to act if:

(1) It is active.
(2) It is not yet injured.
(3) For every $m \in X_j$ and every $\hat{i}, \hat{j}$, if $X_j \subseteq X_{\hat{j}}$, and if $\langle \hat{i}, \hat{j}\rangle + \hat{k} \leq \langle i, j\rangle + k$, where k and $\hat{k}$ are the number of times that $P_{n,i,j}$ and $P_{n,\hat{i},\hat{j}}$ have acted previously, then either $P_{n,\hat{i},\hat{j}}$ has been injured or $\mathcal{M}_m$ is isomorphic to $\mathcal{A}$ on $P_{n,\hat{i},\hat{j}}$.

If it is the case that $P_{n,i,j}$ wishes to act, then first we extend our isomorphisms, then we complete the next stage of our attempted construction in $P_{n,i,j}$. Then, for each $m \in X_j$, we extend our isomorphism between $\mathcal{M}_m$ and $\mathcal{A}$ on $P_{n,i,j}$ in the only way possible. (This extension is unique due to the local rigidity property of our attempted construction.)

Meanwhile, all injured $P_{n,i,j}$ do one additional step in the construction of their computable Fraisse limits. The construction then moves on to the next step.

This completes the construction of $\mathcal{A}$ and of the $\mathcal{B}_n$. As mentioned earlier, the $\mathcal{B}_n$ will not actually satisfy the $\mathcal{D}_n$ requirements. Instead, we will use them to construct a collection of sets $\hat{\mathcal{B}}_n$, each of which is a computable modification of the corresponding $\mathcal{B}_n$. The computable modifications are not uniform, so $\langle \hat{\mathcal{B}}_n : n \in \omega \rangle$ will not be uniformly computable as a sequence.

In our construction, for $k \neq n$, $\mathcal{B}_n$ simply copies $\mathcal{A}$ on every $P_{k,i,j}$. On every $P_{n,i,j}$, one of three things happens: Both $\mathcal{A}$ and $\mathcal{B}_n$ complete their attempted construction, building isomorphic structures by item 3 of the properties of the attempted construction. Both $\mathcal{A}$ and $\mathcal{B}_n$ stop acting at some finite step in their attempted construction. Both $\mathcal{A}$ and $\mathcal{B}_n$ build a computable Fraisse limit. In the first and third cases, we have that $\mathcal{A}$ and $\mathcal{B}_n$ are isomorphic on $P_{n,i,j}$. In the second case, it is possible that $\mathcal{A}$ and $\mathcal{B}_n$ might be frozen with different finite structures.

To compute $\hat{\mathcal{B}}_n$, we rectify this by nonuniformly giving ourselves the function g that describes how and when, for all $m < n$, the various $\mathcal{M}_m$ stop catching up. Let $g : \{\langle n, i, j \rangle\} \to \omega + 1$ be the function such that $g(\langle n, i, j \rangle)$ is number of steps of the attempted construction that are carried out in $P_{n,i,j}$, with $g(\langle n, i, j \rangle)$ being defined to be ω if $P_{n,i,j}$ is ever injured.

A priori it would appear that $g \in \omega^\omega$ and so g cannot be used as a parameter, but we will show that g can be described by a finite collection of nonincreasing functions. Note that every nonincreasing function $h : \omega \to \omega + 1$ can be coded by a natural number which codes the initial value, the number of drops, and also the locations and sizes of the drops.

In particular, we claim that among pairs $\langle i_0, j_0 \rangle$, $\langle i_1, j_1 \rangle$ such that $X_{j_0} = X_{j_1}$, it is actually the case that g is a nonincreasing function. This is because the "wishes to act" property is defined in a way such that if $\langle i_0, j_0 \rangle < \langle i_1, j_1 \rangle$ and if $X_{j_0} = X_{j_1}$, and if neither P_{β,i_0,j_0} nor P_{β,i_1,j_1} is ever injured then P_{β,i_0,j_0} wishes to act at every step where P_{β,i_1,j_1} wishes to act. So, on never injured inputs, g is nonincreasing on such pairs. Also, the way that injuries happen in the construction, if $\langle i_0, j_0 \rangle < \langle i_1, j_1 \rangle$ and if $X_{j_0} = X_{j_1}$, then if P_{β,i_1,j_1} becomes injured, P_{β,i_0,j_0} must have also become injured at that step or a previous step.

Thus, g is nonincreasing on pairs $\langle i_0, j_0\rangle$, $\langle i_1, j_1\rangle$ such that $X_{j_0} = X_{j_1}$. There are finitely many equivalence classes, and so g can be coded by a single parameter.

Knowing g, we compute $\hat{\mathcal{B}}_n$ using $\mathcal{B}_n$ as follows. For $k \neq n$, $\hat{\mathcal{B}}_n$ copies $\mathcal{A}$ on $P_{k,i,j}$. For any $\langle i, j\rangle$ such that $g(\langle i, j\rangle) = \omega$, $\hat{\mathcal{B}}_n$ copies $\mathcal{B}_n$. On these regions, $\hat{\mathcal{B}}_n$ makes sure to copy $\mathcal{B}_n$ using exactly the same elements of ω in its domain, so as to not break any diagonalizations established in $\mathcal{B}_n$. For any $\langle i, j\rangle$ such that $g(\langle i, j\rangle) \neq \omega$, $\hat{\mathcal{B}}_n$ simply runs $g(\langle i, j\rangle)$-many stages of the attempted construction of $\mathcal{A}$ on $P_{n,i,j}$.

This completes our construction of $\hat{\mathcal{B}}_n$. We have just finished arguing that $\hat{\mathcal{B}}_n$ is computable. Also, we have that $\hat{\mathcal{B}}_n \cong \mathcal{A}$ because $\hat{\mathcal{B}}_n$ copies $\mathcal{A}$'s behavior everywhere except on the places where either the attempted construction on $P_{n,i,j}$ is completed or it is injured. In those places $\hat{\mathcal{B}}_n$ copies $\mathcal{B}_n$, which we have already argued is isomorphic to $\mathcal{A}$ on those regions.

This completes the construction of $\mathcal{A}$, and of the $\hat{\mathcal{B}}_n$.

6. Verification

We now verify that the construction as defined above, implemented on every n, will meet all of the $\mathcal{D}_n$ requirements as well as all of the $\mathcal{I}_n$ requirements. For both of these proofs, the following notation will be helpful.

Definition 6.1. *Fix some value of n.*

For each $X \subseteq n$, let S_X be the set of all $P_{n,i,j}$ such that $X_j = X$. Now, consider the set $\mathbb{X} = \{X$ such that there are infinitely many steps in the construction where some $P_{n,i,j} \in S_X$ acts$\}$.

Define $Y = \bigcup_{X \in \mathbb{X}} X$.

This notation implicitly depends on a fixed value of n. We suppress the dependence on n from the notation for clarity. When we use it, there will either be only one value of n that is relevant in context, or we will specify what value of n we are using.

Lemma 6.2. *For any fixed value of n, at most finitely many $P_{n,i,j} \in S_Y$ are ever injured.*

Proof. Note that there is a finite step in the construction after which the only $P_{n,i,j}$ that act are those in S_X for $X \subseteq Y$. Any $P_{n,i,j} \in S_Y$ that becomes active after this step cannot be injured, because of the construction of when injuries occur. □

Lemma 6.3. *For a fixed value of n, if $P_{n,i,j} \in S_Y$ is never injured, then it acts infinitely many times.*

Proof. Assume not. Fix $Z_0, Z_1 \ldots, Z_p \in \mathbb{X}$ such that $\bigcup_{q\leq p} Z_q = Y$. Let i_0, j_0 be such that $P_{n,i_0,j_0} \in S_Y$, and P_{n,i_0,j_0} is never injured. Assume that P_{n,i_0,j_0} acts finitely often, and let k_0 be the number of times that P_{β,i_0,j_0} acts.

Let s_0 be a step of the construction after the step when P_{n,i_0,j_0} is active, such that for every $\langle i,j\rangle \leq \langle i_0, j_0\rangle + k_0$, if $X_j \supseteq Y$, then either $P_{n,i,j}$ has been injured by step s_0, or $P_{n,i,j}$ will never act again after step s_0, or at step s_0, $P_{n,i,j}$ has already acted k times for some k sufficiently large that $\langle i,j\rangle + k > \langle i_0, j_0\rangle + k_0$.

We show that P_{n,i_0,j_0} acts at least once after step s_0, which is a contradiction, because s_0 was chosen to be after the step when P_{β,i_0,j_0} stops acting.

For $q \leq p$, let $s_{q,1}$ be a step after s_0 when some $P_{n,i_{q,1},j_{q,1}} \in S_{Z_q}$ acts for the kth time, where k is sufficiently large that $\langle i_0, j_0\rangle + k_0 \leq \langle i_{q,1}, j_{q,1}\rangle + k$. This must happen because $X_q \in \mathbb{X}$, so either there is some $P_{n,i,j} \in Z_q$ that acts infinitely often, or each acts only finitely often, so eventually some $P_{n,i,j} \in Z_q$ with $\langle i,j\rangle \geq \langle i_0, j_0\rangle + k_0$ must act.

Let et $s_1 = \max_{q\leq p}(s_{q,1})$.

For $q \leq p$, let $s_{q,2}$ be a step after s_1 when some $P_{\beta,i_{q,2},j_{q,2}} \in S_{Z_q}$ acts for the kth time, where k is sufficiently large that $\langle i_0, j_0\rangle + k_0 \leq \langle i_{q,2}, j_{q,2}\rangle + k$, and let $s_2 = \max_{q\leq p}(s_{q,2})$.

We claim then that P_{n,i_0,j_0} must act at step s_1, providing a contradiction. To prove this, we will have to use the fact that by previous assumption P_{n,i_0,j_0} does not act between steps s_0 and s_2. By assumption, P_{n,i_0,j_0} is never injured and is active at step s_1, so we must now show that for every $n \in Y$ and every $\hat{i}, \hat{j}$, if $Y \subseteq X_{\hat{j}}$, and if $\langle \hat{i}, \hat{j}\rangle + \hat{k} \leq \langle i_0, j_0\rangle + k_0$, where $\hat{k}$ is the number of times that $P_{n,\hat{i},\hat{j}}$ had acted before step s_1, then either $P_{n,\hat{i},\hat{j}}$ had been injured by step s_1 or $\mathcal{M}_n$ was isomorphic to $\mathcal{A}$ on $P_{n,\hat{i},\hat{j}}$.

So let $m \in Y$, let $\hat{i}, \hat{j}$ be such that $Y \subseteq X_{\hat{j}}$, and such that $\langle \hat{i}, \hat{j}\rangle + \hat{k} \leq \langle i_0, j_0\rangle + k_0$, where $\hat{k}$ is the number of times that $P_{\beta,\hat{i},\hat{j}}$ had acted before step s_1. Assume that $P_{n,\hat{i},\hat{j}}$ had not been injured at step s_1. Note that by choice of s_0, it must be the case that $P_{n,\hat{i},\hat{j}}$ did not act and was not injured between steps s_0 and s_2. We must show that at step s_1, $\mathcal{M}_n$ was isomorphic to $\mathcal{A}$ on $P_{\beta,\hat{i},\hat{j}}$.

To do this, note that $Y = \bigcup_{q\leq p} Z_q$, so fix $q \leq p$ such that $m \in Z_q$. At steps $s_{q,1}$ and $s_{q,2}$, when $P_{n,i_{q,1},j_{q,1}}$ and $P_{n,i_{q,2},j_{q,2}}$ acted, it must have been the case that $\mathcal{M}_m$ was isomorphic to $\mathcal{A}$ on $P_{n,\hat{i},\hat{j}}$ (because $\langle \hat{i}, \hat{j}\rangle + \hat{k} \leq \langle i_0, j_0\rangle + k_0$). Between these two steps, $P_{n,\hat{i},\hat{j}}$ did not act and was not injured, and so $\mathcal{A}$ did not change on $P_{n,\hat{i},\hat{j}}$. Therefore, it must be the case that $\mathcal{M}_m$ also did not change between steps $s_{q,1}$ and $s_{q,2}$, as it is isomorphic to the same finite structure at both points in time, and computable structures can only change by increasing the number of elements. By construction, $s_{q,1} \leq s_1 \leq s_{q,2}$, and so we have that at step s_1, $\mathcal{M}_m$ was isomorphic to $\mathcal{A}$ on $P_{n,\hat{i},\hat{j}}$.

Thus, at step s_1, P_{n,i_0,j_0} was able to act, and so must have acted, providing the desired contradiction. □

We are ready to prove that our construction satisfies all of our $\mathcal{D}_n$ requirements.

Lemma 6.4. *For every n, $\hat{\mathcal{B}}_n$ is computable, and $\hat{\mathcal{B}}_n \cong \mathcal{A}$ but $\hat{\mathcal{B}}_n \not\cong_{\Delta^0_{\beta_n}} \mathcal{A}$.*

Therefore, all of the $\mathcal{D}_n$ requirements are met.

Proof. During the construction of $\hat{\mathcal{B}}_n$, we argued that $\hat{\mathcal{B}}_n$ is computable, and that $\hat{\mathcal{B}}_n \cong \mathcal{A}$.

To show that $\hat{\mathcal{B}}_n \not\cong_{\Delta^0_{\beta_n}} \mathcal{A}$, we must show that for every i, there is some j such that the attempted construction on $P_{n,i,j}$ is completed. If this is the case, then $\hat{\mathcal{B}}_n$ will be equal to $\mathcal{B}_n$ on those $P_{n,i,j}$ and so, by property 3 of the attempted construction, we will have that for every i, $\hat{\mathcal{B}}_n$ is not isomorphic to $\mathcal{A}$ by the ith $\Delta^0_{\beta_n}$ map, and so $\hat{\mathcal{B}}_n \not\cong_{\Delta^0_{\beta_n}} \mathcal{A}$.

By Lemma 6.2, at most finitely many $P_{\beta,i,j} \in S_Y$ are ever injured and by Lemma 6.3 if $P_{\beta,i,j} \in S_Y$ is never injured, then it acts infinitely many times. For each i, there are infinitely many j such that $X_j = Y$, and so there is at least one $P_{\beta,i,j}$ that acts infinitely many times and is never injured. By item 2 of the properties of the attempted construction, we therefore have that $\hat{\mathcal{B}}_\beta \not\cong_{\Delta^0_\beta} \mathcal{A}$. □

We now verify that the construction meets all of the $\mathcal{I}_n$ requirements.

Lemma 6.5. *For every n, if $\mathcal{M}_n \cong \mathcal{A}$, then $\mathcal{M}_n \cong_{\Delta^0_{\beta_n+1}} \mathcal{A}$.*

Therefore, all of the $\mathcal{I}_n$ requirements are met.

Proof. Fix $n_0 \in \omega$. Assume that $\mathcal{M}_{n_0} \cong \mathcal{A}$. We use $0_{(\beta_{n_0}+1)}$ to compute an isomorphism between the two on each $P_{n_0,i,j}$. Note that every β_n is ≥ 1, so we may ask $0''$ computable questions during the construction.

Let P_{n_1,i_1,j_1} be some fixed $P_{n,i,j}$. We ask first, "is the construction on P_{n_1,i_1,j_1} ever injured?" If yes, then we know that there is a Fraisse limit built there, and we use the computable algorithm for building isomorphisms between Fraisse limits.

If not, we ask "Does the construction on P_{n_1,i_1,j_1} carry out infinitely many stages of its attempted construction?" If not, then, computably in $0''$, we find the step of the construction when A never extends itself again on P_{n_1,i_1,j_1}, we compute A on P_{n_1,i_1,j_1} at that step, we wait for $\mathcal{M}_{n_0}$ to produce an isomorphic structure in its P_{n_1,i_1,j_1}, and we choose an isomorphism between the two finite structures.

Finally, if the construction does carry out infinitely many stages of the attempted construction, then we have two cases. If $n_1 \leq n_0$, then by property 5 of the attempted construction, we have that $0_{(\beta_{n_0}+1)}$ must be able to uniformly in n_1, i_1, j_1 construct an isomorphism between $\mathcal{A}$ and $\mathcal{M}_{n_0}$ on P_{n_1,i_1,j_1}.

If $n_1 > n_0$, then we claim that during the construction of $\mathcal{A}$ on P_{n_1,i_1,j_1}, an isomorphism between $\mathcal{A}$ and $\mathcal{M}_{n_0}$ was constructed, and hence we may use that isomorphism. To prove this, we must show that $n_0 \in X_{j_1}$ for $n = n_1$. (The map $j \mapsto X_j$ depends on n.) Once we have shown this, we will know that $\mathcal{M}_{n_0}$ was one of the structures on which an isomorphism was defined each time that the construction on P_{n_1,i_1,j_1} acted.

The proof of this fact is by the following lemma. □

Lemma 6.6. *Let $n_0, n_1, i_1, j_1 \in \omega$. Assume that $n_1 > n+0$, $\mathcal{M}_{n_0} \cong \mathcal{A}$, and that the attempted construction on P_{n_1,i_1,j_1} is never injured and carries out infinitely many stages of the construction.*

Then $n_0 \in X_{j_1}$, where X_{j_1} is defined for $n = n_1$.

Proof. Let $Y = \bigcup_{X \in \mathbb{X}} X$ be as in Definition 6.1, again for $n = n_1$. Note that $Y \supseteq X_{j_1}$ because P_{n_1,i_1,j_1} acts infinitely often. Assume that $Y \neq X_{j_1}$. Then, as shown above, there are infinitely many $\langle \hat{i}, \hat{j} \rangle$ such that $X_{\hat{j}} = Y$ and $P_{n_1,\hat{i},\hat{j}}$ acts infinitely often. In particular, there is a $\langle \hat{i}, \hat{j} \rangle > \langle i_1, j_1 \rangle$ such that $X_{\hat{j}} \supsetneq X_{j_1}$, and $P_{n_1,\hat{i},\hat{j}}$ has acted at least once.

But then this would suffice for P_{n_1,i_1,j_1} to be injured, which is not the case. Thus it must be the case that $Y = X_{j_1}$. Assume now that $n \notin Y$. We use this assumption to prove that $\{n\} \in \mathbb{X}$, and hence that $n \in Y$, providing a contradiction.

To show this, fix a step s_0 of the construction such that after step s_0, no $P_{n_1,i,j}$ with $X_j \notin \mathbb{X}$ acts again. Fix some $\langle i_2, j_2 \rangle$ such that $X_{j_2} = \{n_0\}$ and P_{n_1,i_2,j_2} becomes active after step s_0. Note that P_{n_1,i_2,j_2} will never be injured because we are assuming that $n_0 \notin Y$, so we have that no $P_{n_1,i,j}$ with $n_0 \in X_j$ will ever act after step s_0.

To show that P_{n_1,i_2,j_2} wishes to act at least once (and hence that $\{n_0\} \in \mathbb{X}$), we must show that there is some step of the construction when for every $\hat{i}, \hat{j}$, if $n_0 \in X_{\hat{j}}$, and if $\langle \hat{i}, \hat{j} \rangle + \hat{k} \leq \langle i_1, j_1 \rangle$, where $\hat{k}$ is the number of times that $P_{n_1,\hat{i},\hat{j}}$ has acted previously, then either $P_{n_1,\hat{i},\hat{j}}$ has been injured or $\mathcal{M}_{n_0}$ is isomorphic to $\mathcal{A}$ on $P_{n_1,\hat{i},\hat{j}}$.

Note that all of the $\hat{\imath}, \hat{\jmath}$ that we are considering have the property that $n_0 \in X_{\hat{\jmath}}$. Therefore, after step s_0, none of them will act again, or be injured again. So the set of $\hat{\imath}, \hat{\jmath}$ that we are considering will not change, and also $\mathcal{A}$ will not change at all on any of those $P_{n_1,\hat{\imath},\hat{\jmath}}$. Furthermore, $\mathcal{A}$ has only finitely many elements in each of those $P_{n_1,\hat{\imath},\hat{\jmath}}$, and there is only a finite number of them, because they are all bounded by $\langle i_1, j_1\rangle$.

By assumption, $\mathcal{M}_{n_0} \cong \mathcal{A}$, and so there must be a step of the construction after which $\mathcal{M}_{n_0} \cong \mathcal{A}$ on each of those $P_{n_1,\hat{\imath},\hat{\jmath}}$. When that happens, P_{n_1,i_2,j_2} will act, proving that $\{n_0\} \in \mathbb{X}$, and providing our contradiction.

Therefore we have that $n_0 \in Y = X_{j_1}$, and hence that every time that P_{n_1,i_1,j_1} acted, it acted because $\mathcal{M}_{n_0}$ had caught up to the construction of $\mathcal{A}$ on P_{n_1,i_1,j_1}, and also that during the course of these actions, an isomorphism was defined by the construction between $\mathcal{A}$ and $\mathcal{M}_{n_0}$ on P_{n_1,i_1,j_1}. For our $\Delta^0_{\beta_{n_0}+1}$ isomorphism, we use that isomorphism. □

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