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EVERY Δ_2^0 POLISH SPACE IS COMPUTABLE TOPOLOGICAL

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Abstract. We show that every Δ_2^0 Polish space admits a computable topological presentation given by an effective indexing of some non-empty open sets in the space.

1. Introduction

The present paper is primarily motivated by the recent works [10, 13, 15] that aim to establish the foundations of the theory of computably presented separable structures. This concept follows a similar pattern seen in computable algebra [7, 9], where the main objects are countable discrete structures. In computable algebra, the classical notions of effective presentability include computably enumerable (c.e.), co-c.e., and computable presentations due to Mal'tsev [21] and Rabin [28]. The study of effectively presented algebraic structures has been a rather successful branch of recursion theory; see the books [1, 7] and the recent monograph [24]. Among many other results, the aforementioned notions of effective presentability have been separated for many algebraic structures including linear orders, broad classes of groups, and Boolean algebras; e.g., [8, 14, 17]. Conversely, in certain cases, it is possible to demonstrate that within a specific class, if a structure possesses a weaker presentation (for example, a c.e. one), then it also has a computable presentation. Such positive results are relatively uncommon and tend to have interesting and sometimes unexpected consequences. For example, Khisamiev [16] showed that every c.e. presented torsion-free abelian group has a computable presentation. Khisamiev was not aware that his theorem implies a positive solution to a question of Baumslag, Dyer and Miller [2] about the integral cohomology of finitely presented groups; see [23] for further details. Another example is the theorem of Downey and Jockusch [6] who showed that every low Boolean algebra is isomorphic to a computable one. Their result has recently found a unexpected application in recursive metric space theory [13].

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In computable structure theory, most of these results comparing different notions of presentability date back several decades and are generally regarded as classical or foundational. In contrast, in computable topology similar results are very recent; see [3, 10, 12, 13, 15, 20]. In fact, several classical notions of effective presentability have not yet been compared up to homeomorphism. The substantial delay may be attributed in part to the additional technical challenges presented by the structures being uncountable. In present paper we prove one such *positive* result that is claimed in the title.

We are now ready to introduce the two main definitions of this paper.

Definition 1.1. A *Polish presentation* of a (Polish) space M is given by a countable metric space $X = ((x_i)_{i\in\omega}, d)$ so that the completion of X is homeomorphic to M. A presentation X is:

- *right-c.e.* if $\{r \in \mathbb{Q} : d(x_i, x_j) < r\}$ is c.e. uniformly in i, j;
- *left-c.e.* if $\{r \in \mathbb{Q} : d(x_i, x_j) > r\}$ is c.e. uniformly in i, j;
- *computable* if it is both left-c.e. and right-c.e.

We call each x_i a *special point* of M.

Right-c.e. and left-c.e. spaces are also called upper- and lower-semicomputable in the literature. Both left-c.e. and right-c.e. spaces form natural subclasses of Δ^0_2 Polish spaces, in which the metric can be computed with the help of the halting problem. All our spaces are Polish, but of course the definition below can be used for a much more broad class of topological spaces.

Definition 1.2. A *computable topological presentation* of a topological space M is given by a sequence $(B_i)_{i\in\omega}$ of non-empty basic open sets of M and a computably enumerable set W such that

$$B_i \cap B_j = \bigcup \{B_k : (i, j, k) \in W\},$$

for any $i, j \in \omega$.

In computable topology, these two classical notions of effective presentability of a Polish space have been around for a long time; we cite Ceitin [4], Moschovakis [25], Nogina [27], and Spreen [30]. It is therefore natural to ask how these classical notions given in Definitions 1.1 and 1.2 are related.

The following fact is folklore.

Fact 1.3. In any right-c.e. Polish space, the basic open balls $\{y : d(x_i, y) < r\}$ with rational parameters form a computable topological space.

Until recently not much was known beyond the elementary Fact 1.3. Remarkably, every computable topological, locally compact Polish group admits a right-c.e. Polish presentation [18]. But of course, this latter result additionally assumes that the group operations are effective. Under the seemingly strong extra assumption of effective regularity, a computable topological space can be effectively metrized [29]. In contrast with these results, there exists a computable topological (locally compact) Polish space not homeomorphic to any hyperarithmetical Polish space [22]. The latter counterexample seems to confirm the intuition that, without additional assumptions, the notion of a computable topological space is very weak. Can we extend Fact 1.3 beyond right-c.e spaces? Given that Definitions 1.1 and 1.2 have been around for over 60 years, one might expect that the answer to the following question should be well-known:

Question. Does every left-c.e. Polish space admit a computable topological presentation?

However, it turns out that the above question is open, and answering the question requires non-trivial effort. What then is the difficulty? In a computable topological space, it is Σ^0_1 to check that the intersection of two basic open sets is non-empty. In a left-c.e. space, this condition becomes Σ^0_2 in general. Thus, one would naturally expect some kind of infinite guessing to be necessary to attack the question. We answer the question stated above in the affirmative.

Theorem 1.4. *Every Δ^0_2 Polish space admits a computable topological presentation.*

The theorem also appears to be the most general *positive* result relating Polish and topological effective presentations known so far. We leave open whether the result can be extended beyond Δ^0_2 Polish spaces, but we conjecture that a much more general fact should hold. However, the methods developed in this article seem to be insufficient to iterate our theorem to cover all arithmetic Polish spaces, or even all Δ^0_3 Polish or all Δ^0_2 topological presentations. The elementary but important adjustments which will be detailed shortly in Section 2 should clarify why our methods seem highly sensitive to even the slightest change in the metric. On the other hand, our presentation exhibits many further properties beyond those required in Definition 1.2, because the interpretation of basic sets in the Δ^0_2 Polish copy is arithmetical. We also conjecture that our methods can be used to show that every Δ^0_2 Polish space has a computable topological presentation with a dense set of points $(x_j)_{j\in\omega}$ such that $x_j \in B_i$ is a c.e. relation (e.g., [30]). We leave this as an open problem.

We conclude this section with some comments about the techniques developed in this paper. The key to constructing a computable topological presentation for a given Δ^0_2 Polish space is to guess whether two given basic open balls intersect. As mentioned before, this is Σ^0_2. Therefore one might expect that Theorem 1.4 can be proved by a standard Π^0_2-argument, perhaps organised as a typical tree argument. However, our proof incorporates several non-standard features. We develop a 'calculus of terms' and the construction does not require the Π^0_2-predicates to fire in any coherent way, dispensing the need for a priority tree. Our technique is loosely related to the e-state methodology used to construct maximal c.e. sets, which we have not yet seen used in this area.

The remainder of the paper is dedicated to a detailed proof of Theorem 1.4.

2. Preliminary analysis

We begin with a brief discussion of Definition 1.2.

Remark 2.1. Up to a change of notation, in Definition 1.2 we can assume that the c.e. open set making up the intersection is always either empty or is a single basic open set, i.e., $B_i \cap B_j = B_k$ for some k. We leave the verification to the reader.

In Definition 1.2 we assume that $B_i \neq \emptyset$ for all i. Up to a change of notation, it is equivalent to saying that $\{i : B_i \neq \emptyset\}$ is computably enumerable. In the literature, Definition 1.2 is typically used with some extra assumption that implies the computable enumerability of $\{i : B_i \neq \emptyset\}$.

In [19] and [31] the non-emptiness of basic sets in Definition 1.2 was not made explicit, but it appears that it was perhaps implicitly assumed there. This assumption was made explicit in, e.g., [11]. Indeed, if we were to drop this assumption, it would imply that all countably based spaces share the same fixed effective topological presentation, hence making the notion meaningless. To see why, declare $B_i \cap B_j = B_k$ for a very large 'fresh' index $k > i, j$. Iterate this procedure to construct an "effective presentation" that is shared among all countably based spaces.

We now turn to a more detailed analysis of Definition 1.1. Moschovakis [26] says that a Polish space is 'recursive' if, in the notation of Definition 1.1, the relations $d(x_i, x_j) > r$ and $d(x_i, x_j) < r$ are actually computable and not merely Σ^0_1. For example, every computable Polish space with the following property is trivially recursive.

Definition 2.2. We say that a computable Polish space is *irrational* if the distance between any pair of distinct special points is an irrational number.

Lemma 2.3 ([10])**.** *Every computable Polish space is (computably) homeomorphic to an irrational one.*

Proof Sketch. First, without loss of generality assume that $d(x_i, x_j) > 0$ for any $i \neq j$; see, e.g., [5, 10]. Use a direct Cantor-style diagonalization to produce a computable real $\gamma > 0$ unequal to any real in the uniformly computable sequence of reals $\dfrac{r}{d(x_i, x_j)}$, where $i \neq j$ and $r \in \mathbb{Q}^+$. Define a new metric $d' = \gamma \cdot d$. $\square$

We write $B(x, r)$ for the basic open ball with rational radius r and centered at a special point x. We write $B^c(x, r) = \{y : d(x, y) \leq r\}$ to denote the respective closed basic ball, and we write $clB(x, r)$ to denote the closure (the completion) of the basic open ball. Note that the latter two closed sets are not equal in general; however, we evidently have $clB(x, r) \subseteq B^c(x, r)$, and $B^c(x, r) \setminus clB(x, r) \subseteq \delta B^c(x, r)$, where

$$\delta B^c(x, r) = \{y : d(x, y) = r\}$$

is the (formal) boundary of $B^c(x, r)$. (This is because $d(x, \xi) < r$ would obviously imply $\xi \in B(x, r) \subseteq clB(x, r)$.) Note that $\delta B^c(x, r)$ does not contain special points. Sometimes we will abuse notation and write $\delta B(x, r)$ in place of $\delta B^c(x, r)$.

Lemma 2.4. *Assume M is an irrational Polish space. Suppose $\mathcal{W}$ is an open set, and let $B_1, \ldots, B_k$ be basic open balls. Then the following are equivalent:*

(1) $\mathcal{W} \setminus \bigcup_{i \leq k} clB_i \neq \emptyset$;
(2) $\mathcal{W} \setminus \bigcup_{i \leq k} B^c_i \neq \emptyset$.

As a consequence, if $\mathcal{W}$ is c.e. open, then the condition $\mathcal{W} \setminus \bigcup_{i \leq k} clB_i \neq \emptyset$ is (uniformly) Σ^0_1.

Proof. (2) $\Rightarrow$ (1) is obvious since $clB_i \subseteq B^c_i$, for each i.

Assume (1). For a set $X \subseteq M$, by $\overline{X}$ we denote its complement $M \setminus X$. We have $\mathcal{W} \setminus \bigcup_{i \leq k} clB_i = \mathcal{W} \cap \overline{\bigcup_{i \leq k} clB_i} = \mathcal{W} \cap \bigcap_{i \leq k} \overline{clB_i}$ is open. Furthermore if it is non-empty, then there is a special point x witnessing this. We also have that $\overline{clB_i} \supseteq \overline{B^c_i}$, and the difference between the two sets lies in δB^c_i, for each i. But δB^c_i

has no special points, and therefore x witnessing (1) has to necessarily witness (2) as well.

To see why $\mathcal{W} \setminus \bigcup_{i \leq k} clB_i \neq \emptyset$ is Σ^0_1 for a c.e. open $\mathcal{W}$, use (1) $\iff$ (2) and the fact that the set

$$\overline{B^c_i} = \{y : d(c_i, y) > r_i\},$$

where c_i is the center of B_i and r_i is its radius, is c.e. open uniformly in c_i, r_i. Clearly, c.e. open sets are closed under taking intersection. It is Σ^0_1 to tell whether a given c.e. open set is not empty: just wait for a basic open ball to be listed in the set. $\square$

Write ϵB to denote the *exterior* of a basic open $B = B(x, r)$. That is, we let $\epsilon B = M \setminus clB$. Naturally, ϵB is an open set. The following notation will be convenient.

Notation 2.5. For a basic open ball B and $\ell \in \{0, 1\}$, write

$$B^\ell = \begin{cases} B & \text{if } \ell = 0, \\ \epsilon B = M \setminus clB & \text{otherwise.} \end{cases}$$

Corollary 2.6. *In an irrational Δ^0_2 Polish space, the relation '$B^{\ell_1}_{i_1} \cap B^{\ell_2}_{i_2} \cap \ldots \cap B^{\ell_k}_{i_k} = \emptyset$' is Π^0_2 uniformly in $k \in \omega$, the indices $i_1, \ldots, i_k$ of basic open balls, and the parameters $\ell_j \in \{0, 1\}$, $j \leq k$.*

Proof. In Lemma 2.4 relativised to $\mathbf{0}'$, let $\mathcal{W}$ be the intersection of those B_{i_k} for which $\ell_k = 0$. $\square$

Another pleasant feature of irrational spaces that follows directly from the elementary proof of Lemma 2.4 is as follows.

Corollary 2.7. *Let M be an irrational space, and let $\tilde{M} = M \setminus \bigcup_{i \in \omega} \delta B_i = \bigcap_{i \in \omega} (M \setminus \delta B_i)$, where B_i range over basic open balls. Then*

$$M \models B^{\ell_1}_{i_1} \cap B^{\ell_2}_{i_2} \cap \ldots \cap B^{\ell_k}_{i_k} = \emptyset \quad \text{if and only if} \quad \tilde{M} \models B^{\ell_1}_{i_1} \cap B^{\ell_2}_{i_2} \cap \ldots \cap B^{\ell_k}_{i_k} = \emptyset.$$

Indeed, $\tilde{M}$ can be replaced with the dense set of special points.

Proof. This is an open set in M, and thus it is non-empty iff there is a special point in the set. Recall special points cannot lie at the formal boundary of any basic closed ball. $\square$

Remark 2.8. In an irrational space, consider $B_{i_1}, B_{i_2}, \ldots, B_{i_k}$. Restrict them and their exteriors to the open set $M \setminus \bigcup_{j \leq k} \delta B_{i_j}$. The operation of taking the exterior of any such B_{i_j} becomes equivalent to taking its complement. If we close these (restricted) sets under union, intersection, and taking the exterior, we obtain a Boolean algebra $\mathcal{B}$. Consider the ideal I generated by the elements of $\mathcal{B}$ that are empty under the interpretation restricted to $M \setminus \bigcup_{j \leq k} \delta B_{i_j}$. Then a finite union of terms of the form $B^{\ell_1}_{i_1} \cap B^{\ell_2}_{i_2} \cap \ldots \cap B^{\ell_k}_{i_k}$ corresponds to a non-zero element of the (finite) quotient Boolean algebra $\mathcal{B}/I$ if, and only if, the respective subset of M is non-empty.

3. Informal discussion

We need to produce a computable topological presentation of a Δ^0_2 Polish space M. By Lemma 2.3 relativised to $\mathbf{0}'$, we can assume that the distance between any two special points in the space is never rational. Recall that to build a computable topological presentation of M, we need to produce a uniform enumeration of a basis of its topology $\mathcal{D} = (D_i)_{i\in\omega}$ and an interpretation $\mu\colon \mathcal{D} \to \mathcal{P}(M)$ mapping each such D_i to a *non-empty* basic open set in M.

3.1. **The elementary case of only two balls.** Imagine we only worry about two basic open balls, B_0 and B_1. It is Π^0_2 to test if $B_0 \cap B_1 = \emptyset$. We can safely set $\mu D_0 = B_0$, and we wish to think that μD_1 equals B_1. If it appears that $B_0 \cap B_1 \neq \emptyset$, we have to declare $D_0 \cap D_1 \supseteq \tilde{D}$, for some basic open $\tilde{D}$. Since necessarily $\mu\tilde{D} \neq \emptyset$, this cannot later be undone.

The idea here is to have many potential versions of B_1. For the sake of this discussion, let us begin with only two D-versions of B_1, denote them D_1 and $\widehat{D}_1$. (But the reader should keep in mind that these sets can later be 'initialised'.) The idea then is to declare $\mu D_1 = B_1$ and $\mu\widehat{D}_1 = B_1 \setminus clB_0 = B_1 \cap \epsilon B_0$, where the latter definitely does not intersect B_0. However, one potential issue with $\widehat{D}_1$ is that $B_1 \cap \epsilon B_0$ can be empty in which case $\widehat{D}_1$ will have to be initialised; we will clarify this shortly. For now, assume $B_1 \cap \epsilon B_0 \neq \emptyset$.

The ball D_1 will "believe" that $B_0 \cap B_1 \neq \emptyset$, which is Σ^0_2. We will immediately introduce a basic $\tilde{D}$ and declare it to be in the intersection of D_0 and D_1. Indeed, to simplify combinatorics, we set $\tilde{D} = D_0 \cap D_1$, and declare $\mu\tilde{D} = \mu D_0 \cap \mu D_1$; recall Remark 2.1.

Recall that $B_0 \cap B_1 \neq \emptyset$ is Σ^0_2. If this predicate 'fires', i.e., a new witness z for the dual Π^0_2 predicate $P(0,1) = \exists^\infty z R(0,1,z)$ is found, we need to redefine μ on D_1 and $\tilde{D}$. In that case, we *initialise* D_1 by setting $\mu D_1 = M$, i.e., to be the entire space, and we redefine μ on $\tilde{D}$ accordingly:

$$\mu\tilde{D} = \mu D_0 \cap \mu D_1 = \mu D_0 = B_0.$$

We introduce a new version D^1_1 of D_1 and set $\mu D^1_1 = B_1$. In a similar way, if the above predicate fires again, we may be forced to re-define $\mu D^1_1 = M$ and introduce a new name D^2_1 for B_1, and so on.

Note that (after resetting μD_1) we will also have to introduce a new basic open D' and declare $D' = \widehat{D}_1 \cap D_1$ and $\mu D' = \mu\widehat{D}_1 \cap \mu D_1 = \mu\widehat{D}_1 = B_1 \cap \epsilon B_0$, assuming that the latter still appears to be non-empty.

Remark 3.1. Even if there is a chance that some set $\mu D_i \cap \mu D_j$ has previously received another D-name, we do not hesitate to put a new name for the set. Recall that equality does not have to be effective in our presentation. It may seem that we are doing way too much extra work, but this approach will indeed simplify the combinatorics in the general case.

Recall that $\mu\widehat{D}_1 = B_1 \cap \epsilon B_0 = B_1 \setminus clB_0$ can potentially be empty. By Corollary 2.6, the predicate $B_1 \cap \epsilon B_0 = \emptyset$ is Π^0_2. Every time the Π^0_2-predicate 'fires', we declare $\mu\widehat{D}_1 = M$ and introduce a new version of $\widehat{D}_1$, say $\widehat{D}^1_1$, and try again with $\mu\widehat{D}^1_1 = B_1 \cap \epsilon B_0$. This may again require us to reset $\mu\widehat{D}^1_1 = M$ and introduce a new $\widehat{D}^2_1$, and so on. If $B_1 \cap \epsilon B_0 \neq \emptyset$ holds, then eventually some such $\widehat{D}^j_1$ will be stable. Every time we 'initialise' $\widehat{D}^j_1$ we set its μ-interpretation equal to M, and we also introduce new D-sets to denote the intersection of this (now maximally large) set with any other set which appears in the construction.

Note the *general pattern* that we follow when we have to redefine μ:

> Every time we redefine μ on a ball that previously followed either B_1 or $B_1 \cap \epsilon B_0$, we set the new interpretation equal to the entire space. We extend μ naturally to intersections of sets. Whenever we have two basic D-sets that appear to intersect non-trivially, and nothing has yet been enumerated into the intersection, we introduce a new D-set with a new fresh name (index) and declare this new D-set equal to the intersection.

More generally, we will later follow this pattern when working with combinations of the form $B^{\ell_1}_{i_i} \cap B^{\ell_2}_{i_2} \cap \ldots \cap B^{\ell_k}_{i_k}$ that appear in Corollary 2.6.

Note that the case when $B_0 \cap B_1 = \emptyset$ and $\epsilon B_0 \cap B_1 = \emptyset$ is impossible by Corollary 2.7, so we end up with only three cases:

Case 1: $B_0 \cap B_1 = \emptyset$, and thus $B_1 \cap \epsilon B_0 = B_1 \neq \emptyset$. In this case D_1 will be initialised infinitely often. If D^j_1 denotes the j-th attempt to define $\mu D_1 = B_1$, then we end up with $\mu D^j_1 = M$ for all $j \in \omega$. On the other hand, $B_1 \cap \epsilon B_0 = B_1 \neq \emptyset$ implies that for some k, $\widehat{D}^k_1$ will never be initialized, and

$$\mu \widehat{D}^k_1 = B_1 \cap \epsilon B_0 = B_1$$

will be stable. Also, $\mu \widehat{D}^j_1 = M$ for all $j < k$, and $\widehat{D}^j_1$ was never introduced for $j > k$. Various finite intersections of these basic sets are defined recursively, according to the strategy.

Case 2: $B_0 \cap B_1 \neq \emptyset$ and $B_1 \cap \epsilon B_0 = \emptyset$. In this case some D^k_1 will never be initialised and the interpretation $\mu D^k_1 = B_1$ will be permanent. Also, $\mu D^j_1 = M$ for $j < k$ and D^j_1 will be undefined for $j > k$. Since $B_1 \cap \epsilon B_0 = \emptyset$, $\mu \widehat{D}^j_1 = M$ for all $j \in \omega$. As before, the intersections are defined according to the strategy, in particular, for some $\tilde{D}$, we will set $D_0 \cap D^k_1 = \tilde{D}$ and thus $\mu \tilde{D} = \mu D_0 \cap \mu D^k_1 = B_0 \cap B_1$.

Case 3: $B_0 \cap B_1 \neq \emptyset$ and $B_1 \cap \epsilon B_0 \neq \emptyset$. In this case for some k and n, D^k_1 and $\widehat{D}^n_1$ will never be initialized, the respective types of basic sets with larger superscripts will never be introduced, and the μ-interpretations of the sets having smaller superscripts will be set equal to the entire space M. In this case we end up with $\mu D^k_1 = B_1$ and $\mu \widehat{D}^n_1 = B_1 \cap \epsilon B_0$, we will also declare $\tilde{D} = D_0 \cap D^k_1$ and $\mu \tilde{D} = \mu D_0 \cap \mu D^k_1 = B_0 \cap B_1$ for some $\tilde{D}$. On the other hand, $D_0 \cap \widehat{D}^n_1$ will be kept empty.

The topology generated by the μ-interpretations of the D-sets will be equivalent to the topology generated by B_0, B_1, and $B_1 \setminus clB_0$ (under intersection and union). This is because every D-set is open, and in any case both B_0 and B_1 appear in the list of D-sets. This finishes the description of the case of only two balls.

3.2. **The general case.** The case of only two balls was rather elementary. However, even with just three balls, the combinatorics can become quite challenging. The 'initialised' balls must still be present in the construction, and so must be their intersections, and the intersections of their intersections (etc.), and the intersections of those with all the other balls throughout the entire proof. The case of only four balls may appear nearly intractable, since there is a great danger to arrive at some logical circularity. Note that this combinatorial difficulty is unrelated to the recursion-theoretic combinatorics of the otherwise straightforward Π^0_2-argument.

The proof could potentially be organised using a Π^0_2-tree of strategies. But in our proof, a node in the tree would do nothing except for measuring a certain Π^0_2-predicate. Furthermore, we will set up our notation so that the stages at which these predicates 'fire' do not have to be synchronised along the current true path, thus making the tree completely redundant; see also (2) below. So we will just keep the predicates and omit the tree. We informally discuss a few further simplifications that will help us in the general case:

(1) We will adjust the outcomes to make the notations easier to handle. For example, in the case analogous to Case 2 above, we will have D_1 initially mapped to $(B_0 \cap B_1) \cup (\epsilon B_0 \cap B_1) \subseteq B_1$ rather than to B_1. Note that it could be that $(B_0 \cap B_1) \cup (\epsilon B_0 \cap B_1) \neq B_1$. To circumvent this, we will define the interpretation of our basic sets in two phases, correcting μ to the final interpretation ν after the construction is done. (See also Remark 4.4.)
(2) There will be very little correlation between different Π^0_2-predicates that measure the non-emptiness of various sets in the construction. We will always measure the predicates on the atoms of the (formal) finite Boolean algebra induced by the (notations of the) first few basic balls that we consider at a stage. Furthermore, if $A = C \sqcup D$ and C is non-empty, we do not have to worry about checking whether A is non-empty; since the subset relation is not in the language, so we do not have to explicitly maintain it at every stage. Furthermore, since C and D cannot possibly intersect (since one is inside the exterior of the other), then we also do not have to coordinate their respective guessing procedures. We also initialise our sets by making them equal to the entire space, so there will be almost no conflict between the strategies.
(3) We will essentially treat our topological presentation as a countable algebraic structure. It will be viewed as a certain term algebra, which will also be a commutative partial monoid. A lot of combinatorics will be handled by the 'calculus of terms' that we present in the section below.

As a consequence of these simplifications and shortcuts, we achieve a surprisingly concise proof, albeit certainly a non-standard one. For instance, the reader may find it a bit unusual that there is no actual construction of the presentation. Instead, there will be a recursive definition that relies on earlier recursive definitions.

4. Proof of Theorem 1.4

4.1. **Calculus of B-terms.** In the case of more than two balls we need a careful choice of notation to handle the combinatorics. Before we proceed, recall Notation 2.5:

$$B^\ell = \begin{cases} B & \text{if } \ell = 0, \\ \epsilon B = M \setminus clB & \text{otherwise,} \end{cases}$$

where B is basic open and $\ell \in \{0, 1\}$. So, for example, $B_0^0 \cap B_1^1 \cap B_2^0$ stands for $B_0 \cap (M \setminus clB_1) \cap B_2$.

Remark 4.1. In what follows, this interpretation of the superscript notation will *never* be applied to D-balls, so in particular D_1^1 will *not* be interpreted as $M \setminus clD_1$.

Instead, D^i will be interpreted as the $(i+1)$-th attempt to define D; this will be clarified later.

Definition 4.2. We call a formal expression of the form $B^{\ell_1}_{i_1} \cap B^{\ell_2}_{i_2} \cap \ldots \cap B^{\ell_k}_{i_k}$ a basic B-term, where $\ell_i \in \{0,1\}$ and $i_1 < i_2 < \ldots < i_k$.

As $t = t(B_0, \ldots, B_{n-1})$ ranges over all basic B-terms in $B_0, \ldots, B_{n-1}$, the (interpretations of) terms $B_n \cap t$ range over subsets of B_n. If we consider the Boolean algebra of subsets of $\bigcup_{i \leq n} B_i$ generated by $B_0, \ldots, B_n$ and extract their boundaries, then the basic B-terms of the form $B_n \cap t(B_0, \ldots, B_{n-1})$ range over all elements in the ideal of subsets of B_n restricted to $M \setminus \bigcup_{i\in\omega} \delta B_i$.

Notation 4.3. Each basic B-term $B^{\ell_0}_0 \cap \ldots \cap B^{\ell_n}_n$ can be uniquely coded by a string $\sigma = \langle \ell_0, \ldots, \ell_n \rangle \in 2^{n+1}$. If σ is a string of this form, we denote the respective basic B-term by t_σ. By Corollary 2.6, it is Π^0_2 to tell whether each such individual set is empty. For each such string σ, uniformly fix a Π^0_2-predicate R_σ that holds iff the subset of M corresponding to t_σ is empty.

We are also interested only in the terms coding subsets of B_n; these are exactly those t_σ ranging over $\sigma \in 2^{n+1}$ having its last coordinate equal to 0. In the definition below, the intended interpretation of $D^i_{n,F}$ is the subset of B_n defined by term $\bigcup_{\sigma\in F} t_\sigma$, where $F \in \mathcal{P}(2^{n+1})$ consisting of strings of the form $\langle \ell_0, \ldots, \ell_{n-1}, 0\rangle$.

Remark 4.4. As we have already mentioned above, the definition of μ below *is not the final interpretation* that will turn our collection of D-sets into a computable topological presentation of M. The final interpretation will be denoted by ν and will be obtained from μ by making one further adjustment.

The intuition behind our definition of $\mu D^i_{n,F}$ below is as follows. We monitor the Π^0_2 predicate measuring whether at least one of t_σ, $\sigma \in F$, is empty. If we believe (up to $(i-1)$ witnesses of the Π^0_2-predicate) that each of these subsets of B_n is non-empty, then we interpret $\mu D^i_{n,F}$ as the union $\bigcup_{\sigma\in F} t_\sigma$. If we see more than $i-1$ witnesses, we expand $\mu D^i_{n,F}$ to be the whole space. We also intend to introduce $D^i_{n,F}$ into the list of our basic open sets only if $\mu D^i_{n,F}$ is defined.

Definition 4.5. For each non-empty $F \in \mathcal{P}(2^{n+1})$ consisting of strings of the form $\langle \ell_0, \ldots, \ell_{n-1}, 0\rangle$ reserve a notation $D^i_{n,F}$ and define

$$\mu D^i_{n,F} = \begin{cases} \bigcup\{\bigcap_{k\leq n} B^{\ell_k}_k : \langle \ell_0, \ldots, \ell_n\rangle \in F\} & \text{if } \bigvee_{\sigma\in F} R_\sigma \text{ fires exactly } (i-1) \text{ times;} \\ M & \text{if } \bigvee_{\sigma\in F} R_\sigma \text{ fires at least } i \text{ times;} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

In the rest of the subsection we develop a notation that will help us to consistently and dynamically extend the definition of μ to arbitrary finite intersections of such D-sets.

4.2. **Calculus of D-terms.**

Definition 4.6. A basic D-term is an expression of the form

$$D^{i_0}_{n_0,F_0} \cap D^{i_1}_{n_1,F_1} \cap \ldots \cap D^{i_j}_{n_j,F_j},$$

where $n_0, \ldots, n_j$ and the respective $F_0, \ldots, F_j$ $(i_k, j \in \omega)$ have the properties described in Definition 4.5 .

In the definition above, the subscripts may have repetitions, i.e., it could be that some fixed n_k, F_k appear in the term with two different superscripts. On the other hand, we can safely assume $D^{i_k}_{n_k,F_k} \cap D^{i_k}_{n_k,F_k} = D^{i_k}_{n_k,F_k}$. We therefore say that a basic D-term is *reduced* if it has no repetitions among the basic D-sets that make up this term. If we additionally (effectively) order the generators (e.g., lexicographically order the terms, finite sets of terms and so on), then such a reduced form becomes unique. We omit the formal definition of a reduced term as it is rather elementary.

If r denotes the operation of taking the reduced form of a given basic D-term, then we define the intersection of two basic reduced D-terms τ and τ' to be $\tau \cap \tau' = r(\tau \cap \tau')$, where the result is of course also a basic D-term. Suppose τ is a reduced basic D-term. We reserve a notation for each reduced τ, but we may never introduce it in the construction. We will stretch our terminology and write D_τ to denote this notation corresponding to a reduced D-term τ. (We thus identify $D^i_{n,F}$ with $D_{D^i_{n,F}}$.)

We also define $D_\tau \cap D_\rho = D_{\tau\cap\rho}$, which we of course intend to put into the list of basic sets only if we can make sure it is not empty.

Remark 4.7. This operation of intersection defined on notations obeys several natural rules. For example, $\cap$ is commutative and associative, and it is also clear that $D_\tau \cap D_\tau = D_\tau$ for any reduced τ. To obtain a commutative monoid, we could introduce (a notation for) the entire space M and set $M \cap D_\tau = D_\tau$ for any reduced τ. We are interested in the c.e. *partial* sub-monoid of this monoid consisting of notations for non-empty subsets of M.

4.2.1. *Extending μ to basic D-terms.* Without loss of generality, we may assume that all our basic D-terms are reduced. Suppose $\tau = D^{i_0}_{n_0,F_0} \cap D^{i_1}_{n_1,F_1} \cap \ldots \cap D^{i_j}_{n_j,F_j}$ is a basic D-term. We reserved (a notation for) the respective basic open D_τ. We also intend to introduce D_τ into the construction only if it corresponds to a non-empty set; i.e., if $\mu(D_\tau) \neq \emptyset$. We *intend* to define μ on D_τ naturally by setting:

$$\mu(D_\tau) = \begin{cases} \mu D^{i_0}_{n_0,F_0} \cap \mu D^{i_1}_{n_1,F_1} \cap \ldots \cap \mu D^{i_j}_{n_j,F_j} & \text{if } \mu D^{i_0}_{n_0,F_0} \cap \mu D^{i_1}_{n_1,F_1} \cap \ldots \cap \\ & \qquad \cap \mu D^{i_j}_{n_j,F_j} \neq \emptyset; \\ \text{undefined} & \text{otherwise,} \end{cases}$$

where the $\mu D^{i_k}_{n_k,F_k}$, $k \leq j$, are defined according to Definition 4.5. Evidently, we assume each individual $\mu D^{i_k}_{n_k,F_k}$ is defined in the first case. We also plan to keep D_τ out of the effective list of our basic sets until $\mu D_\tau \downarrow$, if ever. The obvious obstacle is that for this strategy to work, the condition used in the definition of $\mu(D_\tau)$ has to be Σ^0_1.

Lemma 4.8. *Let $\tau = D^{i_0}_{n_0,F_0} \cap D^{i_1}_{n_1,F_1} \cap \ldots \cap D^{i_j}_{n_j,F_j}$ be a reduced basic D-term. Then the relation $\mu D^{i_0}_{n_0,F_0} \cap \mu D^{i_1}_{n_1,F_1} \cap \ldots \cap \mu D^{i_j}_{n_j,F_j} \neq \emptyset$ is (uniformly) Σ^0_1.*

Proof. Recall that, according to Definition 4.5, $\mu D^i_{n,F}$ is the subset of B_n described by $\bigcup_{\sigma\in F} t_\sigma$, unless some of these t_σ describes the empty set, and the predicate measuring this fact fires at least i times. We also write μ_s to denote the value μ after observing s stages of the respective predicates used in its definition (Definition 4.5).

Suppose s is the first stage at which all $\mu_s D^{i_k}_{n_k,F_k}$ that appear in $\mu_s D^{i_0}_{n_0,F_0} \cap \mu_s D^{i_1}_{n_1,F_1} \cap \ldots \cap \mu_s D^{i_j}_{n_j,F_j}$ become defined. (If such a stage does not exist, we are done.) Some of these $\mu_s D^{i_k}_{n_k,F_k}$ can be already set equal to M; we eliminate then from the term. If there are no terms left after this reduction, then evidently $\mu D^{i_0}_{n_0,F_0} \cap \mu D^{i_1}_{n_1,F_1} \cap \ldots \cap \mu D^{i_j}_{n_j,F_j} = M \neq \emptyset$.

Each $D^{i_k}_{n_k,F_k}$ 'believes' that each of the open sets t_σ (coded by F_k and making up $\bigcup_{\sigma \in F_k} t_\sigma$) is non-empty. If it is not true, or at least if there is more evidence that it might be not true, $\mu D^{i_k}_{n_k,F_k}$ will be set equal to the entire space. Thus, $\mu D^{i_0}_{n_0,F_0} \cap \mu D^{i_1}_{n_1,F_1} \cap \ldots \cap \mu D^{i_j}_{n_j,F_j}$ is indeed equal to the intersection of those $\mu D^{i_k}_{n_k,F_k}$ that will never be redefined.

The basic open sets $B_{n_0}, \ldots, B_{n_k}$ together with their exteriors $B^1_{n_0}, \ldots, B^1_{n_k}$ generate a Boolean algebra on the dense open subset $M \setminus \bigcup_{i \leq k} \delta B^c_{n_i}$ of M, where δB^c_m is the (formal) boundary of the basic closed ball B^c_m; see Remark 2.8.

Each $\mu_s D^{i_k}_{n_k,F_k}$ is currently equal to the subset of B_{n_k} described by $\bigcup_{\sigma \in F_k} t_\sigma$. Interpret $\mu D^{i_0}_{n_0,F_0} \cap \mu D^{i_1}_{n_1,F_1} \cap \ldots \cap \mu D^{i_j}_{n_j,F_j}$ in the finite formal Boolean algebra $\mathcal{B}$ described above (and in Remark 2.8); in this Boolean algebra the operation $X \mapsto X^1$ plays the role of taking the complement of X. (We currently *do not factor* this Boolean algebra $\mathcal{B}$ by the ideal I generated by empty sets, as described in Remark 2.8).

We write $\bigwedge_{k \leq j} \bigvee_{\sigma \in F_k} t_\sigma$ to denote this element. Since $\mathcal{B}$ is a finite Boolean algebra, so it is uniformly decidable to tell whether $\mathcal{B} \models \left(\bigwedge_{k \leq j} \bigvee_{\sigma \in F_k} t_\sigma\right) \neq 0_{\mathcal{B}}$. If it is the case, then we declare $\mu D^{i_0}_{n_0,F_0} \cap \mu D^{i_1}_{n_1,F_1} \cap \ldots \cap \mu D^{i_j}_{n_j,F_j} \neq \emptyset$. Otherwise do nothing and wait for μ to be changed (if ever). If $\mu D^{i_k}_{n_k,F_k}$ is ever adjusted and set equal to M for some k at a later stage, repeat the procedure above to see if we can declare $\mu D^{i_0}_{n_0,F_0} \cap \mu D^{i_1}_{n_1,F_1} \cap \ldots \cap \mu D^{i_j}_{n_j,F_j} \neq \emptyset$.

We argue by induction that the procedure above makes sense, i.e., if it ever declares the intersection to be non-empty then it is indeed non-empty.

Each $\mu D^{i_k}_{n_k,F_k}$ 'believes' that it is composed of non-empty sets. Assume this is indeed true and all these μ-interpretations are final. Then we claim that $\mu D^{i_0}_{n_0,F_0} \cap \mu D^{i_1}_{n_1,F_1} \cap \ldots \cap \mu D^{i_j}_{n_j,F_j} \neq \emptyset$. First, assume n_j is the largest index among $n_0, n_1, \ldots, n_j$. Then $\mu D^{i_0}_{n_0,F_0} \cap \mu D^{i_1}_{n_1,F_1} \cap \ldots \cap \mu D^{i_j}_{n_j,F_j}$ has to be a subset of B_{n_j} composed of basic B-terms. Recall that $D^{i_j}_{n_j,F_j}$ considers combinations of all balls of smaller indices in the definition of its interpretation; $\mu(D^{i_j}_{n_j,F_j}) = \bigcup_{\sigma \in F_j} t_\sigma$, where each t_σ has an instance of B_j and mentions each B_k, $k < j$, or its exterior.

As explained in the second half of Remark 2.8, restricting the interpretation of such terms to $M \setminus \bigcup_{i \leq j} \delta B^c_{n_i}$ preserves the property of being non-empty. On the other hand, in the Boolean algebra $\mathcal{B}$ obtained by means of this restriction, such t_σ correspond to atoms; each of these atoms is known to have a non-empty interpretation (with or without the boundaries). We also see that $\mu D^{i_0}_{n_0,F_0} \cap \mu D^{i_1}_{n_1,F_1} \cap \ldots \cap \mu D^{i_j}_{n_j,F_j}$, interpreted as the element of $\mathcal{B}$, has to be non-zero, and thus has to be equal to a finite subset of these atoms t_σ making up the last set $\mu D^{i_j}_{n_j,F_j}$ in the intersection. Since we have that each such individual atom is non-empty (as stably measured in the definition of $\mu D^{i_j}_{n_j,F_j}$), it makes the intersection non-empty too.

On the other hand, suppose the 'belief' of some $\mu D^{i_k}_{n_k,F_k}$ in not correct, i.e., its interpretation will be eventually set equal to M. In terms of the Boolean algebra (equivalently, in terms of restricting the interpretation μ to $M \setminus \bigcup_{i \leq j} \delta B^c_{n_i}$), this corresponds to replacing the respective $\bigvee_{\sigma \in F_k} t_\sigma$ with $1_{\mathcal{B}}$ in $\bigwedge_{m \leq j} \bigvee_{\sigma \in F_m} t_\sigma$. Clearly, in $\mathcal{B}$ this gives an element $b \geq \bigwedge_{m \leq j} \bigvee_{\sigma \in F_m} t_\sigma$ which also has to be non-zero in $\mathcal{B}$. This means that we can iterate the above argument finitely many times until we arrive at the stage at which every $\mu D^{i_k}_{n_k, F_k}$ that remains in the intersection achieves its final value. We either have all of them equal to M or (as explained above already) $\mu D^{i_0}_{n_0, F_0} \cap \mu D^{i_1}_{n_1, F_1} \cap \ldots \cap \mu D^{i_j}_{n_j, F_j} \neq \emptyset$ (with or without the restriction to $M \setminus \bigcup_{i \leq j} \delta B^c_{n_i}$).

We conclude that the procedure described above indeed gives a correct answer, and thus the lemma follows. □

4.3. **The definition of a computable topological presentation.** Since $\mu_s D^{i_k}_{n_k, F_k}$ can change at most once for each individual $D^{i_k}_{n_k, F_k}$, $\lim_s \mu_s D^{i_k}_{n_k, F_k}$ exists, where μ_s is the value of μ as calculated after approximating the respective predicates for s steps.

Definition 4.9. Let $P(\tau)$ be the Σ^0_1 predicate measuring whether $\tau = \mu D^{i_0}_{n_0, F_0} \cap \mu D^{i_1}_{n_1, F_1} \cap \ldots \cap \mu D^{i_j}_{n_j, F_j} \neq \emptyset$ given by Lemma 4.8. Set

$$\mu_s(D_\tau) = \begin{cases} \mu_s D^{i_0}_{n_0, F_0} \cap \mu_s D^{i_1}_{n_1, F_1} \cap \ldots \cap \mu_s D^{i_j}_{n_j, F_j} & \text{if } P(\tau) = 1; \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Since $\mu_s D^{i_k}_{n_k, F_k}$ (if ever defined) can change at most once for each $D^{i_k}_{n_k, F_k}$, we can define $\mu D_\tau = \lim_s \mu_s D_\tau$, assuming that $P(\tau) = 1$. Also, observe that $\mu D_\tau \downarrow$ iff $\exists s \, \mu_s D_\tau \downarrow$, which is obviously Σ^0_1. In other words, the domain of μ is Σ^0_1. Having this property of μ in mind, we arrive at the following definition.

Definition 4.10. Let $\mathcal{D} = \{D_\tau : \tau \text{ is a basic } D\text{-term such that } \mu(D_\tau) \downarrow\}$, and for any $D_\tau, D_\rho \in \mathcal{D}$ define

$$D_\tau \cap D_\rho = \begin{cases} D_{\tau \cap \rho} & \text{if } \mu D_{\tau \cap \rho} \downarrow; \\ \emptyset & \text{otherwise.} \end{cases}$$

As was mentioned earlier, μ is *not* the final interpretation of $\mathcal{D}$, but it will be used to produce an interpretation that will work.

Proposition 4.11. *There exists an interpretation $\nu \colon \mathcal{D} \to \mathcal{P}(M)$ that turns $\mathcal{D}$ into a computable topological presentation of M.*

Proof. We first discuss several basic properties of μ. Lemma 4.8 ensures that if μ_s is ever defined on D_τ, then the final value $\mu D_\tau = \lim_s \mu_s D_\tau$ is well-defined and is necessarily non-empty. Thus, $\mathcal{D}$ consists of non-empty sets (under μ). Similarly, the definition of $D_\tau \cap D_\rho$ ensures that we put $D_{\tau \cap \rho}$ in $\mathcal{D}$ and set $\mu(D_{\tau \cap \rho}) = \mu D_\tau \cap \mu D_\rho$ only if $\mu D_\tau \cap \mu D_\rho \neq \emptyset$. As we have already mentioned above, the domain of μ is Σ^0_1, which makes the definition of $\mathcal{D}$ effective.

We now need to adjust μ to get a ν so that $\{\nu(D) : D \in \mathcal{D}\}$ forms a (sub)basis of topology compatible with the given metric on M. We would also like to have that $\nu D \neq \emptyset$ iff $\mu D \neq \emptyset$, and that $\nu D_{\tau \cap \rho} = \nu D_\tau \cap \nu D_\rho$. (It should be clear to the reader that μ induces a computable topological presentation of $\tilde{M} = M \setminus \bigcup_{i \in \omega} \delta B_i$).

To see how this can be done, we restrict our attention to D_τ where τ is just a singleton intersection, i.e., a set of the form $D^i_{n,F}$. (Recall Definition 4.5 and recall that we identify $D^i_{n,F}$ with $D_{D^i_{n,F}}$.) Here $F \in \mathcal{P}(2^{n+1})$ is non-empty and consists of strings of the form $\langle \ell_0, \dots, \ell_{n-1}, 0\rangle$. Informally, $\mu D^i_{n,F}$ is the subset of the basic ball B_n described by $\bigcup_{\sigma \in F} t_\sigma$ (and in this case all of them are non-empty), or it is equal to M. Take the collection G of *all* strings of the form $\langle \ell_0, \dots, \ell_{n-1}, 0\rangle$ that describe non-empty subsets of B.

In the space $\tilde{M} = M \setminus \bigcup_{i \in \omega} \delta B_i$, we have that $B_n \restriction_{\tilde{M}} = \mu D^i_{n,G}$ for some i. For this particular set, define $\nu D^i_{n,G} = B_n$. Otherwise keep $\nu = \mu$ for all other versions $D^j_{n,F}$. We go over all basic B_n and all respective D-balls having a correct guess about (non-emptiness of) its subsets, and we adjust the definition of μ to get the definition of ν. This defines $\nu(D_\tau)$ for any singleton τ.

The material of Section 2 (specifically Corollary 2.7) guarantees that, at least when τ is a singleton, $\nu D_\tau \neq \emptyset$ iff $\mu D_\tau \neq \emptyset$. We extend this definition naturally by setting $\nu D_{\tau \cap \rho} = \nu D_\tau \cap \nu D_\rho$, if the latter is non-empty. We claim that it is non-empty if, and only if, $\mu D_{\tau \cap \rho}$ is non-empty. This is again guaranteed by Corollary 2.7. Thus, having or not having a boundary in the interpretation does not affect the procedure described in Lemma 4.8, and we can keep exactly the same Σ^0_1 predicate. This means that we still use μ in the verification and in Definition 4.10, but then use ν to define the interpretation of $\mathcal{D}$. Since every basic open ball is listed among the D-sets, and also every D-set is evidently open in M, we have that $(\mathcal{D}, \nu)$ is a computable topological presentation of M. □

This finishes the proof of Theorem 1.4.

References

1. C. J. Ash and J. Knight, *Computable structures and the hyperarithmetical hierarchy*, Studies in Logic and the Foundations of Mathematics, vol. 144, North-Holland Publishing Co., Amsterdam, 2000.
2. G. Baumslag, E. Dyer, and C. Miller, III, *On the integral homology of finitely presented groups*, Topology **22** (1983), no. 1, 27–46.
3. Nikolay Bazhenov, Matthew Harrison-Trainor, and Alexander Melnikov, *Computable Stone spaces*, Ann. Pure Appl. Logic **174** (2023), no. 9, 103304.
4. G. S. Ceitin, *Algorithmic operators in constructive complete separable metric spaces*, Dokl. Akad. Nauk SSSR **128** (1959), 49–52.
5. R. G. Downey and A. G. Melnikov, *Computably compact metric spaces*, Bulletin of Symbolic Logic **29** (2023), no. 2, 170–263.
6. Rod Downey and Carl G. Jockusch, *Every low Boolean algebra is isomorphic to a recursive one*, Proceedings of the American Mathematical Society **122** (1994), no. 3, 871–880.
7. Yu. L. Ershov and S. S. Goncharov, *Constructive models*, Siberian School of Algebra and Logic, Consultants Bureau, New York, 2000.
8. Lawrence Feiner, *Hierarchies of Boolean algebras*, J. Symbolic Logic **35** (1970), 365–374.
9. S. S. Goncharov and J. F. Knight, *Computable structure and non-structure theorems*, Algebra and Logic **41** (2002), no. 6, 351–373.
10. Vassilios Gregoriades, Tamás Kispéter, and Arno Pauly, *A comparison of concepts from computable analysis and effective descriptive set theory*, Math. Structures Comput. Sci. **27** (2017), no. 8, 1414–1436.

11. Tanja Grubba and Klaus Weihrauch, *On computable metrization*, Electron. Notes Theor. Comput. Sci. **167** (2007), 345–364.
12. M. Harrison-Trainor and A. Melnikov, *An arithmetic analysis of closed surfaces*, Transactions of the American Mathematical Society (2023), published online, doi: 10.1090/tran/8915.
13. Matthew Harrison-Trainor, Alexander Melnikov, and Keng Meng Ng, *Computability of Polish spaces up to homeomorphism*, Journal of Symbolic Logic **85** (2020), no. 4, 1664–1686.
14. G. Higman, *Subgroups of finitely presented groups*, Proc. Roy. Soc. Ser. A **262** (1961), 455–475.
15. M. Hoyrup, T. Kihara, and V. Selivanov, *Degree spectra of homeomorphism types of Polish spaces*, 2020, preprint, arXiv:2004.06872.
16. N. Khisamiev, *The arithmetic hierarchy of abelian groups*, Sibirsk. Mat. Zh. **29** (1988), no. 6, 144–159.
17. ———, *Constructive abelian groups*, Handbook of recursive mathematics, Vol. 2, Stud. Logic Found. Math., vol. 139, North-Holland, Amsterdam, 1998, pp. 1177–1231.
18. H. T. Koh, A. Melnikov, and K. M. Ng, *Computable topological groups*, Journal of Symbolic Logic (2023), published online, doi: 10.1017/jsl.2023.67.
19. Margarita Korovina and Oleg Kudinov, *The Rice-Shapiro theorem in computable topology*, Log. Methods Comput. Sci. **13** (2017), no. 4, Paper No. 30, 13 pp.
20. M. Lupini, A. Melnikov, and A. Nies, *Computable topological abelian groups*, J. Algebra **615** (2023), 278–327.
21. A. Mal′cev, *Constructive algebras. I*, Uspehi Mat. Nauk **16** (1961), no. 3 (99), 3–60.
22. A. G. Melnikov and K. M. Ng, *Separating notions in effective topology*, International Journal of Algebra and Computation (2023), published online, doi: 10.1142/S0218196723500649.
23. Alexander G. Melnikov, *Computable abelian groups*, Bulletin of Symbolic Logic **20** (2014), no. 3, 315–356.
24. Antonio Montalbán, *Computable structure theory—within the arithmetic*, Perspectives in Logic, Cambridge University Press, Cambridge; Association for Symbolic Logic, Ithaca, NY, 2021.
25. Y. N. Moschovakis, *Recursive metric spaces*, Fund. Math. **55** (1964), 215–238.
26. Yiannis N. Moschovakis, *Descriptive set theory*, second ed., Mathematical Surveys and Monographs, vol. 155, American Mathematical Society, Providence, RI, 2009.
27. E. Ju. Nogina, *Effectively topological spaces*, Dokl. Akad. Nauk SSSR **169** (1966), 28–31.
28. M. Rabin, *Computable algebra, general theory and theory of computable fields.*, Trans. Amer. Math. Soc. **95** (1960), 341–360.
29. Matthias Schröder, *Effective metrization of regular spaces*, Computability and Complexity in Analysis, Informatik Berichte **235** (1998), 63–80.
30. Dieter Spreen, *A characterization of effective topological spaces*, Recursion theory week (Oberwolfach, 1989), Lecture Notes in Math., vol. 1432, Springer, Berlin, 1990, pp. 363–387.

31. Klaus Weihrauch and Tanja Grubba, *Elementary computable topology*, J.UCS **15** (2009), no. 6, 1381–1422.

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