PUNCTUAL DEFINABILITY ON STRUCTURES

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Abstract. We study punctual categoricity on a cone and intrinsically punctual functions and obtain complete structural characterizations in terms of model theoretic notions. As a corollary, we answer a question of Bazhenov, Downey, Kalimullin, and Melnikov by showing that relational structures are not punctually universal. We will also apply this characterisation to derive an algebraic characterisation of relatively punctually categorical mono-unary structures.

Keywords: algebraic structures, definability, primitive recursion

1. Introduction

Punctual structure theory studies complexity issues of mathematical structures as we do in computable structure theory, but using primitive recursion as the main notion of computability instead of Turing computability. There are many situations in which primitive recursive functions are preferable over just partial recursive functions, the most obvious advantage being that primitive recursive functions always halt and the time they take to halt can be calculated in advance — they won't keep you waiting, they are punctual. We will use the word *punctual* to refer to notions that involve primitive recursive functions.

There has been a lot of work on punctual structures in the last few years [Mel17, KMN17, BDKM19, DGM$^+$19, DMN, DHTK$^+$] as it turned out to be a much deeper and richer topic than expected. For instance, punctual structures have been used to solve a long standing question of Khoussainov and Nerode [KN08] on automatic structures; see [BHTK$^+$]. We refer the reader to the survey article [DMN] for more motivation and information on the latests developments.

We will restrict ourselves to finite languages. A countable structure is *fully primitive recursive* if its domain is $\mathbb{N}$ and the operations and predicates of the structure are primitive recursive. So far, the study of punctual structure theory has been restricted to the properties of fully primitive recursive structures, or variations of these notion. The purpose of this paper is to study notions that involve presentations of arbitrary complexity and functions that are primitive recursive relative to them. We hope that the ideas and techniques that we develop in this paper to work in this new setting will be useful in further work of relative or on-a-cone notions in punctual structure theory.

We concentrate on two notions: intrinsically punctual functions and punctual categoricity on a cone. The former is about functions in ω^ω that are punctually computable from all presentations of a structure, and the latter is about isomorphism that are punctually computable from a structure. For both notions we find

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interesting structural characterization that are quite different from their Turing computable structure theory counterparts.

Before introducing these two notions, let us mention a corollary of our work that solves a problem posed in [BDKM19]:

Theorem 1.1. *The class of all structures over a finite relational language is not punctually universal.*

When we say that a class of structures is *punctually universal*, we mean that every structure can be coded by a structure in the class preserving its "punctual-theoretic" properties. In the case of Turing computability, it is well known that graphs are universal in the strongest sense possible: every structure is effectively bi-interpretable with a graph and the interpretations are uniform and depend only on the language of the structure. We will prove the theorem above in Corollary 3.8 using results about intrinsically punctual functions.

1.1. **Intrinsically punctual functions.** We consider the information content of a structure that can be decoded in a primitive recursive way. We say that a function $f \in \omega^\omega$ is *intrinsically punctual* in a structure $\mathcal{A}$ if f is primitive recursive relative to $\mathcal{B}$ for every copy $\mathcal{B}$ of $\mathcal{A}$. In Theorem 3.2, we provide a complete structural characterization of the intrinsically punctual functions on a structure. Then, in Corollary 3.6, we show that for relational structures, the only functions that are intrinsically punctual are the ones that are primitive recursive already. We use this result about the limited computational power of relational structures to deduce that they are not punctually universal.

1.2. **Punctual categoricity.** A fully primitive recursive structure $\mathcal{A}$ is said to be *punctually categorial* if for every other fully primitive recursive copy $\mathcal{B}$ of $\mathcal{A}$, there is primitive recursive isomorphism between them. Punctually categorical structures were studied in [KMN17, DHTK$^+$], but no structural characterization was found. Most examples are extremely simple. Maybe the most algebraically interesting example is the infinite dimension vector space over $\mathbb{Z}_p$ for prime p. Nonetheless, with much effort one can construct complex unnatural examples of punctually categorical structures. For instance, much in the spirit of [DKL$^+$15], punctually categorically structures do not even have to be (Turing) Δ^1_1-categorical [DGM$^+$19].

In computable structure theory, the way to get a nice purely-structural characterization of a property is to consider it *on a cone*. The third author has done a lot of work investigating properties in computable structure theory which hold on a cone of Turing degrees, as they often seem to behave better and have better structural characterization than their non-on-a-cone counterparts; see, e.g., [Mon13]. The natural subrecursive analogy of this notion has not yet been studied. In the present paper we use properties which are primitive recursive on a cone with respect to punctual reduction $\leq_{PR}$ which will be defined in the preliminaries.

We thus consider punctual categoricity on a cone: A fully primitive recursive structure $\mathcal{A}$ is *punctually categorical on a cone* if there exists a function $g \in \omega^\omega$ such that for every copy $\mathcal{B}$ of $\mathcal{A}$, there is a isomorphism f between $\mathcal{A}$ and $\mathcal{B}$ such that both f and f^{-1} are primitive recursive relative to $g \oplus \mathcal{B}$. We first consider relational structures, and we find that only trivial structures are punctually categorical on a cone.

Theorem 1.2. *(see Theorem 4.2) Suppose $\mathcal{A}$ is punctual relational structure. Then the following are equivalent:*

(1) *$\mathcal{A}$ is punctually categorical on a cone;*
(2) *$\mathcal{A}$ becomes automorphically trivial after fixing finitely many constants.*

What can be said about relatively punctually categorical structures which are not relational? In Theorem 4.4 we will show that every structure that is punctually categorical on a cone must be $\aleph_0$-categorical. We then extend this to a full model-theoretic characterization of structures punctually categorical on a cone.

Theorem 1.3. *A structure $\mathcal{A}$ is punctually categorical on a cone if and only if*

(1) *$\mathcal{A}$ is $\exists$-atomic;*
(2) *$\mathcal{A}$ is uniformly locally finite;*
(3) *$\mathcal{A}$ is highly local.*

We clarify the notions used in the theorem above. A structure is $\exists$-atomic if every automorphism orbit is defined by a first-order existential formula; see [Mon17]. A structure is uniformly locally finite if there exists a function f which, given the size of a finite subset F of $\mathcal{A}$, bounds the cardinality of the substructure $\langle F \rangle$. Finally, the technical notion of high locality says that for any large enough finite $D \supseteq F$, $\langle D \rangle$ will intersect all automorphism orbits over F; see Section 4 for details. In particular, condition (3) implies that $\exists$-atomicity is witnessed by bounded $\exists$-quantification, but the bound is numerical rather than syntactical. Of course, it follows that every relatively punctually categorical structure satisfies conditions (1)-(3), but in the case of relative punctual categoricity we do not know whether these conditions are sufficient. We leave this as an open question.

As an application of Theorems 1.3 and 4.4, in Theorem 4.10 we will give an algebraic characterisation of relatively punctually categorical unary structures. It essentially says that, up to a finite number of exceptional elements, each punctually categorical unary structure must be composed of infinitely many copies of some fixed finite unary structure so that all these copies are automorphic over the exceptional elements. Already structures with two unary functions are (Turing) computably universal but they are not not punctually universal [DGM$^+$19]. The only known proof of this fact relies on a rather involved analysis of (plain) categoricity of punctual structures [DGM$^+$19]. However, relative punctual categoricity of unary structures has not been explored; it may eventually lead to a more elegant proof of their punctual non-universality.

2. Preliminaries

Informally, a function is primitive recursive if it can be computed by a program that does not use unbounded loops. More formally, one obtain the notion of a primitive recursive function by forbidding unbounded minimisation operator from the inductive definition of a general recursive function. We thus close the elementary basic functions only under recursion schemata and composition; see [Rog87] for more details. It is well-known that we can allow bounded minimisation too.

We will need several relativised versions of primitive recursion.

Definition 2.1. *We say that a function f is punctually reducible to a function g, written $f \leq_{PR} g$, if f can be obtained using primitive recursion and composition from the basic primitive recursive functions and g.*

In the definition of $f \leq_{PR} g$, the primitive recursive schema of g contains only finitely many mentions of g. We could therefore view g as a parameter from ω^ω and thus view f as a *primitive recursive functional*; that is, for some i, we will have

$$f = P_i^g,$$

where $(P_i)_\omega$ is a general recursive uniform enumeration of all primitive recursive schemata with a functional parameter. Primitive recursive functionals will be used to formalise sub-recursive relativisation throughout the paper. Whenever we claim that a procedure is punctual relative to an oracle, then the oracle is always a total function, and we mean that the procedure can be expressed as a primitive recursive schema with this function adjoined to the list of basic functions.

In [DMN], Downey, Melnikov and Ng have suggested another, perhaps more natural, notion of a primitive recursive functional. In this alternate approach we say that a primitive recursive functional is a Turing functional Ψ such that there is a primitive recursive *time function* t with the property that

$$(\forall g \in \omega^\omega)\,(\forall x)\,[\Psi^g(x)\downarrow \text{ in at most } t(x) \text{ steps}];$$

that is, it is a total functional in which the number of steps in which its computation halts is uniformly primitively recursively bounded for *all* oracles.

Alas, these two definitions of a primitive recursive functional are not equivalent in general. For example, the functional $\Phi^g(x) = g(g(0))$ is represented by a primitive recursive scheme (in g) but the use of Ψ will not be uniformly primitively recursively bounded. Nonetheless, if we restrict the input space to a compact $\mathcal{P} \subseteq \omega^\omega$ for which there exists a primitive recursive branching function b with the property

$$(\forall g \in \mathcal{P})\,[g(i) \leq b(i)],$$

then these two definitions become equivalent, as shown in the lemma below. If $\mathcal{P} \subseteq \omega^\omega$ possesses such a primitive recursive b then we say that $\mathcal{P}$ is *punctually branching*. Up to a primitive recursive bijective coding, a punctually branching $\mathcal{P}$ can be identified with 2^ω.

Lemma 2.2. *Suppose $\mathcal{P} \subseteq \omega^\omega$ is punctually branching. Then for a Turing functional Ψ whose oracles range only over $\mathcal{P}$, the following are equivalent:*

1. *Ψ possesses a primitive recursive time function;*
2. *Ψ can be expressed as a primitive recursive scheme in which the oracle function is declared a basic elementary function.*

Proof. (1) $\to$ (2). Simply replace every instance of the μ operator with an instance minimisation operator bounded by $t(x)$.

(2) $\to$ (1). The first naive (an incorrect) idea would be to use a compactness argument of some kind exploiting totality of Ψ; however this would not work in general. Informally, we shall exploit primitive recursive branching of $\mathcal{P}$ as follows. Working from the base of the primitive recursive scheme, produce a primitive recursive list of all possible primitive recursive schemata reflecting all possible computations for all possible values of g. For example, if Ψ^g is $s(g(s(0)))$ and $\mathcal{P}$ is 2^ω then Ψ^g will be replaced with $\{s(0), s(s(0))\}$, the first instance corresponds to $g(1) = 0$ and the second to $g(1) = 1$. Of course, more complex schemata will require a more complex splitting, but the resulting finite list will be uniformly primitive recursive; we can therefore run all these schemata in the list at once and calculate the estimate on the number of steps required to calculate each of them.

Formally, the proof is by induction on the complexity of the primitive recursive scheme for Ψ. We must calculate a primitive recursive bound $t(x)$ on the number of steps required for $\Phi^g(x)$ to halt.

For the base of induction consider the following cases: $\Phi^g = o$, $\Phi^g = s$, $\Phi^g = I^n_m$, and $\Phi^g = g$. The first three cases are evident since they do not refer to g, while in the case when $\Phi^g = g$ take $t = b$, where b is the primitive recursive branching of $\mathcal{P}$. Take $t(x) = \sum_{i \le x} b(i)$ to make t monotonically increasing in its input.

The inductive step splits into two different cases depending on whether the last iteration is composition or an instance of primitive recursion.

Suppose it is composition,

$$\Psi^g(\bar{x}) = \Phi^g(\Theta^g_1(\bar{x}), \dots, \Theta^g_m(\bar{x})),$$

where $\Psi, \Theta_1, \dots, \Theta_m$ there are primitive recursive operators with primitive recursive time bounds $t_0, t_1, \dots, t_m$. As usual, we identify a tuple $\bar{x} = \langle x_1, \dots, x_m \rangle$ with its primitive recursive code. The time functions $t_0, t_1, \dots, t_m$ can be assumed monotonically increasing in their inputs.

Define a primitive recursive time bound for Ψ by the rule

$$t(\bar{x}) = t_0(\langle t_1(\bar{x}), \dots, t_m(\bar{x}) \rangle) + \sum_i t_i(\bar{x}),$$

which can be rewritten into a primitive recursive schema using the standard techniques.

Now suppose that Ψ is defined using an instance of primitive recursion, more specifically

$$\Psi^g(\bar{x}, 0) = \Theta^g(\bar{x});$$
$$\Psi^g(\bar{x}, y+1) = \Phi^g(\bar{x}, y, \Psi^g(\bar{x}, y)),$$

where Φ and Θ are primitive recursive operators which have corresponding primitive recursive time functions t_0 and t_1. Define t by the rule

$$t(\bar{x}, 0) = t_0(\bar{x});$$
$$t(\bar{x}, y+1) = t_1(\bar{x}, y, t(\bar{x}, y)) + t(\bar{x}, y);$$

assuming that t is monotonically increasing in its input this gives the desired upper bound. □

Lemma 2.2 ensures that the definition below is robust.

Definition 2.3. A Turing operator Ψ on 2^ω is *punctual* if there is a primitive recursive function t such that $\Psi^\xi(x) \downarrow$ in at most $t(x)$ steps, for any $\xi \in 2^\omega$.

In the definition, 2^ω can be replaced with a punctually branching $\mathcal{P} \subseteq \omega^\omega$. A typical use of Lemma 2.2 is illustrated in the proof of the simple proposition below.

Proposition 2.4. *Let $\mathcal{A}$ be a countably infinite structure in a finite language. Assume $f \le_{PR} \mathcal{B}$ for every structure $\mathcal{B} \cong \mathcal{A}$ such that the domain of $\mathcal{B}$ is ω. Then f is punctually bounded from above. That is, there is a primitive recursive function $h(x)$ such that $f(x) \le h(x)$ for every x.*

Proof. Note that the number of distinct terms of length n in the free algebra in the language of $\mathcal{A}$ is naturally bounded; this is because the language is finite. Thus, we can find an isomorphic copy $\mathcal{B}$ of $\mathcal{A}$ upon the domain of ω, such that each operation in $\mathcal{B}$ is bounded by a primitive recursive function. This allows us to restrict the space of all possible oracles in the computation to a *punctually bounded space*.By Lemma 2.2, $f \leq_{PR} \mathcal{B}$ implies that all potential computations of $f(i)$ from $\mathcal{B}$ will have a common bound which is primitive recursive in i. □

We also note that we can sub-recursively relativise Lemma 2.2, in the sense that "primitive recursive" can be replaced with "primitive recursive in f" throughout, where f is any total function. In particular, Lemma 2.2 handles the case of an arbitrary total functional $\Psi : 2^\omega \to \omega^{<\omega}$, because each total functional is primitive recursive relative to "itself"; i.e., relative to a function f which imitates its computation on each input, under some natural coding of the domain and the range. Note that f does not have to be computable. Therefore our approach covers the definition from [Kie81, BEY98] in which "online" procedures were assumed to be merely total and not even necessarily (Turing) computable.

3. Intrinsically punctual functions

We identify a countable structure upon the domain ω in a finite functional language $\{f_0, \dots, f_{t-1}\}$ with the the function

$$f_0 \oplus \cdots \oplus f_{t-1}(i, \bar{x}) = \begin{cases} f_i(\bar{x}), & \text{if } i < t; \\ 0, & \text{if } i \geq t \text{ or the size of } \bar{x} \text{ does not coincide with} \\ & \text{the arity of } f_i, \end{cases}$$

where $\bar{x}$ is a finite tuple of integers. In particular, the structure is punctual if, and only if, the function $f_0 \oplus \cdots \oplus f_{t-1}$ is primitive recursive. If a language of a structure contains also some predicates then each predicate is considered as a $\{0,1\}$-valued operation.

Definition 3.1. We say that a family $\{\Phi_i\}_{i\in\omega}$ of first-order formulae is a *punctual family* if there exists a primitive recursive function g such that $g(i)$ is the Gödel number of Φ_i. We write $\{\Phi_g\}$ for a punctual family indexed by a function g. (We allow g to have arity greater than one if necessary.)

Recall that a function $f : \omega \to \omega$ is intrinsically punctual in a structure $\mathcal{A}$ if f is primitive recursive relative to $\mathcal{B}$ for every copy $\mathcal{B}$ of $\mathcal{A}$.

Theorem 3.2. *Let $\mathcal{A}$ be a countably infinite structure over a finite language and $f : \omega \to \omega$ a function. Then f is intrinsically punctual in $\mathcal{A}$ if and only if*

(1) *f is bounded by a primitive recursive function, and*
(2) *there exists a tuple $\bar{a}$ from $\mathcal{A}$ and punctual family $\{\Phi_g\}$ of quantifier-free first-order formulae such that*

$$\begin{aligned} f(x) = y \quad &\iff \quad (\mathcal{A}, \bar{a}) \models \exists \bar{z}(\&_{i<j}(z_i \neq z_j) \ \&\ \Phi_{g(x,y)}(\bar{a}\bar{z})) \\ &\iff \quad (\mathcal{A}, \bar{a}) \models \forall \bar{z}(\&_{i<j}(z_i \neq z_j) \to \Phi_{g(x,y)}(\bar{a}\bar{z})), \end{aligned}$$

where in both cases $\bar{z}$ stands for the free variables in $\Phi_{g(x,y)}$.

Remark 3.3. We feel that the second (universal) property of Φ_g requires some explanation. It says that if a portion of the structure is large enough, it must contain a pattern described by the respective formula. The formula can of course contain many disjunctions, each testing the property of interest on a subset of its the free variables. Another possibility is that the formula may contain no free variables at all. See the examples below.

Example 3.4. For a set $A \subseteq \omega$ let $\mathcal{N}_A = (\omega, 0, s, A)$, where $s(x) = x+1$, and A is viewed as a predicate, i.e., it is the characteristic function of itself. This gives the simplest possible coding of A into the isomorphism type of a structure, and it is clear that A can be punctually extracted from any copy of $\mathcal{N}_A$. Define $A^0 = \neg A$, $A^1 = A$, and $A^y = 0$ if $y > 1$. Then

$$A(x) = y \iff A^y(s^x(0)),$$

where the family $\{A^y(s^x(0))\}$ of quantifier-free sentences is clearly a punctual family.

Example 3.5. For a set $A \subseteq \omega$ let $\mathcal{M}_A = (\omega, 0, \circ, A)$, where

$$x_1 \circ x_2 = \min(x_1 + 1, x_2).$$

In contrast with the previous example, the structure is locally finite. But for $x, y \in \omega$ we can define the term

$$t_x(u) = (\dots(0 \underbrace{\circ\, u) \circ \dots) \circ u}_{x \text{ times}}$$

and the quantifier-free formula

$$\Phi_{x,y}(\bar{z}) = \vee_{u \in \{z_1,\dots,z_x\}}[\&_{i<x}[t_i(u) \neq t_{i+1}(u)] \;\&\; A^y(t_x(u))].$$

with variables $\bar{z} = z_1, \dots, z_x$. Then the condition (2) of the Theorem is satisfied for $A(x) = y$.

Proof. The up direction is quite standard. We prove the down direction. (1) follows from Proposition 2.4. We concentrate in proving (2).

We construct a sufficiently generic $\mathcal{B}$ in which all operations have a primitive recursive bound. Let $\mathcal{P}$ be the punctually branching space of all countable structures in the language of $\mathcal{A}$; the branching function p can be computed from the free algebra in the language of $\mathcal{A}$. In $\mathcal{P}$, every infinite path through $\mathcal{P}$ is a structure in the language of $\mathcal{A}$ upon the domain ω in which every operation is realised by a primitively recursively bounded function (cf. the proof of Prop. 2.4).

As usual in such proofs, given σ (viewed as an initial segment of $\mathcal{B}$) and $\iota : \sigma \to \mathcal{A}$ a partial isomorphism with range $\bar{a}$, we search for an extension $\tau \supseteq \sigma$ and an extension ξ of ι such that $f(x) \neq P_e^\tau(x) \downarrow$ for some x, where P_e is the e-th primitive recursive operator. If we succeed then we consider the next operator, etc. However, if f is equal to $P_e^{\mathcal{B}}$ and e is least such, then our search for τ must have failed for that e; suppose we fail on σ. We claim that this e will allow us to produce a g satisfying the thesis of the theorem.

Apply Lemma 2.2 to punctually compute the finite collection of *all* finite extensions τ of σ for which $P_e^\tau(x) = y$. Punctually calculate a bound N such that $length(\tau) \leq N$ for all such extensions. Taking the disjunction of the atomic diagrams corresponding to all such τ and replacing indices of elements in $dom(\iota)$ with the corresponding $\bar{a}$ in $\mathcal{A}$, we get the desired formula $\Phi_{g(x,y)}$ which satisfies

$$f(x) = y \iff (\mathcal{A}, \bar{a}) \models \exists \bar{z}(\&_{i<j}(z_i \neq z_j) \;\&\; \Phi_{g(x,y)}(\bar{a}\bar{z})).$$

Now, if there was an extension τ' in $\mathcal{P}$ of σ of length at most N that is compatible with $\mathcal{A}$ for which $P_e^\tau(x) \neq y$ then we could diagonalise against P_e. Thus, $P_e^\tau(x) = y$ for *any* extension of σ of length at most N that is compatible with $\mathcal{A}$. Therefore

$$f(x) = y \iff (\mathcal{A}, \bar{a}) \models \forall \bar{z}(\&_{i<j}(z_i \neq z_j) \;\&\; \Phi_{g(x,y)}(\bar{a}\bar{z})),$$

as desired. □

Corollary 3.6. *If $\mathcal{A}$ is relational structure and $f \leq_{PR} \mathcal{B}$ for every $\mathcal{B} \cong \mathcal{A}$ upon the domain ω, then f is primitive recursive.*

Proof. By Proposition 2.4, the function f is bounded by a primitive recursive function. By (2) for every $x, y \in \omega$ we can primitive recursively generate a quantifier-free formula $\Phi_{g(x,y)}$ with free variables $\bar{z} = [z_1, \dots, z_k]$ such that

$$f(x) = y \iff (A, \bar{a}) \models \forall \bar{z}(\&_{i<j}(z_i \neq z_j) \to \Phi_{g(x,y)}),$$

for some fixed tuple of parameters $\bar{a}$.

In this proof the parameters $\bar{a}$ can be completely omitted by replacing the language of $\mathcal{A}$ with a modified language which takes care of the parameters and is punctually bi-interpretable with the original language; see the remark below for a more detailed explanation.

Remark 3.7. Adding more relations which imitate the relations with parameters from $\bar{a}$. For example, if P is a quaternary relation then we add new relations R like $R(x_1, x_2) \iff P(x_1, a_2, x_2, a_1)$ for each possible substitution of the parameters $\bar{a} = a_1, a_2, \dots$ into P. We go through the finitely many relations and parameters until we produce a finite language extending the original one. Note that we can punctually rewrite $P(\bar{a}\bar{x})$ in terms of the new predicates, and thus we can punctually and uniformly transform our formulae Φ into new formulae in the new language. These new formulae will not mention the parameters but will still satisfy the assumed property.

Furthermore, we can bi-interpret each relation in the language with a finite number of (totally) irreflexive relations $R(x_1, \dots, x_k)$; i.e., with relations such that

$$R(x_1, \dots, x_k) \implies \&_{i<j\leq k}[x_i \neq x_j].$$

For example, a predicate $P(x_1, x_2, x_3)$ can be bi-interpreted with the following five predicates

$$\begin{aligned}
P'(x_1, x_2, x_3) &\iff [x_1 \neq x_2] \,\&\, [x_1 \neq x_3] \,\&\, [x_2 \neq x_3] \,\&\, P(x_1, x_2, x_3), \\
P''(x_1, x_2) &\iff [x_1 \neq x_2] \,\&\, P(x_1, x_1, x_2), \\
P'''(x_1, x_2) &\iff [x_1 \neq x_2] \,\&\, P(x_1, x_2, x_2), \\
P^{IV}(x_1, x_2) &\iff [x_1 \neq x_2] \,\&\, P(x_1, x_2, x_1), \\
P^{V}(x_1) &\iff P(x_1, x_1, x_1)
\end{aligned}$$

by the rule

$$\begin{aligned}
&P(x_1, x_2, x_3) \iff \\
&P'(x_1, x_2, x_3) \vee \\
&P''(x_1, x_3) \,\&\, [x_1 = x_2] \vee \\
&P'''(x_1, x_2) \,\&\, [x_2 = x_3] \vee \\
&P^{IV}(x_1, x_2) \,\&\, [x_2 = x_3] \vee \\
&P^{V}(x_1) \,\&\, [x_1 = x_2] \,\&\, [x_2 = x_3].
\end{aligned}$$

Furthermore, we can make the arity of each irreflexive relation equal to the maximal arity K which occurs in the language. For example, if $K = 3$ we can replace a unary relation $P(x)$ by the ternary irreflexive relation

$$P'(x_1, x_2, x_3) \iff [x_1 \neq x_2] \,\&\, [x_1 \neq x_3] \,\&\, [x_2 \neq x_3] \,\&\, P(x_1).$$

We repeat this procedure until all predicates have the same arity and all are irreflexive.

After all these transformations are done, the structure $\mathcal{A}$ can be viewed as a K-multigraph coloured by finitely many colours, as follows. The domain of $\mathcal{A}$ is ω. Although $<$ on ω is not in the language of $\mathcal{A}$, we intend to use the natural ordering relation $<$ on the fixed copy $\mathcal{A}$.

For an irreflexive relation P and a permutation π on the integers $1, 2, \dots, K$, we are adding one specific bit to the color of the K-edge $\{x_1 < x_2 < \dots < x_K\}$ if and only if $P(x_{\pi(1)}, \dots, x_{\pi(K)})$. So the colouring will consist of finite tuples of pairs (P, π), where P range over the predicates and π over the finitely many permutations of $1, 2, \dots, K$. By Ramsey's Theorem there is an infinite monochromatic set $X \subseteq \omega$. In this set, all K-edges have the same color. This means that on the infinite set X the truth of a predicate $P(x_1, \dots, x_n)$ depends only on the natural ordering between the elements $x_1, \dots, x_n \in X$.

Therefore, by Ramsey's Theorem the infinite structure $\mathcal{A}$ in a finite relational language contains an infinite substructure $\mathcal{B}$ isomorphic to a punctual structure. (In fact, the substructure can be chosen isomorphic to a quantifier-free interpretation in the ordering ω, so it has an automatic and thus decidable presentation.) Then the graph of f will be primitive recursive, since

$$f(x) = y \iff \mathcal{B} \models \exists \bar{z}(\&_{i<j}(z_i \neq z_j) \ \& \ \Phi_{g(x,y)})$$

and

$$f(x) = y \iff \mathcal{B} \models \forall \bar{z}(\&_{i<j}(z_i \neq z_j) \to \Phi_{g(x,y)}).$$

Thus, the function f is primitive recursive. □

Recall that we consider only structures in finite languages.

Corollary 3.8. *Relational structures are not punctually universal.*

Proof. It is sufficient to show that Corollary 3.6 fails for structures which are not relational. Indeed, any punctual interpretation of an arbitrary structure $\mathcal{X}$ into a relational structure $\mathcal{A}(\mathcal{X})$ would allow to punctually reconstruct a copy of $\mathcal{X}$ from a copy of $\mathcal{A}(\mathcal{X})$. Thus, if a non-primitive recursive function could be punctually coded into $\mathcal{X}$, then it would also be coded into the relational $\mathcal{A}(\mathcal{X})$, contradicting Corollary 3.6.

It is easy to punctually encode a non-recursive set X into a 1-generated structure $\mathcal{M}$, as follows. Produce an ω-chain using a unary function s, and use a unary predicate P to code the n-th element of the set into the n-th point along the chain. Also, fix the generator by a constant 0. □

4. Punctual categoricity

Recall $f \leq_{PR} g$ if f can be obtained from g and the standard elementary functions using composition, primitive recursion and bounded minimisation. Also, a countable structure $\mathcal{A}$ having domain ω is viewed as a total function under some fixed Gödel coding.

4.1. **Punctual categoricity on a cone.** The definition below is a relativisation of the notion of punctually categorical structure suggested by Kalimullin, Melnikov and Ng [KMN17].

Definition 4.1. *A structure $\mathcal{A}$ is punctually categorical (p.c.) on a cone if there exists a total function g such that for any structure $\mathcal{B} \cong \mathcal{A}$, there exists an isomorphism $f : \mathcal{A} \to \mathcal{B}$ such that $f \leq_{PR} g \oplus \mathcal{B}$ and $f^{-1} \leq_{PR} g \oplus \mathcal{B}$.*

In the definition above, if g is primitive recursive (in which case it can be suppressed) we get the natural notion of relative punctual categoricity [KMN17, KMnN17].

Theorem 4.2. *Suppose $\mathcal{A}$ is a punctual relational structure. Then the following are equivalent:*

1. *$\mathcal{A}$ is relatively punctually categorical;*
2. *$\mathcal{A}$ is punctually categorical on a cone;*
3. *$\mathcal{A}$ becomes automorphically trivial after fixing finitely many constants.*

Proof. The only implication which requires a proof is (2) $\to$ (3).

Proposition 4.3. *Let $\mathcal{A}$ be a punctually categorical structure. Then there is a tuple $\bar{u}$ in $\mathcal{A}$ such that, for every tuple $\bar{x}$, each infinite substructure $\mathcal{X} \subseteq \mathcal{A}$ containing $\bar{u}$ and $\bar{x}$ non-trivially intersects every automorphism orbit of $(\mathcal{A}, \bar{u}, \bar{x})$.*

We first prove (2) $\to$ (3) assuming the proposition, and then we prove the proposition. If a relational structure is not automorphically trivial then every tuple $\bar{u}$ can be extended to a tuple $\bar{x} \supseteq \bar{u}$ such that, for some $y, z \in \mathcal{A}$, we have

$$(\mathcal{A}, \bar{x}, y) \ncong (\mathcal{A}, \bar{x}, z).$$

Without loss of generality we can choose y such that $(\mathcal{A}, \bar{x}, y) \ncong (\mathcal{A}, \bar{x}, z)$ for infinitely many z; taking the infinite substructure (over $\bar{x}$) generated by all such z we arrive at a contradiction with Proposition 4.3.

Proof of Proposition 4.3. Let $\mathcal{A}$ be a structure punctually categorical on the cone above f. Aiming for a contradiction, assume that every tuple $\bar{u}$ extends to a tuple $\bar{x} \supseteq \bar{u}$ such that some substructure $\mathcal{X} \subseteq \mathcal{A}$ contains $\bar{x}$ but does not contain elements automorphic to some $y \in \mathcal{A}$ over $(\mathcal{A}, \bar{x})$. We need to show that $\mathcal{A}$ is not punctually categorical on the cone above f. We will build a structure $\mathcal{B} \cong \mathcal{A}$ such that there is no isomorphism $g : \mathcal{A} \to \mathcal{B}$ such that $g \leq_{PR} f \oplus \mathcal{B}$ and $g^{-1} \leq_{PR} f \oplus \mathcal{B}$.

We say that a pair of functions (g, h) is an isomorphism and its inverse, an ii-pair for short, if g isomorphically maps $\mathcal{A}$ onto $\mathcal{B}$ and $h = g^{-1}$. We need to meet the requirements:

$$R_{\Phi,\Psi} : (\Phi^{f,\mathcal{B}}, \Psi^{f,\mathcal{B}}) \text{ is not an ii-pair,}$$

where (Φ, Ψ) runs over all pairs of primitive recursive operators. We fix some effective enumeration of all such operators, and assign higher priority to the requirements whose pairs appear earlier in this enumeration.

At a stage we will have defined a finite segment $\mathcal{F}$ of the structure $\mathcal{B}$ which is mapped to a finite tuple $\bar{u}$ in $\mathcal{A}$; our plan is to extend this map to an isomorphism from $\mathcal{B}$ onto $\mathcal{A}$. Let $R_{\Phi,\Psi}$ be the highest priority requirement which has not yet been met. As noted above we have an extension $\bar{x} \supseteq \bar{u}$ and an element $y \in \mathcal{A}$ such that $(\mathcal{A}, \bar{x}, y) \ncong (\mathcal{A}, \bar{x}, z)$ for any $z \in \mathcal{X}$, where $\mathcal{X}$ is an infinite substructure of $\mathcal{A}$ containing $\bar{x}$ and corresponding to y.

Let $\mathcal{C}$ be a structure with domain ω which extends $\mathcal{F}$ and which is isomorphic to $\mathcal{X}$. Let $\bar{x}^{\mathcal{C}}$ be the tuple which corresponds to $\bar{x}$ under this isomorphism.

We use the sub-recursively relativised version of Lemma 2.2 throughout the rest of the proof and without explicit reference. Ψ is total, thus there will be a tuple

$\bar{x}' = x'_1 \dots x'_n$ in $\mathcal{A}$ such that $x'_i = \Psi^{f,\mathcal{C}}(x_i^{\mathcal{C}})$ and $\bar{x}^{\mathcal{C}} = x_1^{\mathcal{C}} \dots x_n^{\mathcal{C}}$. We split our further arguments into two substeps below.

(1) If $(\mathcal{A}, \bar{x}) \ncong (\mathcal{A}, \bar{x}')$ then do nothing.
(2) Otherwise, if $(\mathcal{A}, \bar{x}) \cong (\mathcal{A}, \bar{x}')$, then there must exist a y' such that $(\mathcal{A}, \bar{x}, y) \cong (\mathcal{A}, \bar{x}', y')$. Let $z^{\mathcal{C}} = \Phi^{f,\mathcal{C}}(y')$ which is well-defined since Φ is total.

Let $\mathcal{G}$ be the finite portion of $\mathcal{C}$ extending $\bar{x}^{\mathcal{C}}$ and, if applicable, $\bar{x}^{\mathcal{C}} z^{\mathcal{C}}$, which was used in the computations $x'_i = \Psi^{f,\mathcal{C}}(x_i^{\mathcal{C}})$ and $z^{\mathcal{C}} = \Psi^{f,\mathcal{C}}(y')$ above. Let $\mathcal{A}^{\mathcal{C}}$ be the extension of $\mathcal{G}$ isomorphic to $\mathcal{A}$ so that $(\mathcal{A}^{\mathcal{C}}, \bar{x}^{\mathcal{C}}) \cong (\mathcal{A}, \bar{x})$.

Then we have either $(\mathcal{A}, \bar{x}) \cong (\mathcal{A}^{\mathcal{C}}, \bar{x}^{\mathcal{C}}) \ncong (\mathcal{A}, \bar{x}')$ for $x'_i = \Psi^{f,\mathcal{A}^{\mathcal{C}}}(x_i^{\mathcal{C}})$, or

$$(\mathcal{A}^{\mathcal{C}}, \bar{x}_{\mathcal{C}}, z^{\mathcal{C}}) \ncong (\mathcal{A}, \bar{x}', y') \cong (\mathcal{A}, \bar{x}, y),$$

where $z^{\mathcal{C}} = \Phi^{f,\mathcal{A}^{\mathcal{C}}}(y')$. In either case, the pair $(\Phi^{f,\mathcal{A}^{\mathcal{C}}}, \Psi^{f,\mathcal{A}^{\mathcal{C}}})$ cannot be an ii-pair for $\mathcal{B} = \mathcal{A}^{\mathcal{C}}$. Therefore we can find a finite portion $\mathcal{F}' \supseteq \mathcal{G} \supseteq \mathcal{F}$ embeddable into $\mathcal{A}$ which respects our mapping of $\mathcal{F}$ to $\mathcal{A}$ such that $(\Phi^{f,\mathcal{F}'}, \Psi^{f,\mathcal{F}'})$ gives a formal isomorphism disagreement.

To make sure that our isomorphism from $\mathcal{B}$ to $\mathcal{A}$ does not miss any elements of $\mathcal{A}$ by extending $\mathcal{F}'$ by one extra element whose image is the least element of the domain ω of $\mathcal{A}$ which is currently not in the range of our partial isomorphism from $\mathcal{F}$ to $\mathcal{A}$. Now we can proceed with $\mathcal{F}'$ in place of $\mathcal{F}$ to satisfy the next requirement.

At the end of the construction $\mathcal{B}$ will be the union of all these finite segments. It is clear that all the requirements are met. □

Note that in the proof above we used only totality of our operators, so the use of Lemma 2.2 in the proof was an overkill. □

If we knew any non-trivial examples of functional structures punctually categorical on a cone (e.g., encoding a non-computable set) then this would give another, perhaps quite elegant, proof of punctual non-universality of relational structures. Alas, as we shall see in the next section, coding into such structures seems to be a rather challenging task.

4.2. **Characterising punctual categoricity on a cone.** We say that two elements $x, y \in \mathcal{A}$ have the same automorphism type over a tuple $\bar{a}$ in $\mathcal{A}$ if there is an automorphism of $\mathcal{A}$ which fixes $\bar{a}$ and takes x to y. The theorem below says that every punctually categorical on a cone structure must be $\aleph_0$-categorical.

Theorem 4.4. *If $\mathcal{A}$ is punctually categorical on a cone then for every tuple $\bar{a}$ from $\mathcal{A}$, the enriched structure $(\mathcal{A}, \bar{a})$ has only finitely many distinct automorphism orbits.*

Proof. Suppose that $\mathcal{A}$ is punctually categorical on the cone above f. Assume that for some fixed tuple $\bar{a} = (a_0, \dots, a_p)$ from $\mathcal{A}$, there exist infinitely many automorphism types in $(\mathcal{A}, \bar{a})$.

To arrive at a contradiction it suffices to build a copy $\mathcal{B} \cong \mathcal{A}$ upon the domain of ω such that, for every primitive recursive operator Φ, the following requirement is met:

$$R_\Phi : \Phi^{f,\mathcal{B}} \text{ is not an isomorphism from } \mathcal{B} \text{ onto } \mathcal{A}.$$

We will meet the requirements one after another at stages $s > 0$.

At stage 0, we begin the construction of $\mathcal{B}$ with a finite partial structure having domain $\{0, \dots, p\}$ associated naturally with $\bar{a}$ via $i \mapsto a_i$.

At a later stage we will work with some R_Φ, which will be the highest priority requirement which need to be met. At the end of the previous stage we will have constructed a finite partial substructure $\mathcal{F}$ of $\mathcal{B}$ upon $\{0,\ldots,q\}$ which is associated with a finite tuple $\bar{u}=(u_0,\ldots,u_q)$ in $\mathcal{A}$ via a partial isomorphism $\iota : i \mapsto u_i$. Since the tuple $\bar{u}$ contains the tuple $\bar{a}$ there are infinitely many distinct automorphism orbits over $\bar{u}$; indeed, if two points are automorphic over $\bar{u}$ then they are also automorphic over $\bar{a}$.

To meet R_Φ, define a sequence of finite partial structures as follows. Set $\mathcal{F}_0=\mathcal{F}$, and let $\mathcal{F}_{n+1}$ be any finite partial structure such that:

(1) $q+1 \in \mathcal{F}_{n+1}$;
(2) the domain of $\mathcal{F}_{n+1}$ is an initial segment of ω;
(3) $\mathcal{F}_n$ is a proper partial substructure of $\mathcal{F}_{n+1}$;
(4) the result of the application of any function from the language of $\mathcal{A}$ to elements in $\mathcal{F}_n$ lies in $\mathcal{F}_{n+1}$;
(5) the partial isomorphism ι can be extended to a partial isomorphism $\kappa : \mathcal{F}_{n+1} \to \mathcal{A}$ with domain $\mathcal{F}_{n+1}$;
(6) there is no proper partial substructure $\mathcal{G} \subset \mathcal{F}_{n+1}$ satisfying (1)–(5);
(7) the possible values of $\kappa(q+1)$ for all possible κ from (5) consistent with (1)-(4) range over infinitely many distinct automorphism types over $\bar{u}$.

Let n be the least such that

$$\Phi^{f,\mathcal{F}_n}(0)=v_0,\ldots,\Phi^{f,\mathcal{F}_n}(q)=v_q, \text{ and } \Phi^{f,\mathcal{F}_n}(q+1)=w,$$

for some $v_1,\ldots,v_q,w \in \mathcal{A}$; such an n exists since the functional is total.

To meet R_Φ, extend the mapping $\iota : i \mapsto u_{i+1}$ using one of the possible definitions of $\kappa(q+1)$ from (5) such that

$$(\mathcal{A},u_0,\ldots,u_q,\kappa(q+1)) \ncong (\mathcal{A},v_0,\ldots,v_q,w),$$

noting that such a w exists because of condition (7). Set $\mathcal{F}=\mathcal{F}_n$ and go to the next stage.

At the end we will construct $\mathcal{B}\cong\mathcal{A}$, and our actions at the stage at which we considered R_Φ will ensure

$$(\mathcal{B},0,\ldots,q-1,q) \ncong (\mathcal{A},\Phi^{f,\mathcal{B}}(0),\ldots,\Phi^{f,\mathcal{B}}(q-1),\Phi^{f,\mathcal{B}}(q)),$$

and thus R_Φ will be met. □

Recall that a structure $\mathcal{A}$ is locally finite if for every finite subset $F \subseteq \mathcal{A}$ the substructure $\langle F\rangle$ generated by F is finite.

Definition 4.5. *A structure $\mathcal{A}$ is uniformly locally finite if there is a function h such that*

$$\operatorname{card}\langle F\rangle \le h(\operatorname{card} F)$$

for any finite $F \subseteq \mathcal{A}$.

It is well-known that the algebraic closure operator is uniformly locally finite for each $\aleph_0$-categorical M. Since every $a \in \langle F\rangle$ is definable over F, we obtain:

Corollary 4.6. *Suppose $\mathcal{A}$ is punctually categorical on a cone. Then $\mathcal{A}$ is uniformly locally finite.*

Corollary 4.7. *Let $\mathcal{A}$ be a punctually categorical structure. Then there is a finite tuple $\bar{u}$ in $\mathcal{A}$ such that for every finite tuple $\bar{x}$ in $\mathcal{A}$ there is an integer $m_{\bar{x}}$ such that, for every $m_{\bar{x}}$-tuple of pairwise distinct elements*

$$\bar{z} = z_1, \dots, z_{m_{\bar{x}}} \notin \langle \bar{u}, \bar{x} \rangle$$

the substructure $\langle \bar{u}, \bar{x}, \bar{z} \rangle$ intersects all automorphism orbits over $\bar{u}\bar{x}$.

Proof. Suppose not. Then for every $\bar{u}$ there is an $\bar{x}$ such that for every m some substructure $\langle \bar{u}, \bar{x}, z_1, \dots, z_m \rangle$ omits some automorphism orbit over $\bar{u}\bar{x}$. Since there are only finitely many automorphism orbits over $\bar{u}\bar{x}$, we can assume that for all m the structure $\langle \bar{u}, \bar{x}, z_1, \dots, z_m \rangle$ omits one fixed orbit. Consider the infinite collection of finite substructures

$$\{\langle \bar{u}, \bar{x}, z_1, \dots, z_m \rangle : m \in \omega\}.$$

This collection forms an infinite finitely branching tree under the embeddability relation. By König's Lemma there is an infinite substructure $\mathcal{X}$ which contains $\bar{u}$ and $\bar{x}$, completely omits the fixed automorphism orbit over $\bar{u}\bar{x}$. This contradicts Proposition 4.3. □

Definition 4.8. *We say that a structure $\mathcal{A}$ is highly local if it satisfies the condition from Corollary 4.7.*

A natural example of a highly local structure is a vector space over a finite field. Recall that a structure $\mathcal{A}$ is $\exists$-atomic if there is a tuple $\bar{u}$ such that for every tuple $\bar{x}$ there is an existential formula $\Phi_{\bar{x}}$ such that

$$(\mathcal{A}, \bar{u}) \models \Phi_{\bar{x}}(\bar{x}') \iff (\mathcal{A}, \bar{u}, \bar{x}') \cong (\mathcal{A}, \bar{u}, \bar{x})$$

for every tuple $\bar{x}'$ from $\mathcal{A}$.

Theorem 4.9. *A structure $\mathcal{A}$ is punctually categorical on a cone if and only if*

(1) *$\mathcal{A}$ is uniformly locally finite;*
(2) *$\mathcal{A}$ is $\exists$-atomic;*
(3) *$\mathcal{A}$ is highly local.*

Proof. First, suppose that $\mathcal{A}$ is punctually categorical on a cone. Then (1) follows from Corollary 4.6, (3) follows from Corollary 4.7. Also, $\mathcal{A}$ is computably categorial on a cone, thus (2) follows from [Mon17].

Now suppose $\mathcal{A}$ satisfies (1)–(3). Without loss of generality, we can fix a finite tuple $\bar{u}$ which works for both (2) and (3).

Fix a total function f which naturally codes[1]:

- the uniform bound h from (1) (see Definition 4.5);
- the atomic diagram of some fixed copy $\mathcal{A}$ on the domain ω;
- the correspondence $\bar{x} \mapsto \Phi_{\bar{x}}$ from (2) for the fixed copy $\mathcal{A}$;
- the correspondence $\bar{x} \mapsto m_{\bar{x}}$ from (3) for the fixed copy $\mathcal{A}$.

Suppose $\mathcal{B} \cong \mathcal{A}$. We will use a back-and-forth argument to build an isomorphism $g : \mathcal{A} \to \mathcal{B}$ such that $g \leq_{PR} f \oplus \mathcal{B}$ and $g^{-1} \leq_{PR} f \oplus \mathcal{B}$. We begin with mapping $\bar{u}$ to $\bar{u}'$ in $\mathcal{B}$.

Suppose we have already constructed $g : \bar{x} \to \bar{x}'$. We first explain how to define $g(y)$ for the next element $y \in \mathcal{A}$.

[1]That is, for each query we can punctually calculate a number x such that $f(x)$ will give us the desired answer, under some primitive recursive coding.

We need to punctually find a $y' \in \mathcal{B}$ such that $(\mathcal{B}, \bar{u}) \models \Phi_{\bar{x}y}(\bar{x}'y')$ for the existential formula $\Phi_y = \exists \bar{t} \Psi(\bar{t}, \bar{x}, y)$, where Ψ is quantifier-free with parameters $\bar{u}$. It is sufficient to take any $m_{\bar{x}}$-tuple of distinct elements

$$\bar{z}' = z'_1, \ldots, z'_{m_{\bar{x}}} \notin \langle \bar{u}', \bar{x}' \rangle$$

in $\mathcal{B}$ to find representatives all possible automorphism types over $\bar{x}$, including the automorphism type of y'. The size of $\langle \bar{u}', \bar{x}' \rangle$ can be punctually calculated using f using the uniform bound on the size of $\langle \bar{u}', \bar{x}' \rangle$. Therefore, primitively recursively in $f \oplus \mathcal{B}$, we can generate $\langle \bar{u}', \bar{x}' \rangle$ and then pick a $m_{\bar{x}}$-tuple $\bar{z}'$ of distinct elements outside of $\langle \bar{u}', \bar{x}' \rangle$. By our assumption, one of these elements can be set equal to $y' = g(y)$; however, to decide which one we can choose we must also evaluate Φ_y on all these potential images.

To choose a correct y' we restrict our search for t'_0, where $\bar{t}' = t'_0 \ldots t'_k$, to the substructure

$$D_0 = \langle \bar{u}', \bar{x}', y', s'_1, \ldots, s'_m \rangle,$$

where $s'_1, \ldots, s'_m \notin \langle \bar{u}', \bar{x}', y' \rangle$ are arbitrary elements and

$$m = \max\{m_{\bar{x}y} : y \in \langle \bar{u}, \bar{x}, \bar{z} \rangle\},$$

for any $m_{\bar{x}}$-tuple $\bar{z}$ not in $\mathcal{A} \setminus \langle \bar{u}, \bar{x} \rangle$. Note that this computation is punctual in f. Similarly, we can search for each witness for $\exists \bar{t}$ but restricting the search to some $D_1 \supseteq D_1$ which can be calculated punctually in f, etc. Working punctually in f, we will end up with a finite D_k such that $\bar{t}$ can be chosen from D_k. Once we see such a $\bar{t}$ for some y' among $\bar{z}'$. Once this is done we define $g(y) = y'$.

To extend $g : \bar{x} \to \bar{x}'$ to a new element $y' \in \mathcal{B}$, take any $N_{\bar{x}}$-tuple $\bar{z}$ of distinct elements in $\mathcal{A} \setminus \langle \bar{u}, \bar{x} \rangle$ and consider the finite set of existential formulae

$$\{\Phi_{\bar{x}y} : y \in \langle \bar{u}, \bar{x}, \bar{z} \rangle\}.$$

For one of these formulae we should have $(\mathcal{B}, \bar{u}') \models \Phi_{\bar{x}y}(\bar{x}'y')$. We can test the truth of each of these formulae using the same method as we already explained above, and punctually define $g^{-1}(y') = y$ for an element y such that $(\mathcal{B}, \bar{u}') \models \Phi_{\bar{x}y}(\bar{x}'y')$.

It remains to observe that g is an onto isomorphism and both g and g^{-1} are punctual in $\mathcal{B} \oplus f$, as desired. □

As an application of our machinery we derive a characterisation of relatively punctually categorical unary structures.

Theorem 4.10. *Suppose $\mathcal{A}$ is a countably infinite algebraic structure in a finite language consisting of unary functional symbols and predicates. Then the following are equivalent:*

(1) *$\mathcal{A}$ is relatively punctually categorical.*
(2) *$\mathcal{A}$ is punctually categorical on a cone.*
(3) *There are finite substructures E, $D \supseteq E$, and $(F_i)_{i \in \omega}$ of $\mathcal{A}$ such that:*
   (i) *$\mathcal{A} = D \cup \bigcup_{i \in \omega} F_i$;*
   (ii) *for every $i \neq j$, $F_i \cap F_j = E$ and $F_i \cap D = E$;*
   (iii) *every $x \in F_i \setminus D$ generates F_i;*
   (iv) *for every $i \neq j$, F_i is automorphic to F_j over D[2].*

[2]This means that D must be held fixed.

Proof. The only implications that need to be verified are (2) $\to$ (3) and (3) $\to$ (1).

We first check (2) $\to$ (3). We claim that almost all $x \in \mathcal{A}$ will have isomorphic (finite) spans. Suppose F is the smallest cardinality substructure among all the one-generated substructures of $\mathcal{A}$ which appears infinitely often. Let

$$E = \{y \in F : card\,\langle y \rangle < card\,F\};$$

then by the choice of F almost all substructures of $\mathcal{A}$ isomorphic to F must intersect by E. Let $\mathcal{B}$ be the union of all these 1-generated substructures intersecting by E.

We claim that $\mathcal{A} \setminus \mathcal{B}$ must be finite. Suppose not. Then there will be some isomorphism type of a 1-generated substructure which appears in $\mathcal{A}$ infinitely often but does not appear in $\mathcal{B}$ at all; call this isomorphism type exceptional. But then we could build a $\mathcal{C} \cong \mathcal{A}$ and diagonalise against a pair of functionals Φ, Ψ, as follows. At a stage we will have copied only finitely many exceptional substructures into $\mathcal{C}$, say m. To diagonalise against the next ii-pair (Φ, Ψ), copy only $\mathcal{B}$ into $\mathcal{C}$ and wait for the pair of functionals to map $(m+1)$ distinct exceptional substructures into $\mathcal{C}$. Then copy a bit more of $\mathcal{A} \setminus \mathcal{B}$ into $\mathcal{C}$, including one more exceptional substructure, and proceed to the next requirement.

We conclude that $\mathcal{A}$ is equal to the union of a finite substructure D and a family $\mathcal{F}$ of isomorphic finite 1-generated structures such that all structures in $\mathcal{F}$ intersect by some $E \subseteq D$ (which of course could be empty); thus, we have (i). It follows from the definition of E that, for each $F \in \mathcal{F}$, every $x \in F \setminus E$ generates the whole F. This gives (iii). Since D intersects only finitely many members of $\mathcal{F}$ by an element not in E, we can enlarge D if necessary to ensure $D \cap F = E$ for every $F \in \mathcal{F}$; this is (ii).

We must check (iv). By Theorem 4.4, there are at most finitely many automorphism types over D. We claim that almost all F_i lie in the same orbit over E.

Suppose there are (at least) two infinite orbits over E realised by the F_i. In other words, there are two different ways to extend E to F_i which are realised in $\mathcal{A}$ infinitely often; they will of course agree up to an automorphism of E, but this is not good enough because we want E to be held fixed. Note that E is an automorphism-invariant substructure of $\mathcal{A}$ because its definition refers only to internal invariant properties of $\mathcal{A}$. Split $(F_i)_{i\in\omega}$ into two sub-sequences, $(F_i')_{i\in\omega}$ and $(F_i'')_{i\in\omega}$, one corresponding to one way of extending E to F and the rest to all other ways.

Build a copy $\mathcal{C}$ of $\mathcal{A}$. At a stage we will have copied only a finite part $\mathcal{A}$ into $\mathcal{C}$, including E. To diagonalise against a potential isomorphism $\Phi^{\mathcal{C}} : \mathcal{A} \to \mathcal{C}$, wait for $\Phi^{\mathcal{C}}$ to map E in $\mathcal{A}$ to its natural image E' in $\mathcal{C}$. While we wait we keep copying $\mathcal{A}$ into $\mathcal{C}$. If $\Phi(E) \neq E'$ then we have diagonalised. Otherwise, as soon as we see $\Phi = E'$, start copying only $(F_i')_{i\in\omega}$ into $\mathcal{C}$ and wait for Φ to prove that it is not an isomorphism. If Φ is insists that it could be an isomorphism then, from some large enough j, both $\Phi(F_j')$ and $\Phi(F_j'')$ must start hitting the natural images $(\hat{F}_i)_{i\in\omega}$ of $(F_i')_{i\in\omega}$ in $\mathcal{C}$. But $\Phi(E) \subseteq \Phi(F_j')$ and $\Phi(E) \subseteq \Phi(F_j'')$ must correspond to different orbits, while $\Phi(E) \subseteq \hat{F}_i$ will all realise some fixed orbit (which is not necessarily the same as the orbit of F_i' over E). Once we have diagonalised against Φ we resume copying all of $\mathcal{A}$ into $\mathcal{C}$ for a few more stages before attacking the next pair.

Thus, almost all F_i lie in the same orbit over E. Enlarge D to contain the finitely many exceptional F_i. Noting that D and the F_i intersect only by E, we see that condition (iv) holds too.

To prove (3) $\to$ (1) run a straightforward back-and-forth procedure which non-uniformly maps the exceptional set D at the first step. Observe that every search through $\mathcal{A}$ and some other $\mathcal{B} \cong \mathcal{A}$ will be naturally bounded by the size of F. $\square$

5. An application of categoricity to universality

The following rather specific notion will be sufficient for our purposes; it is a relativisation of the natural enough notion of "bottom punctual categoricity" [KMnN17].

Definition 5.1. *A structure $\mathcal{A}$ is bottom punctually categorical (b.p.c.) on a cone if there exists a copy $\mathcal{C} \cong \mathcal{A}$ and a total function f such that for any structure $\mathcal{B} \cong \mathcal{A}$, there exists an isomorphism $g : \mathcal{C} \to \mathcal{B}$ such that $g \leq_{PR} f \oplus \mathcal{B}$.*

In the case when $g \leq_{PR} f \oplus \mathcal{B}$ is witnessed by a single primitive recursive operator,

$$g = \Psi^{f \oplus \mathcal{B}},$$

then we say that $\mathcal{A}$ is uniformly b.p.c. (u.b.p.c.) on a cone.

Any finitely generated structure becomes uniformly bottom categorical (on a cone) after naming a finite tuple which generates it. It is easy to see that the structures from Examples 3.4 and 3.5 are u.b.c. on a cone. The technical notion of u.b.c. on a cone leads to a different (and a bit simpler) proof of punctual non-universality of relational structures. The non-universality will be derived as a corollary of the following technical result.

Theorem 5.2. *Suppose $\mathcal{A}$ is a relational structure. Then $\mathcal{A}$ is uniformly bottom punctually categorical on a cone if, and only if, $\mathcal{A}$ is automorphically trivial.*

Proof. Suppose $\mathcal{A}$ is non-trivial, we show that it cannot be u.b.c. on a cone above f. Fix $n \in \omega$ least such that $(x_1, \dots, x_n, y) \not\cong_{Aut} (x_1, \dots x_n, z)$, for some $x_1, \dots x_n, y, z \in A$. Note that the minimality of n implies that all n-tuples of $\mathcal{A}$ are automorphic.

The key is to use elements from different orbits over $\bar{x}$ for diagonalization purposes. The oracle will be assumed to (punctually) "know" these orbits. Either there is one infinite orbit and finitely many finite orbits, or there are two automorphism invariant (over $\bar{x}$) infinite sets consisting of distinct orbits (over $\bar{x}$). Depending on which of the two cases holds, we use elements from distinct orbits to diagonalise against Φ.

To diagonalise against a potential f-punctual operator Φ, build another copy $\mathcal{B}$ of $\mathcal{A}$ with the help of an oracle. The oracle will help us to deal with the special invariant sets corresponding to $\bar{x}$. The description of the oracle will be incorporated into the description of the structures, but we could *first* describe the oracle and *then* give a formal description of the strategies, so there is no danger of circularity here. We give details below.

Case 1. There exist infinite sets $Y_{\bar{x}}$ and $Z_{\bar{x}}$, both automorphism-invariant over $\bar{x}$, such that any $y \in Y_{\bar{x}}$ and $z \in Z_{\bar{x}}$ are not pairwise automorphic over $\bar{x} = (x_1, \dots x_n)$. These sets will be composed of automorphism orbits over $\bar{x}$. Recall that each such $\bar{x'}$ is automorphic to $\bar{x}$; thus, $Y_{\bar{x'}}$ and $Z_{\bar{x'}}$ are both infinite for any n-tuple $\bar{x'}$. Given any n-tuple $\bar{x'}$ in A, the oracle which we will use can punctually list $Y_{\bar{x'}}$ and $Z_{\bar{x'}}$ and also punctually decides whether a given element is in $Y_{\bar{x'}}$ or if it is in $Z_{\bar{x'}}$ (or neither). The definition of $Y_{\bar{x}}$ and $Y_{\bar{x'}}$ ensures that any automorphism taking $\bar{x}$ to $\bar{x'}$ will take $Y_{\bar{x}}$ to $Y_{\bar{x'}}$ (and similarly $Z_{\bar{x}}$ to $Z_{\bar{x'}}$). To ensure that $\mathcal{B}$ lies in $\mathcal{C}^f_{\mathcal{A}}$, take the join of this oracle with f if necessary. Proceed through the following actions:

(1) Start off by copying $\mathcal{A}$ into $\mathcal{B}$; keep doing so until $\Phi^{\mathcal{B}} : \mathcal{A} \to \mathcal{B}$ converges on $\bar{x}$. Denote the image of $\bar{x}$ by $\bar{x}'$. Since we have been naturally copying $\mathcal{A}$ to $\mathcal{B}$ so far, we can safely identify $\bar{x}$ with the respective tuple in $\mathcal{A}$.
(2) Start copying only elements of $Z_{\bar{x}'}$ into $\mathcal{B}$.
(3) Eventually, for some $y \in Y_{\bar{x}}$ the operator will map y to some $z \in Z_{\bar{x}'}$.
(4) Once this is seen, resume copying $\mathcal{A}$ into $\mathcal{B}$ on all elements.

We claim that in (3) we will have diagonalised against Φ. Recall that both $Y_{\bar{x}}$ and $Z_{\bar{x}}$ were invariant under $\bar{x}$-automorphism, and furthermore were composed of some *specific* orbits which the oracle can produce for the given tuple $\bar{x}$. By the choice of n, $\bar{x}$ and $\bar{x}'$ are automorphic. Any automorphism of the structure taking $\bar{x}$ to $\bar{x}'$ must also take $Y_{\bar{x}}$ to $Y_{\bar{x}'}$. But we have ensured that some element of $Y_{\bar{x}}$ has been mapped $Z_{\bar{x}'}$ via Φ.

Case 2. There exists one cofinite orbit $Y_{\bar{x}}$ over $\bar{x}$; denote the union of the finitely many finite $\bar{x}$-orbits $Z_{\bar{x}}$.

As before, we intend to copy $\mathcal{A}$ to $\mathcal{B}$.

(1) Let N be the maximal size of the use of the f-punctual Φ for all possible oracles. It is crucial that the language of the structure is relational, and therefore the space of possible finite partial maps from $\mathcal{A}$ to $\mathcal{B}$ is punctually bounded and therefore we can apply the sub-recursively relativised version of Lemma 2.2.
(2) Using the oracle, find a subset H of $\mathcal{A}$ of size N with the property that for any $(n+1)$-tuple $\bar{w}$ in H, the tuple is *not* of the form $\bar{x}'z'$, where $z' \in Z_{\bar{x}'}$. (We claim that such a set can be found; to be verified later.) Copy H into $\mathcal{B}$. This ensures that Φ is not an isomorphism from $\mathcal{A}$ onto $\mathcal{B}$, because $\Phi(\bar{x}z)$ must be mapped to $\bar{x}'z'$, where $z' \in Z_{\bar{x}'}$. But H does not contain such tuples of length $n+1$.

It remains to prove that such an H can be found.

Claim 5.3. *Under the assumptions of Case 2 and for every $N \in \omega$ there exists a finite set $H \subseteq \mathcal{A}$ of size N with the property that every $(n+1)$-tuple $\bar{w}$ in H, the tuple is not of the form $\bar{x}'z$, where $z \in Z_{\bar{x}'}$.*

Proof of Claim. We say that $z \in \mathcal{A}$ is exceptional over $\bar{x}$ in A if $z \in Z_{\bar{x}}$. Let $m \in \omega$ be the cardinality of $Z_{\bar{x}}$, and recall that all n-tuples are automorphic in $\mathcal{A}$, by the minimality of n. In particular, $Z_{\bar{x}}$ and $Z_{\bar{x}'}$ will have the same size for any pair of n-tuples $\bar{x}$ and $\bar{x}'$. Given any finite set U of size k, the maximal number of all possible elements exceptional over an n-tuple from U is bounded by $\dfrac{k!}{n!}m$, where $\dfrac{k!}{n!}$ is the number of n-tuples and m the number of exceptional elements over each tuple.

In $\mathcal{A}$, choose disjoint subsets $K_1, \ldots, K_{N-1}$ of sizes

$$\frac{N!}{n!}m + 1,\ \frac{N!}{n!}m|K_0| + 1,\ \ldots,\ \frac{N!}{n!}m|K_0| \cdot \ldots \cdot |K_{N-2}| + 1,$$

respectively. Fix $a_N \in A$ which is not exceptional over any n-tuple in $\bigcup_{i<N} K_i$. We intend to choose $a_1 \in K_1, \ldots, a_{N-1} \in K_{N-1}$ and form H to be the collection of these a_i. Each such choice leads to a *potential* H; there are only finitely many such H.

Choose a_{N-1} from the set K_{N-1} so that it is not exceptional over any tuple $\bar{x}$ from any potential H (all we know about H is that it contains a_N). The size of K_{N-1} is large enough to exceed the maximum number of exceptional elements for all possible choices of smaller a_i, $i < (N-1)$, in the worst possible case. Once a_{N-1} is defined, we proceed to a_{N-2}, etc. □

The claim finishes the verification of Case 2. Since the cases are exclusive, the theorem is proven. □

This gives a second proof of Corollary 3.8:

Proof of Corollary 3.8. Encode a non-recursive set X into a 1-generated structure $\mathcal{M}$ as in the proof of Corollary 3.8. Produce an ω-chain using a unary function s, and use a unary predicate P to code the n-th element of the set into the n-th point along the chain. Also, fix the generator by a constant 0. The resulting structure is u.b.c. on a cone and its Turing degree spectrum is the cone above X.

If relational structures were punctually universal, and $\Psi(\mathcal{M})$ was the relational structure punctually encoding $\mathcal{M}$, then it would have to be u.p.c. on a cone and therefore automorphically trivial. But $\Psi(\mathcal{M})$ must also have the same Turing degree spectrum as $\mathcal{M}$, because punctual universality is a stronger version of Turing computable universality. However, any automorphically trivial structure in a finite relational language has the trivial Turing degree spectrum $\{\mathbf{0}\}$. □

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