COMPUTABLY LOCALLY COMPACT GROUPS AND THEIR CLOSED SUBGROUPS

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Abstract. We show that the Chabauty space $\mathcal{S}(G)$ of a computably locally compact G admits a natural Π^0_1-closed presentation. We construct a computable discrete abelian group H such that $\mathcal{S}(H)$ is not computably closed. Indeed, the only computable points of $\mathcal{S}(H)$ correspond to the trivial subgroups of H, while the full Chabauty space $\mathcal{S}(H)$ is uncountable. In the totally disconnected case, we give an alternate effective characterization of the Chabauty space in terms of meet groupoids, a purely algebraic notion introduced recently by the authors (arXiv: 2204.09878). We apply our results and techniques to establish an arithmetical upper bound on the complexity of the index set of all locally compact abelian groups that contain $(\mathbb{R}, +)$ as a closed subgroup.

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1. Introduction

Computable mathematics probes the algorithmic content of mathematical concepts and results. Our paper contributes to the study of computable Polish groups initiated in [MM18, Mel18]. For early papers concerned with computable topological groups, and specifically profinite groups, we cite Metakides and Nerode [MN79], La Roche [LR81, LR78] and Smith [Smi81, Smi79]. The central notion of such investigation beyond the class of profinite groups is that of a computable Polish space due to Lacombe [Lac59], Ceitin [Cei59], and Moschovakis [Mos64]. In the case of groups, we additionally require the operations to be computable. All these notions will be formally clarified the preliminaries. These investigations follow the general pattern seen in computable algebra [EG00, AK00] that studies algorithmic properties of discrete countable structures. With the formal definitions of computable presentability at hand, one can attack questions of the following sort:

- Which Polish groups in a given class admit a computable presentation?
- If a Polish group has a computable presentation, is it unique?
- Which classical invariants of groups are computable, and in what sense?

The list of potential questions goes on. As we will discuss later, such investigations are closely related to the classification problem in the respective class of groups or spaces. For various recent results that are not restricted to profinite groups, we cite [MM18, GMNT17, Mel18, LMN21, MN22, KMK23] and [PSZ20]. We note that in some of the cited papers, e.g. [LMN21, MM18], computable Polish groups were used to prove theorems that may seem unrelated to topological groups.

This introduction contains an informal overview and the statement of our three results. Most notions will only be defined formally in subsequent sections. We investigate the computability-theoretic aspects of the Chabauty space $\mathcal{S}(G)$ of a given locally compact (Polish) group G. This subtle compact invariant plays a significant role in the theory of locally compact groups, as explained in, e.g., [Cor11]. We establish that or a computably locally compact G, its Chabauty space $\mathcal{S}(G)$ can be uniformly represented as an effectively closed subset of a computably compact space, but this set may be not computable in general. We then apply this effective invariant to derive an effective classification-type result about computably locally compact groups.

1.1. **The Chabauty space.** All our groups are Polish. Assume G is a locally compact (l.c.) Polish group. One defines a topology on the collection of all closed subgroups of G, including those subgroups that are not necessarily compact, as follows. Using the notation of [Cor11, Section 2], a basic open set in the *Chabauty space* $\mathcal{S}(G)$ (of closed subgroups of) a l.c. group G has the form

$$\Omega(K; R_1, \dots, R_n) = \{U \leq_c G \colon U \cap K = \emptyset \wedge \forall i \leq n \, U \cap R_i \neq \emptyset\},$$

where $K \subseteq G$ is compact, and the $R_i \subseteq G$ are open.

If G is a l.c. Polish group, then the Chabauty space $\mathcal{S}(G)$ is a compact Polish space. It is natural to ask whether this compact space is "computable" for a *computably* locally compact group, and if yes, then in what sense.

It is well-known that the Chabauty space $\mathcal{S}(G)$ of a l.c. group can be viewed as a closed subset of the hyperspace of closed subsets $K(G^*)$ of the Alexandroff 1-point compactification G^* of the group G; we will explain this in detail in Subsection 4.1. We will see that, for a computably l.c. G, the hyperspace of closed subsets $\mathcal{K}(G^*)$ of its 1-point compactification G^* is computably compact (Theorem 3.4). A natural question arises for a computably locally compact group G:

Is $\mathcal{S}(G)$ computably closed in $\mathcal{K}(G^*)$?

Our first result answers this as follows.

Theorem 1.1. Let G be a computably locally compact group.

(1) $\mathcal{S}(G)$ is effectively closed (Π^0_1) in the computably compact space $\mathcal{K}(G^*)$.
(2) $\mathcal{S}(G)$ is not in general computably closed in $\mathcal{K}(G^*)$.

(The standard notions of a computably locally compact, computably closed, and effectively closed (Π^0_1) sets will be given in the preliminaries section.) Our construction of the effectively closed presentation of $\mathcal{S}(G)$ has a number of further effective features; the construction will appear in the proof of Proposition 4.2. One such important feature will be stated shortly as our second main result (Theorem 1.2). Also, (2) of Theorem 1.1 is witnessed by a discrete torsion-free abelian group that has no non-trivial computable subgroups but has uncountably many proper subgroups, which makes $\mathcal{S}(G)$ uncountable too; this is Theorem 5.5. In particular, it follows that $\mathcal{S}(G)$ has only two computable points; they correspond to the trivial subgroups of G.

1.2. **The t.d.l.c. case.** Our second result is concerned with the special important case of totally disconnected locally compact (t.d.l.c.) Polish groups. This is the narrowest class of Polish groups extensively studied in the literature that contains both the countable discrete and the profinite Polish groups; recent papers include [Wil15, Wes15, GR17, CCC20, HW15]. In [MN22], the authors have initiated a systematic study of computably t.d.l.c. groups; see also [LMN21]. It follows from [MN22, MN23] (to be explained in Section 6 in detail) that there is a canonical way to pass from a computably locally compact presentation of a t.d.l.c. group to an effective presentation of the discrete countable *meet groupoid* $\mathcal{W}(G)$ of all compact open cosets in G (see Theorem 6.7). A computable duality holds between G and the respective $\mathcal{W}(G)$ [MN22].

Since both $\mathcal{S}(G)$ and $\mathcal{W}(G)$ reflect the subgroup structure of a t.d.l.c. group, it is natural to compare these effective invariants. This is the main motivation behind our next theorem.

Theorem 1.2. *Suppose G is a computably locally compact group. For a closed subgroup H of G, the following are equivalent:*

(1) *H is computably closed;*
(2) *$H^* \in \mathcal{S}(G)$ is a computable point in $\mathcal{K}(G^*)$;*
(3) *If G is totally disconnected, H corresponds to a computable closed-subgroup ideal in the (computable) dual meet groupoid $\mathcal{W}(G)$ of G.*

This correspondence is computably uniform[1].

[1]We identify G with its computable locally compact presentation, and we identify $\mathcal{S}(G)$ with the presentation of the Chabauty space produced in Theorem 1.1. We also identify the meet groupoid $\mathcal{W}(G)$ with its effective presentation that is produced combining the aforementioned results from [MN22, MN23]; the technical details will be explained later in Section 6.

The algebraic notion of a closed-subgroup ideal will be defined below (Def. 6.9); be it sufficient here to say that this is the view of closed subgroups from the data given by the meet groupoid of G. The correspondence in (3) above is (in a certain sense) canonical and is rather natural as well; this will be clarified in Definition 6.8 and the subsequent discussion.

We believe that Theorem 1.2 illustrates that the effective presentation of $\mathcal{S}(G)$ constructed in the present paper is rather (algorithmically) natural and robust. As an immediate consequence of Theorem 1.1(2), we obtain that there is a computably t.d.l.c. G so that only the trivial closed-subgroup ideals of $\mathcal{W}(G)$ are computable.

1.3. **An application to index sets.** The approach to classification problems via index sets is standard in computable (discrete, countable) structure theory; for a comprehensive introduction to this approach, see [GK02]. Early applications of index sets in analysis can be found in [CR99], and for many more recent results we cite [HTM21, BS14] and the recent survey [DM20]. Our third result below essentially states that *it is arithmetical to tell whether a given locally compact abelian group contains a closed subgroup isomorphic to* $\mathbb{R}$*.* We informally explain what we mean, and then we formally state the result. Fix a computable enumeration $(G_i)_{i\in\omega}$ of all partial computable Polish groups. Each such G_i is given by a (potential) computable pseudo-metric on $\mathbb{N}$ and Turing operators potentially representing group operations upon this space. We are interested in measuring the complexity of the index set of P,

$$I_P = \{i : G_i \text{ is a computable Polish group with property } P\}.$$

The general intuition is that 'tractable' properties, such as being compact, tend to have arithmetical index sets, meaning that its complexity lies at some finite level of the arithmetical hierarchy. In contrast, if a given property P is difficult to 'test' for a given space, then its index set tends to be either analytic or co-analytic complete (or beyond). Such estimates reflect that such 'difficult' properties are intrinsically *not local* and certainly *not first-order* even if we additionally allow quantification over effective procedures.

We return to the discussion of our third result. Recall that a metric is *proper* (or Heine-Borel) if all closed balls are compact with respect to this metric. We say that G is properly metrized (or Heine-Borel) if the metric on G is proper. It is well-known that each locally compact Polish space admits a compatible proper metric ([WJ87]) and in Proposition 3.3 we will prove that this is computably true as well. We write $H \leq_c G$ for 'H is (isomorphic to) a closed subgroup of G'. The property '$\mathbb{R} \leq_c G_i$' seems intrinsically *not* first-order and *not* local. Nonetheless, using Theorem 1.1 and the techniques developed in its proof, we establish:

Theorem 1.3. *The index set*

$$\{i : G_i \textit{ is a properly metrized abelian group and } \mathbb{R} \leq_c G_i\}$$

lies at a finite level of the arithmetical hierarchy.

The exact estimate is that it is Π^0_3-complete. The Π^0_3-hardness is easy to verify. Establishing the somewhat unexpected upper bound Π^0_3 takes much more work. We will see that every computable properly metrized group is $0'$-computably locally compact, and this is sufficiently uniform. Once we have access to a $0'$-computably locally compact structure on an abelian G_i, we use a relativized version of Theorem 1.1 to (uniformly) produce a $0''$-computably compact copy of its Chabauty space $\mathcal{S}(G_i)$, and then we will use this presentation of $\mathcal{S}(G_i)$ and a result of Protasov and Tsybenko [PT83] to (indirectly) test whether $\mathbb{R} \leq_c G_i$.

The final section contains several open questions.

2. Computable Polish spaces and groups

We use the standard computability-theoretic terminology and notation that can be found in, e.g., [Soa87]. For instance $0^{(n)}$ stands for the Turing degree of the $(n-1)$th iteration of the halting problem, and $\Sigma^0_n, \Pi^0_n, \Delta^0_n$ denote the finite levels of the arithmetical hierarchy. It is well-known that the Δ^0_{n+1} sets are exactly the sets computable relative to $0^{(n)}$, and Σ^0_{n+1} and Π^0_{n+1} are the classes of $0^{(n)}$-computably enumerable ($0^{(n)}$-c.e.) and the complements of $0^{(n)}$-c.e. sets, respectively.

2.1. Computable Polish spaces.

The notions below are standard and can be found in, e.g., [DM23, IK21].

Definition 2.1. A Polish space M is *computable Polish* or *computably (completely) metrized* if there is a compatible, complete metric d and a countable sequence of *special points* (x_i) dense in M such that, on input i, j, n, we can compute a rational number r such that $|r - d(x_i, x_j)| < 2^{-n}$.

Polish spaces with operations on them, especially computable Banach spaces, have been studied extensively; see books [Wei00, Abe80, PER89]. To define what it means for an operation upon a computable Polish space to be computable, we will need a few more definitions.

A *basic open ball* is an open ball having a rational radius and centred in a special point. Let X be a computable Polish space, and (B_i) is the effective list of all its basic open balls, perhaps with repetition.

Definition 2.2. We call
$$N^x = \{i : x \in B_i\}$$
the name of x (in X).

A (fast) Cauchy name of a point x is a sequence (x_n) of special points such that $d(x_n, x) < 2^{-n}$. It is easy to see that we can uniformly effectively turn an enumeration of N^x into a fast Cauchy name of x, and vice versa. We say that a point x is computable if N^x is c.e.; equivalently, if there is a computable sequence (x_n) of special points that is a fast Cauchy name of x.

We can also use basic open balls to produce names of open sets, as follows. A *name* of an open set U is a set $W \subseteq \mathbb{N}$ such that $U = \bigcup_{i \in W} B_i$, where B_i stands for the i-th basic open set (basic open ball). If an open U has a c.e. name, then we say that U is *effectively open* or *c.e. open*. A closed set C is c.e. if the set $\{i : B_i \cap C \neq \emptyset\}$ is c.e.; equivalently, if the set possesses a uniformly computable dense sequence of points. A closed set is *computable* if both the set and its complement are c.e.; see [DM23, IK21] for further details.

Definition 2.3. A function $f : X \to Y$ between two computably metrized Polish spaces is *effectively continuous* if there is a c.e. family $F \subseteq \mathcal{P}(X) \times \mathcal{P}(Y)$ of pairs of (indices of) basic open sets in such that:

(C1): for every $(U, V) \in F$, $f(U) \subseteq V$;

(C2): for every $x \in X$ and basic open $E \ni f(x)$ in Y there exists a basic open $D \ni x$ in X such that $(D, E) \in F$.

Note that a function is continuous if and only if it is effectively continuous relative to some oracle. The lemma below is well-known.

Lemma 2.4. *Let $f\colon X \to Y$ be a function between computable Polish spaces. The following are equivalent:*

1. *f is effectively continuous.*
2. *There is an enumeration operator Φ that on input a name of an open set Y (in Y), lists a name of $f^{-1}(Y)$ (in X).*
3. *There is an enumeration operator Ψ, that given the name of $x \in X$, enumerates the name of $f(x)$ in Y.*
4. *There exists a uniformly effective procedure that on input a fast Cauchy name of $x \in M$ lists a fast Cauchy name of $f(x)$ (note that the Cauchy names need not be computable).*

Definition 2.5. We say that a map (between computable Polish spaces X and Y) is *computable* if it satisfies any of the items from the lemma above.

Clearly, computable maps are closed under composition, when it is well-defined.

Definition 2.6. A function $f\colon X \to Y$ is *effectively open* if there is a c.e. family F of pairs of basic open sets such that

(O1): for every $(U, V) \in F$, $f(U) \supseteq V$;
(O2): for every $x \in X$ and any basic open $E \ni x$ there exists a basic open $D \ni f(x)$ such that $(E, D) \in F$.

The lemma below is elementary.

Lemma 2.7 ([MM18])**.** *Let $f\colon X \to Y$ be a function between computable Polish spaces. The following are equivalent:*

1. *f is effectively open.*
2. *There is an enumeration operator that given a name of an open set A in X, outputs a name of the open set $f(A)$ in Y.*

In particular, if f is a computable and is a homeomorphism, then it is is effectively open if, and only if, f^{-1} is computable. In this case we say that f is a *computable homeomorphism*. We say that two computable metrizations on the same Polish space are effectively compatible if the identity map on the space is a computable homeomorphism when viewed as a map from the first metrization to the second metrization under consideration. A special kind of self-homeomorphisms are the (left or right) translations of a Polish group by its elements, and also the inverse map. We discuss computable groups next.

2.2. **Computable Polish groups.** In the discrete (at most) countable case, Mal'cev [Mal62] and Rabin [Rab60] define a computable group as follows. A discrete and at most countable group is *computable* (recursive, constructive) if its domain is a computable set of natural numbers and the group operations are computable functions upon this set. A computable presentation (a computable copy, a constructivization) of a group is a computable group isomorphic to is. The study of computable groups, and especially of computable abelian groups, is a central subject in computable structure theory; see the books [AK00, EG00]. (For the well-studied special case of abelian groups, see the surveys [Khi98, Mel14].) The following notion appears to be a rather natural extension of the well-established notion of a (discrete, countable) computable group.

Definition 2.8. [MM18, Mel18] A computable Polish group is a computable Polish space together with computable group operations $\cdot$ and $^{-1}$.

The following rather elementary fact illustrates that the notions actually agree in the discrete case:

Fact 2.9 ([KMK23])**.** For a discrete (at most) countable group G, the following are equivalent:

(1) G has a computable Polish presentation as in Def. 2.8;
(2) G has a computable presentation in the sense of Mal'cev [Mal62] and Rabin [Rab60].

The next fact will be used throughout, often without an explicit reference.

Fact 2.10 ([KMK23])**.** In a computable Polish group, the multiplication and inversion operators are effectively open.

Proof. Given some name for an effectively open set U, in order to enumerate the name for U^{-1}, simply enumerate the preimage of U under $^{-1}$. This must be the name for U^{-1} since $\left(U^{-1}\right)^{-1} = U$. Thus $^{-1}$ is effectively open.

Next, given names for effectively open sets U, V, we wish to computably produce a name for $U \cdot V$. The map $(x, y) \to x^{-1}y$ is computable: enumerate all basic open sets B such that there is some basic open set A with $A \cap U \neq \emptyset$ and

$$A^{-1} \cdot B \subseteq V.$$

Now we claim that the union of all such B is equal to $U \cdot V$. For the inclusion "$\subseteq$", if B is enumerated by the procedure above, let $a \in A \cap U$. For each $b \in B$ we have $b = a \cdot a^{-1}b \in U \cdot A^{-1} \cdot B \subseteq U \cdot V$, and so $B \subseteq U \cdot V$. For the converse inclusion, let $a \in U$ and $b \in V$. Since $a^{-1} \cdot ab = b \in V$, we can let A, B be basic open sets containing a and ab respectively, such that $A^{-1} \cdot B \subseteq V$. Then B will be enumerated by the procedure above, and $ab \in B$. ☐

In the important special cases of compact and (more generally) locally compact groups, it is also natural to assume that the (local) compactness of the group is, in some sense, effective. We discuss this next.

2.3. **Computable compactness.** The following definition is equivalent to many other definitions of effective (computable) compactness that can be found in the literature.

Definition 2.11. A compact computable Polish space is computably compact if there is a (partial) Turing functional that, given a countable cover of the space, outputs a finite subcover.

This is equivalent to saying that, or every n, we can uniformly produce at least one finite open 2^{-n}-cover of the space by basic open balls. For several equivalent definitions of computable compactness, see [DM23] and [IK21]. It is also well-known that, given a computable Polish space C that is compact (but not necessarily computably compact), using $0'$ one can produce a sequence of basic open 2^{-n}-covers of the space, thus making it computably compact relative to $0'$. The following elementary fact is folklore (see, e.g., [DM23]):

Lemma 2.12. *A computable image of a computably compact space is itself computably compact.*

Also, it is well-known that both the supremum and the infimum of a computable function $f : X \to \mathbb{R}$ is computable provided that X is computably compact, and this is uniform. We will also use that a computable closed subset of a computably compact space M is itself computably compact (as a space under the metric inherited from M). The lemma below is elementary.

Lemma 2.13. *Suppose C is co-c.e. closed in a computably compact Polish space K. Then C admits a $0'$-computable dense sequence of points.*

Proof. Use the aforementioned result [BdBP12, Prop. 4.1] to fix computable surjective $f : 2^\omega \to K$. Since f is effectively continuous by Lemma 2.4, the pre-image of the effectively open complement of C is effectively open in 2^ω. Thus, $f^{-1}(C)$ is a Π^0_1 class in 2^ω. With the help of $0'$ we can produce the collection of finite strings that are extendable to a path in $f^{-1}(C)$. This induces a $0'$-computable dense set in $f^{-1}(C)$ and, thus, in C. □

Note the proof of the lemma above is uniform. We cite [DM23, IK21] for further background on computably compact spaces.

2.4. **Computable local compactness.** Recall that the name N^x of a point x is the collection of all basic open sets (balls) of a computable Polish M that contain x. In the definition below, a compact $K \subseteq M$ is represented by the computable index of a procedure enumerating a sequence of finite open 2^{-n}-covers so that each ball in the cover intersects K non-trivially. (It is not difficult to see that, up to a uniform adjusting of the covers, this is equivalent to saying that every ball in is centred in a computable point in K and has a rational radius.) In particular, it makes K a computable closed set and a computably compact subspace of the ambient computable Polish space M. The index of such a procedure enumerating finite covers of K is sometimes refereed to as a *compact name* of K.

The usual definition of a locally compact space M says that for every $x \in M$ there is an open set B and a compact set K such that $x \in B \subseteq K$. We adopt the following notion of effective local compactness which is essentially the approach taken in [Kam01], up to a change of notation.

Definition 2.14. A computable Polish space M is *computably locally compact* if there is an algorithmic procedure that, given (an enumeration of the name N^x of) any point x, outputs a basic open set B and a computable compact set $K \subseteq M$ such that $x \in B \subseteq K$. We call the functional representing the procedure $N^x \to K \supseteq B \ni x$ a *computable locally compact structure* on M.

Remark 2.15. Other closely related definitions can be found in [Pau19, XG09, WZ99]. See [Pau19] for a detailed discussion of the various approaches to effectivizing local compactness in the literature and the subtle differences between them.

Clearly, if the space itself is compact then computable local compactness of the space is equivalent to its computable compactness. Note that x does not have to be a computable point, while the output of the procedure in the definition has to be a computable index of the procedure representing K. If the functional in the definition above is not computable but is Y-computable for some oracle Y (e.g., $Y = \emptyset''$) then we say that M admits a Y-computable locally compact structure.

We will need this relativized version of computable local compactness in the proof of our index set result.

A useful reformulation of the notion of computable local compactness presented below involves the notion of a computable closed ball. If $B = B(\xi, r)$ is an open ball, then we write B^c to denote the respective closed ball $\{x : d(\xi, x) \leq r\}$ which does not have to be equal to the closure $\overline{B}$ of B in general. Recall that we say that $B = B(\xi, r)$ is basic if ξ is special and r is a rational (given as a fraction). If ξ and r are merely computable, we say that the ball is computable.

The following proposition will be rather useful. See [Pau19, Proposition 12] for a similar fact in a different framework. The property established in the proposition below is essentially [WZ99, Definition 3].

Proposition 2.16. *In the notation of Definition 2.14, we can assume that K is equal to B^c for a computable open ball B containing x. In other words, given x we can produce a computable index of a computable point ξ and of a computable real r so that $B(\xi, r) \ni x$ and $B^c(\xi, r)$ is computably compact (and is represented by its compact name). Furthermore, we can additionally assume that $B^c(\xi, r) = \overline{B(\xi, r)}$, where the latter is the closure of $B(\xi, r)$.*

Proof. Proposition 3.30 of [DM23] establishes that we can uniformly produce a system of finite 2^{-n}-covers C_n of a computably compact space K that consist of computable open balls that possess the following strong property. Each ball in C_n is represented by the pair of indices of its computable centre and its computable radius. Given any finite collection of (the parameters describing) balls $B_0, \ldots, B_k \in \bigcup_n C_n$, we can uniformly decide whether

$$\bigcap_{i \leq k} B_i \neq \emptyset.$$

Furthermore (as explained in [DM23, Remark 3.18]), when we decide intersection, we can replace some (or all) balls in the sequence $B_0, \ldots, B_k$ with the respective closures or the closed balls, e.g.,

$$B_0 \cap \overline{B_1} \cap B_2^c \cap B_3 \neq \emptyset$$

iff

$$B_0 \cap B_1 \cap B_2 \cap B_3 \neq \emptyset,$$

where $\overline{B_i}$ are the closure of B_i and B_i^c is the basic closed ball with the same parameters as B_i. Additionally, we can further assume that every $B(a, r) \in C_{n+1}$ is formally included in some $B(b, q) \in C_n$, meaning that $d(a, b) + r < q$.

The following further convenient property of such covers was not mentioned in [DM23].

Claim 1. In the notation above, for any computable ball $B \in \bigcup_n C_n$ we have $\overline{B} = B^c$.

Proof. Suppose $\alpha \in B^c \setminus \overline{B}$. The set $U = K \setminus \overline{B}$ is open, and each $\alpha \in U \cap B^c$ must be contained in U together with some $B' \in C_n$, where n is sufficiently large. But then we have $B' \cap B^c \neq \emptyset$ while $B' \cap \overline{B} = \emptyset$, contradicting the properties of $\bigcup_n C_n$ described before the claim. □

Recall that every ball in $\bigcup_n C_n$ is represented by a pair of indices for its computable centre and its radius.

Claim 2. In the notation above, for every $B \in \bigcup_n C_n$ the closed ball $B^c = \overline{B}$ is a computably compact subspace of K, and this is uniform in the parameters describing B. Indeed, this is witnessed by finite 2^{-n}-covers $C'_n \subseteq C_n$ that can be found uniformly in n.

Proof. We can list a dense sequence of uniformly computable points in B which turns B into a computable Polish space. Since intersection is decidable in $\bigcup_n C_n$ in the strong sense described earlier, we can uniformly effectively restrict each C_n to its finite subset C'_n consisting of $B' \in C_n$ having the property $\overline{B} \cap B' \neq \emptyset$. The resulting uniformly computable system $(C'_n)_{n\in\omega}$ of 2^{-n}-covers of B^c witnesses that B^c is itself a computably compact subspace of K. □

We now return to the proof of the proposition. Let $x \in B \subseteq K$ be as in Definition 2.14, where $B = B(a, q) = \{y : d(a, y) < q\}$. Fix a system of covers (C_n) of K with the strong properties described above, and (using x) search for the first found open ball $D = B(\xi, r) \in \bigcup_n C_n$ such that

$$x \in D \subseteq B$$

as witnessed by $d(\xi, x) < r$ and $d(a, \xi) + r < q$. (Note the latter implies $D^c \subseteq B$.) This search is uniformly effective in the name N^x of x and will eventually terminate.

The closure of D in K is equal to the closure of D in M because both sets are contained in $B \subseteq K \cap M$. By Claim 1, $D^c = \overline{D}$. This makes D^c a computable subspace of M. Using Claim 2, restrict (C_n) to a uniformly computable system (C'_n) of covers of D^c, witnessing that D^c is computably compact. Since we have

$$x \in D = B(\xi, r) \subseteq \overline{B(\xi, r)} = B^c(\xi, r) \subseteq B(a, q),$$

the computably compact ball $D^c = B^c(\xi, r) = \overline{B(\xi, r)}$ satisfies the required properties. □

Definition 2.17. We say that a locally compact Polish group is *computably locally compact* if its domain is computably locally compact and the group operations $\cdot$ and $^{-1}$ are computable upon this domain. (The notion of a computably compact group is defined similarly.)

In the discrete case, Definition 2.8 and Definition 2.17 are actually equivalent; this follows quite easily from Fact 2.9. It is known that a profinite group is "recursive profinite" [MN79, LR81, LR78, Smi81] if, and only if, it is computably compact [DM23]. Further effective properties of locally compact groups have recently been investigated in more detail in [KMK23, MN23]. We will later use the following result from [MN23]. It has been established [MN23] that the natural and robust notion of computable presentability for t.d.l.c. groups [MN22] is in fact equivalent to Definition 2.17; we will give further details in Subsection 6.1. Given the evidence that Definition 2.17 is natural and robust, we choose it as the basic definition of computability for a locally compact Polish group.

Then lemma below was mentioned in the introduction in relation with the effective content of proper metrization results established in [KMK23]. The lemma will be used in the proof of Theorem 1.3.

Lemma 2.18. *Let G be a computable Polish group. Then G is locally compact if (and only if) G is $0'$-computably locally compact.*

Proof. Clearly, if G is $0'$-computably locally compact then it is locally compact. To this end, assume G is locally compact.

Recall that for every element $g \in G$, the translation $x \to x \cdot g$ is a homeomorphism of G onto itself. In particular, G is locally compact if, and only if, the identity element e lies in basic open ball $B(e, r)$ for some positive $r \in \mathbb{Q}$, so that the closure (the completion) of $B(e, r)$ is compact. Without loss of generality, the identity element $e = x \cdot x^{-1}$ can be assumed a computable point in G (here x is the first found special point of G). Any basic open ball $B(e, r)$ contains a c.e. dense sequence of special points of G which turns its closure into a computable Polish space, uniformly in r.

If the closure C of $B(e, r)$ is compact, then using $0'$ we can list all finite covers of C by basic open balls centred at special points inside $B(e, r)$. Recall that we assumed that G was a computable Polish group. In particular, the group operations on G are computable and computably open, by Fact 2.10. Also, the maps $r_y : x \to x \cdot y$ are computable and computably open uniformly in y, where y ranges over all special points in G. If $B = B(e, r)$ has compact closure C, then the uniformly $0'$-computably compact sets

$$\{C \cdot y : y \text{ special in } G\}$$

and the uniformly computably open sets

$$\{B(e, r) \cdot y \subseteq C \cdot y : y \text{ special in } G\}$$

give a $0'$-computably compact structure on G. □

Notice that the lemma above relied heavily on computability of the group operations on G; in particular, that they are total. It is not hard to see that compactness of a computable Polish space is a Π^0_3-property; see [MN13]. Of course, a group is locally compact iff for some $r \in \mathbb{Q}$, $\overline{B(e, r)}$ is compact, where $e = xx^{-1}$ is the identity element. Thus, local compactness of G is arithmetical *assuming G is indeed a computable Polish group.*

3. Computability of the 1-point compactification

In this subsection we will establish several technical facts about computably locally compact spaces that will be sufficient to prove Theorem 1.1 and Theorem 1.2. In contrast with the technical lemmas established earlier, some of these facts seem to hold interest on their own.

3.1. **Computable σ-compactness, a strong form.** Let M be a Polish space with a fixed metric d.

Definition 3.1. For $\epsilon > 0$, we say that a set U is a uniform ϵ-neighbourhood (ϵ-nbhd) of a set V if

$$U \supset V(\epsilon) = \{x \in M : \exists y \in V \, d(y, x) < \epsilon\}.$$

In other words, each point of V is contained in U together with the open ϵ-ball around the point.

There is a danger of confusing computability-theoretic uniformity with topological uniformity (as defined above). In what follows next, we shall avoid saying 'U uniform ϵ-neighbourhood of V' and instead we will use the associated notation '$U \subset V(\epsilon)$'.

We believe that the technical lemma below is new as stated, since it appears to be stronger than other similar results in the literature. Very closely related results can be found in [Pau19]. (Compare this lemma with [Pau19, Proposition 8] that also produces a nice list of compact neighbourhood of a given space. Also, note that the notion of computable σ-compactness as defined in [Pau19, Definition 14] appears to be weaker than the technical property established in the lemma below; see, e.g., [Pau19, Example 17].)

Lemma 3.2 (Strong computable σ-compactness)**.** *Suppose M is a computably locally compact Polish space. There exists a nested sequence $(K_n)_{n\in\omega}$ of uniformly computably compact sets $K_n \subseteq M$ and a uniformly computable sequence of positive reals $c_i \leq 2^{-i}$ with the following properties:*

(1) *$K_{n+1} \supseteq K_n(c_n)$ for all n. (Recall Def. 3.1.)*
(2) *$M = \bigcup_{n\in\omega} K_n$.*
(3) *Each K_n is represented as a finite union of computable closed balls.*
(4) *Given $x \in M$ we can uniformly effectively (in x) calculate some n such that $x \in K_n$, and indeed x is inside one of the open balls whose respective closed balls make up K_n.*

Proof. Suppose the computable local compactness of M is witnessed by an operator Ψ. We can replace Ψ by a functional Ψ that turns a Cauchy name (x_i) of a point to an enumeration of its name N^x and then uses Ψ to produce a compact name of the compact neighbourhood of the point. List all finite sequences of the form $\bar{x} = \langle x_0, \dots, x_k\rangle$, where x_i are special in M and $d(x_i, x_{i+1}) < 2^{-i}$, and calculate the uniform sequence of those $\Psi^{\bar{x}}$ which halt with use at most the length of $\bar{x}$. Let $K_{\bar{x}}$ denote (the computable index of) the computably compact set that is output by the procedure on input $\bar{x}$ if it halts. Observe that any point α of M (special or not) will be contained in some such $K_{\bar{x}}$ that will be listed in the sequence. Further, by Proposition 2.16 we may assume $K_{\bar{x}}$ is a basic computable closed ball B^c with a uniformly computable radius; in the enumeration, it will be represented by its computable index. Let $(D_i)_{i\in\omega}$ be the resulting sequence of (computable indices of) uniformly computably compact closed balls.

To make sure that all conditions of the lemma are satisfied, we need to modify the sequence as follows. Set $K_0 = D_0$. If K_n has already been defined, uniformly fix a finite 2^{-n}-cover $L_1, \dots, L_k$ of K_n so that the respective closed balls L_i^c are (uniformly) computably compact, and set

$$K_{n+1} = D_{n+1} \cup \bigcup_i L_i^c.$$

(These balls are the first found balls in the sequence $(D_i)_{i\in\omega}$; they must exist so we just search for the first found ones.) It should be clear that conditions (2), (3), and (4) of the lemma are satisfied by the sequence (K_i) of uniformly computably compact subspaces.

It remains to define the parameters c_n required in (1). In the notation above, use [DM23, Remark 3.18] to uniformly produce a finite cover of K_n that formally refines $L_1, \dots, L_k$. This means that for each ball in this cover $B(a, r)$ there is some ball $L_j = B(b, q)$ containing it so that $d(a, b) + r < q$. For each such $B(a, r)$ effectively choose a positive rational $l = l_j$ so that $d(a, b) + r + l < q$ still. It follows from the triangle inequality that for any $z \in K_n$ we have that the l-ball around z is contained in $L_j = B(b,q)$. (If $w \in B(z,l)$ then $d(a,w) \leq d(a,b)+d(b,z)+d(z,w) \leq d(a,b)+r+l < q$.) Choose c_n to be smaller than the least among all these l_j, $j = 1,\dots,k$, and also is at least twice smaller than c_{n-1} (to make sure the sequence (c_n) satisfies $c_n \leq 2^{-n}$). Then we have that $K_n(c_n)$ (see Def. 3.1) has to be included into $\bigcup_i L_i^c$, and thus

$$K_n(c_n) \subseteq D_{n+1} \cup \bigcup_i L_i^c = K_{n+1}.$$

This gives (1). □

The subsequent proposition, which can be readily deduced from the above lemma, was already mentioned in the introduction. It appears to be new. The proposition will not be directly used in the rest of the paper, however, it seems to be sufficiently important for the framework of computable Polish spaces.

Proposition 3.3. *Every computably locally compact Polish space admits a computable regular (i.e. Heine-Borel) metric. Furthermore, δ and d are effectively compatible, i.e., the map $Id : (M,d) \to (M,\delta)$ is a computable homeomorphism*[2].

Proof. We use Lemma 3.2 to implement an idea similar to that suggested in [WJ87], which in turn the authors attribute to H. E. Vaughan. In the notation of Lemma 3.2, define

$$f_n : M \to [0,1]$$

to be

$$f_n(x) = \begin{cases} d(x,K_n)/c_n & d(x,K_n) \leq c_n, \\ 1 & d(x,K_n) \geq c_n, \end{cases}$$

where the two cases are not mutually exclusive but evidently agree at when $d(x,K_n) = c_n$. Since K_n is computably compact uniformly in n, $(f_n)_{n\in\omega}$ is a uniformly computable sequence of functions. Note also that, for each fixed $x \in M$, the sequence $(f_n(x))_{n\in\omega}$ is eventually zero. Define

$$f(x) = \sum_{n\in\omega} f_n(x),$$

which is well-defined and is computable. Finally, define

$$\delta(x,y) = d(x,y) + |f(x) - f(y)|.$$

It is clear that δ is indeed a metric; we claim it is also equivalent to d, i.e. that the metrics share the same converging sequences. Since $\delta \geq d$, we need only to show that every d-converging sequence also converges with respect to δ; but this follows from computability (thus, continuity) of f. In particular, we may use the same dense sequence (x_i) in M as we used for d, and we get that $\overline{((x_i),\delta)}$ is a computable Polish space such that $\overline{((x_i),\delta)} = \overline{((x_i),d)} = M$.

We show δ is proper. If a closed δ-ball $B^c(x,r)$ is not fully contained in one of the compact sets K_n, then there will be an infinite sequence (y_m) so that $y_m \notin K_{m+1}$, and thus $f(y_m) = \sum_n f_n(y_i) \geq m$, contradicting that we must have

$$\delta(x,y_m) = d(x,y_m) + |f(x) + f(y_m)| \leq r.$$

it follows that each basic closed ball has to be compact.

[2]In particular, the space remains computably locally compact under δ.

To see why δ and d are effectively compatible, first recall that $\overline{((x_i), \delta)} = \overline{((x_i), d)} = M$. The pre-image of each basic open δ-ball $B(x, r)$ under the identity map is simply

$$\{y : d(x, y) + |f(x) - f(y)| < r\}$$

which is clearly an effectively open set with respect to d. On the other hand, if (x_i) is a fast Cauchy sequence with respect to δ,

$$\delta(x_i, x_{i+1}) = d(x_i, x_{i+1}) + |f(x_i) - f(x_{i+1})| < 2^{-i},$$

then clearly (x_i) is also fast Cauchy for d. It remains to interpret the latter as the image of $x = \lim_i x_i$ under id^{-1}, where the limit can be taken with respect to either metric. □

See the last section for a further discussion of proper metrics.

3.2. **Effective 1-point compactification.** Theorem 3.4 will be a key step in the definition of the Π^0_1-presentation of the Chabauty space of a computably locally compact group.

Theorem 3.4. *Given a computably locally compact Polish space M that is not already compact, we can uniformly effectively produce a computable homeomorphic embedding f of M into its computably compact 1-point compactification $M^* \cong M \cup \{\infty\}$. Furthermore, f^{-1} is computable (everywhere except for ∞), and this is also uniform.*

A brief discussion. In order to prove the theorem, we combine Lemma 3.2 with the metric on M^* suggested in [Man89]. There are other potential ways to prove the theorem, e.g., using some effective version of the Urysohn's metrization theorem along the lines of [GW07]. However, the rather explicit construction in [Man89] provides us with some extra information about the constructed metric, and this will be rather convenient in establishing the computable compactness of $M \cup \{\infty\}$. We strongly suspect that the assumption that M is not compact can be dropped without any effect on the uniformity of the procedure, however, we will not verify this claim.

Proof. Given a computably locally compact (M, ρ), adjoin a point ∞ to M and declare the point to be special in M^*. Fix a sequence of uniformly computably compact neighbourhoods (K_n) for M and the corresponding computable sequence (c_n) given by Lemma 3.2. For any $x \in M$, let

$$h(x) = \sup\{c_i - \rho(x, K_i) : i \in \omega\}.$$

Claim 3. The function $h\colon M \to \mathbb{R}$ is computable.

Proof of Claim. Given x, we can uniformly find some n so that $x \in K_n$; this is (4) of Lemma 3.2. It follows that, for such an n, $h(x) = \sup\{c_i - \rho(x, K_i) : i \leq n\}$, because (c_i) form a decreasing sequence and since $x \in K_m$ for all $m \geq n$. Since K_i are uniformly computable, $\rho(x, K_i)$ are reals uniformly computable relative to x. This makes $h(x)$ a uniformly x-computable real. □

Define d on $M \cup \{\infty\}$ by the rule

$$d(x, y) = \begin{cases} \inf\{\rho(x, y), h(x) + h(y)\}, \text{ if } x, y \in M, \\ h(x), \text{ when } x \in M, y = \infty, \end{cases}$$

and set $d(x,x) = 0$ for any $x \in M \cup \{\infty\}$. It is clear that for any $x, y \in M$ we have

$$d(x,y) \leq \rho(x,y),$$

and thus the identity map maps fast ρ-Cauchy names to fast d-Cauchy names. Thus, we can set $f = Id_M$. It is shown in [Man89] that this definition of d gives a metric compatible with the topology in the one-point compactification of M; the metric is clearly complete (by compactness). By the claim above, $d(x,y)$ is uniformly computable for any pair of special points x, y in $M \cup \{\infty\}$.

Recall that $K_n(c_n) \subseteq K_{n+1}$, where the latter is viewed as a computable Polish space w.r.t. the new metric d. Recall that we set $f = Id_M$. We can view $Id_M(K_n) = K_n$ as the computable image of a computably compact space K_n (w.r.t. ρ) inside a computable Polish space K_{n+1} (w.r.t. d). It thus follows that (K_n, d) is computably compact; see, e.g., [DM23, Lemma 3.31], [Wei03, Theorem 3.3] and [Pau16, Proposition 5.5]. This is also computably uniform (in n). It follows from the definition of d that, when $x \in (M \cup \{\infty\}) \setminus K_{n+1}$, we have

$$\rho(\infty, x) = h(x) \leq c_{n+1} < c_n,$$

see [Man89] for an explanation and for further details. We conclude that

$$M \cup \{\infty\} = B(\infty, c_n) \cup \bigcup_{i \leq n+1} K_i.$$

To find a finite 2^{-n}-cover of $M \cup \{\infty\}$, recall that $c_n < 2^{-n}$. Effectively uniformly fix a finite 2^{-n} cover of $\bigcup_{i \leq n+1} K_i$ (in the new metric d). Together with $B(\infty, c_n)$, this gives a finite 2^{-n}-cover of $M \cup \{\infty\}$.

We now show that f^{-1} is also computable on M. (Recall that f is the identity map, but viewed as a map between two different metrizations of M.) Assume $x \neq \infty$. Calculate $d(x, \infty)$ to a precision sufficient to find some n so that $d(x, \infty) > c_n$. By the definition of the metric d and the sequence (c_n), it must be that $x \in K_n$. Since f is just the identity map, it is clearly guaranteed that $f^{-1}(x) \in (K_n, \rho)$. It is well-known that the inverse of a computable homeomorphic map between computably compact spaces is computable; see, e.g., [DM23, Theorem 3.33] or [BdBP12, Corollary 6.7]. This is also clearly effectively uniform. Apply this fact to $f : (K_n, \rho) \to (K_n, d)$ to calculate $f^{-1}(x)$. □

Let M be a computably locally compact space, $M^* = M \cup \{\infty\}$ be its computably compact 1-point compactification, and $f : M \to M^*$ a computable embedding having computable inverse; see Theorem 3.4.

The Hausdorff distance between finite sets of computable points is clearly computable. The hyperspace of compact subspaces of a computably compact space is itself computably compact; this is folklore. (To see why, note that, given a finite ϵ-cover, we can restrict ourselves to finite subsets of the centres of the balls involved in the cover. These finite subsets will give rise to an ϵ-cover of the hyperspace.) Since M^* is computably compact, we conclude:

Fact 3.5. For a computably locally compact Polish space M, the space $\mathcal{K}(M^*)$ is computably compact as well, and this is uniform.

In a different framework and using different methods, a similar fact was established in [Pau19, Subsection 5.2].

4. Proof of Theorem 1.1(1): $\mathcal{S}(G)$ is effectively closed in $\mathcal{K}(G^*)$.

We begin our proof with an explanation of how the Chabauty space of a l.c. G can be viewed as a closed subspace of the hyperspace of closed sets of the 1-point compactification of G. In the next subsection we will use the results from the previous section, specifically Fact 3.5, to prove that this presentation is indeed Π^0_1.

4.1. The Chabauty space of G and the 1-point compactification of G.

The fact below appears to be folklore among the experts in topological group theory.

Fact 4.1. If G is locally compact Polish group, then there exists a homeomorphic embedding for the Chabauty space $\mathcal{S}(G)$ into the hyperspace $\mathcal{K}(G^*)$ of closed (thus, compact) subsets of the 1-point compactification G^* of G. The embedding is given by the map $i : C \to C \cup \{\infty\}$.

It is difficult to find a reference where this fact is thoroughly explained. Cornulier mentions it at the very beginning of his paper [Cor11], but in terms of convergence of nets. A more general fact appears already in [Fel62]; see (II) on p. 475. It is stated much more explicitly, but still not quite in the form that we need it, in the unpublished lecture notes by Pierre de la Harpe that can be found on arXiv [dlH08]; see "2. Prop. (v)" on pages 2-3. To make our presentation more self-contained, we include a detailed verification below.

Proof of Fact 4.1. Recall that the topology of the 1-point compactification $G^* = G \cup \{\infty\}$ of G is generated by:
- the open sets of G, and
- the sets $(G \setminus K) \cup \{\infty\}$, where K ranges over compact subsets of G.

Now recall that Vietoris topology on the set of its closed (thus, compact) subsets of G^* is generated by

$$\{F : F \cap V \neq \emptyset\} \text{ and } \{F : F \subseteq U\}$$

where U, V are open in G^*. It is well-known that the topology induced by the Hausdorff metric and the Vietoris topology are equivalent. The Chabauty topology on the closed subgroups of G is generated by the sub-basic sets

$$\{F : F \cap K = \emptyset\} \text{ and } \{F : F \cap V \neq \emptyset\}$$

where K is compact and V open in G.

Given a closed subgroup (more generally, subset) $C \subseteq G$, define

$$i : C \to C^* = C \cup \{\infty\}$$

which can be viewed as a map from the Chabauty space $\mathcal{S}(G)$ of X to the hyperspace $\mathcal{K}(G^*)$ of the 1-point compactification of G. The map is evidently injective. We show that it is continuous, and thus a homeomorphism. ($\mathcal{S}(G)$ is known to be compact and thus so is its i-image, so establishing continuity will suffice.)

When V is open in $G \cup \{\infty\}$ and does not contain ∞, then it is open in G and

$$i^{-1}\{F^* : F^* \cap V \neq \emptyset\} = \{F : F \cap V \neq \emptyset\}$$

which is open in Chabauty topology. Otherwise, if $\infty \in V$, the pre-image is the whole space $S(G)$.

Now consider an open set of the second kind, specifically $\{F^* : F^* \subseteq U\}$. If U does not contain ∞, then the set is empty and thus so is its i-preimage. Otherwise, if U is an open set containing ∞, then it has to be of the form $(G \setminus K) \cup \{\infty\}$, where K is compact in G; in particular, $\infty \notin K$. (This K is the intersection of $C_i = (G \cup \{\infty\}) \setminus U_i$, where $U = \cup U_i$ and each U_i is a finite intersection of sub-basic sets in the topology of $G \cup \{\infty\}$. At least one such $U_i = U_{i_1} \cap U_{i_2} \cap \ldots \cap U_{i_k}$ has to contain ∞, so necessarily each of its sub-basic U_{i_j} contains ∞. The complement K_{i_j} of U_{i_j} is compact in G, for each $j \leq k$, and thus $K_i = K_{i_1} \cup K_{i_2} \cup \ldots \cup K_{i_k}$ is compact too. The complement of $U = \cup U_i$ is the intersection of K_i with the closed complements of $U_m, m \neq i$; it is compact.)

We arrive at

$$i^{-1}\{F^* : F^* \subseteq U\} = \{F : F \subseteq (G \setminus K)\} = \{F : F \cap K = \emptyset\}$$

which is open in Chabauty topology. □

4.2. $\mathcal{S}(G)$ **as a** Π^0_1**-subset of** $\mathcal{K}(G^*)$**.** Recall that an open set U is computably enumerable if there is a computably enumerable sequence of basic open balls making up the set. A closed set is Π^0_1 or effectively closed if its complement is computably enumerable open. For a computably l.c. G, we identify $\mathcal{K}(G^*)$ with its computably compact presentation established in Fact 3.5.

Proposition 4.2. *For a computably locally compact G, the Chabauty space $\mathcal{S}(G)$ of G is Π^0_1 (effectively closed) in the computably compact hyperspace $\mathcal{K}(G^*)$ of compact subsets of the 1-point compactification of G.*

Proof. In $\mathcal{K}(G^*)$, we can distinguish between special points that contain ∞ from those that are generated by points coming from G; recall that the point $\{\infty\}$ is special in G^*. We suppress "H" in "d_H" throughout and write simply "d" for the Hausdorff metric. Recall also that $f : G \to G^*$ is computable, and f^{-1} is computable everywhere except for ∞ where it is undefined.

A non-empty closed $H \subseteq G$ is a subgroup iff $xy^{-1} \in H$, for all $x, y \in H$. For the corresponding $K = H \cup \{\infty\}$ in $\mathcal{K}(G^*)$, this condition corresponds to

$$\forall x, y \in K \ \inf\{d(x, \infty), d(y, \infty)\} > 0 \to f(f^{-1}(x) \cdot [f^{-1}(y)]^{-1}) \in K,$$

for any $x, y \in K \setminus \{\infty\}$. Recall that f is effectively continuous and effectively open, and the group operations are effectively continuous and effectively open as well. It follows that the function

$$g(x, y) = f(f^{-1}(x) \cdot [f^{-1}(y)]^{-1})$$

is computable on its domain. Thus, the condition above *fails* if, and only if, for some small enough basic balls B, D, V (in the metric of G^*), all not containing ∞, we have that

$$[B \cap K \neq \emptyset] \wedge [D \cap K \neq \emptyset] \wedge [V \cap K = \emptyset] \wedge [g(B, D) \subseteq V],$$

where the basic open balls not containing ∞ and satisfying $g(B, D) \subseteq V$ can be effectively enumerated. Say that a triple of basic open balls (B, D, V) is *wrong* if $g(B, D) \subseteq V$.

We are ready to effectively enumerate the complement of $\mathcal{S}(G)$ in $\mathcal{K}(G^*)$. Given an wrong triple (B, D, V), define the effectively open set (in $\mathcal{K}(G)$) to be all closed (compact) subsets K that satisfy:

(1) $B \cap K \neq \emptyset$;
(2) $D \cap K \neq \emptyset$;
(3) $d(K, cntr(V)) > r(V)$,

where $cntr(V)$ and $r(V)$ are the distinguished rational centre and the radius of the basic open $V = B(cntr(V), r(V))$ in the metric of G^*. (Note that the former two intersections, if they hold, will be witnessed already by a sufficiently close finite approximation to K, in $\mathcal{K}(G^*)$, by a finite collection of special points in G^*. The same can be said about condition (3). In other words, the conditions are Σ^0_1.) Let $\mathcal{W}$ be the collection of all such K, where (B, D, V) ranges over all wrong triples. Since $(1) - (3)$ are uniformly Σ^0_1, $\mathcal{W}$ is effectively open in $\mathcal{K}(G^*)$.

If a compact set does not contain $\{\infty\}$, then it is separated from $\{\infty\}$ by a non-zero distance. Also, any subgroup of G has to contain the identity element. It is clear that

$$\mathcal{V} = \{K \in \mathcal{K}(G^*) : d(\infty, K) > 0 \text{ or } d(f(e), K) > 0\}$$

is an effectively open set as well.

Then $K \in \mathcal{K}(G^*)$ corresponds to a closed subgroup of G if, and only if, there is no wrong triple for K, and $f(e), \infty \in K$. It follows that

$$\mathcal{S}(G) = \mathcal{K}(G^*) \setminus (\mathcal{V} \cup \mathcal{W}),$$

where $\mathcal{V} \cup \mathcal{W}$ is a c.e. open set in $\mathcal{K}(G^*)$. We conclude that $\mathcal{S}(G)$ is effectively closed in the computably compact space $\mathcal{K}(G^*)$. □

5. Proof of Theorem 1.1(2): $\mathcal{S}(G)$ is not computably closed in general.

We will show more than is stated in Theorem 1.1(2). Recall that that for a Π^0_1-closed subset of a computable space, being computable is equivalent to having a computable dense sequence of points; see Subsection 2.1.

We will prove that there exists a computable abelian discrete G so that the *only* computable points of $\mathcal{S}(G)$ are the ones corresponding to $\{0\}$ and G, yet G has uncountably many subgroups. Clearly, $\{0\}$ and G have to be computable subgroups; in this sense our result is also optimal. We split the proof into several subsections.

5.1. **The effective correspondence lemma.** In the notation of the previous section, we have:

Lemma 5.1. *Let G be a computably locally compact group, and $\mathcal{S}(G) \subseteq \mathcal{K}(G^*)$ its effectively closed Chabauty space. Then the following are equivalent:*

(1) *$H \leqq_c G$ is computably closed.*
(2) *$H^* = H \cup \{\infty\}$ is a computable point in $\mathcal{K}(G^*)$.*

(The lemma above essentially states that (1) $\leftrightarrow$ (2) in Theorem 1.2.)

Proof. The proof is not difficult, but it relies heavily on various properties of c.e. closed and open sets, computable and effectively open maps (see Subsection 2.1) and the technical results established earlier (e.g., Theorem 3.4). We will also use the following, seemingly well-known, fact.

Fact 5.2. A closed subset C of a computably compact space K is computably closed if, and only if, C is a computable point in the space $\mathcal{K}(K)$ (under the Hausdorff metric).

Proof. We have that C is computably closed in K if, and only if, it is a computably compact subspace of K; see, e.g., [DM23, Proposition 3.29].

Assume C is computably closed. Since C is computably compact, we can uniformly compute the Hausdorff distance $d(x, K) = \inf_{y \in C} d(x, y)$ to any special point $x \in K$. (This is because taking the infimum of a computable function over a computably compact set gives a computable real, and this is uniform, as mentioned in Subsection 2.1.) If follows that we can also uniformly calculate the Hausdorff distance between C and any finite set of special points. The finite sets of special points of K are the special points in $\mathcal{K}(K)$. Given n, search for a finite set of special points D_n at distance 2^{-n} from C. The sequence $(D_n)_{n \in \omega}$ is a computable fast Cauchy name of C.

Conversely, suppose C is a computable point in $\mathcal{K}(K)$. A basic open ball $B(x, r) \subseteq K$ does not intersect C if (and only if) $d(x, C) > r$, which is Σ^0_1. On the other hand, $C \cap B(x, r) \neq \emptyset$ is equivalent to $d(x, C) < r$, which is also Σ^0_1. The latter is equivalent to C having a computable dense sequence of points (folklore; see [DM23, Lemma 3.27]). □

Assume (1). By the fact above, it is sufficient to show that H^* is computably closed. Fix f given by Theorem 3.4. Since $f : G \to G \cup \{\infty\}$ is computable, it uniformly maps computable points to computable points. In particular, it maps the dense computable set X of H to a uniformly computable sequence that we denote $f(X)$. The set $f(X) \cup \{\infty\}$ is dense in H^*. It follows that H^* is c.e. closed. To conclude that H^* is computably closed, we need to show that it is Π^0_1, i.e., its complement is c.e. open. By Theorem 3.4, f^{-1} is computable when defined. Since f is a homeomorphism of G onto its image in $G \cup \{\infty\}$ (and in particular is injective), this is equivalent to saying that f is effectively open. Since f is effectively open, it maps the c.e. complement of H (in G) to a c.e. open set. But this set is the complement of H^* in G^*. Thus, H^* is computable closed in G^*, and (2) follows from (1).

Now assume (2). As noted above, (2) is equivalent to saying that H^* is computable closed in G^*. Recall that f^{-1} is computable (and defined) everywhere except ∞. Since ∞ is a special point in G^*, we can effectively list those points in the computable dense sequence of H that are not equal to ∞. The f^{-1}-images of these points make up an effective dense set of H in G. It shows that H is c.e. closed. Let U be the c.e. complement of H^* in $\mathcal{K}(G^*)$; note $\infty \notin U$. Since f is computable it is effectively continuous, and thus $f^{-1}(U)$ is c.e. open in G. Since we have $H = G \setminus f^{-1}(U)$, it follows that H is both c.e. and Π^0_1, and therefore computably closed. □

We view a computable discrete group (in the sense of Rabin and Malcev) as a computable Polish group w.r.t. the discrete metric:

$$d(x, y) = 1 \text{ whenever } x \neq y.$$

Any such computable discrete group is clearly computably locally compact. Furthermore, a subset of such a group is c.e. iff it is c.e. as a closed or open subset of G. In particular, we have:

Fact 5.3. For a discrete computable G, a subgroup $H \leq G$ is a computable subgroup iff it is computably closed w.r.t. the discrete metric defined above.

By Fact 5.3 above, in the discrete case we can 'forget' about topology and simply work in the standard, discrete recursion-theoretic setting. This approach is taken in the next section where computable topology is not mentioned at all.

5.2. **Computably simple groups.** All groups in this subsection are discrete, abelian, and at most countable. A lot is known about computable presentations of such groups, however, the definition below is new. (For the foundations of computable abelian group theory, we cite the surveys [Khi98, Mel14].)

Definition 5.4. A computable group is *computably simple* if it has no non-trivial computable proper normal subgroups.

Since all our groups are abelian, all subgroups are trivially normal.

Theorem 5.5. *There exists a computably simple presentation of the free abelian group of rank ω.*

Proof. We build a computable presentation H of the free abelian group

$$A = \bigoplus_{i\in\omega} \mathbb{Z}$$

of countably infinite rank. We view it as a module over $\mathbb{Z}$. The 'natural' computable copy of the group A, which we identify with A, is (freely) spanned by the computable sequence $(b_i)_{i\in\omega}$ over $\mathbb{Z}$. At every stage we will have only a finite part H_s of the group H enumerated. Initially, we will just copy A into H, unless we have to diagonalise. At some stage we may want to declare some of the b_j dependent on $b_i, i < j$, using large coefficients never seen before that stage; the exact choice will depend on our strategy. (Equivalently, we introduce a relation on the generators; for instance, it could be of the form $mb_i = b_k$ for a large $m \in \mathbb{N}$ and $k < i$. Our construction can be reformulated in terms of building a computable subgroup X of A and then setting $B = A/X$, but this point of view would not be too helpful.) This way we will build the not-so-natural copy $H = \bigcup_s H_s$ of the group $A = \bigoplus_{i\in\omega} \mathbb{Z}$ that is not computably simple.

5.2.1. *The requirements.* Fix the uniformly effective enumeration $(W_j)_{j\in\omega}$ of all computably enumerable (c.e.) sets. We identify elements of H with their $\mathbb{N}$-indices. Every potential computable subgroup will be represented by a pair (W_e, W_j) of c.e. sets, where W_e is thought of as listing the subgroup and W_j its complement. The requirements are:

$R_{e,j}$: (W_e, W_j) does not represent a proper subgroup of H.

For (W_e, W_j) to represent a proper subgroup, both W_e and W_j have to contain non-zero elements, and W_e has to be closed under $+$ and $-$.

5.2.2. *The strategy for $R_{e,j}$.* Recall W_e is a potential subgroup and W_j is its complement. We shall assume that at every stage s, $W_{e,s}$ is closed under the group operations (whenever they are defined) in H_s, otherwise we do not have to worry about W_e at all. Recall that we initially begin building B to be freely generated by $(b_i)_{i\in\omega}$, but later we may opt to make the current value of some of these b_i linearly dependent.

The set W_e does not have to contain (the potential) basic elements b_i. However, since the group is free abelian, at least one such b_i has to be put into W_j (which is supposed to list the complement), for otherwise W_e will have to be equal to the whole of H, and (W_e, W_j) will not represent a proper subgroup.

In the strategy below, we say that an integer k is 'large' at stage s if $k > s$, and thus, it is larger than any parameter seen so far in the construction. We identify elements of H with their $\mathbb{N}$-indices, and we also fix some computable bijection $\langle \cdot, \cdot \rangle : \mathbb{N}^2 \to \mathbb{N}$. The strategy is as follows.

Suppose the strategy becomes active at stage t the first time after its initialization (to be clarified).

(1) Wait for some b_v to be put W_j and some non-zero x enumerated into W_e.
(2) Wait for b_w, where $w > \max\{x, v, 2^{\langle e,j \rangle}, t\}$ to be put in either W_e or W_j.
(3) If $b_w \in W_j$, then declare $b_w = mx$ for some large m.
(4) If $b_w \in W_e$ then:
  (4.a) Declare $b_w = mb_v$ for a large integer m.
  (4.b) Wait for some b_u, where $u > w$, to be put in either W_e or W_j.
  (4.c) If $b_u \in W_j$ then declare $b_u = kx$ for a large integer k.
  (4.d) If $b_u \in W_e$ then declare $b_u = qb_v$ for a large prime $q > m$.

5.2.3. *The verification of one strategy in isolation.* We argue that the strategy guarantees that the pair (W_e, W_j) cannot represent a proper subgroup of H. As we explained above, if we never find b_v and x in (1) then (W_e, W_j) cannot represent a proper subgroup, and thus we do not have to worry about $R_{e,j}$. Also, our search in (2) must eventually terminate, otherwise $H \neq W_e \cup W_j$. If $b_w \in W_j$, then we act in (3) and set $b_w = mx$. But $x \in W_e$, so either W_e is not a subgroup or W_j is not its complement. If $b_v \in W_j$, then our search for b_u in (4.b) has to terminate, otherwise $H \neq W_e \cup W_j$. If $b_u \in W_j$ then we declare $b_u = kx$ in (4.c) for a large integer k. In this case, just as in (3) explained above, we have that either $W_e \ni x$ is not a subgroup or $W_j \neq H \setminus W_e$. If $b_u \in W_e$, then in (4.d) we declare $b_u = qb_v$ for a large prime $q > m$. In particular, $(q, m) = 1$. Recall earlier in (4.a) we set $b_w = mb_v$. We have that $b_w, b_u \in W_e$, and $b_w = mb_v$ together with $b_u = qb_v$ $\mathbb{Z}$-generate b_v because $(q, m) = 1$. If W_e were a subgroup, it would imply $b_v \in W_e$. However, $b_v \in W_j$ (see (1)), and therefore either $W_e \neq H \setminus W_j$ or W_e is not a subgroup.

5.2.4. *Putting the strategies together. Movable markers.* In presence of many strategies we will need to redefine the interpretation of b_i whose values in H have been declared dependent over some other b_j, $j < i$. Then we later argue that this process of re-defining b_i has to eventually settle for each i. In the limit, the final values of b_i will indeed freely generate the group. Of course, in presence of only one strategy this was not necessary.

We arrange the strategies into a priority order, with $R_{e,j}$ declared higher priority than $R_{e',j'}$ if $\langle e, j \rangle < \langle e', j' \rangle$. Additionally, in the construction we shall redefine the interpretation of the parameters b_i in a way that at the end of every stage these parameters correspond to linearly independent elements of H_s that generate H_s freely. We write $b_{i,s}$ to denote the value of b_i at the beginning of stage s. Think of b_i as being a 'movable marker' $\boxed{\text{i}}$ that at every stage is placed at some element of H. Then $b_{i,s}$ is the element h of H_s so that currently $\boxed{\text{i}}$ is placed at h. If $b_{i,s+1} \neq b_{i,s}$ then we have 'moved' the 'marker'. We usually suppress s in $b_{i,s}$ if it is clear that we mean the value of b_i at some stage, where the exact index of the stage is not important or can be reconstructed from the context. We now explain how exactly we move the markers.

At every stage s, at most one strategy will act according to its instructions. Assume some b_j (where $j = u$ or w in the notation of the strategy) is declared to be inside the $\mathbb{Z}$-span of some other elements b_i, $i \neq j$. In this case we say that the strategy *acts*. It must be that we declare $b_{j,s} \in Span_{\mathbb{Z}}(b_{0,s}, \dots, b_{j-1,s})$ because the strategy always chooses its witnesses b_u and b_w to be large. In particular, we can assume that x and b_v are so that $x = \sum_{i<j} m_i b_{i,s}$ and $v < j$. After the relation witnessing $b_{j,s} \in Span_{\mathbb{Z}}(b_{0,s}, \dots, b_{j-1,s})$ is introduced, also declare

$$b_{k,s+1} = \begin{cases} b_{k,s} \text{ when } k < j \\ b_{k+1,s} \text{ when } k \geq i. \end{cases}$$

Because strategies of higher priority will have their parameters x and b_v too small compared to j, they will be in $Span_{\mathbb{Z}}(b_0, \dots, b_{j-1})$ and therefore will not have to be updated. We also *initialize* all weaker priority strategies by setting all their parameters undefined.

Before we proceed to the construction, we also mention that some other requirement R' could be accidentally met because of the actions of the R-strategy. We make this possibility implicit. In this case we do not have to do anything to meet R'.

5.2.5. *The construction.* We build $H \cong A$ in stages. At every stage,

$$H_s = \bigoplus_{i \leq s} \mathbb{Z} \restriction_s b_{i,s}.$$

At stage 0, we initialize all strategies by setting all their parameters undefined.

At stage s, we let the highest priority instruction act according to its instructions, but only if there is a strategy $R_{e,j}$ with $\langle e, j\rangle \leq s$ that needs to act. Also, redefine the values of $b_{i,s}$ explained in the previous subsection. In any case, let each $R_{e,j}$ with $\langle e, j\rangle \leq s$ do s more steps in their search for suitable parameters (unless they are already defined) according to their instructions. If any of these strategies needs to act, delay this until the next stage.

5.2.6. *The verification.* By induction, every strategy can be initialized only finitely many times, and also that $\lim_s b_{i,s}$ exists, i.e., the sequence stabilizes. Consider the strategy corresponding to $R_{e,j}$. Suppose it becomes active at stage t the first time after its last initialization, and suppose it will never be initialized again. It will then choose its parameters b_u, b_w large, and the same is true for any strategy corresponding to a $R_{e',j'}$ having its priority weaker than $R_{e,j}$. Consequently, the values of b_j for $j < t$ will remain unchanged after stage t. If the strategy for $R_{e,j}$ will eventually act, then requirement for $R_{e,j}$ will be met as explained in Subsection 5.2.3. Otherwise, if it never acts, $R_{e,j}$ is also met because either W_e is not closed under the group operations or the strategy fails to find its parameters, or perhaps it will be accidentally met due to other strategies of weaker priority acting. In any case, the strategy will act only once, and the requirement will be met. Note that we simultaneously illustrated that $\lim_s b_{i,s} = b_{i,t}$ for all $j < t$. The group H is obviously computable, and since each b_i eventually settles, it is also free abelian upon the generating set $(\lim_s b_{i,s})_{i \in \omega}$. □

5.3. **Finalizing the proof of Theorem 1.1(2).** By Theorem 5.5, there exists a computable presentation of the free abelian group of countably infinite rank that is computably simple. Clearly, it has uncountably many proper subgroups. Fact 5.3 implies that the group can be viewed a computably locally compact that has no non-trivial computable closed subgroups. By Lemma 5.1, its Chabauty space $\mathcal{S}(H)$

will have uncountably many points, but only two computable points, specifically H^* and $\{0\}^*$. Thus, $\mathcal{S}(H)$ does not have a computable dense sequence (in $\mathcal{K}(H^*)$).

6. The Chabauty space of a t.d.l.c. group via its meet groupoid

In this section we establish an effective correspondence between the Chabauty space $\mathcal{S}(G)$ of G and the meet groupoid $\mathcal{W}(G)$ of all compact open subsets of a totally disconnected locally compact (t.d.l.c.) group G, which is defined below.

Recall that a groupoid is given by a domain $\mathcal{W}$ on which a unary operation $(.)^{-1}$ and a partial binary operation, denoted by "$\cdot$", are defined. These operations satisfy the following conditions:

(a) $(A \cdot B) \cdot C = A \cdot (B \cdot C)$, with either both sides or no side defined;
(b) $A \cdot A^{-1}$ and $A^{-1} \cdot A$ are always defined;
(c) if $A \cdot B$ is defined then $A \cdot B \cdot B^{-1} = A$ and $A^{-1} \cdot A \cdot B = B$.

A *meet groupoid* [MN22] is a groupoid $(\mathcal{W}, \cdot, (.)^{-1})$ that is also a meet semilattice $(\mathcal{W}, \cap, \emptyset)$ of which $\emptyset$ is the least element. Writing $A \subseteq B \Leftrightarrow A \cap B = A$ and letting the operation $\cdot$ have preference over $\cap$, it satisfies the conditions

(d) $\emptyset^{-1} = \emptyset = \emptyset \cdot \emptyset$, and $\emptyset \cdot A$ and $A \cdot \emptyset$ are undefined for each $A \neq \emptyset$,
(e) if U, V are idempotents such that $U, V \neq \emptyset$, then $U \cap V \neq \emptyset$,
(f) $A \subseteq B \Leftrightarrow A^{-1} \subseteq B^{-1}$, and
(g) if $A_i \cdot B_i$ are defined $(i = 0, 1)$ and $A_0 \cap A_1 \neq \emptyset \neq B_0 \cap B_1$, then
$$(A_0 \cap A_1) \cdot (B_0 \cap B_1) = A_0 \cdot B_0 \cap A_1 \cdot B_1.$$

Definition 6.1. [MN22] Let G be a t.d.l.c. group. We define the meet groupoid $\mathcal{W}(G)$ of G, as follows. Its domain consists of the compact open cosets in G (i.e., cosets of compact open subgroups of G), as well as the empty set. We define $A \cdot B$ to be the usual product AB in case that $A = B = \emptyset$, or A is a left coset of a subgroup V and B is a right coset of V; otherwise $A \cdot B$ is undefined.

Notice that the groupoid defined above also carries a natural partial order $\subseteq$ of set-theoretic inclusion under which it is also forms a meet semilattice, hence the name. It has been established in [MN22] that a duality holds between t.d.l.c. groups and their respective meet groupoids. This duality is also fully effective in the sense that it gives a computable 1-1 correspondence between the respective computable meet groupoids and computably t.d.l.c. groups.

6.1. Computable setting.

A separable t.d.l.c. group is homeomorphic to a locally compact subspace of ω^ω. It therefore makes sense to represent such groups using trees. Our (rooted) trees are viewed as sets of strings in $\omega^{<\omega}$ closed under prefixes. Finite strings can be viewed as 'nodes' of the tree. A tree has no dead ends if every finite string on the tree is extendible to an infinite path through the tree. The space of paths through a tree T is denoted $[T]$. The space T is a metric space under the shortest common initial segment ultrametric. A computable tree T with no dead ends evidently induces a computable Polish presentation on $[T]$.

Definition 6.2. We say that a computable tree is *nicely* computably locally compact if it has no dead ends, only its root can be ω-branching, and given a node one can compute the number of its successors.

Clearly this implies that $[T]$ is a computably locally compact space.

Definition 6.3 ([MN22])**.** Let G be a Polish t.d.l.c. group. A *computable Baire presentation* of G is a topological group $\widehat{G} \cong G$ of the form $\widehat{G} = ([T], \cdot, ^{-1})$ such that

(1) T is a nicely effectively locally compact tree;
(2) $\cdot\colon [T] \times [T] \to [T]$ and $^{-1}\colon [T] \to [T]$ are computable (as operators).

We have already mentioned the following result:

Theorem 6.4 (M. and Ng [MN23])**.** *For a t.d.l.c. G, the following are equivalent:*

(1) *G has a computably locally compact presentation, and*
(2) *G has a computable Baire presentation.*

The implication (2) $\to$ (1) in the theorem above is of course obvious. It is important for us that (1) $\to$ (2) is witnessed by a computable Baire presentation of G that is *computably homeomorphic* to the given computably locally compact copy of the group. (Recall this in particular means both the homeomorphism f and its inverse f^{-1} are computable.) Thus, the closed subgroups of these presentations will be in a 1-1 effective correspondence induced by this effective homeomorphism. It follows that, in the special case of t.d.l.c. Polish groups we can use the rather convenient Definition 6.2 above in place of the (seemingly) more general notion of a computably locally compact presentation. In particular, for a fixed computable tree with no dead ends, the notions of a computable open and closed sets become rather explicit:

Definition 6.5. An open subset R of $[T]$ is called computable if $\{\sigma \in T\colon\ [\sigma] \subseteq R\}$ is computable. A closed subset S of $[T]$ is called computable if $[T]-S$ is computable. Equivalently, the subtree $\{\sigma \in T\colon\ [\sigma] \cap H \neq \emptyset\}$ corresponding to S is computable.

For the dual meet groupoid, we shall use the following:

Definition 6.6. [MN22] A meet groupoid $\mathcal{W}$ is called *Haar computable* if

(a) its domain is (indexed by) a computable subset D of $\mathbb{N}$;
(b) the groupoid and meet operations are computable; in particular, the relation $\{\langle x, y\rangle\colon\ x, y \in D \wedge x \cdot y \text{ is defined}\}$ is computable;
(c) the partial function with domain contained in $D \times D$ sending a pair of subgroups $U, V \in \mathcal{W}$ to $|U : U \cap V|$ is computable.

The key result relating the definitions above is the following:

Theorem 6.7 ([MN22])**.** *A group G has a computable Baire presentation if, and only if, its meet groupoid $\mathcal{W}(G)$ has a Haar computable copy.*

This effective duality result also enjoys a number of further effective properties. For instance, the elements of the Haar computable copy of $\mathcal{W}(G)$ produced based on a Baire presentation of G are in an effective 1-1 correspondence with computable compact open cosets of the respective computable Baire presentation, and vice versa. Note however that $\mathcal{W}(G)$ captures only *compact* open subgroups. What about closed subgroups of G? How are they reflected in $\mathcal{W}(G)$? We answer this question (non-effectively) in the following section, and then in the subsequent section we establish an effective correspondence in order to prove Theorem 1.2.

6.2. **Ideals in a meet groupoid.** To this end, G will always denote a t.d.l.c. group, and $\mathcal{W} = \mathcal{W}(G)$ its meet groupoid. Recall that the compact open cosets form a basis for the topology of G. For $A, B_1, \ldots, B_n \in \mathcal{W}$, the relation $A \subseteq \bigcup_i B_i$ is first-order definable in $\mathcal{W}$, because its negation is given by the formula $\exists C \subseteq A \bigwedge_i [C \cap B_i = \emptyset]$. So $\mathcal{W}$ determines a set of ideals in the following sense.

Definition 6.8. We say that a set $\mathcal{J} \subseteq \mathcal{W}$ is an *ideal* of $\mathcal{W}$ if $\emptyset \in \mathcal{W}$ and

$$(B_1, \ldots, B_n \in \mathcal{J} \ \wedge\ A \subseteq \textstyle\bigcup_i B_i) \Rightarrow A \in \mathcal{J}.$$

Similar to Stone duality between Boolean (Stone) spaces and Boolean algebras, open subsets R of G naturally correspond to ideals $\mathcal{J}$ of $\mathcal{W}$ via the maps

$$R \mapsto \{A\colon\ A \subseteq R\} \text{ and } \mathcal{J} \mapsto \bigcup \mathcal{J}.$$

We also write $S_{\mathcal{J}} = G - \bigcup \mathcal{J}$, which is a closed set in G.

Definition 6.9. Let $\mathcal{J}$ be an ideal closed under inversion $A \mapsto A^{-1}$.
(i) We say that $\mathcal{J}$ is an *open-subgroup ideal* if

$$A, B \in \mathcal{J} \ \wedge\ A \cdot B \text{ defined} \Rightarrow A \cdot B \in \mathcal{J}.$$

(ii) We say that $\mathcal{J}$ is a *closed-subgroup ideal* if

$$A \cdot B \text{ defined } \wedge A \cdot B \in \mathcal{J} \Rightarrow A \in \mathcal{J} \vee B \in \mathcal{J}.$$

It is immediate that $\mathcal{J}$ is open-subgroup ideal $\Leftrightarrow$ $R_{\mathcal{J}}$ is an (open) subgroup of G. Note that the set $\mathcal{CSI}(\mathcal{W})$ of closed-subgroup ideals is closed in the product topology on $\mathcal{P}(\mathcal{W})$.

Lemma 6.10. *$\mathcal{J}$ is closed-subgroup ideal $\Leftrightarrow$ $S = S_{\mathcal{J}}$ is a closed subgroup of G.*

Proof. $\Leftarrow$: Immediate.
$\Rightarrow$: Let $g, h \in G$. If $g^{-1} \notin S$ then there is $A \in \mathcal{J}$ such that $g^{-1} \in A$. Since $\mathcal{J}$ is closed under inversion, this implies $g \notin S$.

If $g, h \in S$ but $gh \notin S$ then pick $C \in \mathcal{J}$ such that $gh \in C$. By continuity there are $A, B \in \mathcal{W}$ such that $g \in A, h \in B$ and $AB \subseteq C$ (the product of subsets in G). Let A be left coset of U and B be right coset of V. Let $L = U \cap V$. Replacing A by $\widehat{A} = gL$ and B by $\widehat{B} = Lh$, we have $\widehat{A} \cdot \widehat{B} \subseteq C$ and hence $\widehat{A} \cdot \widehat{B} \in \mathcal{J}$. But $\widehat{A} \cap S \neq \emptyset \neq \widehat{B} \cap S$, so neither $\widehat{A}$ nor $\widehat{B}$ are in $\mathcal{J}$, contrary to the assumption that $\mathcal{J}$ is a closed-subgroup ideal. ☐

We now look at the Chabauty space of a t.d.l.c. G. Using the notation of [Cor11, Section 2], a basic open set in $\mathcal{S}(G)$ has the form

$$\Omega(K; R_1, \ldots, R_n) = \{U \leq_c G\colon U \cap K = \emptyset \wedge \forall i \leq n\, U \cap R_i \neq \emptyset\}, \tag{1}$$

where $K \subseteq G$ is compact, and the $R_i \subseteq G$ are open. (If G is t.d.l.c. then $\mathcal{S}(G)$ is also totally disconnected. So unless $\mathcal{S}(G)$ has isolated points, it will be homeomorphic to Cantor space. We cite Cornulier [Cor11] for further background.)

Proposition 6.11. *The map $\Gamma\colon \mathcal{CSI}(\mathcal{W}) \to \mathcal{S}(G)$, given by*

$$\mathcal{J} \mapsto S_{\mathcal{J}} = G - \bigcup \mathcal{J},$$

is a continuous bijection of compact spaces, and hence a homeomorphism.

Proof. Fix a closed-subgroup ideal $\mathcal{J}$. To show that Γ is continuous at $\mathcal{J}$, suppose that $S = \Gamma(\mathcal{J}) \in \Omega(K; R_1, \ldots, R_n)$, a basic open set as defined in (1). Since the compact open cosets form a basis of the topology of G, we may assume that each R_i is a compact open coset. Since $K \subseteq G - S$ which is open, there are compact open cosets $B_1, \dots, B_m$ such that $K \subseteq \bigcup_k B_k \subseteq G - S$. Let $\mathfrak{L}$ be the basic open set of $\mathcal{CSI}(\mathcal{W})$ consisting of the subgroup ideals $\mathcal{A}$ such that $B_k \in \mathcal{A}$ for each $k \leq m$, and $R_i \notin \mathcal{A}$ for each $i \leq n$. Clearly $\mathcal{J} \in \mathcal{A}$, and $\mathcal{H} \in \mathfrak{L}$ implies $\Gamma(H) \in \Omega(K; R_1, \dots, R_n)$. ☐

6.3. **Concluding the proof of Theorem 1.2.** Recall that (1) $\leftrightarrow$ (2) of Theorem 1.2 was already verified in Lemma 5.1. To establish (1) $\leftrightarrow$ (3), we first prove the following lemma.

Lemma 6.12. *Suppose that a t.d.l.c. G is given as a computable Baire presentation based on a tree T, and let $\mathcal{W}$ denote the corresponding Haar computable copy of $\mathcal{W}(G)$. For an ideal $\mathcal{J} \subseteq \mathcal{W}$,*

$$\mathcal{J} \text{ is computable} \Leftrightarrow \text{the open set } R_{\mathcal{J}} \subseteq [T] \text{ is computable.}$$

Proof. As we already noted after Theorem 6.7, elements of the groupoid are in a 1-1 effective correspondence with the respective cosets in the group. We elaborate how exactly this leads to the claimed effective correspondence between $\mathcal{J}$ and $R_{\mathcal{J}} \subseteq [T]$, and give references to the technical claims in [MN22] that we use.
$\Rightarrow$: Given $\sigma \in T$, we may assume that $length(\sigma) > 0$, and thus $[\sigma]_T$ is compact. Hence by [MN22, Lemma 2.6] one can compute $A_1, \dots, A_n \in \mathcal{W}$ such that $\bigcup_i A_i = [\sigma]$. Then $[\sigma] \subseteq R$ iff $A_i \in \mathcal{J}$ for each i.
$\Leftarrow$: By the definition of $\mathcal{W}$ from the computable Baire presentation based on the tree T, each $A \in \mathcal{W}$ is given in the form $\mathcal{K}_u = \bigcup_{\eta \in u} [\eta]_T$ (as in [MN22, Def. 2.5]) where u encodes a finite set of nonempty strings on T. Then $A \in \mathcal{J}$ iff $[\eta] \subseteq R$ for each η in u. ☐

Corollary 6.13. *In the notation of the above, it follows that an open-subgroup ideal [closed-subgroup ideal] $\mathcal{J}$ is computable iff $R_{\mathcal{J}}$ [resp., $S_{\mathcal{J}}$] is computable.*

By Theorem 6.4 and the discussion after it, given a computably locally compact presentation C of a t.d.l.c. G we can produce a computable Baire presentation of the group via a tree T and a computable group-homeomorphism $f : C \to [T]$ with computable inverse. By Lemma 5.1, computable closed S_j correspond to computable points in the effective closed presentation of $\mathcal{S}(G)$ based on $[T]$ or, equivalently, based on C. Let $\mathcal{W}$ be the Haar computable presentation of the dual meet groupoid given by Theorem 6.7. Lemma 6.10 illustrates that closed-subgroup ideals $\mathcal{J}$ are in 1-1 correspondence with closed subgroups $S_{\mathcal{J}}$ of G. By the corollary above, the *computable* closed-subgroup ideals are in a 1-1 correspondence with computable closed subgroups $S_{\mathcal{J}}$ in the presentation given by $[T]$. This is also clearly uniform, and so is the proof of Lemma 5.1. This gives (1) $\leftrightarrow$ (3) of Theorem 1.2.

Remark 6.14. There is nothing special about computability of points and ideals in the proof above. Indeed, we can computably uniformly transform (names of) X-computable points into X-computable ideals, and vice versa. Thus, as we already pointed out in the introduction, the proof above really gives a 'computable homeomorphism' between $\mathcal{CSI}(\mathcal{W})$ and $\mathcal{S}(G)$. However, formalising this statement would involve various notions that we did not define in the present paper. This is because neither of these objects is a computable Polish space. Thus, we leave the exact formulation of the more general fact (and the formal verification of it) as a strong conjecture.

7. Complexity of index sets

Recall from Def. 2.14 that an *X-computably compact structure* on a computable Polish space is an X-computable functional representing the procedure $N^x \to K \supseteq B \ni x$ that works for any point x of the space. Recall that Theorem 1.3 states that

$$\{i : G_i \text{ is a properly metrized abelian group and } \mathbb{R} \leq_c G_i\}$$

is arithmetical.

7.1. Proof of Theorem 1.3.

We have already discussed the following fact in the introduction.

Proposition 7.1. *The index set of properly metrized Polish groups is arithmetical*[3].

Proof. Fix $G = G_i$ given by a computable Polish space and two operators acting on the space. Since we have that

$$\overline{B(x,r)} = cl\{y : d(x,y) < r\} \subseteq B^c(x,r) = \{y : d(x,y) \leq r\} \subseteq \overline{B(x,r')}$$

whenever $r' > r$, G is properly metrized iff $\overline{B(x,r)}$ is compact, for all x, r. First, view G as a computable Polish space ignoring the group operations; the computable index of the space can be computably reconstructed from i. We claim that it is arithmetical to tell that for every special x and each positive $r \in \mathbb{Q}$, the closure $\overline{B(x,r)}$ of $B(x,r)$ is compact. The (c.e.) collection of special points in $B(x,r) = \{y : d(x,y) < r\}$ makes $\overline{B(x,r)}$ a computably Polish space, uniformly in x, r. It is Π^0_3 to tell whether a given computable Polish space is compact; see [MN13].

Fact 7.2. Every properly metrized Polish space admits a $0'$-computably locally compact structure, and this is uniform.

Proof. Since every compact Polish space is $0'$-computably compact (and this is uniform), the basic balls and their closures induce a $0'$-computably locally compact ($0'$-c.l.c.) structure on the space. ☐

Assume G is properly metrized (as a space). Using Fact 7.2, fix $0'$-c.l.c. structure on the domain of the group. We claim that it is arithmetical to tell whether the functionals representing the (potential) group operations on a $0'$-c.l.c. group are well-defined (total). This is because they are total if, and only if, for every n their restrictions to K_n are total, where the K_n are the compact sets from Lemma 3.2 (relativized to $0'$) making up the $0'$-c.l.c. G. The totality on each K_n is uniformly Π^0_3; this is essentially the aforementioned [Mel18, Lemma 4.2]. Thus, we conclude that totality of the (potential) group operation is an arithmetical property, uniformly in the index of G.

Assuming that the operations are witnessed by total functionals, the collection of all points that satisfy the group axioms form an effectively closed set. Thus, if there are some points that fail the axioms, then there are special points that fail the axioms. It follows that the group axioms can be tested on special points, which makes the test (uniformly) Π^0_1 in the index of the group and the names of the group operations. ☐

[3]The bound in Π^0_3, and it follows from the proof of [Mel18, Proposition 4.1] that this is indeed Π^0_3-complete within the class of totally disconnected groups. The same upper bound holds for the index set of properly metrized Polish spaces, and this is also complete (via the same proof).

Remark 7.3. In view of the effective proper metrization result Proposition 3.3, the proposition above allows to apply the index set approach to the class of computable compact groups in a meaningful way. Also, it contrasts greatly with the Π^1_1-completeness of being locally compact Polish space proved by Nies and Solecki [NS15].

We return to the proof of Theorem 1.3. Of course, being abelian is a Π^0_1-property and can be tested on special points as well. In summary, using several Turing jumps we can ensure that $G = G_i$ is indeed a properly metrized (thus, locally compact) abelian group. Recall that we can arithmetically check whether it is compact [MN13]. (If G_i is compact, obviously it cannot contain $\mathbb{R}$ as a closed subgroup.) Thus, we can assume G_i is not compact. By Lemma 2.18, $G = G_i$ admits a $0'$-computable locally compact structure whose index can be uniformly recovered form i using $0'$.

Using Theorem 3.4 relativized to $0'$, uniformly pass to its $0'$-computably compact 1-point compactification G^*_i, and uniformly effectively define the $0'$-computably compact hyperspace $\mathcal{K}(G^*)$. Using Theorem 1.1, uniformly produce the $\Pi^0_1(0')$-presentation of $\mathcal{S}(G)$ inside $\mathcal{K}(G^*)$. By Lemma 2.13, $\mathcal{S}(G)$ has a $0''$-computably compact presentation.

We need the following:

Lemma 7.4 (E.g., [Mel18, DM23])**.** *It is arithmetical (Π^0_1) to tell whether a given computably compact set is connected.*

Proof. By the elementary [DM23, Lemma 4.21], we can effectively list all clopen non-trivial splits of the space. Thus, we simply need to state that such a split will never be found, which is Π^0_1. □

Relativising the lemma above to $0''$, we conclude that the following is an arithmetical (Π^0_3) property:

$$\mathcal{S}(G) \text{ is connected.}$$

In summary, it is arithmetical (Π^0_3) to tell whether G_i is an abelian properly metrized group so that its Chabauty space $\mathcal{S}(G)$ is connected. To finish the proof of the theorem, apply the following result is due to Protasov and Tsybenko [PT83]: *Suppose G is a locally compact group. If its Chabauty space $\mathcal{S}(G)$ is connected, then some subgroup of G is topologically isomorphic to $(\mathbb{R}, +)$. For abelian groups, the converse holds.*

Remark 7.5. To establish Π^0_3-completeness, take the effective sequence (C_i) totally disconnected abelian groups from [Mel18, Proposition 4.1] which indeed witness the Π^0_3-completeness in Proposition 7.1, and consider $C_i \oplus \mathbb{R}$.

8. A few open questions

In Theorem 1.1 we constructed a Π^0_1 presentation of the Chabauty space of a computably locally compact group, and we argued that this presentation is (effectively) natural. However, following the general pattern of computable mathematics, we wonder if this presentation is (in some sense) computably unique for some groups.

Question 8.1. Can we describe those groups G for which $\mathcal{S}(G)$ is *computably unique*?

The computable uniqueness here should perhaps be interpreted using homeomorphisms that are (in some sense) effective; we are not sure what the right effectiveness notion would be in the context.

In Theorem 5.5 we constructed a *fixed* computable presentation H of the free abelian group $\mathbb{Z}^{<\omega}$ so that $\mathcal{S}(H)$ is not computable in $\mathcal{K}(H^*)$. Clearly, the 'natural' computable presentation of this group does not have this pathological property. Thus, we leave open:

Question 8.2. Is there a computable discrete group G so that for *any* computable presentation H of G, $\mathcal{S}(H)$ is not computable in $\mathcal{K}(H^*)$?

Recall that Theorem 5.5 established that there exists a computable group that is not simple but has no computable proper normal subgroups. We called such groups 'computably simple'. Can this pathological property hold in *any* computable presentation of some (non-simple) group? It is not difficult to see that, if such a group exists, it cannot be abelian. In the class of abelian groups, the following similar question can prove to be challenging.

Question 8.3. Can we describe computable discrete abelian groups that admit computably simple presentations?

We used discrete groups as a tool, but clearly such investigations do not have to be restricted to discrete groups. For instance, we wonder whether it is possible to produce examples similar to Theorem 1.1(2) among, say, profinite groups.

Recall that we relied on proper metric in our proofs, and that in Proposition 3.3 we showed that every computably compact Polish space admits an effectively equivalent proper metric. Struble [Str74] showed that every locally compact Polish group admits a proper left- (or right-) invariant metric; see also [HP06]. It has been shown in [KMK23] that, with much extra work, the result can be effectivized for computably locally compact groups, but it produces a right-c.e. metric in general. (Interestingly, the right-c.e. space is effectively locally compact in the same sense as Π^0_1-classes in 2^ω are effectively compact; see [KMK23].) It is currently open whether a computably locally compact group admits a proper left-invariant metric that would also be computable.

We move on to index sets. Recall that every locally compact Polish group admits a proper (left- or right-invariant) metric; see also [HP06], and this result has been effectivized in [KMK23]. Combined with these results, Proposition 7.1 entails that various index set questions can be attacked in a meaningful way for the class of locally compact groups using proper metric presentations. For example, we conjecture that the index set of properly metrized totally disconnected groups is arithmetical. Further properties of $\mathcal{S}(G)$ (e.g., [Cor11]) could be of significant assistance.

We also wonder whether "Pontryagin-Chabauty duality" established in [Cor11] is computable. Various effective dualities have recently been established in [BHTM21, LMN21, DM23, HTMN20, HKS20, MN23].

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