COMPUTABLY COMPACT METRIC SPACES

RODNEY G. DOWNEY AND ALEXANDER G. MELNIKOV

ABSTRACT. We give a systematic technical exposition of the foundations of the theory of computably compact metric spaces. We discover several new characterizations of computable compactness and apply these characterizations to prove new results in computable analysis and effective topology. We also apply the technique of computable compactness to give new and less combinatorially involved proofs of known results from the literature. Some of these results do not have computable compactness or compact spaces in their statements, and thus these applications are not necessarily direct or expected.

CONTENTS

1. INTRODUCTION

1.1. **Compactness.** Compactness plays a central role in classical analysis. We don't have space to talk about all the applications of compactness in analysis, but refer the reader to the survey [62] for a detailed discussion. Compactness allows us to have an intrinsic connection between the infinite and the discrete finite instances of problems. In the words of Hewitt [62],

> "A great many propositions of analysis are:
>   - trivial for finite sets;
>   - true and reasonably simple for infinite compact sets;
>   - either false of extremely difficult to prove for noncompact sets."

If the reader thinks about any basic course in analysis, they will be struck about how many elementary theorems about real analysis rely on compactness in some form or another. When we turn to analysis on more general spaces, again compactness plays a central role. Classically, compact metric spaces are perhaps the most well-understood separable spaces after the discrete ones. Metrizable compact manifolds and compact topological groups have been studied extensively.

1.2. **Goals.** The concern of this paper is computable analysis; specifically computability aspects of Polish spaces (complete separable metric spaces). In this paper we will hope to achieve three goals.

- We will give a unified and smooth account of uses of compactness in computable analysis. This will involve the *unification* of a number of disparate approaches offered by many authors down through the years. We will describe and improve the fundamental techniques associated with computable compactness that are scattered throughout the literature.
- We will apply this machinery to prove several new results. This includes a characterization of recursive profinite groups in terms of computable compactness and a new computation of Čech cohomology. We will also give new simplified proofs of some known results (more details below).
- We will offer an answer to the following question:
     "What is the 'correct' notion of computability for a compact Polish(able) topological space?"
  In contrast with the situation for computable *discrete* algebra, the situation for Polish spaces is not clear *especially if we are willing to view them up to homeomorphism.*

1.3. **Historical context.** The roots of computable analysis go back to the early 20th century; see, e.g., Borel [12]. In his seminal papers [144, 145], Alan Turing clarified this early intuition. He gave the first universally accepted formal definition of a computable function. Turing used this definition to solve the Hilbert's *Entscheidungsproblem.* But Turing also introduced computable analysis on $[0, 1]$. He analysed computable functions on the field of computable real numbers. He defined a real ξ to be computable if there is a Turing machine that, on input i, outputs a rational $r = \frac{m}{n}$ such that $|\xi - r| < 2^{-i}$. This approach was pursued especially in Russia by Markov and his school, culminating in Aberth's book [1]. Strangely, Turing's definition of a computable function (i.e. on the computable reals) is now usually referred to as *Markov computability.* Our paper lies in the tradition of what has become known as "type 2" computable analysis. This tradition goes back to the work of Grzegorczyk [57–59] and Kleene [81]. In this approach, we view effective functions as computable *operators* that are not restricted to computable reals and work for arbitrary reals. Avigad and Brattka [4] give an excellent overview of the development of computable analysis from the work of Turing.

The majority of early research was restricted to computability on the real line and in $\mathbb{R}^n$. In these spaces, the rationals and the tuples of rationals can be used to define computable points. This idea can be extended to more general spaces, as follows. For an abstract Polish space, we fix a dense sequence and require that we have a distance

function which is a computable on the dense set. For example, as above, we could use polynomials with rational coefficinets in $C[0,1]$. (Precise definitions will be found in Section 2.) Within this setting, computability-theoretic aspects of metric and normed spaces have also been studied for many decades, some of the earlier references include [113, 125, 126].

Computability-theoretic aspects in the Euclidean and the totally disconnected ultrametric cases, particularly Cantor space 2^ω, have been of central importance. The Euclidean case – in the sense that the space is actually a subset of $\mathbb{R}^n$ for some n – is treated in Pour-El and Richards [127], Ko [83], Braverman and Yampolsky [22], and Weihrauch [148]. The ultrametric case – and more specifically the study of effectively closed subsets of Cantor space – is a well-developed classical subject of computability theory with surveys such as [27, 28]. This subdivision is of course a bit artificial since Cantor space can be viewed as a subset of $[0,1]$, but the metric will no longer be an ultrametric.

In the past decade or two there has been an increasing interest in the computability-theoretic aspects of abstract metric spaces. The central questions in such investigations include:

- When does a space admit a computable presentation, and in what sense is it computable?
- Can we compute certain invariants or objects associated with this space (e.g., the space of probability measures on the space), and if 'yes' then in what sense?
- Can we establish computable analogues of the classical topological results?
- Can we classify computable points in computable topological spaces?
- Can we classify computably presentable spaces in a given class? etc.

The computability-theoretic study of abstract metric and topological spaces is developing hand in hand with work on reverse mathematics [111], algorithmic randomness [65], enumeration degree theory [121], and (to some extent) effective descriptive set theory [110]. The notion of a computable presentation of a space is central in such investigations. Many classical spaces such as 2^ω and $L^2[0,1]$ are equipped with a 'natural' computability structure which is usually *fixed*; the theory is then developed for the fixed computability structure. The two classical texts [127, 148] essentially take this approach. Even though both books talk about ways to compare different computability structures on a fixed space, the space under consideration is usually equipped with a few 'natural' computability structures that can be compared. For example, in $C[0,1]$ one could use polynomials with rational coefficients or, alternatively, piecewise linear functions with rational parameters; these turn out to be equivalent (in a rather strong sense). But of course, not every space has a computable presentation simply from cardinality-theoretic observations. Can we describe those spaces that do admit computable presentations, at least from some common classes? For instance, for which compact Polish K does the Banach space $C(K)$ have a computable presentation? What about the space of probability measures on K? etc.

To attack these and similar questions we will often have to depart from classical computable analysis (that deals with fixed 'natural' computable presentations) and use methods of computable topology. Although we can point at earlier initiatives such as, e.g., [75–77], [117, 118], and [140–142], most of the related work in computable topology is more recent and includes [50, 64, 78, 87, 89, 146, 150]. Computable topology is notorious for its zoo of various notions of computability for a topological space. In contrast with effective algebra [3, 43] where all standard notions of computable presentability had been separated more than half-a-century ago (e.g., Feiner [44]), some of the key notions of computable presentability in topology have been separated only very recently [8, 61, 64, 95]; these results will be discussed in detail later in the paper.

In the recent years there has been a tendency to focus on the three main notions of computable presentability of a (compact) Polish space: a computable Polish space, a computable topological space, and a computably compact space. Our paper is focused on one of these three important notions of computable presentability, namely *computable compactness* that is defined and discussed below. Computable compactness is clearly restricted to compact Polish spaces. Nonetheless, we will see that the notion and the techniques associated with it have far-reaching applications in computable analysis that are not restricted to compact spaces.

1.4. **Computable compactness.** Recall that we mentioned that the notion of a computable Polish space, or a computably metrized space, seems to be the most well-established notion of computable presentability for a Polish(able) space. The early classical works on computable metric spaces include Lacombe [92], Ceitin [26], and Moschovakis [109]. A Polish space is *computable* or *computably completely metrized* if there is a complete, compatible metric d and a dense subset of *special* or *ideal* points $(x_i)_{i\in\omega}$ of the space such that $d(x_i, x_j)$ are computable reals

uniformly in i and j. If we view spaces up to isometry, we fix the metric; if we study them up to homeomorphism then we assume d is compatible. In the present article, we assume the metric to be complete, since in all examples that we care about it will indeed be such. For instance, compact metric spaces are complete and separable[1]. However, the issue is that in a computably metrized compact space, we do not necessarily have *computable* access to its finite covers.

Classical uses of compactness do not need an understanding of how finite covers are obtained. For classical purposes, it is sufficient that the finite covers exist. Thus, when we consider computability aspects of compact spaces, it is natural to quantify what we mean by this. There are many definitions of a space being computably compact throughout the literature. Remarkably, as we prove below, they – as well as some new useful ones – are all equivalent. For instance, Mori, Tsujii, and Yasugi [108] say that a computably metrized space is *computably compact* (or *effectively compact*) if there is a computable function which takes n and produces a finite 2^{-n}-cover of the space by open balls centred in special points and having rational radii. (Such balls are called basic.)

The notion has proven to be extremely useful, and the techniques associated with computable compactness tend to be elegant. Indeed, it is not uncommon that a tedious and technical proof in computable analysis becomes transparent and compact (pun intended) after a thoughtful application of computable compactness.

In spite of the usefulness of computable compactness and its numerous applications in the literature, it seems there is not many "standard" references that would contain a systematic exposition of the most fundamental results and techniques associated with computable compactness. Even though these are some excellent papers and Ph.D. theses written on related subjects (e.g., [21, 69, 107, 120]), many results and proofs are scattered throughout the literature. As a result, it seems that some fundamental facts about computably compact spaces keep being rediscovered over and over again. Proofs of some results in the literature (including some recent ones) can be significantly simplified via choosing a more careful set-up in which computable compactness can be used to simplify combinatorics. It seems that some of the standard techniques associated with computable compactness are not necessarily uniformly known, and perhaps even that the theory itself is a bit under-appreciated.

Thus, as mentioned above, our *first main goal* of this article is to fill this apparent gap in the literature, at least partially. Once we accumulate enough techniques and develop new ones, we will apply this machinery to prove new results and improve known proofs; this is our *second main goal*. Recall that our *third goal* is to try to suggest a correct notion of computable compact Polish space. The potential candidates for the 'correct' computability notion include: a computable topological space, a computably metrized space, and computably compact space, and some other perhaps more exotic notions – such as a right-c.e. (upper-semicomputable) metric spaces – that can be found in the literature and some of which will be mentioned later. We suggest that the following might be true:

Computable compactness is the right notion of computability for compact Polish spaces.

Even if the reader will disagree with this thesis after looking at the results that we present here, the definition of computable compactness is certainly *robust*. More formally, Theorem 1.1 contains seven equivalent formulations of computable compactness some of which are *new*. Many of the applications that we discuss in this article – perhaps most notably the recently discovered effective Stone and Pontryagin dualities – strongly suggest that our thesis should not be dismissed even if we view spaces up to homeomorphism. We will also explain why all three standard definitions of computable presentability for a compact Polish space – computably compact, computably metrized, computable topological – differ up to homeomorphism.

The study of computably compact spaces is very closely related to the investigation of effectively closed subsets of computably metrized Polish spaces, especially when the set happens to be computable closed. As we will see in Proposition 3.29, under some mild restrictions computable closed sets can be viewed as computably compact spaces, and vice versa by (vi) of the theorem below. We cite [69] for an excellent recent survey. The cited survey however does not really have many proofs or proof sketches, so we felt that including proofs should be a good idea; this is done in Subsection 3.5 that includes the necessary facts that we will need in the present paper.

---

[1]There are various notions in the literature which do not assume the metric to be complete. For more details about these closely related notions of effective presentability, such as 'recursive' and 'computable' metric spaces, we cite [56, 149]. We note that the aforementioned two notions of 'recursive' and 'computable' spaces are equivalent up to scaling the metric by a computable real [56].

Another classical and closely related subject in computability theory is the area of Π^0_1 classes. This area can be viewed as a special case of the theory of effectively closed sets but restricted to 2^ω. Of course, more can be said about 2^ω than about an arbitrary space. Unlike the theories of computably compact spaces and computable closed sets, there is no shortage of excellent surveys and papers about Π^0_1-classes (e.g., [29, 30, 35]), and the draft of a book "Effectively Closed Sets Π^0_1 Classes" by Cenzer and Remmel that is available online (as of early 2022). Thus, we will not include many proofs, we just state a few results in Subsection 2.6 that we will need, and will give references.

These three topics – computably compact spaces, computable closed sets, Π^0_1-classes – are closely related, and no firm line can be drawn between them.

Before we proceed, we should admit that giving a complete and comprehensive survey of the existing literature and results on the subject is *not* among the main goals of this article, but nonetheless we will provide many useful references. This is not a survey paper in the usual sense, it is mainly a technical semi-survey paper with many new results, and it should be treated as such. We also chose to spread further discussion and references to the literature throughout the paper (where it is relevant) rather than to write a giant introduction.

1.5. **The Main Theorem.** The following theorem will be proven over the subsequent sections. The fundamentals of computable compactness theory will be developed simultaneously with the proof. We will discuss each clause of the theorem in detail shortly; for now we note that $(iii), (iv)$, and $(viii)$ are *new*.

Theorem 1.1. *For a computably, completely metrized Polish space M, the following are equivalent:*
   (i) *Given n, we can effectively compute (the finite set of parameters describing) a finite 2^{-n}-cover K_n of M by basic open balls.*
  (ii) *We can effectively enumerate all finite basic open covers (each given at once as a finite set of parameters) of the space.*
 (iii) *Same as (i), but additionally in each finite cover n we can uniformly decide whether any finite collection of basic open balls in K_n intersect.*
 (iv) *In the notation of (iii), we can additionally uniformly decide (non)emptiness of intersection for any finite collection of balls in $\bigcup_{n\in\omega} K_n$, but balls may have merely computable radii (and, thus, are not necessarily basic).*
  (v) *There is a computable sequence of computable reals $(\epsilon_n)_{n\in\omega}$ such that $\epsilon_n \leqslant 2^{-n}$ so that, for every n, we can compute the maximal number of points in the space that are at least ϵ_n-far from each other.*
 (vi) *M is computably homeomorphic to a computable closed subset of the Hilbert cube.*
 (vii) *M is a computable surjective image of 2^ω.*
(viii) *The full continuous diagram of M is decidable.*

As we will note later, it is easy to see that (ii) is also equivalent to the approach that is standard throughout reverse mathematics (Simpson [136]): there exists an enumeration functional that, given a countable cover composed of basic open balls outputs (an index of) some finite subcover of the cover. This approach is perhaps the most familiar one to a working mathematician, while (i) is rather an effective analogy of total boundedness. Of course, a complete metric space is compact iff it is totally bounded, but this elementary fact is perhaps not quite as well-known as the standard definition of compactness. There are also other characterizations of computable compactness, e.g., for subsets of a fixed space in terms of Hausdorff distance. We will mention this characterisation later when we talk about computable closed sets and the Hilbert cube (vi); see Fact 3.39. There are also other characterisations of computable compactness in special classes, e.g., in the class of profinite groups. We will discuss some such characterizations in due course.

In view of this theorem, we have the following definition, which is a characterization established in this paper:

Definition 1.2. A computable Polish space is called *computably compact* or *effectively compact* if it satisfies (the equivalent conditions in) Theorem 1.1.

The equivalence of (i) and (ii) is well-known, and usually one of the two is taken as the standard definition of a computably compact space. See Theorem 3.3.

It seems that (iii) or something similar might be also known, but perhaps in some other form; e.g., see [64] for some related informal discussions. Also, Section 8.4 of [69] contains a discussion of spiritually similar results. Perhaps, the most closely related material can be found in [122]. However, we were unable to find any explicit proof of the equivalence of (i) and (iii) anywhere in the literature. For a proof, see Theorem 3.13.

As far as we know, (iv) is new. Characterization (iv) will be very useful in several applications greatly reducing the combinatorial complexity of proofs in many cases. It will be especially useful in computing cohomology groups of spaces, but some further (perhaps, less expected) applications will also be presented. This equivalence is stated and then proven in Theorem 3.16.

Iljazovič [67] was the first to discover the equivalence of the fifth formulation (v) with the standard definition; it has recently been re-discovered in [123]. Its significance is the remarkable fact that computable compactness is an isometric invariant of the space. In other words, computable compactness is not a property of some specific nice presentation, but it is an intrinsic property that holds for *all* isometric presentations. The result will appear as Theorem 3.21.

The sixth version (vi) is well-known, we are not sure who was the first to observe its equivalence to computable compactness. See Theorem 3.36 for a proof.

The seventh item (vii) is an effective version of the classical Hausdorff-Alexandroff Theorem; see Theorem 3.40. The result is due to Brattka, de Brecht, and Pauly; see Proposition 4.1 of [18]. Also, working independently, Couch, Daniel, and McNicholl [32] proved the result for the special case of closed subspaces of $\mathbb{R}^n$. Around the same time, Day and Miller [33] independently discovered another important special case of this result, specifically for probability spaces. Interestingly, in Remark 3.23 of his large unpublished survey "Algorithmic randomness, martingales and differentiability" Jason Rute refers to this property (being a computable image of 2^ω) as being a stronger version of computable compactness[2]. And indeed it may seem at first glance that it should be stronger than, e.g., (i). We give two new and substantially different proofs of the result, one using (iv) discussed above, and the other one using Π_1^0 classes and the Hilbert cube. Both proofs serve as a fine illustration of the techniques that we develop in the paper. The effective Hausdorff-Alexandroff Theorem will be rather useful in several applications that will be discussed in due course.

Finally, the last item $(viii)$ of the theorem is new; it is inspired by [10, Definition 9.9] and the very recent paper [24]. It says that any formula of continuous logic formed in the language of pure metric uniformly defines a computable function $M^n \to [0, 1]$, where n is the number of free parameters in the formula. This pleasant, unexpected, but quite elementary result will be stated formally in Theorem 3.7. Theorem 3.7 will be restricted to spaces of diameter $\leqslant 1$, but as noted in the remarks preceding this theorem, it is just a mere notational convenience. The result perhaps confirms one's suspicion that in the case of compact metric spaces, continuous logic is not particularly expressive.

1.6. **Summary of applications.** We now discuss several applications (of computable compactness) that can be found in Section 4. We are mainly focused on the applications that are either new, or give new proofs of known results. We also mention several applications that are very recent and are related to our research interests. We also pose several open questions.

Our list of applications is not even close to being exhaustive, but we will discuss the literature where more results of this sort can be found. Here is a summary:

(1) In Subsection 4.1 the reader can find several useful standard results most of which are at least half-a-century old. They serve as a mere illustration of some of the basic techniques.

(2) The next Subsection 4.2 contains an unpublished result of Nies and Melnikov that states that Π_1^0 classes can be used to represent isometric isomorphisms between effectively compact spaces. The result is not difficult, but its consequences are fairly powerful; in particular, combined with several standard results about Π_1^0-classes, this method gives elegant and much more 'compact' proofs of some results from the literature.

(3) Subsection 4.3 contains an application of computable compactness to constructing basic sequences in Banach spaces. The application is elementary but is neat. The subtlety is that, in classical Banach space theory,

---

[2]As of 2022, the survey is still available at the personal homepage of Jason Rute.

one routinely uses dual spaces and Hahn–Banach Theorem to construct basic sequences (e.g., [25]), but it is known that Hahn–Banach Theorem is not computable [16, 106]. Computable compactness gives a way to circumvent this obstacle. The result is very recent and can be found in Long's M.Sc. thesis [94].

(4) The next Subsection 4.4 applies the techniques developed in Section 3 to Stone spaces; these results are very recent and can be found in [61, 64]. For instance, we will see that a Stone space is computably compact iff it is computably metrizable iff the dual Boolean algebra is computably presentable. Among many applications, we will explain why the isomorphism problem for Stone spaces is Σ_1^1-complete. The result is 'known' but it seems it has never been stated in the literature; we include it for future reference. We will also see that these techniques can be used to produce an example of a computable topological Polish space not homeomorphic to any computably metrized space.

(5) In Subsection 4.5 we prove that a profinite group is recursively presented (in the sense of [90, 138]; to be defined) iff it is computably compact; this result is new.

(6) Subsection 4.6 contains a new algorithm for computing Čech cohomology of a computably compact space. The algorithm is new, but the result itself is not new (though it is very recent [95]). In the subsection we also discuss several applications of Čech cohomology in computable topology.

(7) Subsection 4.7 applies computability of Čech cohomology established in the previous subsection to produce examples of computably metrized compact spaces that are not *homeomorphic* to any computably compact space. It is not hard to find a computably metrized space that is not *isometrically* isomorphic to any computably compact space (just take the interval $[0, \Omega]$, where Ω is Chaitin's omega[3] or some other left-c.e. real that codes $0'$.) However, the situation becomes more complex if we view spaces up to homeomorphism. The result is not new but is very recent, and the proof that we give is a new combination of modern and classical techniques some of which we introduce in the preceding subsections. Our new proof is perhaps the simplest one known so far.

(8) Subsection 4.8 contains a new proof of computable universality of $C[0, 1]$ among computable Polish spaces up to (computable) isometry. The issue is that the standard proofs of universality of $C[0, 1]$ rely on Hahn–Banach Theorem; as we have already mentioned above, it is not computable in general. Sierpinski [135] suggested a more direct proof that he thought was 'effective'; however, his proof gives a merely $0'$-computable embedding. We use tools of computable compactness to produce a computable embedding of any computable Polish space to the standard presentation of $C[0, 1]$. The result is not new ([5]), but the proof that we here give is new. Our new proof is *much* less combinatorially involved than the one in [5]; the latter does not use computable compactness tools. (However, it is not necessarily clear that our proof holds primitively recursively, while the proof from [5] gives a primitive recursive embedding.)

(9) In Subsection 4.9 we prove that every computably compact space of finite covering dimension can be computably embedded into a finitely dimensional Euclidean space. This is an improved version of a very recent result of Harrison-Trainor and Melnikov [60] that establishes that there is an arithmetical embedding. Our result is stronger and the technique that we use is different from what has been used in [60]. The new version heavily relies on one of the new characterizations of computable compactness that we prove in the paper. It will allow us to effectivize one of the standard proofs from the classical literature with only minor modifications.

(10) Subsection 4.10 contains the proof of the fact that, for a computably compact X, the space of probability measures $\mathcal{P}(X)$ on X is a computable homeomorphic image of 2^ω. This is known, even though the standard reference [33] does it only for the special case of $X = 2^\omega$ and via an explicit construction of a computable map from 2^ω onto $\mathcal{P}(X)$. But it is actually easier to establish computable compactness of $\mathcal{P}(X)$ directly (using covers), and *then* apply (vii) of Theorem 1.1. The result is very recent and is due to Marcone and Valenti [96]. Computable compactness of $\mathcal{P}(X)$ can be used to show that a compact incomputable group is computably compact iff it admits a computable Haar probability measure. This is a known result and we will discuss it more fully in Subsection 4.10.

(11) The final Subsection 4.11 contains several open questions that are related to the material contained in the previous subsections. Most of these questions are directly or indirectly related to compactness.

---

[3]While it is not central to this paper, Ω is the Lebesgue measure of the domain of a universal prefix-free Turing machine (see Downey and Hirschfeldt [36]). It is a "natural" example of a left-c.e. real which is not computable, in the same way that the halting problem is a natural example of a c.e. non-computable set.

To make our exposition smoother, we shall often define notions when we need them. The most commonly known basic notions of computable metric space theory can be found in the preliminaries.

2. Preliminaries

All of our spaces are Polish (separable and completely metrizable) spaces. Such spaces are also sometimes called Polishable. All spaces are also compact, unless stated otherwise. There will be only very few exceptions towards the end of the paper where the spaces considered will not be compact, namely the Urysohn space and the space of continuous functions on the unit interval $C[0, 1]$.

We will almost never consider the empty space, even though it is actually possible to include this rudimental case into our framework. However, many proofs become more uniform and definitions more convenient if we exclude this case.

We remind the reader that a real α is computable if there exists a computable sequence $(q_{f(i)} \mid i \in \mathbb{N})$ of rational numbers such that $|\alpha - q_{f(i)}| < 2^{-i}$. If we have a computable sequences of rationals but only know that $q_{f(i)} \to \alpha$, but not a computable modulus of convergence, then we would say that α is a Δ_2^0-real; and if the sequence $q_{f(i)}$ is monotonically increasing (resp. decreasing), then α is said to be left-c.e. or lower semi-computable (resp. right c.e. or upper semi-computable).

Much work in computable analysis from recent years has been concerned with the theory of representations [148]. In the type 1, countable case, when we talk about functions acting on, for example, polynomials, we really mean functions acting on numbers or strings "representing" the objects. In the type 2 case, a representation is a way of assigning an infinite string α in Baire space ω^ω with the object we wish to run algorithms upon; and to do so in a computationally meaningful way. However, in the case of Polish spaces, Cauchy sequences provide a natural and effective way to represent elements. Thus, we stick throughout with the notation as presented in the subsections below.

2.1. Effective metrizations of Polish spaces.
A Polish space (M, d) is *right-c.e. presented* or *admits a right-c.e. metric* if there exists a sequence $(\alpha_i)_{i \in \omega}$ of M-points which is dense in M and such that for every $i, j \in \omega$, the distance $d(\alpha_i, \alpha_j)$ is a right-c.e. real, uniformly in i and j. (In particular, we always assume that the metric is complete.) More formally, there is a c.e. set $W \subseteq \omega^2 \times \mathbb{Q}$ such that for any i and j,

$$\{q \in \mathbb{Q} : d(\alpha_i, \alpha_j) < q\} = \{q : (i, j, q) \in W\}.$$

Note that the sequence $(\alpha_i)_{i \in \omega}$ may contain repetitions; equivalently, it is possible that $d(\alpha_i, \alpha_j) = 0$ for some i, j. We call points α_i from the sequence *special* or *ideal*. For instance, an undirected (simple) graph with the shortest path metric is a right-c.e. metrized space. We will see that Π_1^0 classes can also be viewed as right-c.e. metrized spaces.

The definition of a left-c.e. Polish space is obtained from the notion of a right-c.e. Polish space using the notion of a left-c.e. real, mutatis mutandis.

Definition 2.1. A Polish space is *computably presented* or, perhaps more descriptively, *computably metrizable* if there is a (complete) metric on the space which is both right-c.e. and left-c.e.

Strictly speaking, a computable or a right-c.e. metrization of a space is a countable object $(\alpha_i)_{i \in \omega}$, but we will usually identify a computable metrization $(\alpha_i)_{i \in \omega}$ of space M with its completion $\overline{(\alpha_i)_{i \in \omega}}$.

Remark 2.2. Note that we intentionally did not emphasise whether we consider Polish spaces up to isometric isomorphism or under some other notion of similarity, such as, e.g., quasi-isometry or homeomorphism. Indeed, these will lead to non-equivalent notions. For example, for a real ξ, the space $[0, \xi]$ is isometrically isomorphic to a computably metrized space if, and only if, ξ is left-c.e. However, for any real ξ this space is homeomorphic to the unit interval $[0, 1]$ which is of course computably metrizable. In this paper we usually consider Polish spaces under homeomorphism, that is, a Polish space has a right-c.e. presentation if it is *homeomorphic* to the completion of a right-c.e. metrized space. Nonetheless, we will emphasise this in most of the theorems and lemmas that we prove to make sure that there is no conflict of terminology.

2.2. **Computable topological spaces.** There are several definitions of a computable topological space that can be found in Kalantari and Weitkamp [75], Korovina and Kudinov [85], and Spreen [140]. We will use the following.

Definition 2.3 (see, e.g., Definition 3.1 of [86] or Definition 4 of [150])**.** A *computable topological space* is given by a computable, countable basis of its topology for which the intersection of any two basic open sets ("basic balls") can be uniformly computably listed. More formally, it is a tuple (X, τ, β, ν) such that

- (X, τ) is a topological T_0-space,
- β is a base of τ,
- $\nu: \omega \to \beta$ is a computable surjective map, (i is called an *index* of $\nu(i)$) and
- there exists a c.e. set W such that for any $i, j \in \omega$,

$$\nu(i) \cap \nu(j) = \bigcup \{\nu(k) : (i, j, k) \in W\}.$$

Let (X, τ, β, ν) be a computable topological space. For $i \in \omega$, by B_i we denote the open set $\nu(i)$. As usual, we identify basic open sets B_i and their ν-indices. There are many versions of this notion above in the literature; see, e.g., [140]. All of these notions are essentially Definition 2.3 with some extra assumption. For example, one can also additionally require that there is a computable dense sequence $(x_i)_{i \in \omega}$ such that $\{\langle i, j \rangle : x_i \in B_j\}$ is computably enumerable. See [118, 140] for *many* other extra assumptions, some of which definitely seem ad hoc. We thus stick with the basic Definition 2.3. Perhaps, the most natural example of a computable topological Polish space is given by the proposition below.

Proposition 2.4 (cf. Theorem 2.3 of [86])**.** *Every right-c.e. Polish space is a computable topological space.*

Proof. Let (M, d) be a right-c.e. Polish space, and let $(\alpha_i)_{i \in \omega}$ be its sequence of special points. By τ we denote the metric topology of (M, d). As usual, the base β of τ contains basic open balls

$$B(\alpha_i, q) = \{x \in M : d(\alpha_i, x) < q\}, \quad i \in \omega, \ q \in \mathbb{Q}^+.$$

For $i \in \omega$ and $q \in \mathbb{Q}^+$, we put $\nu(i, q) = B(\alpha_i, q)$.

We prove that the tuple (M, τ, β, ν) is a computable topological space. It is sufficient to establish the following: for any $i, j \in \omega$ and $q, r \in \mathbb{Q}^+$, we can (uniformly) effectively enumerate a set $X \subseteq \omega \times \mathbb{Q}^+$ such that

$$(1) \qquad B(\alpha_i, q) \cap B(\alpha_j, r) = \bigcup \{B(\alpha_k, t) : (k, t) \in X\}.$$

Our set X is defined as follows: X contains all pairs (k, t) such that

$$d(\alpha_i, \alpha_k) < q - t \text{ and } d(\alpha_j, \alpha_k) < r - t.$$

Since the space (M, d) is right-c.e., it is not hard to see that the set X is c.e., uniformly in i, j, q, r. If $(k, t) \in X$, then by using the triangle inequality, we can easily show that $B(\alpha_k, t)$ is a subset of $B(\alpha_i, q) \cap B(\alpha_j, r)$.

Let x be an arbitrary point from $U = B(\alpha_i, q) \cap B(\alpha_j, r)$. Choose positive rationals ϵ and δ such that $\epsilon < q - d(\alpha_i, x)$ and $\delta < r - d(\alpha_j, x)$. Since U is open, we can find $k \in \omega$ and $t \in \mathbb{Q}^+$ such that $x \in B(\alpha_k, t) \subseteq U$ and $t < \min(\epsilon/2, \delta/2)$. Then we have

$$d(\alpha_i, \alpha_k) \leqslant d(\alpha_i, x) + d(\alpha_k, x) < (q - \epsilon) + t < q - \epsilon/2 < q - t.$$

Therefore, (k, t) belongs to X, and the set X satisfies (1). Hence, (M, τ, β, ν) is a computable topological space. $\square$

For instance, every computably metrized Polish space is a computable topological space. We shall return to computable topological and right-c.e. spaces later, when we talk about Stone duality. Effective compactness can also be defined for computable topological and right-c.e. spaces, but we will not need this degree of generality until Subsection 4.4. (This will be clarified in the remarks before Theorem 4.28.) Until Subsection 4.4 we restrict ourselves to computable, completely metrized spaces.

2.3. **The definition of computable compactness.** We have already explained what it means for a Polish space to be computably metrized. We usually assume that all our spaces are Polish metric and non-empty. Recall that a complete metric space M is compact iff for every $\varepsilon > 0$, there exists a finite set F of points such that every point has distance less than ε to F. For now, we say that a space is computably compact if it satisfies (i) of Theorem 1.1:

Definition 2.5. A computably metrized space is called *computably compact* if there exists a computable function that, given n, outputs the index of a finite tuple of basic open balls of radii $< 2^{-n}$ that cover M.

We should explain a bit more carefully all the terms that we use in the definition above. A ball centred in a special point is *basic* if its radius is a rational number. When we consider finite covers, we usually say that we can *compute* a finite cover by basic open balls if we can compute the index of a finite set that codes the indices of the finitely many centres and the rational radii of basic open balls in the cover. This should not be confused with *enumerating* a finite cover, i.e, listing one ball after another in a c.e. fashion.

Remark 2.6. Most proofs in this article do not actually need the radii to be rational numbers, but would work with balls of radius a computable real. However, we cannot list computable reals effectively. Therefore, we cannot hope to have an effective base of topology consisting of all basic balls with computable radii and computable points.

Suppose a compact space is computably metrized. How much computational power do we need to make it computably compact?

Definition 2.7. Let $M = (M, d, (p_i)_{i \in \mathbb{N}})$ be a compact computable metric space. A *compactness modulus* of M is is any function that bounds

$$h(n) = \min\{j \colon \forall i \exists k < j \, d(p_i, p_k) \leqslant 2^{-n}\}$$

from above. We call h the least modulus of compactness.

Note that if $h(n) = j$, then the 2^{-n+1}-basic open balls centred in $p_0, \ldots, p_j$ cover the space. Since $d(p_i, p_k) \leqslant 2^{-n}$ is a Π_1^0 condition and the quantifier $\exists k < j$ is bounded, and the space is compact, h is computable relative to $0'$. It is not difficult to show that there exists a computably metrized compact space in which the least modulus of compactness computes $0'$, and indeed, any modulus of compactness as well. As we mentioned in the introduction, the interval $[0, \Omega]$ defines a computably metrized space that is not isometrically isomorphic to any computably compact space (by Theorem 3.21), and its modulus of compactness (for any computable presentation) computes $0'$.

It what will follow, we will not necessarily need *the* (least) modulus of compactness. Indeed, it is sufficient to calculate *some* (and not necessarily the least) j such that $\forall i \exists k < j \, d(p_i, p_k) \leqslant 2^{-n}$. One way to state this would be to require j to be 'the first found' (in some Δ_2^0 approximation sense) that works. A space is computably compact if for every n we can compute *some* j that works. It is not difficult to manufacture a pathological example of a space where some j can be computably found for a given n, but one cannot compute the least such j. (See the next subsection for a similar counterexample.)

2.4. More about basic open balls.

Notation 2.8. For a basic open B, write B^c for the basic closed ball with the same centre and radius as B.

The reader should keep in mind that the space can be very strange, quite unlike $\mathbb{R}^n$. For instance, the closure $\overline{B}$ of B does not have to be equal to B^c in general (think of an isolated point in $B^c \backslash B$). Also, in general we cannot decide whether two basic open or closed balls intersect or not, as is illustrated by the example below.

Example 2.9. *There exists a computably compact subspace of the unit square such that there is no uniformly computable procedure deciding whether two given basic open or basic closed 2^{-n}-balls intersect.*

To make sure that the non-emptiness of intersection of open balls is undecidable, for every n create a gadget consisting of two points x_n and y_n at distance $2^{-n} - 2^{-n-2}$ from each other, and also put a third point z_n at distance exactly 2^{-n} from each of x_n and y_n. The point is at the intersection of the 2^{-n}-circles centred at x_n and y_n.

- Wait for the nth potential procedure to declare that $B(x_n, 2^{-n}) \cap B(y_n, 2^{-n}) = \varnothing$.
- If this ever happens at some stage s, take $m = s + n + 1$ and put a new point w_n at distance exactly 2^{-m} from z_n so that $w_n \in B(x_n, 2^{-n}) \cap B(y_n, 2^{-n})$.

To make sure that the non-emptiness of intersection of basic closed balls is undecidable, for every n create a similar gadget, but this time keep z_n out of the space at every finite stage. Instead, initiate the enumeration of a sequence $(\xi_{i,n})$ of points in the complement of $B^c(x_n, 2^{-n}) \cup B^c(y_n, 2^{-n})$ rapidly converging to z_n, i.e., $d(z_n, \xi_{i,n}) = 2^{-n-i}$. At stage s, put $\xi_{s,n}$ into the space.

- Wait for the nth potential procedure to declare that $B^c(x_n, 2^{-n}) \cap B^c(y_n, 2^{-n}) \neq \varnothing$.
- If this ever happens at some stage s, stop putting points $\xi_{s,n}, \xi_{s+1,n} \ldots$ into the space.

It should be clear that, for each gadget, the diagonalization works. We also note that the gadgets are uniformly computably compact. We can fit all these gadgets into the unit square and get a computably compact metric space (of Cantor-Bendixson rank 2) with the desired property. □

Thus, we see that the non-emptiness of set-theoretic intersection of basic open balls in not c.e. in general. We will return to this issue in Subsection 3.3 where we will see that there are enough balls for which this property actually is decidable; and hence we can get a characterization of computable compactness where the basic balls used in covers have decidable intersections.

A similar example can be produced to show that inclusion is also not c.e. in general. The following stronger notion is c.e.; it will be very useful throughout the paper. We write $r(B)$ for the radius of a basic ball B and we use $cntr(B)$ to denote its (distinguished) center.

Definition 2.10. A basic open ball U is said to be *formally included* into a basic open W, written $U \subseteq_{form} W$, if

$$r(U) + d(cntr(U), cntr(W)) < r(W).$$

This notion has been around for many decades; see, e.g., [140] where it is called strong inclusion. If the centres and the radii are computable (not necessarily special and rational, respectively), formal inclusion remains c.e. The same can be said about formal s-disjointness defined as follows.

Definition 2.11. Two basic open balls U and W are *formally s-disjoint* if $r(U) + r(W) < d(cntr(U), cntr(W))$ and this can be seen after calculating the radii and the distance with precision 2^{-s}. We say that U and W are *formally disjoint* if the are formally s-disjoint for some s.

We note that formal inclusion remains c.e. in the context of right-c.e. metric spaces, while formal disjointedness remains c.e. in left-c.e. metric spaces.

2.5. **Effectively continuous maps.** Let X be a computable topological space. For a point $x \in X$, its *name* is the set

$$N^x = \{i \in \omega : x \in B_i\}.$$

We say that a map $f : X \to Y$ between computable topological spaces X and Y is *computable* if there exists an enumeration operator that, given the name of $x \in X$, outputs the name of $f(x) \in Y$.

An *open name* of an open set $U \subseteq X$ is a set $W \subseteq \omega$ such that

$$U = \bigcup_{i \in W} B_i.$$

Definition 2.12. Let X and Y be computable topological spaces. A function $f \colon X \to Y$ is *effectively continuous* if there is a c.e. family $F \subseteq \mathcal{P}(X) \times \mathcal{P}(Y)$ of pairs of (indices of) basic open sets such that:

(C1) for every $(U, V) \in F$, we have $f(U) \subseteq V$;
(C2) for every point $x \in X$ and every basic open E in Y such that $f(x) \in E$, there exists a basic open D in X with $(D, E) \in F$ and $x \in D$.

The elementary fact below is well-known and can be traced back to, e.g., [26]. In this specific form it can be found in [100]. The lemma essentially says that a map is computable if, and only if, it is effectively continuous.

Lemma 2.13. *Let $f \colon X \to Y$ be a function between computable Polish spaces. Then the following conditions are equivalent:*

(1) *f is effectively continuous.*
(2) *There is an enumeration operator Φ that on input an open name of an open set V in Y lists an open name of the set $f^{-1}(V)$ in X.*
(3) *There is an enumeration operator Ψ that given the name of a point $x \in X$, enumerates the name of $f(x) \in Y$.*

(We remark that the proof below works for right-c.e. spaces. It also works for computable topological spaces with c.e. formal (strong) inclusion that can be defined abstractly without any reference to a metric; see, e.g., [100, 140].)

Proof. (1) $\Rightarrow$ (2). Suppose $V = \bigcup_{i \in W} B_i$. Note that (C2) implies that

$$f^{-1}(V) = \bigcup \{D \in X : (D, E) \in F \,\&\, \exists i \in W \, E \subseteq_{form} B_i\},$$

and thus the name of $f^{-1}(B_i)$ can be listed using only positive information about W, with all possible uniformity.

(2) $\Rightarrow$ (3). Note that $B \in N^{f(x)}$ if and only if $f^{-1}(B)$ contains a basic open set in N^x.

(3) $\Rightarrow$ (1). Define a collection F of pairs (D, E) of (indices of) basic open sets in $X \times Y$ as follows. Fix a basic open E in Y. Enumerate all basic open D in X, and for each such D, enumerate all finite collections $D, A_1, \ldots, A_k$ of basic open sets (in X) such that $D \subseteq_{form} \cap_{i \leqslant k} A_i$ (meaning that D is formally contained in each A_i). Feed these finite collections to Φ and wait for some E to be enumerated in the output. When E is enumerated (if ever), put (D, E) into F.

We claim that F defined above satisfies (C1) and (C2). We check (C1). If $(D, E) \in F$ then $f(D) \subseteq E$. Indeed, suppose $d \in D$. There exists a sequence $D, A_1 \ldots A_k$ such that $\Phi^{\{D, A_1, \ldots, A_k\}}$ enumerates E. Recall $D \subseteq_{form} \cap_i A_i$ implies $D \subseteq \cap_i A_i$, thus for any $d \in D$ the sequence listed by Φ^{N^d} will contain E, and therefore $f(D) \subseteq E$. We now check (C2). Fix $x \in X$ and a basic open $E \ni f(x)$. We must show that for some basic open $D \ni x$, $(D, E) \in F$. By assumption, Φ^{N^x} enumerates $N^{f(x)}$ that contains E. Suppose E is listed with use $A_1, \ldots, A_k$. Since the A_i all contain x, there exists a basic open $D \ni x$ that is formally included into their intersection. Since the operator uses only positive information about its oracle, it will list E on input $\{D, A_1, \ldots, A_k\}$ as well, and thus (D, E) will be enumerated into F. $\qquad\square$

2.6. Π_1^0**-classes.** Π_1^0 classes will be important for some of the work to follow. Thus we give a brief reminder of the basic definitions and results.

We fix the standard computable presentation of 2^ω under the usual shortest common initial segment ultra-metric. The space 2^ω can be viewed as $[2^{<\omega}]$ the set of infinite paths through the complete binary tree, so points are paths. A closed subset C of 2^ω is called a *a Π_1^0 class* if we can computably enumerate the basic clopen sets whose union make up the complement of C in 2^ω. That is, C is the set of paths $[T]$ through a computable subtree T of $2^{<\omega}$.

A moment's thought reveals that in computable mathematics Π_1^0 classes occur everywhere. One of the fundamental correspondences are Π_1^0 classes and degrees of theories, pioneered by Jockusch and Soare in the early 1970's [72–74], and even earlier by Kreisel [88] and Shoenfield [134]. That is, Π_1^0 classes effectively correspond to (completions of) axiomatizable theories under Stone duality.

We shall need the following elementary fact:

Fact 2.14. An isolated point in a Π_1^0-class is computable.

Thus, for instance, if a Π_1^0 class is countable, it must have a computable point. Another well-known but less elementary fact that we will refer to is the following result.

Theorem 2.15 (The Low Basis Theorem [74]). *A non-empty Π_1^0 class contains a member P of low Turing degree, that is $P' \equiv_T \varnothing'$.*

Since we are concerned with computable compactness, it seems reasonable to see what computable compactness means in this context.

Example 2.16. Let $C = [T] \subseteq 2^\omega$ be a Π_1^0 class such that $\text{Ext}(T) = \{\sigma \in T \mid \exists \alpha \in 2^\omega (\sigma \alpha \in [T])\}$ is computable[4]. Then $C = [T]$ can be viewed as a computably compact space. To see why, first use computability of $\text{Ext}(T)$ to define a computable sequence of (uniformly computable) strings $D = \{\alpha_i \mid i \in \omega\} \subseteq 2^\omega$ so that D is dense in C; we omit the details. (The example will be generalised in Subsection 3.5 where we will give a complete proof of a more general result.) Clearly, being the completion of D, C forms a computably metrized space under the ultra-metric inherited from 2^ω. Given n, we compute a 2^{-n} cover. Compute $E_n = \{\sigma \mid \sigma \in \text{Ext}(T) \wedge |\sigma| = n + 1\}$. Let $E_n = \{\tau_1, \ldots, \tau_k\}$. For each i we can use the fact that $\text{Ext}(T)$ is computable to calculate the leftmost extension α_i of τ_i in C, and then the balls of radius 2^{-n} around the $\alpha_i \cap C$ cover C.

Many other facts about Π_1^0 classes can be found in Cenzer [27], Cenzer and Remmel [28] and Chapter 2 of Downey and Hirschfeldt [36].

---

[4]Such Π_1^0 classes (with $\text{Ext}(T)$ computable) have been given several names historically: *recursive, recursively closed,* and *decidable.* See, e.g., LaRoche [91] and Downey [37].

3. The definition of computable compactness is robust

3.1. The other two standard definitions of computable compactness. Our convention is that all of our spaces are nonempty Polish spaces. Definition of computable compactness says that for every n we can compute a cover of the space by basic 2^{-n}-balls. This definition seems a weak form of compactness, as it seems that having a fixed cover for each 2^{-n} does not seem quite as good as having access to all finite covers. The following definition would seem to give a stronger notion of computable compactness.

Definition 3.1. We define computably metrized space to be $*$-*computably compact* if the collection of *all* finite covers of M by basic open balls can be given as a c.e. collection of explicit finite sets.

We also note that Definition 3.1 is equivalent to:

Definition 3.2. We say that a computably metrized space is *computably countably compact* if there is a partial computable operator that, on input any potential c.e. open basic cover, halts if it is a cover and outputs some finite sub-cover.

It is easy to see that a space is effectively $*$-compact iff it is computably countably compact. To see why computable countable compactness implies $*$-computable compactness, we can enumerate all finite collections of basic open balls and apply the algorithm: if the procedure halts output its subcover. This enumerates a collection of finite covers, and to enumerate them all, we consider the union of this collection with the collection of all finite sets of balls. The other direction is also straightforward: if the space is $*$-computably compact, given a c.e. cover wait till we enumerate a finite subcover. These versions of countable compactness is essentially the approach used in reverse mathematics (e.g., Simpson [136]).

Interestingly, the two potential definitions suggested above (and a few more) turn out to be equivalent.

Theorem 3.3 (Folklore)**.** *For a computably metrized (compact) Polish space M, TFAE:*

(1) M is computably compact;

(2) M is $*$-computably compact.

Proof. The implication $(2) \rightarrow (1)$ is obvious.

Assume (1), we prove (2). Take a finite collection (B_i) of basic open sets and assume it is a cover. We must argue that eventually we will be able to effectively recognise that it is indeed a cover. The idea is that there exists an $\epsilon = 2^{-n}$ so small that every ϵ-cover of M is formally contained in this given cover. (This will be the Lebesgue number of the cover, in particular.) This will also be true for the ϵ-cover that will be given to us according to the definition of computable compactness. Since formal inclusion is c.e., we will be able to recognise that this formal inclusion has occurred. Noting that formal inclusion does imply set-theoretic inclusion, so if some ϵ cover is formally included in some other finite collection of basic open balls, then this other collection must also be a cover. Thus, if we succeed, it will show that (B_i) is equivalent to saying that, for some n, every ball in the 2^{-n}−cover given to us by the definition of computable compactness (and indeed, any other 2^{-n}-cover) is formally included in one of the B_i. This is, of course, a Σ_1^0-property.

It remains to prove that such an ϵ exists. We argue non-computably. Let c_i be the center of B_i, and r_i be its the radius. Define for every i, a function $f_i(x) = r_i - d(x, c_i)$ if x is in the ball B_i, and 0 otherwise. This function is continuous. Now take the supremum of the finite family (f_i) to define a new continuous g; $g(x) = \sup_i f_i(x)$. If (B_i) indeed was a cover, then the function g would be strictly positive, because each x is inside one of the B_i.

Let v be its infimum that is achieved somewhere, by compactness. Take a rational $\epsilon = 2^{-m}$ less than $v/2$. Then for every point y, we have $\epsilon < r_i - d(y, c_i)$; that is

$$d(y, c_i) + \epsilon < r_i,$$

equivalently, $B(y, \epsilon) \subset_{form} B_i$. This inclusion will still hold if we replace ϵ with an even smaller ϵ'. Thus, in particular, every basic open ϵ'-ball is formally included in one of the B_i. Consequently, (1) implies (2). $\qquad\square$

Remark 3.4. The proof of $(1) \rightarrow (2)$ above additionally tells us that, for any given finite basic cover there is an ϵ small enough so that any ϵ-cover formally refines the given cover. Also note that to recognize formal inclusion in a c.e. way, we do not need the radii r_i to be rational numbers; (uniformly) computable r_i will suffice.

In view of Theorem 3.3, henceforth we use computably compactness and $*$-computably compactness interchangeably, and without further comment.

Elementary properties of computably compact spaces. Examples of computably compact spaces are the unit interval $[0,1]$, the unit circle that can be viewed as the set of complex numbers having norm one: $\{\xi \in \mathbb{C} : ||\xi|| = 1\}$, the Hilbert cube, cantor space 2^ω, and also 'natural' (rational) geometric realisations of finite simplicial complexes that are central to algebraic topology. Simplicial complexes will be used as a tool later in the paper, and indeed will be discussed in the next subsection. We shall give much more intricate examples of computably compact spaces in due course.

There are several properties of computably compact spaces that are immediate from the definitions. These, for instance, include those summarised in the following:

Proposition 3.5 (Folkore).   *(1) Let $f : M \to \mathbb{R}$ be computable and M be computably compact. Then $\sup_{x \in M} f(x)$ and $\inf_{x \in M} f(x)$ are computable real numbers. Furthermore, this is uniform.*

(2) The class of (non-empty) computably compact spaces is closed under taking (finite or computably infinite) direct products. More specifically, if $(M_i)_{i \in I}$ is a uniformly computable sequence of spaces, where $I \in \omega \cup \{\omega\}$, then the direct product

$$\prod_{i \in I} M_i$$

under (say) the metric

$$\sum_{i \in I} 2^{-i} \frac{d(x_i, y_i)}{1 + d(x_i, y_i)},$$

where x_i denotes the ith projection of $x \in \prod_{i \in I} M_i$, is a computably compact metric space[5]. (See, e.g., Lemma 3 of [129].)

(3) If $f, g : M^n \to \mathbb{R}$ are computable and M is computably compact, then the following functions are also computable:
- *$\sup_{x \in M} f(x, x_2, \ldots, x_n)$ and $\inf_{x \in M} f(x, x_2, \ldots, x_n)$;*
- *$\max\{f, g\}$ and $\min\{f, g\}$;*
- *$f - g, f + g, \alpha g$ for any computable real α.*

This is also uniform in the strongest sense possible.

We omit the elementary proof. We remark that in (2), the choice of a dense computable sequence is not canonical. One way of choosing a dense computable sequence it is to fix *some* (e.g, the first found) sequence of special points α in the product, and then using elements that are "eventually α". That is the dense subset will be given by the collection of sequences of special points that are equal to α for cofinitely many coordinates (projections). There are other potential metrics that we can use instead of the one suggested above, but the natural choices will be effectively equivalent (meaning that the identity map will be computable with respect to one and the other metric under consideration).

3.2. **Continuous logic and decidability.** In this subsection we discuss continuous model theory and continuous logic. For a smooth introduction to this subject, we cite [9]. We will not need the definitions of continuous model theory in their full generality. Our structures are compact metric spaces of diameter at most 1. The restriction on the diameter can be removed using linear scaling of the metric. We view such spaces as structures in the signature $\{d\}$, where $d : M^2 \to [0,1]$.

The idea is that every formula ϕ of continuous logic is (associated with) a uniformly continuous function, and $[\phi] : M^n \to [0,1]$. In classical logic the truth values $\{T, F\}$ are built into the language, and continuous logic has $[0,1]$ similarly built into the signature. We therefore fix the standard computably compact copy of $[0,1]$ given by $\mathbb{Q} \cap [0,1]$ and view it as a part of the language, not a part of our structure. Traditionally, $\phi(\bar{x})$ is interpreted as 'true' if $[\phi(\bar{x})] = 0$, and false if $[\phi(\bar{x})] = 1$. But of course, there are continuum many possibilities in-between. Having in mind this intended interpretation of formulae, we write $f, g, h \ldots$ to denote our continuous formulae.

---

[5]As was noted by one of the referees, it is not clear whether this observation holds in absence of the metric. Specifically, it is not known whether X^ω is computably compact for an arbitrary computably compact represented space. For more about effective compactness in general represented spaces we cite [120].

Because of the aforementioned somewhat unusual interpretation of the truth values, it makes sense to define the analog $\dot{-}$ of implication as follows:

$$[f \rightarrow g] = [g \dot{-} f] = [g] \dot{-} [f] = \max\{0, [f] - [g]\}.$$

In particular, 1 (which is the 'ultimate false') implies any g. Also, we interpret the disjunction as the minimum:

$$[f \vee g] = \min\{[f], [g]\},$$

and we interpret the conjunction as the maximum of the two functions. The analog of the negation of f is $1 \dot{-} f$. Also, we include $\frac{1}{2}f$ to mean $\frac{1}{2}[f]$. Finally, instead of quantifiers we use sup and inf. For example, if $f(x, y)$ has already been defined, we can define $g(x)$ to be

$$\sup_y f(x, y),$$

and in this case clearly

$$[g] = [\sup_y f(x, y)] = \sup_{y \in M}[f(x, y)],$$

which is (uniformly) continuous if f was[6].

Since our language is merely $\{d\}$, the atomic formulae are just $d(x, y)$ and the constant functions 0 and 1. We close these formulae under finite iterations of $\sup, \inf, \wedge, \vee, \frac{1}{2}\cdot$, and $\dot{-}$ to define *the (full) continuous diagram* of (M, d). The definition below is inspired by [10, Definition 9.9] and [24, Definition 3.2]. Recall that we fixed the usual computable presentation on the unit interval.

Definition 3.6. We say that a Polish space of diameter $\leqslant 1$ is *continuously decidable* if its continuous diagram is uniformly computable. That is, given (the Gödel number of) a continuous formula $\phi(\bar{x})$ (from the full continuous diagram of M) we can uniformly produce an index for a Turing operator that computes the function

$$[\phi(\bar{x})] : M^n \rightarrow [0, 1],$$

where $\bar{x} = x_1, \ldots, x_n$.

In the definition above, we could allow M to have an arbitrary diameter $1 \leqslant \delta \leqslant m \in \mathbb{N}$ and use $m^{-1}[\phi(\bar{x})]$ as an interpretation of our formulae. Alternatively, we could scale the metric and use $d'(x, y) = m^{-1}d(x, y)$. Thus, the theorem below is not really restricted to spaces of diameter $\leqslant 1$; it is a mere notational convenience.

Theorem 3.7. *Suppose X is a computable Polish space that is compact and has diameter $\leqslant 1$. Then the following are equivalent:*

(1) X is continuously decidable;
(2) X is computably compact.

Proof. $(2) \rightarrow (1)$. The supremums and infimums of computable functions are uniformly computable (see Proposition 3.5(1)), and so is $f \dot{-} g = \sup\{0, f - g\}$ for every computable f and g. The proof then proceeds by induction on the complexity of continuous formulae (with parameters).

(1) The distance function is computable by our assumption, and so are the constant functions 1 and 0.

(2) If $f(x, \bar{y}), g(x, \bar{y})$ are computable, then so are $\sup_x f(x, \bar{y}), f(x, \bar{y}) \dot{-} g(x, \bar{y})$, and $\frac{1}{2}f(x, \bar{y})$; see Proposition 3.5(3).

The latter also includes the case when there are no parameters, which means that the function is just the constant function (a computable real).

$(1) \rightarrow (2)$. Let (x_i) be the computable dense sequence. There is a formula of continuous logic saying that

$$U_n(x_0, ..., x_m) = \sup_{x \in X} \min_{i \leqslant m} \left(d(x, x_i) \dot{-} 2^{-n}\right)$$

---

[6] In general, we should also include the moduli of uniform convergence into the definition of our formulae, but in the language of just pure metric this is not needed. Since M is compact, and indeed will be computably compact, we can potentially make the moduli implicit even in more general signatures.

which we can computably evaluate with precision 2^{-n}. If we discover that

$$U_n(x_0, ..., x_m) < 2^{-n},$$

then every point is at distance at most 2^{-n+1} from one of the x_i, by the triangle inequality. Since we can effectively list such formulae, and for every n there exists an m for which the formula holds up to 2^{-n} (by compactness), we conclude that we can effectively produce at least one 2^{-n}-cover for every n. $\qquad\square$

The result above should hold for any metric compact structures, e.g., compact Polish groups. Of course, $(1) \to (2)$ remains the same, but $(2) \to (1)$ should be carefully verified. We leave this as a conjecture.

Remark 3.8. Note that in the proof, we only need to decide formulae with parameters special points. So it follows that a presentation is decidable with parameters special points iff it is decidable with arbitrary parameters.

3.3. **Deciding the intersection.** The results in this section appear to be new as stated, however, similar arguments and some informal explanations closely related to what we do here can be found in the literature (e.g., in [64, 122]). One standard way of using a (finite) cover of a compact space in dimension theory and algebraic topology is to use Alexandroff's notion of a *nerve*.

Definition 3.9 (Alexandroff [2])**.** A *nerve of a cover* is a simplicial complex in which the faces are the collections of basic open sets that have a non-trivial intersection; i.e., each basic open set is a 0-dimensional simplex (a node), and balls $\{B, C, D\}$ form a 2-dimensional face if $B \cap C \cap D \neq \varnothing$.

From the computability-theoretic standpoint, the issue with this definition is that, for a fixed finite open cover, the non-emptiness of each specific intersection is merely Σ_1^0; recall Example 2.9 in the preliminaries. Intuitively, most problems arise when the notions 'closed ball' and 'closure of ball' disagree. This in fact can happen only for countably many radii, and we can find sufficiently small "acceptable" radii" [122]. To state the result formally, we push the notion of computable compactness to its limits.

Definition 3.10. A sequence of basic open balls is $\cap$-decidable if for every finite sequence of balls $B_0, \dots, B_k$ from the sequence, we can computably decide whether $\bigcap_{i=0,\dots,k} B_i = \varnothing$.

Before we proceed, we state and prove one elementary but important lemma. Recall that, for a basic open B, we write B^c for the basic closed ball with the same centre as B, and that the closure $\overline{B}$ of B does not have to be equal to B^c in general.

Lemma 3.11. *Suppose M is computably compact. Then, for basic closed balls B_i^c and B_j^c, the property $B_i^c \cap B_j^c = \varnothing$ is c.e. uniformly in i, j. The same is true for any finite collection of basic closed balls.*

Proof. The open set $M \backslash B_i^c$ is c.e. Indeed, we just list all the basic open balls that are formally disjoint from B_i^c via a standard argument[7]. Thus, the union of the complements, which is the complement of the intersection $B_i^c \cap B_j^c$, is also c.e. open. It covers the space if, and only if, the intersection is empty. By computable compactness of M, this is c.e. The case of finitely many balls is similar. $\qquad\square$

Definition 3.12. A computably metrized (compact) M is *nerve-decidable*, or **-*computably compact*, if for every $n > 0$ we can computably find a finite 2^{-n}-cover K_n (represented as a finite tuple of basic open balls) of M so that K_n is $\cap$-decidable uniformly in n.

Theorem 3.13. *A computably metrized M is nerve-decidable (**-computably compact) if, and only if, it is computably compact.*

Proof. Obviously, **-computably compactness implies computable compactness. To this end, we assume computable compactness of M. We will use the equivalence of computable compactness and *-computable compactness throughout the rest of the proof without explicit reference.

We need to show that, for every $\epsilon > 0$, there exists a finite basic open ϵ-cover K of the space. Fix a finite $\epsilon/2$-cover of the space by basic open balls, and replace each ball in the cover with a ϵ-ball with the same centre. Let S be this new ϵ-cover. Recall that B^c denotes the basic closed ball with the same centre as B. For each each basic open $B_1, \dots, B_k \in S$, (exactly) one of the possibilities is realized:

---

[7]Every point in $M \backslash B_i^c$ has the property $d(cntr(B_i), y) > r(B_i) = r$, and if we take $B(y, q)$ where $0 < q < \frac{d(cntr(B_i), y) - r(B_i)}{2}$ then $d(cntr(B_i), y) > r + q$.

(a) $\bigcap_{i \leqslant k} B_i^c = \varnothing$, or

(b) $\bigcap_{i \leqslant k} B_i \neq \varnothing$, or

(c) $\bigcap_{i \leqslant k} B_i^c \neq \varnothing$ but $\bigcap_{i \leqslant k} B_i = \varnothing$.

Note that there are only finitely many conditions like that in total.

If we shrink the radii of all $B \in S$ by a $\delta < \epsilon/2$ (but keep the same centres), then the conditions of the form (a) will still hold, and the smaller balls will still cover the space because the ϵ-balls do. If δ is small enough, then the conditions of the form (b) will still be satisfied, since there are only finitely many conditions like that involved[8]. The third alternative (c) must be witnessed only by points y such that, for some $B_i = B(c, r)$, $d(y, c) = r$. This means that, after the shrinking by $\epsilon/2 > \delta > 0$ so small that the alternative (b) still holds for each tuple of balls, we completely exclude the third alternative for the new cover.

This argument shows that such a cover exists. Since the conditions are c.e. by Lemma 3.11, it remains to search for a cover such that each finite collection of basic balls in the cover satisfies either (a) or (b). $\qquad \square$

By Remark 3.4, we can additionally assume that K_{n+1} formally refines K_n. Clearly, we get the following corollary which will be useful later:

Corollary 3.14. *Every computably compact space admits uniformly computable sequence of 2^{-n}-nerves (one nerve for each n), where the latter are represented as a finite combinatorial simplices. Furthermore, the formal inclusion between covers K_{n+1} and K_n induces a simplicial map between the respective nerves; these maps are uniformly computable in n. (We cite Munkres [112] for the standard definitions from algebraic topology that we omit here.)*

A stronger condition. It will be convenient to have a system of covers (K_n) so that not only each K_n is $\cap$-decidable but the whole collection $\bigcup_n K_n$ is $\cap$-decidable. (We strongly conjecture that there is an elementary counterexample.) For instance, we will see soon that having such a stronger system of covers will allow us to computably map 2^ω onto the space; this is (vii) of Theorem 1.1.

We are not sure whether such covers can be uniformly constructed for basic open balls with rational radii (represented as a pair of integers), but we can contract such a system for balls with centres in special points and uniformly computable radii[9]. We call such balls *basic computable open*.

Definition 3.15. A computably metrized M is strongly computably compact if M admits a system of 2^n-covers K_n, $n \in \omega$, by basic computable open balls such that $\bigcup_n K_n$ is $\cap$-decidable.

Theorem 3.16. *A computably metrized M is computably compact if, and only if, it is strongly computably compact.*

Proof. By slightly increasing the radii of all the balls in a cover, we can ensure their radii are rational. Thus, every strongly computably compact space is computably compact. To this end, we assume the space is computably compact.

The idea behind the proof is as follows. We would like to argue that the idea from the proof of the previous Theorem 3.13 can be iterated. For example, suppose we have come up with a $\cap$-decidable K_0 and need to find K_1 so that $K_0 \cup K_1$ is $\cap$-decidable. But to find such a cover, we might have to slightly shrink the radii of the balls that we have already put into K_0. This is because it could be that for some $B \in K_0$ and C that we attempt to put into K_1, there is a point at distance $r(B)$ from the centre of B that lies in C and is isolated, so there is nothing in $B \cap C$.

Suppose we iterate the strategy form the proof of Theorem 3.13 and allow the procedure to slightly shrink all the balls in K_0, thus updating the radii of balls in K_0. But note K_0 must still satisfy the closed properties $\bigcap_{i \leqslant k} B_i^c = \varnothing$ and finitely many open properties $\bigcap_{i \leqslant k} B_i \neq \varnothing$. The former is not an issue since the radii will decrease. The latter however needs to be maintained more carefully. When we first discover the finitely many open relations of the form of finitely many strict inequalities (when K_0 is first introduced), we also *compute* a rational parameter $\delta_0 > 0$ such

---

[8]Each such intersection is witnessed by a (special) point which must be at distance strictly less than 2ϵ to all centres, say, γ-less. If we shrink the radii by a value less than the minimum of these γ (which is a positive value since there are only finitely many B_i involved) then the inequalities will still hold. Another way to think about it is as follows: $B \cap D \neq \varnothing$ is an 'open property' of the parameters (radii and centres), and a finite conjunction of open properties is also open because open sets are closed under taking finite intersections.

[9]The radii can likely be made rational if necessary, but they will be represented via Cauchy sequences, not as a fraction.

that the relations will still hold if we decrease the radii of the balls by δ_0. This is possible since the conditions build down to finitely many strict inequalities involving the radii and computable numbers:

$$d(c_i, x_j) < r_i,$$

where c_i are centres of the balls and x_i are special points witnessing that a certain intersection is not empty.

We then define $\delta_{0,n} = 2^{-n-2}\delta_0$ and note that $\sum_i \delta_{0,i} < \delta_0$. We intend to shrink the radii of each ball in K_0 by at most $\delta_{0,s}$ at stage s. This will make the radii in the balls computable while maintaining the finitely many conditions that K_0 needs to satisfy.

We also iterate this. When we define K_1, we will have more open conditions to maintain for $K_0 \cup K_1$. We compute a $\delta_1 > 0$ and set $\delta_{1,n} = 2^{-n-2}\delta_1$. We also ensure that $\delta_{1,n} \leqslant \delta_{0,n}$, for every n. When we define (our first approximation to balls in) K_2 at stage t, we will allow balls in K_0 to shrink by at most $\delta_{1,t} \leqslant \delta_{0,t}$ and balls in K_1 by at most $\delta_{1,t}$. All the finitely many conditions will still be satisfied.

We iterate this process until, in the end of the construction, we finally get a collection of computable balls $\bigcup_n K_n$. At no stage we are stuck. By the choice of the parameters, all of the open conditions still hold, while the closed conditions will be satisfied vacuously.

Hopefully, the explanation above is sufficiently convincing, but we shall give a formal proof for completeness.

Formal proof.

Lemma 3.17. *For every $\epsilon > 0$ and $\delta > 0$ and any finite collection K' of basic open balls, there exists a finite basic open ϵ-cover K of the space and a collection of basic open balls K'' such that:*

(i) *Every ball in K'' has the same centre as some ball in K' but its radius is at most δ-smaller;*
(ii) *for each basic open $B_1, \ldots, B_k \in K'' \cup K$, either $\bigcap_{i \leqslant k} B_i^c = \varnothing$ or $\bigcap_{i \leqslant k} B_i \neq \varnothing$ holds.*

Of course, $\bigcap_{i \leqslant k} B_i^c = \varnothing$ implies $\bigcap_{i \leqslant k} B_i = \varnothing$, and thus K in the lemma has computable nerve, and for the same reason $K \cup K'$ is $\cap$-decidable.

Proof of Lemma 3.17. Fix a finite $\epsilon/2$-cover of the space by basic open balls, and replace each ball in the cover with a ϵ-ball with the same centre. Let S be this new ϵ-cover. Recall that B^c denotes the basic closed ball with the same centre as B.

For each tuple of basic open $B_1, \ldots, B_k \in S \cup K'$, (exactly) one of the possibilities is realized:

(a') $\bigcap_{i \leqslant k} B_i^c = \varnothing$, or
(b') $\bigcap_{i \leqslant k} B_i \neq \varnothing$, or
(c') $\bigcap_{i \leqslant k} B_i^c \neq \varnothing$ but $\bigcap_{i \leqslant k} B_i = \varnothing$.

Note that there are only finitely many conditions in total.

If we shrink the radii of all $B \in S$ by a $\delta' < \min\{\delta, \epsilon/2\}$ (but keep the same centres), then the conditions of the form (a') will still hold, and the smaller balls will still cover the space because the ϵ-balls do. If δ' is small enough, then the conditions of the form (b') will also still be satisfied, since there are only finitely many conditions like that involved. Note that the third alternative must be witnessed only by points y such that, for some $B_i = B(c_i, r)$, $d(y, c_i) = r$. This means that, after we shrink the radii by $\epsilon/2 > \delta' > 0$, (b') will still holds for each tuple of balls, but we completely exclude the third alternative. Define K'' to be the balls in K' after the shrinking, and K is the shrunken balls from S. $\qquad\square$

The rest of the proof proceeds by induction; we iteratively apply Lemma 3.17 to produce a system of covers that satisfies the properties required in the definition of a strongly computably compact presentation. We produce a sequence (K_n) of covers, as follows.

At stage 0, search for a finite collection of basic open balls K satisfying conditions of Lemma 3.17 with $\epsilon = \delta = 1$ and $K' = \varnothing$. Define $\hat{K}_{0,0}$ equal to the first found such K. Also, for the finite collection of strict inequalities that witness non-emptiness of intersections in $K_{0,0}$ calculate a parameter $\delta' \in \mathbb{Q}$ such that the inequalities would still hold if we decrease the radii by δ'. Set $\delta_0 = \delta'$ and $\delta_{0,i} = 2^{-i-2}\delta_0$.

At stage $s > 0$, suppose $K_{i,s-1}$ $(i < s)$ and $\delta_{s-1,s-1}$ have already been defined. Search for a finite collection of covers K that satisfies the lemma with $K' = \bigcup_{i<s} K_{i,s-1}$, $\epsilon = 2^{-s}$, and $\delta = \delta_{s-1,s-1}$. This will give a finite collection

of balls K'' having the same centres as balls in K' but perhaps having radii at most δ-smaller than the radii of the respective balls in K'. For $i < s$, define $K_{i,s}$ to be the collection of those balls in K'' that have the same centre as some ball in $K_{i,s-1}$. Define $K_{s,s} = K$, where K is first found satisfying the conditions of the lemma. Compute a rational $\delta' > 0$ so small that the finitely many strict inequalities that witness non-emptiness of finite collections of balls in $\bigcup_{i \leqslant s} K_{i,s}$ will still hold if we decrease all the radii of all balls in $\bigcup_{i \leqslant s} K_{i,s}$ by δ'. Set $\delta_s = \min\{\delta_s, \delta'\}$ and define $\delta_{s,s} = 2^{-s-2}\delta_s$. Proceed to the next stage.

The verification boils down to noting that at no stage of the construction we are stuck, so $K_{i,s}$ and $\delta_{s,s}$ are defined for every s. This is because of Lemma 3.11 implying that the conditions (a') and (b') are uniformly c.e., and because of Lemma 3.17 saying that balls and parameters with the needed properties exist. Thus, we just search for the first found balls and parameters.

At every stage s, the radii of the balls in $K_{i,s}$ shrink by at most $\delta_{s,s} \leqslant 2^{-s}$, which makes each of the radii uniformly approach a computable real number as s goes to infinity. Set K_i equal to these balls that have their radii equal to the limit of the radii of the balls with the same centre in $K_{i,s}$. When compared to the radius of the ball in $K_{s,s}$ when it was first introduced, the radius of the respective ball in K_s will be smaller by at most $\sum_{n>s} \delta_{n,n} < \sum_{n>s} \delta_{s,n} \leqslant \delta_s$, and δ_s is not greater than the parameter δ' that was calculated at stage s and that was sufficient to maintain the non-emptiness of finite intersections in $K_{s,s}$. Since the radii can only decrease, the conditions that say that the closed balls do not intersect will be preserved from a stage to a stage, and in the limit.

It follows therefore that $\bigcup_n K_n$ consists of uniformly computable collection of basic computable balls and is $\cap$-decidable. $\square$

Remark 3.18. The reader should note that, instead of using basic open covers, we could just as well used basic closed covers in the proof of Theorem 3.16. For instance, for any finite collection of basic computable balls $C_0, \ldots, C_k$, we have $\bigcap_{i \leqslant k} C_i = \varnothing \iff \bigcap_{i \leqslant k} C_i^c = \varnothing$, where C_i^c is the basic closed ball with the same centre and radius as C_i. Also, by Remark 3.4, we can always assume that K_{n+1} is formally contained in K_n, and this will still be true if we choose working with closed covers. For that, define a new system of (closed or open) covers $K_{f(n)}$ where $f(n)$ is a computable monotone function that grows sufficiently fast so that $K_{f(n+1)}$ formally refines $K_{f(n)}$.

However, even when we are working with closed basic computable balls, the intersection can always be witnessed by a special point, because the respective open balls intersect too.

Recall that $\overline{C}$ denotes the closure of a basic computable open ball C, and recall that

$$C \subset \overline{C} \subset C^c,$$

and, in general, both inclusions can be strict. But these inclusions also guarantee that we can use closures of the open balls to form covers in Theorem 3.16 and still decide intersection. We can also come up with any combination of open, closed and closures (e.g., decide whether $B_i \cap \overline{B_j} \cap B_k^c = \varnothing$) for any computable balls in K constructed in Theorem 3.16.

Definition 3.19. If a computable sequence (K_n) of finite 2^{-n}-covers of computable balls satisfies the properties described in Definition 3.15 then we say that (K_n) is a *fully $\cap$-decidable system of covers* of the space. Such a K is given by a uniformly computable sequence of (finite sets K_n of) indices of radii and special points, and we can choose whether we want to consider open, closed, or closures of open balls that have these parameters (see Remark 3.18).

For instance, when we say "$B^c(r, q)$ is in K_n" or "$\overline{B(r, q)}$ in K_n", or the same for $B(r, q)$, we really mean that parameters $\langle r, q \rangle$ are listed in K_n (where q is given as an index of a computable real).

The following lemma will be useful later.

Lemma 3.20. *Let $K = \bigcup_n K_n$ be a fully $\cap$-decidable system of covers of a space M. Then, for every closed ball D^c in K we can enumerate all basic open B in M such that $B \cap D^c \neq \varnothing$.*

Proof. Suppose $B \cap D^c \neq \varnothing$ and let x be any (not necessarily special) point in the intersection. Suppose the radius of B is δ, and let c_1 be the centre of B, and r_1 its radius. For some positive δ, we have

$$d(x, c_1) = r_1 - \delta.$$

Fix n so that $2^{-n} < \delta/2$, and consider the finite set K_n. Since K_n is a (closed or open) cover of the whole space, there must exist some $C \in K_n$ such that $x \in C$. Since $x \in D^c$, it must be that

$$C \cap D^c \neq \varnothing,$$

and (by our assumption) this can be recognised computably. We claim that for this C, we have that C is formally included into B.

Indeed, if c_2 is the centre of C and r_2 is it radius, then we have that $d(x, c_2) < r_2 \leqslant 2^{-n} < \delta/2$, and therefore

$$d(c_1, c_2) + r_2 \leqslant d(x, c_1) + d(x, c_2) + \delta/2 < r_1 - \delta + \delta/2 + \delta/2 = r_1,$$

which is the same as to say that C is formally included into B.

It follows that B intersects D^c if, and only if, there is an $n > 0$ and a ball $C \in K_n$ such that $C \cap D^c \neq \varnothing$ and C is formally included into B. This is a Σ_1^0-property. $\square$

3.4. **Isometry-invariance of computable compactness.** Iljazovič [67] discovered that the notion of computable compactness admits another characterization that entails that it is isometry-invariant, i.e., every isometrically isomorphic computable metrization of the space must also be computably compact. This property has recently been independently rediscovered in [123].

Theorem 3.21. *Suppose M is computably compact and N is a computably metrized space isometrically isomorphic to M. Then N is computably compact as well.*

We claim that the theorem can be derived as a consequence of condition (v) in Theorem 1.1:

> There is a computable sequence of computable reals $(\epsilon_n)_{n \in \omega}$ such that $\epsilon_n \leqslant 2^{-n}$ and so that, for every n, we can compute the maximal number of points in the space that are at least ϵ_n-far from each other (in the sense of strict inequality).

We first prove Theorem 3.21 assuming that $(i) \leftrightarrow (v)$ in Theorem 1.1, and then we prove $(i) \leftrightarrow (v)$.

Proof of Theorem 3.21. Assume that, in M, we can compute $(\epsilon_n)_{n \in \omega}$ and the maximal number $D(n)$ corresponding to ϵ_n. In N, search for $D(n)$-many spacial points that are ϵ_n-far from each other. The ϵ_n-balls around these points will give a finite cover of the space. We can slightly enlarge the radii to make sure that they are rational. $\square$

Proof of $(i) \leftrightarrow (v)$ in Theorem 1.1. The proof of Theorem 3.21 above essentially shows that (v) implies (i). Thus, we assume computable compactness and prove (v).

The reader perhaps wonders why we did not use $\epsilon_n = 2^{-n}$ in (v). To clarify this subtlety, we shall *attempt* to show that, in a computably compact M, the following invariant $D(M, 2^{-n})$ is uniformly computable in n:

> $D(M, 2^{-n})$ is the maximal number of points of the space M that are $> 2^{-n}$-apart from one another.

The issue that we will face will clarify why we need to adjust our ϵ_n's. Also, it will be easy to modify this naive attempt and obtain a procedure that actually works for some $\epsilon_n \leqslant 2^{-n}$.

We describe our attempt. Given $\bar{x} \in M^m$, we can calculate $\inf_{i<j\leqslant m} d(x_i, x_j)$ and then

$$\sup_{\bar{x} \in M^m} \inf_{i<j\leqslant m} d(x_i, x_j).$$

If this supremum is $< 2^{-n}$, then m is too large, i.e., $m > D(M, 2^{-n})$. Note that this is a c.e. event. On the other hand, by searching through all possible m-tuples we can bound the maximal number of such points from below. The issue is that the supremum could be exactly equal to 2^{-n}, so we may end up with a pair of integers $n_0, n_0 + 1$ each of which can potentially be equal to the invariant $D(M, 2^{-n})$; here $n_0 + 1$ corresponds to the situation when there are $n_0 + 1$ points at distance exactly 2^{-n} from one another.

In this case we shall wait long enough so that any $(n_0 + 2)$-tuple has at least one pair of points at distance $> 2^{-n}$, and for some small $\xi_n < 2^{-n-1}$ there exist $n_0 + 1$ points at distance $2^{-n} - \xi_n$. Then

$$D(M, 2^{-n} - \xi_n) = n_0 + 1.$$

This allows us to compute a computable sequence of rationals

$$\epsilon_n = 2^{-n} - \xi_n,$$

where clearly $\epsilon_n \leqslant 2^{-n}$, such that $D(M, \epsilon_n)$ is a computable sequence of natural numbers. Indeed, for this ϵ_n there is a $n_0 + 1$ tuple $\bar{y}_n$ of points that satisfy the desired properties. We can assume these points are special. □

It follows from the proof above that these ϵ_n can be chosen rational and indeed, computed in the strong sense (as fractions). Note that we also obtain:

Corollary 3.22. *Suppose that M and N are isometrically isomorphic computable metrized spaces. Then both admit the same modulus of compactness up to Turing degree.*

3.5. **Calculus of effectively closed sets.** In this subsection we present some well-known basic results about effectively closed sets, and we also derive several pleasant consequences of these results that will be important in the sequel. The notion of an effectively closed set is a generalisation of a Π_1^0 class, and it is especially useful if the ambient space is effective compact. We will need some basic facts of this generalised theory, but of course a lot more is known; see the very recent large survey [69].

Definition 3.23. A closed subset C of a computably metrized M is effectively closed if $M \backslash C$ is c.e. open.

It should be clear that effectively closed sets are closed under finite unions and arbitrary computable intersections (meaning that the effective procedures listing the complements must be uniformly indexed). The following lemma is also an immediate consequence of the definition:

Lemma 3.24 (Folklore). *Suppose $f : A \to B$ is a computable surjection, and assume C is effectively closed in B. Then $f^{-1}(C)$ is effectively closed (in A).*

Proof. This is because $A \backslash f^{-1}(C) = f^{-1}(B \backslash C)$ is c.e. open. To see why, recall that f is computable if, and only if, it is effectively open. If a basic open D is enumerated into $B \backslash C$, then we will list all pairs (D', D'') in the continuous name of f such that D'' is formally included in D. Since every $x \in D$ is contained in such a D'' (by surjectivity), this will give an enumeration of $f^{-1}(D)$. Putting these enumerations together for all such D in $B \backslash C$, we will list its preimage. □

Another observation is an easy generalization of a well-known fact about Π_1^0 classes.

Fact 3.25 (Folklore). *Suppose $P = \{p\}$ is effectively closed singleton in a computably compact space X. Then the only point p of P is (uniformly) computable[10].*

(This can of course be pushed to show that isolated points can also be computed, though non-uniformly.)

Proof. Given n, wait for a basic open ball D of radius 2^{-n} and finitely many basic open $B_1, \ldots, B_n \in X \backslash P$ such that $D, B_1, \ldots, B_n$ cover X. Then $p \in D$. □

The fact above admits various generalisations, but we will encounter one such generalisation (specifically, in the proof of Corollary 4.11). More generally, effectively closed sets, Π_1^0-classes, computable functions, and computably compact spaces are closely technically related. To make this relationship explicit, we need one more definition. As usual, we identify basic open balls with their indices.

Definition 3.26. A closed subset of a computably metrized M is c.e. if $\{B : B \text{ basic open and } B \cap C \neq \varnothing\}$ is c.e.

Sets that satisfy the definition above are sometimes called *computably overt* in the literature. For more about effectively overt spaces, we cite [21, 34]. The fact below is well-known; we are not sure who was the first to observe this. We cite, e.g., [21, Corollary 3.14(1)].

Lemma 3.27. *A closed subset C of a computably metrized space M is computably enumerable if, and only if, C possesses a uniformly computable (in M) dense sequence of points.*

Note that the dense sequence makes C a computable Polish space under the induced metric.

---

[10]As has been pointed out by one of the referees, this seemingly obvious and tame fact is indeed rather useful in many applications of effective compactness. We will discuss some of these applications in due course.

Proof of Lemma 3.27. Suppose C possesses such a computable sequence $(\alpha_i)_{i\in\mathbb{N}}$. Then the density of the sequence in C implies that $B_i \cap C \neq \varnothing$ iff $\exists j \alpha_j \in B_i$, which is a uniformly Σ_1^0 statement.

Now suppose C is a computably enumerable closed subset of M. Our goal is to construct a uniformly computable (finite or infinite) sequence of points $(\alpha_i)_{i\in I}$ that is dense in C. The proof below does not have to be non-uniform, but for notational convenience we split it into two cases, namely, when C is finite or infinite.

If C is finite, then it clearly contains only computable points. To see why, assume it is not empty (in this case there is nothing to prove) and let x be any point in C. Take a ball small enough so that $\{x\} \in B \cap C$. To get an 2^{-n}-approximation to x, wait for a basic open B' of radius $< 2^{-n}$ so that $B' \cap C \neq \varnothing$ and additionally B' is formally contained in B.

Without loss of generality, we assume C is infinite. We uniformly approximate a computable sequence by stages. Before we describe stage s, recall that two basic open balls U and W are formally s-disjoint if $r(U) + r(W) < d(cntr(U), cntr(W))$ and this can be seen after calculating the radii and the distance with precision 2^{-s}. Then U and W are formally disjoint if the are formally s-disjoint for some s.

At stage 0, search for a basic open ball $B_{0,0}$ of radius < 1 such that $B_{0,0} \cap C \neq \varnothing$. If such a ball is never found then do nothing. If it is every found, go to the next stage.

At stage $s > 1$ first check whether there exists a basic open ball with index $< s$ which is formally s-disjoint from $B_{0,s-1}, \ldots, B_{s-1,s-1}$. If such a basic open B exists, then choose the first fund $B_{s,s} \subseteq_{form} B$ and $B_{i,s} \subseteq_{form} B_{i,s-1}$, $i < s$ such that $B_{j,s} \cap C \neq \varnothing$, the $B_{j,s}$ are pairwise formally disjoint and $r(B_{j,s}) < 2^{-s}$, $j = 0, \ldots, s$. Otherwise, if no such B exists, fix the first found pairwise formally disjoint $B_{0,s}, \ldots, B_{s,s}$ that intersect C, have radii $< 2^{-s}$, and such that $B_{i,s} \subseteq_{form} B_{i,s-1}$ for $i < s$ (note there is no further restriction on $B_{s,s}$). This ends the construction.

Let α_i be the unique point of the Polish space such that $\{\alpha_i\} = \bigcap_{j\geqslant i} B_{i,j}$. Since the construction is uniform and the radii of balls are rapidly shrinking, the points α_i form a uniformly computable sequence. Since each of the $B_{i,j}$ $(j = i, i+1, \ldots)$ intersects C and C is closed, each $\alpha_i \in C$. It remains to check that $(\alpha_i)_{i\in\mathbb{N}}$ is dense in C. Let $\overline{(\alpha_i)_{i\in\mathbb{N}}}$ be the completion of $(\alpha_i)_{i\in\mathbb{N}}$.

Suppose $c \in C$. We claim that $c \in \overline{(\alpha_i)_{i\in\mathbb{N}}}$. Assume $c \notin \overline{(\alpha_i)_{i\in\mathbb{N}}}$, and there is a ball U centred in c which is outside $\overline{(\alpha_i)_{i\in\mathbb{N}}}$. There will be a basic open ball $B' \ni c$ of radius at most 2^{-n} and which is formally contained in U with precision 2^{-n}:

$$d(cntr(U), cntr(B')) + r(B') < r(U) + 2^{-n}.$$

Then at every stage $s > n + 4$ the balls $B_{i,s-1}$, $i = 0, \ldots, s-1$ will be formally s-disjoint from B, as will be readily witnessed by the metric. At some late enough stage s' we will get a confirmation that $B \cap C \neq \varnothing$. There exist only finitely many basic balls that have their index smaller than the index of B. Therefore, eventually B will be used to define $B_{t,t} \subseteq_{form} B$, contradicting the assumption that $U \cap \overline{(\alpha_i)_{i\in\mathbb{N}}} = \varnothing$. $\square$

Definition 3.28. A closed subset of a computably metrized M is *computable* if it c.e. and effectively closed.

As we mentioned immediately after the statement of Lemma 3.27, a c.e. closed subset of a computable metric space M can be viewed as a computably metrized space under the induced metric. It thus makes sense to ask when this subspace is computably compact. If M is computably compact, then it is both computable compact subset of itself and an effectively closed subset of itself. Interestingly, this trivial example is not misleading. The proposition below is also folklore; see, e.g., [21, Corollary 4.14(1)].

Proposition 3.29. *For a closed subset C of a computably compact M, the following are equivalent:*
  1. C is a computably compact subspace of M;
  2. C is computable.

Before we proceed to the proof, the reader might well wonder what is wrong with the following analog of the classical argument that closed subsets of compact spaces are compact:

Suppose that C is a effectively closed subset of a computably compact space P, then C is computably compact. Apply computable compactness we can compute a finite subcover and then attempt to 'throw away' $\overline{C}$ (that can be listed). One obvious problem with this idea is that we can never be sure whether a basic open ball B (in P)

intersects C, and thus we can never be sure whether we can keep B in our cover of C. By Lemma 3.27, this is equivalent to locating a computable dense sequence in C. This problem cannot be circumvented as C might not contain a computable dense sequence. Indeed, as Kleene showed in the 1950's there are effectively closed subsets of 2^ω containing no computable points at all. We discuss more about this issue in Section 4. We turn to the proof of Proposition 3.29.

Proof. Assume (1). It is clear that C is c.e. (by Lemma 3.27). To list its complement, fix $x \in M\backslash C$. Let $\delta = \inf_{c \in C} d(x,c)$. Then any $\delta/4$ -cover of C must be formally disjoint from any ball centred in y with $d(y,x) < \delta/4$. For every n, fix a finite 2^{-n}-cover $\mathcal{K}$ of C. It follows that $M\backslash C$ is equal to the union of the (uniformly) effectively open sets U_n, where

$$\{B : B \text{ basic open and } B \text{ is formally disjoint from every ball in } \mathcal{K}\}.$$

It follows that (2) holds.

Now assume (2). C is computably metrized by Lemma 3.27; let (y_j) be the computable dense subsequence. Fix $\epsilon = 2^{-n}$. We need to find an ϵ-cover of C by basic open balls[11]. Regardless of whether the balls involved are basic or not, as long as their centres and radii are computable, the relation of formal containment remains c.e.

If a finite collection $\mathcal{K}$ of basic open (in C) balls formally contains a cover $\mathcal{K}'$ by basic open (in M) balls, then clearly $\mathcal{K}$ is a cover of C. We claim that this condition is also necessary (for $\mathcal{K}$ to cover C).

From the proof of Lemma 3.3 we know that, for a given cover $\mathcal{K}$ of C by (basic or not) open balls there is a small enough ϵ such that every ϵ-cover of C will be formally contained in at least one ball of $\mathcal{K}$.

Take $\delta = \epsilon/4$. Fix a finite δ-cover $\mathcal{K}'$ of C by balls that are centred in special points of M, not C. Every $B' \in \mathcal{K}'$ intersects C at some point x, and by the choice of δ, $d(x, cntr(B')) + \delta < \epsilon$, thus $B(\epsilon, x) \supset_{form} B'$. By transitivity of formal inclusion[12], we have that B' must be formally contained in some ball in $\mathcal{K}$.

By computable compactness of M and computability of C, we can produce at least one δ-cover $\mathcal{K}'$ of C by basic open balls of M, uniformly in δ. (To see why, replace every basic open ball in the c.e open name of $M\backslash C$ by the effective union of balls of radii at most δ that are formally contained in it. This gives a new c.e. enumeration of the complement of C but with balls of radii at most δ. Then take the c.e. collection of all basic δ-balls that intersect C. Together these sets of balls cover M. Initiate the combined enumeration of these two c.e. sets and wait until at some finite stage we discover that we have a cover of M.)

Since formal inclusion is c.e., this gives a procedure of listing covers of C by basic balls (in C). $\square$

We see that computable compactness and computability of a closed set are very closely related notions. We have already mentioned above that an effectively closed set does not have to be computable, in general. However, suppose C is an effectively closed (Π^0_1) subset of, say, $\mathbb{R}^3$, and suppose further that we know that it is a sphere or a ball. Is it computably closed? J.Miller [107] used algebraic topology to answer this question in the affirmative (in fact, in any dimension). The idea is that, roughly, we can non-uniformly localise it to a compact box in $\mathbb{R}^n$ and then use that a computably compact ball will eventually contained in a simplex that "looks like the ball"; algebraic topology helps to make this formal. The results of Miller have been extended (e.g., to compact manifolds under some extra conditions) in [23, 66, 68, 70]. But of course, if we are interested in *presentations* of spaces and especially up to homeomorphism, then a sphere or a compact surface is clearly homeomorphic to a computably compact space (e.g., given by a geometric realisation of its triangulation).

In general, there is no good reason why a basic closed ball (or the closure of a basic open ball) in an abstract Polish space needs to be computable closed; pathological examples similar to Ex. 2.9 can be constructed. Interestingly, it follows that there are always *enough* closed balls with computable radii that are computable closed as

---

[11]Note that y_j does not have to be special in M, and thus basic open balls in C do not have to be basic open in M. Nonetheless, an open ball of M centred in a computable point y and having a computable (more generally, left-c.e.) radius r is effectively open:

$$B(y,r) = \bigcup \{B(x,q) : d(x,y) + q < r\},$$

that is, $B(y,r)$ it is the union of basic open balls formally contained in it. Note effective openness of $B(y,r)$ is uniform in y and r.

[12]This is because $d(x,y) + r_2 < r_1$ and $d(y,z) + r_3 < r_2$ (together with the triangle inequality) imply $d(x,z) + r_3 < r_1$.

sets, and indeed uniformly so[13]. More specifically, an immediate consequence of the proposition above is that, in Theorem 3.16, we can additionally state that the basic closed balls in the covers are computable closed sets:

Proposition 3.30. *Suppose $K = \bigcup_n K_n$ is a fully $\cap$-decidable system of covers of a computably metrized M. Then each computable closed ball D^c in K is a computable closed set (thus, is a computably compact subspace of M), and this is uniform.*

Proof. Lemma 3.20 says that each such D^c is c.e. closed. If $x \in M \backslash D^c$, then x inside an open ball C that is formally disjoint from D^c, and such balls can be computably enumerated[14]. Thus, the c.e. union of such open balls formally disjoint from D^c makes up the complement of D^c. $\qquad\square$

The fact above will be useful when we talk about the universality of 2^ω. More generally, it seems to be useful in any iterated recursive argument in which a space is eventually replaced by its compact subset, and then a subset of this subset, etc. The lemma below will also be useful throughout the rest of the paper. It is also well-known; see [149, Theorem 3.3] and [120, Proposition 5.5].

Lemma 3.31. *Suppose $f : X \to Y$ is a computable map, and assume X is computably compact. Then $f(X)$ is c.e. closed (in Y) and computably compact.*

Proof. Let (x_i) is a computable dense sequence in X. Then $(f(x_i))$ is dense in $f(X)$. (Every $\alpha = \lim_j x_j$ for some subsequence (x_j) and, by continuity, $f(\alpha) = \lim_j f(x_j)$, so $f(X) \subseteq cl(f(x_i))$. Suppose $\xi \in cl(f(x_i))$, say $\xi = \lim_j f(x_j)$. By compactness, (x_j) has a convergent subsequence (x_{j_k}), so let $z = \lim_k x_{j_k} \in X$ be its limit. Then $f(z) = \lim_k f(x_{j_k}) = \lim_j f(x_j) = \xi$.)

Given a cover B_j of $f(X)$ by basic open (in $f(X)$) balls of radius 2^{-n} centred in $f(x_j)$, calculate the c.e. names of each B_j in Y and begin enumeration of $f^{-1}(C)$ for each such open basic C; note it could be that some of these $f^{-1}(C)$ will be undefined. At some stage the preimages must cover the whole X. We can see which B_j included the basic open (in Y) balls whose images were sufficient to cover X. This gives a way of producing at least one 2^{-n}-cover of $f(X)$ uniformly in n; now apply Lemma 3.3. $\qquad\square$

Combining Lemma 3.31 with Proposition 3.29, we get:

Corollary 3.32. *Suppose $f : X \to Y$ is computable and X is computably compact.*
- *If f is surjective then $Y = f(X)$ must be computably compact.*
- *If Y is computably compact then $f(X)$ is a computable closed subset of Y.*

In computable algebra, the inverse of a computable bijective map is clearly computable as well. In contrast, there is no reason why the inverse of a computable bijection between spaces has to be computable even if its inverse is continuous (we mention here that this is actually true for isometric maps). The theorem below is elementary and is folklore (e.g., [18, Corollary 6.7]), but it is rather important because it tells us that effectively continuous maps are the right morphisms in the category of computably compact spaces.

Theorem 3.33. *Suppose $f : X \to Y$ is a computable bijection between computably metrized spaces, and assume X is computably compact. Then Y is also computably compact, and f^{-1} is computable.*

It is easy to see that f is indeed a homeomorphism[15]. Our task is to produce a more subtle computable version of this observation.

Proof. computable compactness of Y follows from the corollary above. Given a (not necessarily) special point $y \in Y$, act computably relative to y. The set $Y \backslash \{y\}$ is effectively open relative to y. Indeed, for every $z \neq y$ there must exist formally disjoint basic open $B \ni y$ and $D \ni z$. Thus, it is sufficient to list, effectively in y, all basic open balls formally disjoint from some ball in the name of y.

---

[13]Issues of this sort are investigated in detail in [122].

[14]To see why, let c be the centre of D^c and r its radius, and assume $d(c, x) = r + \delta$. There must be a special x_i such that $d(x_i, x) < \delta/2$. Take the basic open ball $C = B(x_i, \delta/2)$. Then the distance between their centres is $d(c, \alpha) > r + \delta - d(x_i, x) > r + \delta - \delta/2 = r + \delta/2$, which is the sum of their raddii. So the balls are formally disjoint.

[15]Fix any for every open Z in X, its complement is closed and thus is compact. Since the continuous image of a compact set is compact and therefore closed, $f(X \backslash Z)$ is closed. Since f is a bijection, $f(X \backslash Z) = f(X) \backslash f(Z) = Y \backslash f(Z)$, which makes $f(Z)$ open.

Since f is computable, $f^{-1}(Y\backslash\{y\})$ is effectively open relative to y. Since f is bijective, we have that

$$X\backslash f^{-1}(Y\backslash\{y\}) = \{f^{-1}(y)\}.$$

To list a basic open D into the name of $x = f^{-1}(y)$, wait for finitely many basic open balls $B_1, \ldots, B_k$ in $X\backslash C_0 = f^{-1}(Y\backslash\{y\})$ such that $D, B_1, \ldots, B_k$ cover X. Note that, for each $D \ni x$, such a finite collection must exist by compactness. Since X is computably compact, the process described above is uniformly computable in y, and thus f^{-1} is computable. $\square$

One useful consequence says that partial inverses also exist under some conditions.

Corollary 3.34. *Suppose $f : C \to M$ is a computable injective embedding of a computably compact C into a computably compact M. Then f^{-1} is computable on $f(C)$ (when viewed as a map between the induced computable structure on $f(C)$ and C).*

Proof. Let N be the computably compact induced computable metrization of $f(C)$ that exists because of Lemma 3.31 and which is furthermore computably compact by Corollary 3.32 and Proposition 3.29. The map $f : C \to f(C)$ can be viewed as a computable map from C to the induced computable metrication on C, as follows. When (B, C) is enumerated in the name of f, find a basic open ball D in $f(C)$ that formally contains C. The basic open balls in C are balls with centres that are special in $f(C)$ but are computable in M, but formal containment is still c.e. So we enumerate (B, D) into the new name of f.

Another way to view this is to replace an ϵ-approximation $x_i \in M$ to $f(y)$ by an 2ϵ-approximation $c_i \in C$ to $f(y)$; it must exist.

To compute f^{-1}, apply the previous theorem. $\square$

We also include another nice fact connecting computable compactness with computable closed sets proved by Brattka [17]. The result will be used later (in Corollary 4.3) to clarify one of the well-known applications of computable compactness.

Theorem 3.35. *Let X and Y be computably compact spaces and $f : X \to Y$. Then f is computable if and only if $graph(f)$ is effectively closed and if and only if $graph(f)$ is computable closed.*

Proof. We know that $X \times Y$ is computably compact. Suppose f is computable. Then $f(x) = y$ is clearly a Π^0_1 property.

Now assume $graph(f)$ is effectively closed. The subspace $\{x\} \times Y$ is effectively closed relative to x, thus $graph(f) \cap \{x\} \times Y$ is an effectively closed singleton relative to x. By Fact 3.25 relativized to x, given x we can compute $(x, f(x))$ and, thus $f(x)$.

It remains to note that $graph(f)$ is actually c.e. closed for a computable f, because if a basic open B (in $X \times Y$) intersects the graph then we will eventually recognise it. $\square$

We did not really have to assume that Y is computably compact; the proof would still work. But of course, by Theorem 3.33 the space $f(X)$ has to be computably compact, so we can always replace Y with $f(X)$.

3.6. **Computable universality of the Hilbert cube.** We now discuss another way of looking at computably compact spaces, using the *Hilbert cube* H. In $H = [0,1]^\omega$, we define the distance as $d((x_i), (y_i)) = \sum_i 2^{-i} d(x_i, y_i)$. A canonical dense sequence is given by rational sequences that are eventually zero.

The Hilbert cube is a universal space for computably compact spaces. To see this, recall that all our spaces are complete with respect to their metric. We embed a given compact computably metrized space M into the Hilbert cube, as follows. Assume the diameter of M is at most one[16]. Map a point $x \in M$ to the sequence $(d(x, x_i))_{i \in N}$ in $H = [0,1]^\omega$.

It is easy to see that this embedding is computable and its image is c.e. closed. We will see that the image does not have to be effectively closed, and hence the image does not have to be computable by Proposition 3.29. This embedding gives yet another characterisation of computable compactness, but this time up to homeomorphism:

---

[16]If it is > 1, then replace the metric with the new metric $\frac{1}{n}d(\cdot, \cdot)$, where n is a large enough positive integer. You can also redefine the metric to be equal to one on a pair x and y if the original distance between x and y is greater than 1. The latter method is computably uniform and gives a metric computably compatible with the original one, i.e., the identity map is a computable homeomorphism between the old metrized space and the newly metrized one.

Theorem 3.36. *For a computably metrized compact M, the following are equivalent.*

> (1) *M is homeomorphic to a computably compact space;*
> (2) *M is homeomorphic to a computable closed subset of H.*

Proof. Note that H being a (computable) product of computably compact spaces is itself computably compact by Proposition 3.5.

If the space has a computably compact presentation, then its image under the canonical embedding will also be computably compact (Proposition 3.29, Lemma 3.31), and thus the image will be computable closed by Corollary 3.31. On the other hand, if a closed subset of H homeomorphic to the space is a computable closed subset, then it gives a computably compact homeomorphic presentation of the space by Proposition 3.29 because H, being a (computable) product of computably compact spaces, is itself computably compact. □

Remark 3.37. We note that, by Theorem 3.33, if (the fixed computable complete metrization of) M is computably compact, then it is computably homeomorphic to $f(M)$, meaning that both f and f^{-1} (when restricted to M) have to be computable; see Corollary 3.34.

In other words, we can always effectively reduce the study of computably compact spaces up to homeomorphism to the investigation of computable closed subsets of H. One pleasant and well-known (e.g., [21, Corollary 3.14]) characterization of computably closed sets in H is given below.

Lemma 3.38. *An closed subset C of H is computable if, and only if, $D(x) = d(x, C)$ is a computable function.*

Proof. We can list the basic open balls $B(x_i, r)$ for which $D(x_i) > r$, and they must cover $M \backslash C$. But we can also list the balls $B(x_j, q)$ such that $D(x_i) < r$, and these are exactly the basic open balls that intersect C. □

Note that we can uniformly list r such that $D(x_i) > r$, i.e. D is left-c.e. (lower semi-computable) iff C is effectively closed, and we can uniformly list r such that $D(x_i) < r$, i.e., D is right-c.e. (upper semi-computable) iff C is c.e. closed. We omit details. We also note that there is really nothing special about the choice of H in the lemma above; it could as well be some other computably compact space.

The closed subsets of H give us yet another way to look at computably compact spaces. Recall that compact spaces correspond to c.e. closed subsets of H, and computably compact ones to computable closed subsets of H. Note that the space of all compact (or closed) subsets $\mathcal{C}(H)$ of H is a Polish metric space in which the metric is given by the Hausdorff distance and the countable dense set is given by finite discrete subsets of special points of H.

Fact 3.39. [Folklore] For a c.e. closed subset C of H, C is computable iff C is a computable point in $\mathcal{C}(H)$.

Proof. If C is computable then it is computably compact by Proposition 3.29. If (B_i) is a finite 2^{-n} cover of C and x_i is a special point in B_i, then every point of C is at most 2^{-n+1}-far from one of the x_i.

On the other hand, assume $c \in B_i \cap C$; indeed c is contained in B_i together with an ϵ-ball. Thus, there is ϵ so small that any D that is ϵ-close to C in $\mathcal{C}(H)$ contains a special point in B_i. This will be eventually recognised, and thus such B_i can be computably enumerated, making C c.e. closed. □

Of course, there is again nothing really special about H is the fact and the remark above, and it can be replaced by some other computably compact space if necessary. For instance, one could look at computability of graphs of functions $f : [0, 1] \to [0, 1]$ and see that f is computable iff it can be approximated by piecewise linear functions iff the graph is a computable closed set (cf. Theorem 3.35). Generalizations of this fact can be found in [17].

3.7. **Computable universality of Cantor space.** We assume that our spaces are non-empty Polish ones, and so that our metrics are complete.

We identify 2^ω with its standard computable presentation of Cantor space by infinite strings under the usual ultrametric. It is well-known that every compact metric space is a homeomorphic image of 2^ω; this is the classical Hausdorff-Alexandroff theorem. The following computable version of this fact fully characterises computable compactness.

Theorem 3.40 ([18]). *A computably metrized (non-empty) compact M is computably compact if, and only if, there is a computable continuous surjective $f : 2^\omega \to M$.*

If M is a computable image of 2^ω then it has to be computably compact by Theorem 3.33. To this end, we therefore assume M is computably compact. We give two proofs of the harder direction of the theorem. The first proof exploits the strongest so far combinatorial characterisation of computable compactness given in Subsection 3.3 (specifically, Theorem 3.16) and basically follows the standard textbook argument pretty closely. The second proof is more indirect. It handles the combinatorics differently using a space-filling curve and a technical Lemma 3.41 that is of independent interest.

The first proof. Fix a fully $\cap$-decidable system of covers (K_n) of M that exists by Theorem 3.3. By Remark 3.18, we can use finite basic computable *closed* covers throughout.

We follow the standard classical topological proof very closely, and we use $\cap$-decidability of $\bigcup_n K_n$ throughout. Suppose $K_0 = \{D_1, \ldots D_k\}$, and fix $D_i \in K_0$. Then K_1 contains finitely many balls that cover D_i, denote them $D_{i,j}$. These can be computed and indeed, these $D_{i,j}$ are exactly the closed balls in K_1 that intersect D_i, because the rest are disjoint from it. Define $\hat{D}_{\langle i,j \rangle} = D_i \cap D_{i,j}$.

We can computably proceed by recursion and define $\hat{D}_\sigma$, which is a finite non-empty intersection of basic closed balls, where σ ranges over a computably branching tree T with no dead ends (observe that $\sigma \neq \tau$ does not necessarily imply that $\hat{D}_\sigma \neq \hat{D}_\tau$). Equip the set of all infinite paths through $[T]$ with the standard (longest common prefix) ultrametric; then $[T]$ is computably metrized space in which the dense sequence is given by $\sigma 1^\omega$, $\sigma \in T$. We identify σ with the basic clopen ball of $[T]$ consisting of all strings with prefix σ.

Also recall that, by the construction of (K_n), without loss of generality we can assume that basic open balls intersect whenever the respective closed balls intersect, and thus we can always calculate a special point x_σ in each $\hat{D}_\sigma$ (see Remark 3.18). We could view x_σ to be an ϵ-approximation to any path in $[T]$ extending σ, where $\epsilon = 2^{-n+1}$ for the largest n such that a ball from K_n is mentioned in $\hat{D}_\sigma$. So we define $f(\sigma) = x_\sigma$ with precision 2^{-n+1}.

For an infinite path $\xi \in [T]$, $\hat{D}_{\xi \upharpoonright n} \subseteq \hat{D}_{\xi \upharpoonright m}$ whenever $m \leq n$ are prefixes of ξ, and since the diameter of $\hat{D}_{\xi \upharpoonright n}$ is at most 2^{-n+1} and it is non-empty, we conclude that

$$\bigcap_n \hat{D}_{\xi \upharpoonright n} = \{\alpha\}$$

for some $\alpha \in M$. So we set $f(\xi) = \alpha \in M$. It is routine to show that the procedure above defines a computable and surjective $f : [T] \to M$. It is not difficult to see that $[T]$ is a computable image of 2^ω; as we promised in the preliminaries, we include a proof of this well-known fact.

Claim 1 (Folklore). (1) If T is a computable, computably branching tree with no dead ends then there is a computable surjective map from 2^ω onto $[T]$.
(2) For every computable, non-empty Π_1^0 class C there is a surjective computable map from 2^ω onto C.

Proof. We first reduce (2) to (1). Realise C as the set of (infinite) paths through a computably branching tree T without terminal nodes, as follows. Computably rearrange T into a new tree Γ such that $[T]$ is computably homeomorphic to $[\Gamma]$ and Γ is (at most) binary. To do so, split a node only if both basic clopen sets associated with the two successors of the node in T contain elements of C. (Note that, given a basic clopen set, we can *decide* whether it intersects C.) Neither Γ nor T has terminal nodes, and there is a computable homeomorphism between C and $[\Gamma]$.

For (1), we can also reduce the case of an arbitrary computably branching tree to the case when the tree is at most binary. If a node splits into n successors, where $n > 2$, replace it with a gadget in which every node has at most two successors. This gives a computable and (at most) binary tree Γ with no dead ends such that $[\Gamma]$ is computably homeomorphic to $[T]$.

Thus, it remains to prove (1) for such a Γ. Define the map g from 2^ω onto $[\Gamma]$ by recursion. We define the name of g by mapping clopen sets (on)to clopen sets. We also identify finite strings with the respective clopen sets in both trees.

At a stage s, assume g has already been declared on paths/basic clopen sets of length s-1. Suppose $g(\sigma) = \tau \in \Gamma$ such that $|\tau| = |\sigma| = n - 1$. If $\tau 0$ and $\tau 1$ both exist in Γ, then set $g(\sigma 1) = \tau 1$ and $g(\sigma 0) = \tau 0$. Otherwise, without loss of generality, only $\tau 0$ exists. In this case, set $g(\sigma i) = \tau 1$ for $i = 0, 1$. Do that for every string of length n, and then go to the next stage. The map is clearly computable and surjective, and thus induces a computable surjective map of 2^ω onto C. $\square$

In combination with f defined above, this gives a computable surjection of 2^ω onto M. $\qquad\square$

The second proof. Recall that we assumed that M is computably compact. Without loss of generality, we can assume it is a computable closed subset of H; see Remark 3.37. Recall that there is a computable map from 2^ω onto $[0, 1]$; for instance, map every infinite sequence σ in 2^ω into the binary expansion $\sum_{i\in\omega} 2^{-i-1}\sigma(i)$. Also, the famous Hilbert's curve computably maps $[0, 1]$ onto $H = [0, 1]^\omega$; see, e.g., [131] for a *primitive recursive* version due to Schoenberg. (See [5] for a detailed explanation of computability of this particular construction. We also cite [32] for more about computability of space-filling curves.) This gives a computable surjective $f : 2^\omega \to H \supseteq M$, and we know from Lemma 3.24 that $f^{-1}(M)$ is a Π_1^0-class which, unfortunately, does not have to be computable in general. Recall that, in the beginning of the subsection, we assumed that M (thus, P) is non-empty. If it *were* a computable Π_1^0 class, then we would be able to computably surjectively map 2^ω onto it by Claim 1. But first, we must be in the position to apply the claim. The lemma below helps.

Lemma 3.41. *Suppose $f : 2^\omega \to K$ is computable, surjective, and K is computably compact. If $P \subseteq K$ is (non-empty) computable closed, then there is a computable $\tilde{f} : 2^\omega \to K$ and a computable Π_1^0-class $C \subseteq 2^\omega$ such that $\tilde{f}(C) = P$.*

Furthermore, $\tilde{f}$ can be chosen so that $\tilde{f}$ agrees with f on $f^{-1}(P)$ and $\tilde{f}(2^\omega\backslash C) \subseteq K\backslash P$; we will also include the verification of these properties in our proof below. Also, we will see that the simultaneous construction of $\tilde{f}$ and C is uniform. The proof below is somewhat informal: after all, we have already proven the theorem. We hope that the elementary formal details that are missing should be easy to reconstruct.

Proof. This is done as follows. We use a c.e. formal open name of f and computable compactness of K and computability of P throughout. In particular, we will use that f is uniformly computably continuous. This is done by listing 2^{-n}-covers K_n of K and using the formal name of f to find a cover S of 2^ω such that, for every $\sigma \in S$ there is a $D \in K_n$ so that (σ, D) is in the name.

Since P is computable and thus is a computably compact subspace (see Proposition 3.29), we can assume that we know which open sets in K_n intersect P and which do not. To see why, create two lists: one is the list of all finite covers of P, and the second list includes finite collections of basic open balls in K that are formally disjoint from *some* cover from the first list. For every ϵ, there is a finite ϵ-cover $K' \cup K''$ of K, where K' is from the first list and K'' is from the second list[17]. The finite set of balls K'' can be empty, but K' is never empty. If $\epsilon = 2^{-n}$ then we can set $K_n = K' \cup K''$.

The construction of C and $\tilde{f}$ proceeds as follows.

Suppose at a stage we see that, for some basic open set σ in 2^ω, $f(\sigma) \subseteq B$ for some ϵ-ball B such that $B \cap P = \varnothing$. Then declare σ outside of the closed set C that we build, and let (the name of) $\tilde{f}$ copy (the name of) f.

Now suppose, using the same notation and premises, $B \cap P \neq \varnothing$. It still possible that $f(\sigma) \cap P = \varnothing$ (because all we know is that $f(\sigma) \subseteq B$, so it can miss P), but we do not know that yet. In this case declare that σ intersects C and proceed to the next stage. (As we made sure above, for any such B and σ that we use at the stage we can decide whether B intersects P or not.)

At the next stage we will find a refined cover of K and 2^ω and, in particular, of σ. If at least one such σ' refining (extending) σ has the property that $f(\sigma') \subset B'$, where $B' \cap P \neq \varnothing$ (here B' is taken from the more refined cover of K and B') then we let (the name of) $\tilde{f}$ copy (the name of) f.

Of course, it is entirely possible that we discover that for *all* of the finitely many extensions σ' of σ, the respective ball B' (such that $f(\sigma') \subseteq B'$) does *not* intersect P. However, we have already declared that $\sigma \cap C \neq \varnothing$.

In this case, go to the previous stage and find a computable point $x \in B \cap P$ (that can be uniformly extracted from the proof of Lemma 3.27) and declare

$$\tilde{f}(\xi) = x$$

---

[17]Every point in $K\backslash P$ is contained in a basic ball formally disjoint from some finite cover of P by basic open balls. If we take the collection of all such balls around all such $x \in K\backslash P$ of radius at most ϵ, then, together with any (finite) cover of P, they must cover the whole space K. Thus, there is a finite subcover in which we can keep all the finitely many ϵ-balls that cover P and intersect P. By our assumption, K is computably compact / computable closed, and thus all these conditions are c.e.; we just wait until such a finite ϵ-cover of K is found.

for every $\xi \in B$. In this case we also say that the clopen ball B is declared *artificially in* C. (This is also the only case when $\tilde{f}$ does not agree with f; we set f and $\tilde{f}$ equal in all other cases.)

By construction, $\tilde{f}$ is well-defined on all of 2^ω and is computable. We need to argue that it additionally has the properties claimed in the statement of the lemma.

If $\xi \in B$ then clearly $\tilde{f}(\xi) \in P$. If $\xi \in f^{-1}(P)$ then at no stage it can be declared in a ball which is out or artificially in C, so it must be that $\tilde{f}$ and f agree on ξ.

If ξ is such that it is never in any ball that is artificially in or out, it means that for every 2^{-n} there must be a point $\alpha_n \in P$ that is 2^{-n}-close to $\tilde{f}(\xi)$, and we see that $\tilde{\xi} = \lim_n \alpha_n \in P$ because P is compact and thus closed.

Finally, $f(2^\omega \backslash C) \subseteq K \backslash P$ follows from the fact that if a basic clopen σ is declared out of C, we let $\tilde{f}$ follow f in its definition within σ. $\qquad\square$

It follows that M is a computable surjective image of a computable Π_1^0-class. It remains to map 2^ω to such class surjectively using Claim 1. This finishes the second proof. $\qquad\square$

Note that, in the second proof, the combinatorics is handled using Lemma 3.41 rather than using $\cap$-decidable covers; we did not need them here.

4. Applications

4.1. **A few elementary applications of Π_1^0-classes.** In this subsection we briefly discuss several applications of effectively closed sets and Π_1^0 classes in classical computable analysis. Most of these applications are well-known and are not difficult. However, they serve as a good illustration of the convenience of the techniques we have described in the previous sections. Indeed, in each case we can come up with a brute-force direct proof which would however be much less pleasant and often would give a less general result too.

Consider a computable function $f : [0,1] \to \mathbb{R}$. Then the set of zeroes $Z_f = \{x \in [0,1] : f(x) = 0\}$ is effectively closed in $[0,1]$. Following result shows that any Π_1^0 class can be 'realized' as the set of zeros of a computable function. This allows for simple applications of Π_1^0 classes in real analysis.

Theorem 4.1 (Nerode and Hwang [114]). *Given a Π_1^0 class C (C is thought of a subset of the Cantor set) there is a computable function $f : [0,1] \to \mathbb{R}$ whose zeroes $Z_f = \{x : f(x) = 0\}$ exactly the members of C.*

Sketch. Define a computable function by stages on the Cantor set (linear elsewhere) so that, while an interval is not yet declared out of C, f keeps getting closer to 0, say, from below. If the interval is declared out, freeze the function at this interval and, thus, keep it away from zero. It shall approach zero but only at points that correspond to paths in C. $\qquad\square$

Ignoring the Π_1^0 coding, this method of diagonalization sketched above goes back to Specker [139] and is a mainstay of Aberth [1], and arguably has roots in the work of Bishop [11]. We also cite [18, Corollary 3.14] for a more general version of Theorem 4.1.

Here are well-known (cf. [18, Theorem 11.8.3, Corollary 11.8.4]) easy applications:

Corollary 4.2. *For a function $f : [0,1] \to \mathbb{R}$, let $Z_f = \{x : f(x) = 0\}$ denote the set of its zeros.*

(1) There is a computable function $f : [0,1] \to \mathbb{R}$ with uncountably many zeroes and no computable zeroes.
(2) If $f : [0,1] \to \mathbb{R}$ is computable and $Z_f \neq \varnothing$, then then Z_f contains a low point.
(3) If $Z_f \neq \varnothing$ is finite then all of its members are computable.
(4) If Z_f is infinite and countable, then it contains infinitely many computable points.

Proof. (1) Fix an uncountable Π_1^0-class P without computable points and the apply Theorem 4.1; for example, some Π_1^0 class consisting of only Martin-Löf random reals (e.g., Downey-Hirschfeldt [36]).

(2) By Theorem 3.40, there is a computable surjection $g : 2^\omega \to [0,1]$, and by Lemma 3.41 the pre-image of the effectively closed $X = f^{-1}Z_f$ is also effectively closed in 2^ω, that is, it is a Π_1^0 class. Using this argument, we easily see that if Z_f is nonempty then it has a low point, by the Low Basis Theorem.

(3) This is because isolated members of effectively closed sets are computable.

(4) If Z_f is infinite and countable, and thus it has no perfect kernel. In particular, there must be infinitely many isolated points; apply (3). $\qquad\square$

The elementary corollary below entails that the intersection of two computable closed sets is not computable in general. This unfortunate property of computable closed sets is well-known. The issue is that, although we can list balls that intersect *both* sets, some balls like that can have no points from the intersection of the closed sets.

To see why the corollary implies this counter-intuitive property of closed sets, note that every non-empty computable closed set has at least one computable point (since it has to be effectively overt).

Corollary 4.3. *There exist two computable compact subsets of the unit square $[0,1]^2$ that intersect but have no computable points in the intersection.*

Proof. In view of Theorem 3.35, it is sufficient to take $C_1 = graph(f)$ for f in (2) of the theorem above, and $C_2 = [0,1]$ (the "x-axis") which is clearly computable too. $\square$

Another application of Π_1^0-classes, similar technically to (1) of Theorem 4.2 above, it is concerned with Markov computability. As mentioned in the introduction, the original definition of computable function used by Turing in [144], was that f is computable if there is a Turing operator taking any computable index of a computable real ξ to some index of the computable real $f(\xi)$. Note we could define this notion of computability for a function that is not even defined on non-computable reals, let alone continuity. It is of course much more interesting to have a continuous counter-example.

Theorem 4.4 (Folklore; see Remark 4.5). *There is a continuous, Markov computable $f : [0,1] \to R$ such that $\sup_{x \in [0,1]} f(x)$ is not computable and, thus, f is not computable (by, e.g., (1) of Proposition 3.5).*

The function constructed below, when viewed as a function on computable reals only, has a unique continuous extension on the whole interval $[0,1]$ that cannot be (uniformly, type II, Kleene) computable. Thus, another way to state the theorem is that there is a Markov computable function (which, by definition, can be merely defined only for computable reals) that has a continuous extension on $[0,1]$ but has no *computable* continuous extension on $[0,1]$.

Sketch. This is somewhat similar to the proof of Theorem 4.2(1). Fix a Π_1^0 class without computable members and define f on the Cantor set (and linearly elsewhere), but this time make its value approach a left-c.e. non-computable real α. Note that, if a point x is computable, then it has to be either on a segment of the Cantor set that was declared out at some stage, or it is on a linear segment connecting two such points that have been declared out. Observe also that each linear segment connects computable points. In either case, we can wait for the point to be listed in the effectively open complement of the homeomorphic image of the Π_1^0 class, go to the stage of the construction where that happened, and compute the index of the image. $\square$

Remark 4.5. The first example of a Markov computable function not extendible to a continuous computable one was provided by Aberth; see [1, Theorem 7.3]. We are not sure who was the first to construct such an example with domain restricted to $[0,1]$; this seems to be folklore. The history according to Aberth is in the back of his book [1, pp. 178-179]. Zaslavskii [151] was certainly among the first to study various basic properties of Markov computable functions. Similar ideas and constructions are routinely used in reverse mathematics. For example, see [136, Theorem IV.2.3(2)], which can be viewed as a slight modification of the construction found in Specker [139]. Specker used a Π_1^0 class with no computable members to construct a Markov computable function that approaches a computable real from below but does not attain the supremum at any computable point. Simpson uses the same idea to (essentially) construct an example of a Markov computable function on $[0,1]$ that is not even bounded, but the key idea remains the same. We simply slightly modify this construction and make the supremum approach a left-c.e. non-computable real instead of driving it to ∞ or to a computable value; the rest remains the same.

Another more recent application of Π_1^0 classes in computable analysis is by Barrett, Downey and Greenberg [6], and concerns Cousin's Lemma. This is a core lemma in the theory of the Denjoy integral. Recall that a *gauge* is a function $\delta : [0,1] \to \mathbb{R}^+$. A *tagged partition* is a partition $0 = a_1 < a_2 < \ldots a_n = 1$ together with a sequence $z_i \in (a_i, a_{i+1})$. For a gauge δ we say that a tagged partition is δ-*fine* iff for all i, $[a_i, a_{i+1}] \subset (z_i - \delta(z_i), z_i + \delta(z_i)))$. Cousin's Lemma states that for any gauge δ, there is a δ-fine partition. Using similar codings to those of Theorem 4.2, we have the following theorem.

Theorem 4.6 (Barrett, Downey, and Greenberg [6]). *There is a computable gauge δ with no computable δ-fine partition.*

Actually this was stated in [6] as "RCA$_0$ proves that 'Cousin's Lemma for continuous functions' is equivalent to WKL$_0$." Many results in Reverse Mathematics concerning WKL$_0$ correlate to coding Π_1^0 classes. (See Simpson [136].) In passing we remark that in [6], the authors looked also at Borel gauges δ. We say that f is effectively Baire 0 if it is computable, and effectively Baire $n+1$, if it is the pointwise limit of a computable collection of effectively Baire n functions. In [6] it is shown that if α is a computable ordinal, then there is a effective Baire 2 function f such that any δ-fine partition computes $\varnothing^{(\alpha)}$.

4.2. **The space of isometries.** An isometry is a metric-preserving map. It is clearly continuous. Note that an isometry is always injective, and if f is surjective then we say that it is an isometric isomorphism. Using a brute-force search, we can easily show that the inverse of a computable isometric isomorphism is always a computable map even if the spaces are not computably compact. In particular, we do not need to refer to Theorem 3.33 to compute the inverse of an isometric isomorphism.

Remark 4.7. However, we can argue that its proof can be used to find a more satisfying way to compute it. For instance, we might be able to argue that a primitive recursive procedure might be possible under the right choice of definitions. The subject of primitive recursive or "punctual" analysis (see [7, 39]) is largely unexplored. Moreover, connections between polynomial time analysis (Ko and Friedman [84], Ko [83]) and compactness also remain to be analysed.

The following result was stated in [103] without proof. As correctly noted in [46] the space of isometries between any computable Polish spaces can be viewed as a Π_1^0-class in Baire space, but since we are only interested in compact spaces and classes we shall not explain this. The details can be found in [45].

Theorem 4.8 (Melnikov and Nies). *Suppose X is computably compact and Y is computably metrized and is isometrically isomorphic to X. Under an appropriate coding, the collection of all isometric isomorphisms $Iso(X,Y)$ can be viewed as a Π_1^0-class*[18].

Proof. Unfortunately, the space $C(X,Y)$ of all continuous maps from X to Y does not have to be compact. Nonetheless, there are still ways to appeal to the theory of effectively closed and computably compact sets. But it seems easier to just do the coding explicitly.

Recall that an ϵ-isometry is a function f such that $|d(x,y) - d(f(x),f(y))| \leqslant \epsilon$, where $\epsilon > 0$ and x,y range over the space. By Theorem 3.21, Y has to be computably compact as well. Let h be a computable compactness modulus of Y as defined in Definition 2.7. The idea is that we have at most $h(n)$-many essential refinements of a given partial 2^{-n+1}-isometry from X to Y since any other choice will be within a 2^{-n}-error. We will also use the fact that every isometric embedding X in Y has to be onto, by compactness, and because X is isometric to Y. Suppose the special points of Y are given by the sequence $(r_i)_{i \in \mathbb{N}}$, and let $(p_i)_{i \in \mathbb{N}}$ be the dense computable sequence in X. We define a computably bounded tree $B \subseteq \omega^{<\omega}$.

Definition 4.9 (The definition of the tree). The n-th level of B is given by Gödel numbers of (some) tuples from

$$\{r_0, \ldots, r_{h(n)}\}^n$$

that satisfy the Π_1^0 condition

$$|d_Y(r_i, r_k) - d_X(p_i, p_k)| \leqslant 2^{-n+1}$$

for each $i < k < n$. (Recall h is a computable compactness modulus for Y.)

We view these tuples $\bar{r} = \langle r_0, \ldots, r_{n-1} \rangle$ as possible isometric images of $\langle p_0, \ldots, p_{n-1} \rangle$, up to an error of 2^{-n+1}. Thus, we require the Π_1^0 condition that $|d_Y(r_i, r_k) - d_X(p_i, p_k)| \leqslant 2^{-n+1}$ for each $i < k < n$. For a tuple $\bar{u}$ at level n and a tuple $\bar{v}$ at level $n+1$, we posit as a further Π_1^0 condition that v is a child of u if $d(u_i, v_i) \leqslant 2^{-n}$ for each $i < n$. We let B consist of all strings σ such that for each $n < |\sigma|$, $\sigma(n)$ is on level n, and if $n > 0$ then $\sigma(n)$ is a child of $\sigma(n-1)$. Then B is Π_1^0; furthermore, clearly there is a function $\hat{h} \leqslant_{\mathrm{T}} h$ that bounds any $f \in [B]$.

---

[18]More formally, $Iso(X,Y)$ admits a natural representation as a Π_1^0 class $[B]$ in the sense that there is a Turing functional turning infinite paths through B into isometries from X to Y, and so that every member $Iso(X,Y)$ has a code in $[B]$.

We claim that $[B]$ codes $Iso(X, Y)$, in the sense that there is a map from $[B]$ onto $Iso(X, Y)$. Furthermore, we claim that the map is computable in the sense that there is a computable functional that turns any infinite path through B into an isometry from X to Y.

This is verified below.

Suppose there is an isometric embedding $\Theta : X \to Y$. Then let $\pi(n)$ be a tuple of special points on level n that is at distance less than 2^{-n} from $\langle \Theta(p_0), \ldots, \Theta(p_{n-1}) \rangle$. Then $\pi \in [B]$, and using π we can effectively reconstruct Θ.

Now suppose $f \in [B]$. We claim that f uniformly computes an isometric embedding $\Theta_f : X \to Y$. For each i, we have a Cauchy sequence $s_n^i = f(n)_i$ (where $n > i$). Thus f uniformly computes the function Θ_f given by $\Theta_f(i) = \lim_{n > i} f(n)_i$. For each $i < k < n$ we have

$$|d_Y(s_n^i, s_n^k) - d_X(p_i, p_k)| \leqslant 2^{-n+1}.$$

Thus, Θ_f is an isometric embedding. Note that this is all uniformly effective. □

We can now appeal to facts about Π_1^0 classes.

Corollary 4.10. *For a computably compact space X, if $Y \cong_{iso} X$ then there is a low isometric isomorphism witnessing this.*

Using different methods, Iljazović [67] proved a special case of the next corollary for the case when $Aut_{iso}(M)$ is finite.

Corollary 4.11. *Suppose a computably compact X has only at most countably many self-isometries. If $Y \cong_{iso} X$ then there is a computable isometric isomorphism witnessing this.*

Proof. This follows from the fact that $Iso(X, Y)$ must contain an isolated point. To see why, fix an arbitrary $\psi : X \to Y$. Then every isometry ϕ from X to Y gives an automorphism $\psi^{-1}\phi\psi$ of X, and since there are only countably many isometric isomorphisms, there could be only countably many members in $Iso(X, Y)$. Thus, it cannot be a perfect space, so it must contain an isolated point Θ.

Take any neighbourhood U in $C(X, Y)$ under the metric

$$d(f, g) = \sup_{x \in X} d_Y(f(x), g(x))$$

so that it contains a unique member $\Theta \in Iso(X, Y)$. If $F : [B] \to Iso(X, Y)$ is the computable functional from Definition 4.9, then $F^{-1}(\Theta)$ does not have to be a singleton in $[B]$. On the other hand, $C(X, Y)$ does not have to be compact. Thus, we cannot appeal to Fact 3.25 directly. However, this is not really an issue, because all paths through $F^{-1}(\Theta)$ eventually give very close approximations to Θ and, thus, we can use any extension that looks good so far (to be clarified). As was pointed out by one of the referees, Fact 3.25 and the argument that we present below admit a generalisation that would suffice, but we give a direct proof.

Fix an n so that 2^{-n} is smaller than the diameter of U. Fix a sufficiently long $\sigma \in B$ that:

(1) σ is extendible to an infinite path in $[B]$, and
(2) the F-images of all its extensions in $[B]$ lie in U.

Note that for any such extension $\pi \in [B]$, we must have $F(\pi) = \Theta$.

Given $m > n$, wait for a late enough stage s so that all extensions of σ that have not yet been declared out of B have their potential F-images at distance at most 2^{-m} from each other. Since Θ is isolated in $U \subseteq Iso(X, Y)$ and $[B]$ is a Π_1^0 class, it follows that such a stage must exist. Choose any such ρ_m extending σ that has not yet been declared out of B. It determines a finite partial map that can be used to calculate Θ to precision 2^{-m+1}. It follows that Θ is computable. □

It is natural to ask whether the isomorphism in the corollary above can be reconstructed uniformly from a given pair of compact presentations of the space. The answer to this question is (perhaps, not surprisingly) negative; an intricate example can be found in [55]. The cited paper also contains a subtle definability-theoretic analysis of

computably unique metric spaces, i.e., up to computable isometry. It is also known that every compact Polish space admits a Scott sentence of very low complexity; see [103].

In their very recent paper [71], Iljazović and Validžić prove two further interesting results generalising Corollary 4.11 and the techniques that Iljazović used in [67]. Both results from [71] can be derived from Theorem 4.8. For example, using clever combinatorial techniques, they show:

Corollary 4.12. *In a computably compact metric space, the orbit of a computable point under the action of the self-isometry group is Π^0_1.*

We give a proof that uses Theorem 4.8.

Proof. Suppose X is computably compact and consider the presentation $P = [B]$ of $C(X, X)$ as a computably bounded Π^0_1 class given in Definition 4.9. Without loss of generality, we can assume that the computable point that we care about is actually special; if it is not, add it to the computable dense sequence. We can therefore assume $x = p_0$.

We examine the class P of all potential partial 2^{-n}-isometries and see whether there is an $f \in P$ such that $f(p_0) = y$. By compactness, this is the same as to say that, for every n, there is a $\sigma \in P$ of length n that codes some 2^{-n+1}-isometry g_n mapping p_0 to some $r_i \in Y$ with $d(y, r_i) \leqslant 2^{-n}$. Note that the existential quantification over such σ is computably bounded, since there are only finitely many such σ at level n of P, and we can computably bound this number. Also, the condition $d(y, r_i) \leqslant 2^{-n}$ is Π^0_1 uniformly in n and y. Thus, the overall property of being in the orbit of $x = p_0$ is Π^0_1, uniformly in y and n. In other words, the orbit is effectively closed[19]. $\square$

We see that a careful use of Π^0_1 classes and effective compactness can help to simplify proofs. Nonetheless, we emphasise again that in this subsection we view spaces up to isometric isomorphism. When we study spaces up to homeomorphism, the situation usually becomes more complex. We will discuss spaces up to homeomorphism a bit later; here we only cite [115] where it is shown that the closure of the set of orientation-preserving self-homeomorphism of the unit square forms a computable closed set in $C([0,1]^2, [0,1]^2)$.

We now discuss another corollary that uses index sets to estimate the complexity of the classification problem for compact spaces, up to isometry. Fix an effective listing (M_i) of all (partial) computable Polish spaces. Each such M_i is given by a dense sequence that can be identified with ω and a (partial) computable metric on it. (The space M_i represents is the completion $\overline{M_i}$ of M_i.) We could list all partial computably compact spaces in a similar way, but this approach is not standard and has never been used in the literature.

Corollary 4.13 (Melnikov and Nies [103]). *The following index sets are arithmetical:*

(1) *The characterisation problem $\{i : \overline{M_i}$ is compact$\}$.*
(2) *The isometric isomorphism problem $\{\langle i, j \rangle : \overline{M_i} \cong_{isom} \overline{M_j}$ & $\overline{M_i}, \overline{M_j}$ are compact$\}$.*

Sketch. For (1), say that the metric is total, is indeed a metric, and for every n there is a 2^{-n}-cover of the space by closed basic balls.

To see why (2) holds, note that (by Corollary 4.10) it is sufficient to state that there exists a $0'$-computable isometry. All conditions that express that it 'works' are arithmetical. $\square$

In contrast with compact spaces, the characterisation problem for locally compact Polish spaces is Π^1_1-complete as conjectured in [103] and then formally clarified in [116] (the sketch contained in [103] is incorrect). We will return to index sets later when we discuss computable Stone duality.

We now discuss potential converses to Theorem 4.8. J. Miller (personal communication with Nies) suggested the following example.

Proposition 4.14. *Let $A, B \subseteq \mathbb{N}$ be disjoint c.e. sets. There are isometric computably compact computable metric spaces L, R such that any representation of an isometry computes a set S such that $A \subseteq S$ and $B \cap S = \varnothing$.*

---

[19]As was pointed out by one of the referees, one of the many convenient features of computable compactness is that in certain scenarios quantification over a computably compact set can be treated as computably bounded search. For instance, c.e openness (or being Π^0_1) is typically preserved under such quantification.

Let A and B be disjoint c.e. sets. The collection of *separating sets* $P = \{X : X \supseteq A \text{ and } X \cap B = \varnothing\}$ is a non-empty Π_1^0 class. A pair of c.e. sets A and B is effectively inseparable if there is no computable set C, such that $C \supseteq A$ and $\overline{C} \supseteq B$. For example, by the proof of the incompleteness theorem, for Peano Arithmetic (PA), the c.e. sets of (Gödel codes for) provable formuale $A = \{\#\psi : PA \vdash \psi\}$ and $B = \{\#\psi : PA \vdash \neg\psi\}$ form an effectively inseparable pair. (For a more straightforward example, consider $A = \{e : \varphi_e(0) = 0\}$ and $B = \{e : \varphi_e(0) = 1\}$.) If we choose A, B effectively inseparable, then this shows that the Π_1^0 class of isometries from X to Y is Medvedev complete within the Π_1^0 classes, meaning that whenever there is a member of some other class of degree $\mathbf{d}$ then this Medvedev-complete class also has a member of that degree. Thus, the isometries are as complicated as allowed by Theorem 4.8. In particular, there exist isometric computably compact spaces that are not computably isometric.

Proof. We describe the metric spaces by giving a connected graph of special points. If there is an edge the distance is defined directly. All the other distances between pairs of special points p, q will be given indirectly as the path distance. We ensure in the construction that this distance function is consistent and computable. There will be only one limit point in the space. Thus the space has Cantor-Bendixson rank 2.

Let $\delta_n = 4^{-n}$. For each n, there are special points a_n, u_n, v_n, a_{n+1} in L that have pairwise distance δ_n. We call this basic configuration the n-th diamond. The space R looks similar with special points $a_n', u_n', v_n', a_{n+1}'$ sharing the same properties.

At any stage of the construction, the procedure $\mathsf{mark}(x, \gamma)$, where x is a special point already introduced, adds a new special point y with $d(x, y) = \gamma$.

Construction of L, R. We may assume at most one number enters $A \cup B$ at any stage.

If n enters A at stage s, call $\mathsf{mark}(u_n, 3^{-s})$ and $\mathsf{mark}(u_n', 3^{-s})$.

If n enters B at stage s, call $\mathsf{mark}(u_n, 3^{-s})$ and $\mathsf{mark}(v_n', 3^{-s})$.

It is clear that L, R are isometric. Each space is computably compact: let $h(n)$ be so large that the special points up to $h(n)$ include the first $n + 1$ diamonds, and the points used for marking up to stage n. Then in either space the special points up to $h(n)$ form a 2^{-n}-net. Now suppose that $\Theta \colon L \to R$ is an isometry. The special points a_n and a_n' $(n > 0)$ are singled out by having three points at distance δ_n, and three points at distance δ_{n+1}. Thus, $\Theta(a_n) = a_n'$. Therefore $\Theta(u_n) \in \{u_n', v_n'\}$. Let $S = \{n \colon \Theta(u_n) = u_n'\}$. Then $A \subseteq S$ and $B \cap S = \varnothing$ as required.

Since $d(u_n', v_n') = \delta_n$, using a term sufficiently far out in the Cauchy name for $\Theta(u_n)$, we can decide whether $n \in S$ using the representation as an oracle. $\square$

We conjecture the following.

Conjecture 4.15. Every Π_1^0-class can be realised (e.g., up to m-degree) as $Iso(M, N)$ for two isometric computably compact spaces.

Conjecture 4.15 holds for computable discrete algebraic structures in the place of effective compact spaces [46] and we conjecture that it should be possible to turn such structures into compact spaces. Also, as was noted by one of the referees, the simple construction above should have implications at the level of the Weak König's Lemma, under the appropriate coding and in the spirit of the results found in, e.g., [51, Section 6]. Specifically, among other things [51] studies the computability-theoretic and reverse mathematical strengths of the Hahn-Banach Theorem in Banach spaces. The cited paper also utilises the technique of effective compact sets, but we will not discuss these results here. Instead, we shall discuss the recent results about bases in Banach spaces.

4.3. **Basic sequences in Banach spaces.** Some of the most used Polish Spaces are Banach spaces, and there is a reasonably well-developed theory of effective Banach spaces beginning with Pour-El and Richards [126]. We note that Banach spaces are usually viewed under isometric isomorphism, and it is well-known that every isometric isomorphism has to be affine (this is Mazur-Ulam Theorem).

Definition 4.16. A *computable Banach space* is a computably metrized Banach space in which the Banach space operations are computable.

This definition means that any computable Banach space needs to be separable, since it needs a computable dense set. We regard this as presented with a computable norm $||\cdot||$, and for simplicity will consider the space as a complete normed vector space B over the reals, although the results also work if the field is the complex numbers.

Some consequences of computable compactness, such as and effective version of the open mapping theorem, can be found in [15]. We will give a couple of recent applications of computable compactness to the theory of computable Banach spaces. A simple application of computable compactness is Pour-El Richards [127] result that linear independence in a Banach space is Σ^0_1. To see this, we note that $\{x_1, \ldots, x_n\}$ is independent iff

$$\min_{\lambda_i \in \mathbb{R} \setminus \{0\}} \|\sum_{i=1}^n \lambda_i x_i\| > 0.$$

But by normalizing we can consider the ball

$$S = \{(\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \setminus \{0\} \mid \sum_{i=1}^n |\lambda_i| = 1\},$$

for the quantification, and this is computably compact, meaning that the minimum $\min_{(\lambda_1, \ldots, \lambda_n) \in S} \|\sum_{i=1}^n \lambda_i x_i\|$ is Σ^0_1.

Almost all elementary linear algebra works via the fact that vector spaces have bases. However, the analog for Banach spaces is not so easy. The most accepted candidate of a basis for a Banach space is called a *Schauder Basis*.

Definition 4.17. A *sequence* $X = \{x_i \mid i \in \omega\}$ is called a Schauder Basis for B iff for each $z \in B$ there is a *unique* sequence $\{\lambda_i \mid i \in \omega\}$ such that

$$\sum_{i=1}^\infty \lambda_i x_i = z.$$

Brattka and Dillhage [19] showed that the theory of computable Banach spaces with well-behaved computable Schauder bases[20] is relatively well-behaved. In particular, many theorems using duality lift quite smoothly to have computable versions.

However, one of the unfortunate aspects of Banach space theory is that not only don't (computable) Banach spaces necessarily have computable Schauder bases, but in fact some don't have Schauder bases *at all*. This is a remarkable result of Enflo [42] : There is a separable Banach space with no Schauder basis. (This result solved a question of Banach that had been open for 45 years.) Bosserhoff [13] proved that Enflo's construction could be made computable to give a computable Banach space.

The fundamental fact about Schauder bases is the following characterization by Banach.

Lemma 4.18 (Banach). *Let $X = x_1, x_2, \ldots$ be a sequence of elements of B. Then this sequence forms a Schauder basis iff*

(1) $x_i \neq 0$ for all i

(2) The finite span of $\{x_i \mid i \in \omega\}$ is dense in B.

(3) There is a $K \in \mathbb{R}$ such that for all $m < n$, and all sequences of scalars λ_i,

$$\|\sum_{i=1}^m \lambda_i x_i\| \leqslant K \|\sum_{i=1}^n \lambda_i x_i\|.$$

K in (3) above is called the *Basis Constant* $bc(X)$ of the Schauder basis. The hard direction of this lemma is to suppose that X is a basis and consider the projections $S_k(\sum_{i=1}^\infty \lambda_i x_i) = \sum_{i=1}^k \lambda_i x_i$. We need to prove that $\sum_j \|S_j\|$ is finite, and this is achieved by considering the equivalent norm $\|\cdot\|'$ defined by $\|\sum_{i=1}^\infty \lambda_i x_i\|' = \sup_m \|\sum_{i=1}^m \lambda_i x_i\|$. This is bounded by the Open Mapping Theorem.

Note that we can also define the basis constant of a *space* as the infimum of the basis constants of Schauder bases for the space. We remark that there are even finite dimensional spaces without a basis with $K = 1$. But in the finite dimensional case, we can at least get a computable basis constant for the whole space. That is, a simple application of computable compactness yields the following:

Lemma 4.19. *Let X be a computable Banach space.*

(1) (Bosserhoff [13]) Assume $x_1, \ldots, x_n \in X$ is a (computable) independent sequence. Then $bc(x_1, \ldots, x_n)$ is a computable real.

---

[20]Specifically, those with "monotone" or "shrinking" bases.

(2) (Downey, Long and Greenberg [94]) For a finite dimensional X, $bc(X)$ is a computable real.

Notice that Lemma 4.19 means that each finite dimensional projection in the proof of the existence of K must have a computable basis constant. Therefore the sup K is a computable sup of computable reals. That is K is a left c.e. real for a computable Schauder basis X. In fact, in Long's MSc thesis [94] it is shown that any left c.e. real can be the basis constant of a computable Banach space. It is presently unknown what can be said about the basis constants of computable spaces.

The one theorem we will look at is the following. It provides a counterpoint to the theorem of Metakides and Nerode [104] where they constructed a computable vector space over $\mathbb{Q}$ where every c.e. independent set is finite.

Theorem 4.20 (Downey, Long and Greenberg [94]). *If B is an infinite-dimensional computable Banach space, then B has an infinite dimensional subspace with a computable Schauder basis $Z = z_1, z_2, \ldots$.*

Sketch. Let $E = \{e_i \mid i \in \omega\}$ be an effective dense set for B. We begin with a lemma of Mazur: If B is an infinite dimensional Banach space and Y a finite dimensional subspace, $\epsilon > 0$ then there is $x \in B$ with $||x|| = 1$ and

$$||y|| \leqslant (1 + \epsilon)||y + \lambda x||,$$

for all $y \in B$ and $\lambda \in \mathbb{R}$. Since E is dense, it is not hard to show that we may choose $x = e_i$ for some i, by playing with the triangle inequality, choosing one close to x. Now we can follow the classical argument of Banach. Choose a sequence of reals ϵ_i with $\prod_{i=1}^{\infty}(1 + \epsilon_i) < \infty$. Then construct the basic sequence in stages. Having constructed $z_1, \ldots, z_n$, find an e_i in the effectively dense sequence E with $bc(x_1, \ldots, x_n, e_i) \leqslant \prod_{i=1}^{n+1}(1 + \epsilon_i)$, and we know by computable compactness, that this procedure is computable. $\square$

We will return to Banach spaces a bit later, when we discuss the effective content of Banach-Stone duality that establishes a 1-1 correspondence between computable presentability of Banach spaces in a broad class with computably compact presentability of totally disconnected compact spaces.

4.4. **Computable Stone duality with applications.** Recall that two basic open balls $B(c_1, r_1)$ and $B(c_2, r_2)$ are formally disjoint if $r_1 + r_2 < d(c_1, c_2)$. Two sets of basic open balls are formally disjoint if any pair of basic open balls, one coming from the first set and the other from the second, are formally disjoint. A *clopen split* of M is a pair of (cl)open sets X, Y such that $X \sqcup Y = M$.

Computable Stone duality. Various versions of the elementary lemma below can be found in [20, 61, 64, 99], but is some of the cited papers the proof contains minor but misleading errors, thus we give a proof.

Lemma 4.21. *Suppose an oracle X can effectively enumerate all basic finite covers of M. Then X can also effectively enumerate all clopen splits of M.*

Proof. Suppose $M = X \sqcup Y$ is a split, and let δ be the infimum-distance between these compact open sets

$$\delta = \inf_{(x,y) \in X \times Y} d(x, y).$$

(Since $X \times Y$ is compact and d is continuous, it attains its infimum at some pair (x_0, y_0). In particular, $\delta > 0$.)

Suppose $0 < \epsilon < \delta/4$. Then every finite ϵ-cover will consist of two formally disjoint subsets of basic open balls. Indeed, every ball covering a point in X cannot contain a point in Y, and every ball covering a point in Y cannot contain a point in X. If a basic open B has its centre in X and D has its centre in Y, then the distance between their centres is at least δ, while the sum of their radii is at most $\delta/2 < \delta$, making them formally disjoint.

On the other hand, if a finite open cover of M consists of two formally disjoint subcovers, then these subcovers induce a split of M into clopen components. Since the property of being formally disjoint is a c.e. property, X is able to list all such covers. $\square$

Another way to state the lemma above is that any modulus of compactness of M can computably enumerate the clopen splits of M. Also note that we could have used $\cap$-decidable covers in the proof of the lemma above, and this way we can additionally assume that, for the clopen sets that we list, we can additionally decide whether they intersect or not. This will be very convenient in the proof of the next result.

Theorem 4.22 ([64]). *Let M be a computably compact Stone space (a totally disconnected compact Polish space). Then the Boolean algebra of its clopen subsets admits a computable presentation.*

Proof. Fix a $\cap$-decidable system of covers $K = \bigcup_{n \in \omega} K_n$. Using the previous Lemma 4.21, effectively list all clopen splits of M into (open, formally disjoint names of) pairs of clopen sets. Let (X_i, Y_i) be the enumeration of these clopen splits. Note that we can also wait and see whether both X_i and Y_i are non-empty; just wait for a special point to appear in one of the two. In this case we say that the split is *proper*. This is a c.e. event because X_i and Y_i are both given by their open as well as their closed covers, whichever is more convenient. Thus, without loss of generality we can assume that we list only the proper splits. Write X_i^{-1} for the corresponding Y_i in a proper split; and let X_i^0 be another notation for X_i.

The Boolean algebra is generated by the empty set and arbitrary finite *non-empty* conjunctions of the form $\bigwedge_i X_i^{\epsilon_i}$, where $\epsilon_i \in \{0, 1\}$. Since the system of covers K is $\cap$-decidable, we can indeed decide whether such a finite intersection is empty. In other words, if F is the free Boolean algebra generated by the X_i, then the Boolean algebra of clopen sets of M is isomorphic to $\frac{F}{I}$, where I is a computable ideal. This makes the Boolean algebra computable. $\square$

Remark 4.23. For a computably compact Stone space, we can *uniformly* produce the dual Boolean algebra of its clopen sets. (As usual, we assume our spaces are non-empty. Alternatively, we can stretch our terminology a bit and view the 1-element lattice as a Boolean algebra.) It is also not difficult to see that the construction is also *locally uniform* in the following sense. Using clopen components we can produce a computably branching, computable tree T without dead ends such that the space is homeomorphic to $[T]$. If we take the usual ultrametric on the infinite paths through T, then this metric will be computably compatible with the original metric in the sense that $Id : M \to [T]$ is a computable map.

This result allows us to establish the following representation theorem.

Theorem 4.24 ([61]). *Let M be a computably metrized Stone space. Then the Boolean algebra of its clopen subsets admits a computable presentation.*

Proof. Let M be the computable space. Recall that $0'$ can compute a modulus of compactness of the space. Relativize the previous theorem to get $0'$-computable presentation of the Boolean algebra of clopen sets. Given an element of the Boolean algebra, we can use its representation via a finite union of basic computable balls and ask whether there exist two unequal special points x, y that are contained in this clopen set; this is a Σ_1^0 property. This element is an atom if, and only if, no such pair of points exists. Thus, the atom relation is also $0'$-computable. It is well-known that every $0'$-computable Boolean algebra in which the atom relation is also $0'$-computable has a computable presentation; see [82] for the explicit statement and [38] for the first implicit use of this property. It follows that the Boolean algebra of clopen sets has a computable copy. $\square$

Given a computable Boolean algebra B, it is not difficult to represent its dual Stone space $\widehat{B}$ as the collection of infinite paths $[T]$ through a computably branching, computable tree T without dead ends; see [52] for the details. Thus, we have:

Theorem 4.25 ([61, 64]). *For a countable Boolean algebra B, B has a computable presentation iff its dual Stone space $\widehat{B}$ can be computably metrized iff $\widehat{B}$ has a computably compact metrization.*

Corollary 4.26 ([61]). *Every computably metrized Stone space is homeomorphic to a computably compact Stone space.*

Computable topological vs. computable Polish spaces. The following result is folklore; see, e.g., [143] and [31].

Theorem 4.27. *For a countable Boolean algebra B, the following are equivalent:*

(1) *B admits a c.e. presentation.*
(2) *The dual Boolean algebra $\widehat{B}$ is homeomorphic to a Π_1^0-class.*

Idea. To see why $(1) \Rightarrow (2)$ should hold, think of a $B = A/I$, where A is atomless and I is a c.e. ideal. One can view A as a full binary tree in which some nodes can be declared equal. Instead of making them equal, we can declare them to be *outside* of a Π_1^0 tree T such that the Stone space of B is $[T]$. To understand why $(2) \Rightarrow (1)$ should hold, define a c.e. ideal by setting a node x to be equal to the left-most node y at the same level that still remains in the tree (representing the Π_1^0 class). $\square$

Working independently, Bazhenov, Melnikov and Harrison-Trainor [8] proved another closely related and much more subtle computable version of Stone duality. To state it, note that the basic notion of computable compactness (Definition 3.1) can be defined without the assumption that the metric is computable. Definition 3.1 also works for computable topological spaces: just say that we can effectively enumerate all finite covers of the space by basic open sets. In this case we say that the computable topological space is effectively compact. Recall that right-c.e. spaces are computable topological spaces, where the base is given by the basic open balls; see Proposition 2.4. An example of such a right-c.e. "effectively compact" space is any non-empty Π^0_1 class. (This requires a relatively straightforward inductive argument that can be found in, e.g., [8].)

Theorem 4.28 ([8]). *For a countable Boolean algebra B, the following are equivalent:*

 (1) *B admits a c.e. presentation.*
 (2) *The dual space $\hat{B}$ is homeomorphic to an effectively compact right-c.e. completely metrized space.*

Idea. The proof of $(1) \Rightarrow (2)$ is similar to the proof of $(1) \Rightarrow (2)$ the previous theorem. (A non-empty Π^0_1 class can be viewed as a right-c.e. metrized space.)

To prove $(2) \Rightarrow (1)$ use a theorem a Odintsov and Selivanov [119] who showed that a Π^0_2-presented Boolean algebra admits a c.e. presentation. It is therefore sufficient to show that the Boolean algebra of clopen sets has the form β/I, where β is the atomless algebra and I is its Π^0_2 ideal. This can be done using methods similar to the techniques described in this section; we omit the details.   □

Recall again that every right-c.e. space is a computable topological space (Proposition 2.4). Recall also that Feiner [44] constructed an example of a c.e. presented Boolean algebra that does not admit a computable presentation. Thus, Theorems 4.27 and 4.28 imply:

Corollary 4.29 ([8]). *There exists a computable topological Polish space that is not homeomorphic to any computably (completely) metrized space.*

In spite of appearing as a standard classical result, Corollary 4.29 is very recent. We really have a stronger consequence. In view of Theorem 4.27, it follows also that effective compactness for computably metrized and right-c.e. metrized spaces differ up to homeomorphism.

Applications to classification problems. The next corollary measures the classification problem for compact computable Polish spaces up to homeomorphism. Recall that we fixed an effective listing $(M_i)_{i \in \omega}$ of all (partial) computable Polish spaces. (We identify M_i with the completion of the respective dense computable sequence.)

As was pointed out by Selivanov (in personal communication with the second author), it seems that the corollary below has actually never been explicitly stated in the literature. The corollary is, of course, not really new. Compare the corollary below with Corollary 4.13.

Corollary 4.30. *The homeomorphism problem $\{\langle i, j \rangle : M_i \cong_{hom} M_j \,\&\, M_i, M_j \text{ are compact}\}$ for computable compact Polish spaces is Σ^1_1-complete.*

Proof. The Σ^1_1-hardness follows from the Σ^1_1-completeness of the isomorphism problem for computable Boolean algebras [53] and the fact that Stone duality is computably uniform.

We need to argue that the index set is Σ^1_1. We give only a sketch since this fact seems to be well-known. A rather similar fact is folklore in descriptive set theory [49]. A detailed proof in the lightface case (and in the harder context of topological groups) can be found in [99]. A very closely related (and, in some sense, stronger) argument is Lemma 3.6 of [60].

It is arithmetical to say that M_i is a (presentation of a) compact Polish space; see (1) of Corollary 4.13. To say that there is a homeomorphism $f : M_i \to M_j$, it is sufficient to state that there exist continuous surjective $f_1 : M_i \to M_j$ and $f_2 : M_j \to M_i$ such that $f_1 \circ f_2 = Id_{M_i}$. Every $g : X \to Y$ between compact X and Y can be represented by, e.g., a pair $(\tilde{g}, m)$ where $\tilde{g} : \omega^2 \to \omega$ and $m : \omega \to \omega$, where the function $\tilde{g}(n, k)$ is interpreted as the image of the nth special point with precision 2^{-k}, and m as the modulus of uniform continuity.

It is arithmetical to say that $(\tilde{g}, m)$ represents a continuous function $\lim_k g(\cdot, k) : X \to Y$. This is because totality is arithmetical, and also one can express that m is a modulus of continuity that works for $\tilde{g}$ as a closed property. Thus, as before, if it fails then it must fail for some special points. Since the continuous image of a compact space

is closed, it is arithmetical to say that $(\tilde{g}, m)$ represents a surjective function. (If it does not, then again there is a special point in the complement witnessing this.)

This allows to state the existence of f_1 and f_2 in a Σ_1^1 way. Finally, to say that $f_1 \circ f_2 = Id_{M_i}$, it is sufficient to say that it is true for special points because the property is (again) closed. This can be expressed arithmetically (in the presentations of) f_1 and f_2. □

Of course, the result above can be relativized to any oracle, thus implying the classical descriptive-theoretic result saying that the homeomorphism problem for compact Polish spaces is analytic complete.

Remark 4.31. The Σ_1^1-completeness in the proof above is witnessed by computably compact Stone spaces. It is also not difficult to see that the index set of Stone spaces is arithmetical, and thus the homeomorphism problem is actually complete within the class of Stone spaces (that can further be assumed computably compact). To see why saying that M_i is a Stone space is an arithmetical property, iterate the process of splitting the space and search for a non-trivial connected component using Lemma 4.21. Every connected component can be expressed as a finite union of basic open balls, and thus the existence of a non-trivial connected component can be expressed as a first-order, arithmetical property.

A substantially different proof of Corollary 4.30 can be extracted from a closely related result for topological groups established in [99]. A computably metrized topological group is a computable Polish space with computable group operations defined on the space. We say that it is computably compact if the space is furthermore computably compact.

Theorem 4.32 ([99])**.** *The topological isomorphism problem for computably metrized connected compact abelian groups is Σ_1^1-complete.*

The proof of the result above uses a computable version of *Pontryagin duality* that was proved in [99] and has recently been extended in [95]; we omit the definitions. In fact, it follows from the version of computable duality established in [95] that the Σ_1^1-completeness is witnessed by computably compact groups. We now explain how the theorem above gives a different proof of Corollary 4.30. As we shall discuss later in the paper (see Subsection 4.7, the second proof of Theorem 4.40), using algebraic topology it is possible to show that two compact connected abelian Polish groups are homeomorphic (as spaces) if, and only if, they are isomorphic as topological groups. Assuming this result, Corollary 4.30 follows from the theorem above, but this time the corollary is witnessed by connected computably compact spaces, not totally disconnected computably compact spaces.

To conclude the subsection, we mention one more application of computably compact Stone spaces, this time to Banach spaces. Recall that *computable Banach space* is a computably metrized Banach space in which the Banach space operations are computable. Recall that Banach-Stone duality states that $C(K_0; \mathbb{R})$ and $C(K_1; \mathbb{R})$ are linearly isometrically isomorphic if, and only if, the respective compact spaces K_0 and K_1 are homeomorphic. In [8], a computable version of Banach-Stone duality has been established:

Theorem 4.33. *For a countable Boolean algebra B, B has a computable copy iff the space $C(\widehat{B}; \mathbb{R})$ is a computable Banach space.*

In view of Theorems 4.27 and 4.28, this correspondence gives an explicit application of computable compactness to computable Banach space theory. In particular, we obtain the following corollary. Fix a computable list $(B_i)_{i \in \omega}$ of all (partial) computable linear spaces over $\mathbb{Q}$ with a computable norm, and write $\overline{B_i}$ for the completion of B_i with respect to its norm.

Corollary 4.34. *The linear isometric isomorphism problem $\{\langle i, j \rangle : \overline{B_i} \cong_{iso} \overline{B_j}\}$ for computable separable Banach spaces is Σ_1^1-complete.*

Proof. We have that $C(\widehat{B_0}; \mathbb{R}) \cong_{iso} C(\widehat{B_1}; \mathbb{R})$ iff $\widehat{B_0} \cong_{hom} \widehat{B_1}$ (iff $B_1 \cong B_2$). The proof of aforementioned result from [8] is uniform when passing from computably compact Stone spaces to the respective Banach spaces. So Σ_1^1-hardness follows from the previous corollary.

It remains to note that the upper bound is also Σ^1_1. It is sufficient to state that there is an isometry that works for special points, maps zero to zero, and is, furthermore, surjective (these properties are closed). The well-known Mazur-Ulam theorem asserts that every isometry with these properties has to be linear. □

Further results and open questions related to index sets in analysis, and to compact spaces and compact groups more specifically, can be found in [40].

4.5. **Profinite groups.** Profinite groups are the Galois groups. They are inverse limits of finite groups. The study of effective profinite groups began with Metakides and Nerode [105], La Roche [90, 91] and Smith [137, 138], where they defined the group to have a *co-r.e. presentation* if it was isomorphic to a computably bounded Π^0_1 class $[T]$ (in Baire space) where the group operations were computable. La Roche and Smith defined the group to be recursively presentable (computable) if the set of extendible nodes in T forms a computable set. If F is a computable (countable) field, and K is a c.e. subfield of F, then the Galois group $G(F\backslash K)$ is a co-r.e. profinite group. As Smith [138] observed, Waterhouse's result [147] can be effectivized to show that each co-r.e. profinite group is effectively isomorphic to $G(F\backslash K)$ for some computable F and K.

A profinite group is *recursive* if it can be represented as the projective limit of computable linear sequence of finite groups (F_i) given by their strong indices and computable *surjective* $f_i : F_{i+1} \to_{onto} F_i$. Smith [138] showed that a profinite group is recursive if, and only if, it it is isomorphic to a decidable computably bounded Π^0_1 class $[T]$ where the group operations were computable. Since every infinite profinite group is homeomorphic to 2^ω, and any two computably compact presentations of 2^ω are computably homeomorphic (e.g., [8]), without loss of generality we can assume that $[T] = 2^\omega$.

A natural question arises what happens when the metric on the group is not necessarily an ultrametric. For example, it is sometimes convenient to think of abelian profinite groups as subgroups of an infinite direct power of the unit circle group.

We give a classification of profinite groups with an arbitrary compatible metric that have a recursive presentation. The result below is new. A similar result that establishes an arithmetical bound without the assumption of computable compactness can be found in [99]. Recall that the notion of a computably compact group was defined before Theorem 4.32.

Theorem 4.35. *For a profinite group G, the following are equivalent:*

(1) *G has a recursive presentation;*
(2) *G has a computably compact presentation.*

Proof. Clearly, every recursive presentation is computably compact. Now assume we are given a computably compact presentation. Using Lemma 4.21, computably list all clopen components of the group. At this stage there are two ways we can proceed to prove the theorem.

The first possibility is to use the materials of the previous section to construct a computably branching tree T with no dead ends such that the domain of G is $[T]$, and note that the natural shortest-prefix ultrametric inherited from the tree is computably compatible with the original metric (in the sense that the identity map $G \to [T]$ is computable; see Remark 4.23). Then we can use the aforementioned result of Smith and conclude that G admits a recursive presentation (i.e., via a surjective linear computable inverse system). We will not give any further details.

The second possibility is to directly calculate the recursive presentation without any reference to effective compatibility and the result of Smith. To make the paper self-contained, we give the details below.

To say that a clopen component is a normal subgroup, use the fact that every clopen component is a computable subspace of the group, and thus is computably compact, by Proposition 3.29. To see if a clopen C is a subgroup, search for a pair of finite covers, say (B_i) and (D_j), of C such that for every i, j there is a k with the property

$$B_i \cdot B'_j \subseteq D_k$$

and for every i there is a k such that

$$B_i^{-1} \subseteq D_k.$$

We also search for a finite cover (U_n) of G such that for all n, m and i there is a k with

$$U_n^{-1} \cdot B_i \cdot U_m \subseteq D_k.$$

We argue that such a cover exists, and this will imply that we can computably list all clopen subgroups of G. Then we explain how to use these subgroups to build a recursive presentation of the group.

Since the clopen component C can be expressed as a (finite) union of open balls, the preimage of the clopen component under the computable maps $x, y \to xy$, $x \to x^{-1}$ and $z, x \to z^{-1}xz$ in the respective product space (respectively, $C \times C$, C, and $G \times C \times G$) can be uniformly listed. If C were not a normal subgroup then there will be special points witnessing this, and these would be witnessed together with sufficiently small basic open balls containing them. On the other hand, if C is a normal subgroup then every equation of the form, say,

$$z^{-1}xz = y,$$

where $x, y \in C$ and $z \in G$, would be witnessed by small enough basic open balls containing these points, i.e.,

$$U^{-1} \cdot B \cdot U \subseteq D,$$

where $z \in U$, $x \in B$, and $y \in D$. These products of these balls would give a cover of the respective compact product space (in the case of conjugation and in the notation above, $B \times U \times D$ cover $G \times C \times G$.) It follows that we can find a finite subcover.

We conclude that we can list all clopen normal subgroups of G. Note that, by the uniform computable compactness of each such clopen C, we can compute the diameter of C, which is $\sup_{x,y \in C} d(x, y)$. Using the techniques of Lemma 4.21 and Theorem 4.22 – that basically can be summarised by saying that we take the next cover by very small balls – we can furthermore produce a nested sequence of (finite open names of) clopen normal subgroups $\{C_i : i \in \omega\}$ such that:

(1) $C_{i+1} \subseteq C_i$ is formal[21],
(2) $diam\, C_i < 2^{-i}$,
(3) for every i there exists a computable finite tuple $(x_{i,j})$ of special points (given by its strong index) such that $(x_{i,j}C_i)$ is a cover of G.
(4) For every i, j, n, if $x_{i,j}C_{i+1} \subseteq x_{i+1,n}C_{i+1}$ then this inclusion is formal.
(5) When $j \neq j'$, $x_{i,j}C_{i+1} \cap x_{i,j'}C_{i+1} = \varnothing$.

If we succeed, then $\bigcap_{i \in \omega} C_i = \{0\}$, so it is a uniformly computable basis of clopen normal subgroups of G. We will then use the cosets to calculate the finite G/C_i and the homomorphisms from G/C_{i+1} onto G/C_i.

More formally, we proceed by recursion. Assume C_{i-1} has been defined. We search for a C_i that satisfies all these four conditions. If we drop 'formal' in all these conditions, then it should be clear that such a C_i and $x_{i,j}$ must exist. Then fix such a C_i.

By Lemma 4.21 and the analysis of normality above, a normal clopen C_i will eventually be found, and furthermore both C_i and the finitely many cosets mod C_i will be represented as a finite collections of balls. Our task it to show that we can effectively recognise that these finite parameters describing the cosets define what we need. For that, we might need to adjust the finite covers by refining them so that, for instance, the inclusion is witnessed by formal inclusion of covers. This is done as follows.

We satisfy (1) by choosing the radii of a finite cover describing C_i small (see Remark 3.4), and we satisfy (2) by evaluating the computable diameter of the clopen set (this is again essentially done by further refining the cover). Here we use that C_i is indeed a computable closed set because of Lemma 4.21, so we can apply Proposition 3.29.

We elaborate why we will eventually find special points $(x_{i,j})$ and will eventually recognize that they satisfy (3). For that, note that each coset of C_i is open, and thus in particular contains a special point, say x. In particular, every coset mod C_i has the form xC_i. Since for every special x its coset xC_i is the image of C_i under the computable map $y \to xy$ and C_i is computably compact with all possible uniformity, by Lemma 3.31 we conclude that xC_i is also computably compact, and with all possible uniformity. By refining the cover of xC_i (see Remark 3.4), we can ensure that all set-theoretical inclusions of xC_i into the clopen sets seen so far in the construction hold formally. We can also ensure that if two cosets do not intersect then this is also witnessed formally[22]. This gives a way

---

[21]Meaning that each ball from the open cover of C_{i+1} is formally included into some basic open ball in the fixed cover of C_i. Similar for condition (4) below.

[22]Just take the radii of open balls much smaller than the pairwise distances between the finitely many clopen sets to see that it can be done. It is crucial here that the sets are clopen, see, e.g, Corollary 4.3 for a potential issue in general.

of computably recognising condition (5). We can also wait for finitely many such special points $x_{i,j}$ so that the respective cosets $x_{i,j}C_i$ cover the whole space.

To reconstruct the computable operation on G/C_i, calculate the product and the inverse on the special points $x_{i,j}$ with a sufficient precision until you see that the result is in one of the cosets modulo C_i. This is all computable because the cosets $x_{i,j}C_i$ are (uniformly) given by their finite open covers, and the operations on G are computable.

Finally, use effectiveness of condition (4) to calculate the surjective group-homomorphism $\phi_i : G/C_{i+1} \to G/C_i$ that maps every $x_{i+1,j}C_{i+1}$ to the unique coset $x_{i,j'}C_i$ that contains it. This gives a computable surjective inverse system

$$(G/C_i, \phi_i)_{i\in\omega}$$

the (inverse, projective) limit of which is topologically isomorphic to G. Since the system is uniformly computable (in the sense of strong indices of finite sets), this gives a recursive presentation of G. $\qquad\square$

Remark 4.36. In view of the results in the previous subsection that connect c.e.-presented Boolean algebras with Π^0_1-classes, we conjecture that co-c.e. presented profinite groups should correspond to computably compact right-c.e. metrized groups. Melnikov [99] gives the first example of a profinite computably metrized group that does not admit a recursive presentation. In view of our theorem above and the results of Smith, the notion of a recursive profinite group seems to be the "right" notion of computability for profinite groups. See [99] for a complete description of computably categorical profinite abelian groups and an effective version of Pontryagin duality that works for such groups.

4.6. **Computability of Čech cohomology.** The earliest application of simplicial homology in computable analysis we are aware of can be found in J. Miller's thesis [107] (this application has already been discussed above). Simplicial (co)homology is computable in its nature, and this can be made formal. For example, Chapter 1 (§11) of [112] contains a careful verification of the computability of the homology groups for finite simplicial complexes. This of course entails computability of cohomology groups as well. More specifically, given a (strong index of a) simplicial complex, we can uniformly compute its ith homology group represented as $\bigoplus_{i\leqslant k}\langle a_i\rangle$, where $a_0,\ldots,a_k$ are the generators of the group such that the orders of the cyclic $\langle a_i\rangle$ are also uniformly computable. Since $A^i = Hom(A_i, \mathbb{Z})$, we can easily observe that respective cohomology groups are also computable in this strong sense.

In this section we extend these results to arbitrary computably compact spaces and their Čech cohomology groups that will be defined shortly. For a finite simplicial complex, its Čech cohomology is isomorphic to its simplicial cohomology; see that last chapter of [112]. One of the convenient features of Čech cohomology is that it does not rely on triangulation and works for an arbitrary compact metric space.

Background from algebraic topology. Given a compact M, let $\mathcal{N}$ be the directed set of all its finite open covers (under refinement). Since the covers by basic ϵ-balls, where ϵ ranges over positive rationals, are cofinal among all covers, without loss of generality we can restrict ourselves only to covers by basic open balls with rational radii. For instance, $\mathcal{N}$ could be the $\cap$-decidable system of covers nested under formal refinement instead of the usual refinement (at this stage, computability of these conditions is not important).

For each member C of $\mathcal{N}$, recall that its *nerve* $N(C)$ is the collection of all sets in the cover that intersect non-trivially. One can view $N(C)$ as a (finite) simplicial complex in which the n-dimensional faces are exactly the n-element subsets X of $N(C)$ such that $\bigcap\{Y : Y \in X\}$ is a non-empty set. For these finite simplicial complexes we can define their *cohomology groups* $H^*(N(C))$ (with coefficients in $\mathbb{Z}$) as follows.

We follow §73 of [112] and define the Čech cohomology group of a compact metrized space as follows. For a fixed finite set of basic open balls $C \in \mathcal{N}$ and the respective simplex $N(C)$, define the simplicial chain complex as usual:

$$\ldots \to_{\delta_3} A_2 \to_{\delta_2} A_1 \to_{\delta_1} A_0$$

where A_i are finitely generated free abelian groups and δ_i are boundary homomorphisms, and then define the associated cochain complex $A^i = Hom(A_i, \mathbb{Z})$ and define $d_i : A^i \to A^{i-1}$ to be the dual homomorphism of δ_{i+1}. Then $H^i(N(C)) = Ker(d_i)/Im(d_{i-1})$ is the ith cohomology group[23] of the simplex $N(C)$ which is a finitely

---

[23]We are not really interested in the case when $i = 0$ when d_{-1} is not really defined; just assume $Im(d_{-1}) = \{0\}$.

generated abelian group which can be thought of as given by finitely many generators and relations. Let $H^*(M)$ be the direct limit of $H^*(N(C))$ induced by the inverse system $\mathcal{N}$ under the refinement maps.

The result. Recall that a (discrete, countable) group is c.e.-presented if it is isomorphic to a factor of a computable free group by its computably enumerable subgroup. In other words, the operations of the group are computable by equality is c.e., thus the name.

Theorem 4.37 ([95]). *For a computably compact M, its ith Čech cohomology group admits a c.e. presentation uniformly in i.*

A version of this proof for computably compact spaces can be found in [95], and similar result for computable Polish spaces (and with a simpler proof, but giving merely $0'$-computable nerves) is contained in [98]. The proof in [95] relies on a new constructive version of Čech cohomology that was designed to circumvent the following obvious obstacle: for a given cover, we cannot (in general) compute its nerve. However, Theorem 3.16 tells us that this difficulty can be circumvented even if we use the standard notion of a nerve. Thus, we do not need to out-source to the notationally heavy apparatus of algebraic topology compared to which the somewhat tedious proof of Theorem 3.16 looks rather tame.

Proof of Theorem 4.37. As we noted above, we can assume that we a given a system of 2^{-n} covers $\mathcal{N}$ that is linearly nested under formal inclusion and is $\cap$-decidable; by Theorem 3.16 and Remark 3.18 this can be done computably. We say that a sequence of finitely generated uniformly computable abelian groups (A_j) is *strongly completely decomposable* if each A_i uniformly splits into a direct sum of its cyclic subgroups, and furthermore the sets of generators of the cyclic summands are given by their strong indices.

Fix a $\cap$-decidable finite cover C.

Claim 2. The groups $H^i(N(C))$ are strongly completely decomposable (uniformly in C and i).

Proof. The finite complex $N(C)$ is computable because the cover C is $\cap$-decidable. A close examination of the definitions shows that, given C (as a finite set of parameters) and i, we can compute the generators of $A^i = Hom(A_i, \mathbb{Z})$ and compute d_i. We will need the fact below which is well-known; see [47] for a proof.

Fact 4.38. Let $G \leqslant F$ be free abelian groups. There exist generating sets $g_1, \ldots, g_k$ and $f_1, \ldots f_m$ ($k \leqslant m$) of G and F, respectively, and integers $n_1, \ldots, n_k$ such that for each $i \leqslant k$, we have $g_i = n_i f_i$.

We can computably find the set of generators (a_j) of $Ker(d_i)$ and a set of generators (b_s) of $Im(d_{i-1})$ such that for each s there is an integer m and an index i such that $ma_i = b_s$; we know that such generators exist so we just search for the first found ones. It follows that the factor $H^i(N(C)) = Ker(d_i)/Im(d_{i-1})$ is strongly completely decomposable with all possible uniformity. $\square$

Recall that a group admits a Σ^0_1 presentation if it is isomorphic to a factor of a computable group by a Σ^0_1 subgroup.

Claim 3. The direct limit $\lim_{C \in \mathcal{N}} H^i(N(C))$ admits a Σ^0_1 presentation.

Proof. We can list the ϵ-covers and decide whether two given basic open balls intersect in the listed covers. The refinement relation between two covers $C \subseteq_{form} C'$ in $\mathcal{N}$ induces a simplicial map between the respective nerves $N(C)$ and $N(C')$, and this induces a homomorphism between the respective cohomology groups $H^i(N(C)) \to H^i(N(C'))$. By Claim 2, these finitely generated abelian groups are effectively completely decomposable uniformly in C and i. Note that $Im\,\phi$ is generated in $H^i(N(C'))$ by the images of the generators of $H^i(N(C))$. Similarly to the proof of Claim 2, choose new generators of $H^i(N(C'))$ and $Im\,\phi$ so that the latter are integer multiples of the former. In particular, it is easy to see that $Im\,\phi$ is a computable subgroup of $H^i(N(C'))$. This means that we can augment $Im\,\phi$ with extra generators in a computable way to expand it to $H^i(N(C'))$. It follows that

$$\lim_{C \in \mathcal{N}} H^i(N(C)) = H^i(G)$$

can be consistently defined as the "union" of the $H^i(N(C)), C \in \mathcal{N}$, to obtain a group in which the operations are computable and the equality is Σ^0_1. (The equality is merely Σ^0_1 because an element $a \in H^i(N(C))$ can be mapped to 0 in some $H^i(N(C''))$ which appears arbitrarily late in the directed system.) $\square$

This finishes the proof of the theorem. $\square$

We will also see that in Theorem 4.37 "uniformly c.e. presented" cannot be improved to "uniformly computably presented"; this is Corollary 4.43. (In fact, in the example given in the next subsection each individual cohomology group will actually have a computable presentation, just not uniformly so.)

Applications. Perhaps the most significant application to date is computability of Pontryagin duality for computably compact connected groups; see [95]; we omit the definitions. Further applications of computability of cohomology include various index set results in topology; see [95]. One sample result is:

Corollary 4.39 ([95])**.** *The index set of solenoid spaces*[24]*is arithmetical among all compact Polish spaces.*

It follows that in any class of compact manifolds in which non-homeomorphic members have non-isomorphic cohomology groups (for some i), the homeomorphism problem is arithmetical provided that the respective cohomology groups have arithmetical isomorphism problem. The significance of this fact is that, in general, even if a computably metrized manifold admits a triangulation, it is not known whether it always admits an arithmetical triangulation[25]. Indeed, even in the seemingly trivial case of compact surfaces, producing an arithmetical triangulation based entirely on the given metric takes some 18 Turing jumps [60]. It is believed that complexity is likely close to being optimal. In contrast, calculating Čech cohomology groups allows to completely avoid triangulation (that does not even have to exist, let alone an arithmetical triangulation). Some further discussion can be found in [60].

In the next subsection we discuss how Čech cohomology can be used to find a computably metrized space not homeomorphic to a computably compact one.

4.7. **Computably metrized spaces not homeomorphic to computably compact ones.** As we noted in the introduction, finding an example of a computably metrized compact space not isometric to a computably compact one is easy: just take $[0, \alpha]$ for a left-c.e. non-computable real α. The situation is significantly more difficult if we want to work up to homeomorphism. For instance, we have seen that every computably metrized Stone space is homeomorphic to a computably compact one, so no such example can be found among totally disconnected compact Polish spaces.

It seems that constructing such an example necessarily requires some relatively advanced techniques. A few years ago, a closely related result was established by Bosserhoff and Hertling [14]: For any $n \geqslant 2$ there exists a c.e. compact subset $C \subseteq \mathbb{R}^n$ such that $\phi(C)$ is not computable compact for any self-homeomorphism ϕ of $\mathbb{R}^n$. However, the result and the techniques that were used to establish it are restricted to $\mathbb{R}^n$. Hoyrup, Kihara and Selivanov [64] were the first to announce a general construction of a computably metrized space that is not homeomorphic to any computably compact space. Using completely different techniques, a connected example has recently been suggested in [95].

We outline two constructions of a compact computable space not homeomorphic to any computably compact space. The first proof is more similar to what Hoyrup, Kihara and Selivanov [64] announced; it will be given in almost complete detail. The second proof can be found in [95]. It produces a relatively natural example using Pontryagin- van Kampen duality; because too much background is necessary to fully explain the proof, we will only briefly sketch it here. Both proofs rely heavily on Čech cohomology. Finally, we will briefly discuss whether we can completely avoid cohomology to prove the theorem below.

Theorem 4.40. *There exists a computably (completely) metrized compact Polish space not homeomorphic to any computably compact space.*

First proof of Theorem 4.40. The proof is very similar to the one given in Hoyrup, Kihara and Selivanov [64]. The proof that we give here replaces the most complex definability part of their proof with an argument that involves computability of Čech cohomology first established in [95] and then improved in the proceeding subsection.

The definition below "encodes" a set into a space. We view an isolated point as a 0-sphere S^0.

---

[24]A solenoid (space) is a compact connected topological space which the inverse limit of a system (S_i, f_i) with $f_i : S_{i+1} \to S_i$, where each S_i is a circle and f_i is the map that uniformly wraps S_{i+1} $n_i \geqslant 2$ times around S_i. These constructions are important in the area of hyperbolic dynamical systems.

[25]Classically, classification of manifolds works via trangulations.

Definition 4.41. For a set $X \subseteq \omega$, let $CP(X)$ be the one-point compactification of the disjoint union of spheres S^k, with infinitely many copies for each $k \in X$.

One way to think about $CP(X)$ is as follows. Fix a fishbone (a 1-atom) in the Cantor space 2^ω in which every isolated path (an atom) is replaced with a copy of S^k for some $k \geqslant 0$.

Proposition 4.42.

(1) X is Σ_3^0 if, and only if, $CP(X)$ is computably (completely) metrizable.
(2) X is Σ_2^0 if, and only if, $CP(X)$ admits a computably compact presentation.

Proof. (1) Recall that using a modulus of compactness we can find a split of a computable Polish space into two clopen subspaces, as explained in Subsection 4.4. Recall also that $0'$ can compute a modulus of compactness. Also, a space is connected if it does not have a non-trivial clopen split. Thus, $0''$ can produce a uniform list of computable indices of the clopen connected subspaces of $CP(X)$. Each such index is a finite collection of (open or closed) open balls that contain (only) the component. For a (finite) simplicial complex, Čech cohomology groups are isomorphic with the respective simplicial cohomology groups; see the final chapter of [112]. For the n-sphere we have:

$$H^p(S^n) \cong \mathbb{Z} \text{ iff } p = 0, n,$$

and it vanishes otherwise.

Suppose that we know that a computably mertrized M is homeomorphic to S^k, $k > 0$, but we do not necessarily know what this k is. The modulus of compactness of each connected component of M is $0'$-computable uniformly in the finite set of parameters that isolates this component. Since the Čech cohomology groups are uniformly Σ_2^0-presentable (by Theorem 4.37 relativized to $0'$), in this case $M \cong S^k$ is equivalent to saying that $\check{H}^k(M)$ contains at least one non-zero element, which is a Σ_3^0 property (the equality in the group is Σ_2^0). It follows that $0''$ can list the components for which the kth cohomology group is non-trivial, and it can also list such $k > 0$. The set of these k is equal to X.

Now we prove that, given $X \in \Sigma_3^0$, we can produce a computable metric on $CP(X)$. Represent Σ_3^0 as the set of all k such that $\exists x \exists^\infty y R(x, y, k)$, for some computable predicate $R \subseteq \omega^3$; see [130]. Say we are testing whether $k \in X$, $k > 0$. For each existential witness x corresponding to k, create a new component and do more steps in making it look like S^k. This is done by enumerating more points into the components when more $\exists^\infty$-witnesses are found for the given existential witness x; abandon the finitely many points until the next expansionary stage.

This gives a uniformly computable sequence of spaces, each space is either finite discrete or is equal to S^k. We can make sure there are infinitely many copies S^k for each $k \in X$. Put them together (uniformly shrink the ith component by 2^{-i} and use an ultra-metric of the Cantor space to define the distance between different components).

(2) It follows from the material in Subsection 4.4 that, using computable compactness we can list indices (names) clopen connected components of a (compact) space using $0'$. It also follows from Proposition 3.30 that each component can be viewed as a computable closed subset of M (being a finite union of computable sets), and thus it is also computably compact uniformly in its name.

computable compactness makes the Čech cohomology groups uniformly c.e.-presented, and saying that it is not trivial is now merely Σ_2^0. This makes X a Σ_2^0-set.

For the other direction, given a Σ_2^0 infinite X, represent it via $\exists \forall$. Assume we are guessing whether $k \in X$. For each $\exists$-witness keep building a computably compact copy of S^k unless a counterexample to the universal quantifier is found (in which case abandon the component). Then put the spaces together as before, but this time observe the space is computably compact since each component can be easily made uniformly computably compact. $\square$

It remains to fix a Σ_3^0-complete set X. $\square$

Corollary 4.43. *Theorem 4.37 cannot be improved to state that the Čech cohomology groups are uniformly computably presented.*

Proof. Let X be Σ_2^0-complete and consider $CP(X)$ defined above. Then $CP(X)$ admits a computably compact presentation. In fact, the construction of $CP(X)$ is based on the uniform construction of a sequence of computably compact disjoint components, each being either S^k or a finite union of isolated points.

Assume that the Čech cohomology groups were uniformly computably presented in general (and for these components in particular). It is well-known that the cohomology of a finite disjoint union is the direct sum of the cohomologies of the components; this follows from Mayer–Vietoris sequence calculations [112]. Thus, the components that are a finite union of isolated points will have trivial cohomology groups for $i > 0$. Also, by assumption, saying that the ith *computable* cohomology group has a non-zero element is now Σ_1^0. Thus, to decide if $i \in X$ ($i > 0$) it is sufficient to ask whether there is a component whose ith cohomology group is non-trivial, and this is also Σ_1^0 contradicting the choice of X. □

As we already mentioned above, Khisamiev [80] showed that every c.e. presented torsion-free abelian group has a computable presentation. All known proofs of this result are non-uniform, but the only non-uniformity comes from deciding whether there is a non-zero element on the group. As a biproduct of the proof of the corollary above, it follows that this obstacle cannot be circumvented.

The basic idea of the first proof above was to code information into connected components of the space. Producing a connected example seems to necessarily require some advanced techniques that are outside the scope of this paper. We outline an argument that relies on the recent results from [95, 98] and which gives a connected example of covering dimension 1. We leave many terms and notions undefined, see [95, 98] for the definitions and more explanation.

Second proof (sketch). For a discrete torsion-free abelian group G, the 1st Čech cohomology group of the space of its compact *connected* Pontryagin - van Kampen dual $\widehat{G}$ is isomorphic to G; see, e.g., Part 5 of Chapter 8 of [63]. Using the aforementioned result of Khisamiev and two new computable versions of Pontryagin – van Kampen duality from [95] and [98] we can conclude that, for some broad enough class of groups, namely q-divisible groups, G has a Δ_2^0-presentation iff $\widehat{G}$ is computably metrizable, and G has a computable presentation iff $\widehat{G}$ has a computably compact presentation. As there are plenty of Δ_2^0 q-divisible groups that have no computable presentation (including examples having X-computable copies iff X is non-low [97, 98]), the result follows. Indeed, we can find a subgroup of the rationals with this property; this will give a connected example of a solenoid space that satisfies the theorem. In particular, there exist examples like that having covering dimension 1. □

We *conjecture* that one can completely avoid using homological algebra to prove Theorem 4.40. We suspect that one way to do this would rely on a combinatorially involved construction similar to one that can be found in [61]. We outline a plan of this argument; a detailed verification would take too much space (if it works).

Cohomology-free proof idea (for Theorem 4.40). An n-*star* is a the Wedge sum of n-copies of the unit interval (identify the left most-points of n copies of $[0, 1]$). The basic idea here is to replace n-spheres with n-stars in the previously discussed proof of Theorem 4.40. A 0-star is just an isolated point. One can use a fairly basic technique of ϵ-chains to produce a Σ_4^0-enumeration of the set of $n \in \mathbb{N}$ such that the space has an n-star component. As has been suggested by Ng (personal communication with the second author), under the assumption of computable compactness, this definition should become Σ_3^0. It remains to prove that, given a Σ_4^0-set, we can produce a computably metrized (compact) space that codes the set into its n-star components. This requires a relatively involved priority construction that can be viewed as a $0'''$-argument; see [61]. In [61], we can find a construction of this sort that produces a locally compact space. As explained in [61], we can use the 1-point compactification of this space to produce a compact space. □

It is not clear at all whether this approach (if it works) is any simpler than the approach that uses cohomology since it relies on a $0'''$-argument.

4.8. **Computable universality of C[0,1].** Fix the standard computable presentation of $C[0, 1]$ under the supremum metric given by piecewise linear functions with finitely many rational breaking points. We should note that there are also "non-standard" computable presentations of this space that are isometric but not computably isometric to the standard one. We also cite [8, 102] for further results about the computability-theoretic aspects of this space. The theorem below has recently been established in [5] using a direct combinatorial argument. We give a new proof that uses computable compactness to sort out the combinatorics.

Theorem 4.44. *Every computably (completely) metrized Polish space can be computably isometrically embedded into $C[0,1]$.*

It is known that $C[0,1]$ is actually not computably unique up to computable linear isometry [101, 102]. We prove the theorem for the 'natural' computable presentation of the space given by, e.g., piecewise linear functions with rational parameters.

Proof. It is sufficient to computably embed the Urysohn space $\mathbb{U}$ into the natural computable presentation of $C[0,1]$. It is known (and is not hard to show) that it is computably universal in the sense that every computable Polish space can be computably isometrically embedded into the Urysohn space; see [5, 79]. There is no ambiguity here because the Urysohn space admits a unique computable presentation, up to computable isometry [101]. It is also known that the original construction of Urysohn is actually *primitively recursively* universal [5], but unfortunately the space is no longer primitively recursively unique (categorical). For the present proof, we shall only need the (general recursive, Turing) computable universality of $\mathbb{U}$. For a related construction of the Urysohn metric space in constructive setting without choice principles, we cite [93].

We return to the proof. Recall that $\mathbb{QU} = (p_i)_i$, the rational Urysohn space, is dense in $\mathbb{U}$. It is also known that the distances between special points p_i and p_j are rational numbers uniformly computable from i, j (as fractions).

In the Hilbert cube, basic open box is a product of intervals only finitely many of which are open rational sub-intervals of $[0,1]$ and the rest are $[0,1]$. It is clear that basic open boxes are effectively open. We computably adjust the metric in the Hilbert cube and view it as $H = [-d(p_0,p_n), d(p_0,p_n)]^\omega$, and thus adjust the notion of a basic open box accordingly.

Observe that

$$-d(p_0,p_n) \leqslant d(p,p_n) - d(p,p_0) \leqslant d(p_0,p_n),$$

for any p_n. Say that a point ξ in $H = [-d(p_0,p_n), d(p_0,p_n)]^\omega$ corresponds to p_j if the projections $\pi_n(\xi)$ of the point to the edges of the cube are exactly the

$$\gamma_n(p_j) = d(p_j,p_n) - d(p_j,p_0),$$

and let U be the collection of all such points. We can effectively enumerate U as a sequence of computable points.

Lemma 4.45. $P = cl(U)$ *is computable closed.*

Proof. Since we can list U which is dense in $cl(U)$, by Lemma 3.27 it is sufficient to show that $cl(U)$ is effectively closed. One way to do this is as follows.

Say that reals $d_1, ..., d_k$ are *legit* if there is a real d such that

$$\{d(p_i,p_j), d, d_1 + d, ..., d_k + d, i < j \leqslant k\}$$

is a diagram of a metric space (on points p_i, p), where $d_i = d(p,p_i) - d(p,p_0)$ and $d = d(p,p_0)$. Recall also that $\mathbb{U}$ has the extension property, in other words if there is a finite metric space extending $p_0, \ldots, p_k$ then it is isometrically embeddable to $\mathbb{U}$ over $p_0, \ldots, p_k$. The existence of a 1-point extension $p, p_0, \ldots, p_k$ is equivalent to saying that $d_0, \ldots, d_k$ are legit, where $d_i = d(p,p_i) - d(p,p_0)$, as witnessed by $d = d(p,p_0)$.

Claim 4. We can decide whether a given basic open box in H contains a legit tuple.

Proof. There is a first-order formula in the language of $(R, +, \times, 0)$ that says that, for some real d and reals $d_1, \ldots, d_k$ that range between some *fixed* rational parameters (describing the intervals in a given open box), $\{d(p_i,p_j), d, d_1 + d, ..., d_k + d : i, j < k\}$ is a diagram of a metric space. Recall that $d(p_i,p_j)$ are rational. Using Tarski's elimination of quantifiers, we can computably find an equivalent quantifier-free formula with rational parameters. It follows that the property is decidable because equality and order are decidable for rational numbers. $\square$

Recall that $\mathbb{QU} = (p_i)_i$, the rational Urysohn space, is dense in $\mathbb{U}$. Since $\mathbb{QU}$ is dense in the Urysohn space, a basic open box of H (determined by the projection onto the first k coordinates) contains a tuple of legit reals if, and only if, it contains $d(p_j,p_i) - d(p_j,p_0)$, where p_j is sufficiently close to p, where $p \in \mathbb{U}$ are such that $d_i = d(p,p_i) - d(p,p_0)$ and $d = d(p,p_0)$ witness that the tuple is legit.

Since we can decide whether a basic open box is free of legit tuples, it follows that we can decide which basic open box contains no $(\gamma_i(p_j))_i$ for any j. Since the collection of such sequences U is dense in $cl(U)$, and since basic

open boxes are uniformly effectively open and form a basis of topology in H, we conclude that we can effectively enumerate the complement of $cl(U)$. □

We effectively identify 2^ω with the ternary Cantor set $\mathcal{C}$ in $[0, 1]$ (in the sense that we do not distinguish between these computably homeomorphic spaces; see also Theorem 3.33). This is where we get to use effective compactness (which is implied by Claim 4 and the effective compactness of the Hilbert cube). Using Theorem 3.40 and Proposition 3.29, fix a computable surjective $g : \mathcal{C} \to P$, and let $g_i = \pi_i g$, where π_i is the ith projection in the cube. Define a computable f_i to be equal to g_i on the Cantor set, and to be linear otherwise (this standard technique was discussed in Subsection 4.1). We define a computable embedding of $\mathbb{U}$ into $C[0, 1]$ by mapping p_i into f_i.

To show that the map $p_n \to f_n$ is isometric, first note that

$$d\left(p_i, p_k\right) = (d(p_k, p_i) - d(p_k, p_0)) - (d(p_k, p_k) - d(p_k, p_0)) = \gamma_i\left(p_k\right) - \gamma_k\left(p_k\right) = f_i\left(t_k\right) - f_k\left(t_k\right),$$

where t_k is any pre-image (under g) of the point in H corresponding to p_k. It thus follows that

$$d\left(p_i, p_k\right) \leqslant d_{sup}\left(f_i, f_k\right).$$

On the other hand, for any $i, k \in \omega$ and any $t \in \mathcal{C}$ with $g(t) = \lim_j (\gamma_s(p_j))_s \in cl(U)$, we have

$$f_i\left(t\right) - f_k\left(t\right) = \pi_i(\lim_j(\gamma_s(p_j))_s) - \pi_k(\lim_j(\gamma_s(p_j))_s) = \lim_j \gamma_i\left(p_j\right) - \lim_j \gamma_k\left(p_j\right) = \lim_j(d\left(p_j, p_i\right) - d\left(p_j, p_k\right)).$$

By the triangle inequality we get $|d\left(p_j, p_i\right) - d\left(p_j, p_k\right)| \leqslant d\left(p_i, p_k\right)$ for every j, and therefore

$$|f_i\left(t\right) - f_k\left(t\right)| \leqslant d\left(p_i, p_k\right),$$

for any $t \in \mathcal{C}$. By the definition of f_n, any maximum difference must be attained on $\mathcal{C}$. Thus

$$d_{sup}\left(f_i, f_k\right) \leqslant d\left(p_i, p_k\right).$$

Since the maps $p_i \to f_i$ are uniformly computable in i, it follows that the isometry defined above for $(p_i)_{i \in \omega}$ induces a computable isometric embedding $\mathbb{U} \to C[0, 1]$. □

4.9. Covering dimension and embeddings into $\mathbb{R}^n$. Fix a compact Polish space M.

Definition 4.46. The covering dimension of M is the least $n \in \mathbb{N} \cup \{\infty\}$ such that every open cover of M has a refinement of order $n + 1$, i.e., each point belongs to at most $n + 1$ sets.

We know that every compact space is homeomorphic to a subspace of the Hilbert cube $[0, 1]^\omega$, and we have seen that this also holds effectively. It is well-known that a compact space of covering dimension n can be homeomorphically embedded into $\mathbb{R}^{2n+1}$ (and indeed, into $[0, 1]^{2n+1}$). Is this also computably true? One pleasant application of $\cap$-decidable covers is the following theorem that answers the question in the affirmative:

Theorem 4.47. *Let M be a n computably compact Polish space of covering dimension n. Then there is a computable homeomorphic embedding of M into $\mathbb{R}^{2n+1}$.*

The proof is an improved version of a result of Melnikov and Harrison-Trainor [60] that states that every computably metrized compact Polish space of covering dimension n can be $0'$-computably homeomorphically embedded into $\mathbb{R}^{2n+1}$. The proof that we give below is more subtle and relies heavily on a classical argument from [124] but with some modifications. It uses computability of nerves, so strictly speaking $**$-computable compactness would be enough to run the proof.

Proof. We say that a continuous $f : M \to \mathbb{R}^{2n+1}$ is an ϵ-homeomorphism if $f^{-1}(x)$ has diameter at most ϵ for every x in the range. We will need to prove an computable version of the following well-known fact:

Fact 4.48. *The set of ϵ-homeomorphisms form a dense open set in $C[M, \mathbb{R}^{2n+1}]$.*

Let's first explain how at least one ϵ-homomorphism can be found. The proof below is an adaptation of the argument that can be found in [124] (see Theorems 4 and 5), however, our definition of an ϵ-homomorphism is a bit different.

Fix $\epsilon = 2^{-m}$ for some m. Construct a computable ϵ-homeomorphism of M to $\mathbb{R}^{2n+1}$ as follows. Use Theorem 3.16 and fix a strongly $\cap$-decidable basis of computable balls K in M, recall that this means that the non-emptiness of

intersection (of finite families) is decidable. (By Remark 3.18, we could alternatively use computable closed balls with the same centres and same radii.)

(1) Find an open ϵ-cover $C_1, ..., C_k$ of M having order $n+1$, where each C_i is a finite union of open computable balls from K.
(2) Compute the nerve N of $C_1, \ldots, C_k$.
(3) Find special points $c_1, ..., c_k$ in $\mathbb{R}^{2n+1}$ and a (geometrical) simplicial complex on vertices $c_1, ..., c_k$ isomorphic to N via a simplicial map which maps vertices to vertices.
(4) Define $d_i(x)$ as follows. First, assume $C_i = \cup_{j \in J_i} B(k_{i,j}, r_{i,j})$ and set $d_{i,j}(x) = \sup\{r_i - d(x, k_{i,j}), 0\}$. (In Theorem 4 of [124] Pontryagin uses the distance from x to the complement of C_i.) Then define

$$d_i(x) = \sup_{j \in J_i} d_i(x),$$

let $u(x) = d_1(x) + \ldots + d_k(x)$. Noting that u is strictly positive (because $C_1, \ldots, C_k$ cover the space), set $\theta_i(x) = d_i(x)/u(x)$, and finally define

$$f(x) = \sum_i \theta_i(x) c_i.$$

We will argue that f is an ϵ-homomorphism of M to $\mathbb{R}^{2n+1}$ with some additional properties. But first, we argue that the steps above can be performed computably. The first step is possible because *some* ϵ-cover $C'_1, \ldots, C'_r$ having order $n+1$ exists. By compactness, each C'_i can be replaced with a finite union C_i of (closed or open) basic computable balls from K of radii at most ϵ and that are contained in C'_i, and so that together $C_1, \ldots, C_r$ cover the space. The new cover $C_1, \ldots, C_r$ has order at most the order of $C'_1, \ldots, C'_r$ because $C_i \subseteq C'_i$. We conclude that, in (1), such a cover *exists* among finite subsets of K. We can decide intersection for computable balls in K, and therefore we can also decide which finite families (representing C_i) intersect. The diameter of each C_i can be easily computably estimated from above[26]. Together with $\cap$-decidability of K this implies that, given $\epsilon = 2^{-m}$, we can computably search for such an ϵ-cover satisfying (1). This also implies that the nerve formed in the second step is uniformly computable (as a finite object, i.e., is uniformly given by its strong index).

The third step is possible because in topology we prove that finite simplicial complexes are realizable in Euclidean spaces, this is done using points in a general position, and being in a general position is an open property[27]. If the (combinatorial) dimension of the simplex is n then this can be done in $\mathbb{R}^{2n+1}$ using points that are in general position, which is a c.e. property. See, e.g., Theorem 3 of [124]. There is much freedom in the choice of points in general position, in particular, those can be found in any collection of open balls $V_1, \ldots, V_k$ in $\mathbb{R}^{2n+1}$; this is Theorem 2 of [124]. Theorem 1 of [124] also implies that $c_1, \ldots, c_k$ can be chosen special. Searching for a simplicial map is a finitistic task and can be done in finite time.

The third step uses elementary properties of computable functions to define a computable f.

The proof that f is an ϵ-homeomorphism is essentially literally the same as the proof of the analogous property of the ϵ-homeomorphism constructed in the proof of Theorem 4 of [124], because our function satisfies the same properties (sufficient to prove that it is an ϵ-homeomorphism) as the function built in the proof of the aforementioned Theorem 4 of [124]. More specifically, θ_i is continuous and has support C_i, and also for every x we have that $\sum_i \theta_i(x) = 1$. For instance, it follows that a face in the nerve of the cover is mapped to the corresponding face in the geometric complex on $c_1, \ldots, c_k$. We refer the reader to Theorem 4 of [124] for further details.

It is rather important that in step (3) of the definition of f we only needed that $c_1, \ldots, c_k$ were in general position. By the aforementioned Theorem 2 of [124], such points can be found in any collection of open neighbourhoods of $\mathbb{R}^{2n+1}$. In particular, this is exploited in the proof of Theorem 5 of [124] to show that the set of ϵ-homeomorphisms form a dense open set in $C[M, \mathbb{R}^{2n+1}]$. Although our definition of f is different from that in [124], it shares all the

---

[26]This can be done using, e.g., the distances between the centres and the radii. Alternatively, replace all computable balls in C_i with the respective closed balls that are computable closed sets by Proposition 3.30. Then use that the union of finitely many computable closed sets is computable, and that the diameter is a supremum of a computable function defined on the computably compact space; see Proposition 3.5. This gives an arbitrary tight upper estimate on the diameter of C_i, with all possible uniformity.

[27]Points $c_1, \ldots, c_k \in \mathbb{R}^d$ are in general position of any subset of at most d-many points of $\{c_1, \ldots, c_k\}$ is linearly independent.

properties needed from an ϵ-isomorphism in the proof of Theorem 5 of [124]; in particular, all that is needed is that it maps a face of the nerve to the respective face of the geometric complex.

In other words, we have:

Fact 4.49. For every $g \in C[M, \mathbb{R}^{2n+1}]$ and every $m, k > 0$, there exists a computable 2^{-k}-homomorphism f such that

$$\sup_{x \in M} ||f(x) - g(x)|| < 2^{-m}.$$

Such an f from the fact above will have a rather clear definition given by the construction described above, and for some specific choice of special points $c_1, \ldots, c_k$.

We thus iterate this process. Given an 2^{-n}-homeomorphism f_n, search for an 2^{-n-1}-homeomorphism f_{n+1} (according to (1)-(4) above), such that $\sup_{x \in M} ||f_n(x) - f_{n+1}(x)|| < 2^{-n}$. The limit of the process exists and gives a computable injective continuous embedding of M into $\mathbb{R}^{2n+1}$, thus, it is a homeomorphic embedding (as injective continuous functions on compacta are homeomorphisms). $\qquad \square$

4.10. **Probability spaces and Haar measure.** For a computable compact space X, the space of all probability measures $\mathcal{P}(X)$ is a computably metrized space under the Wasserstein metric defined to be

$$d_w(\mu, \nu) = \sup |\int f d\mu - \int f d\nu|,$$

where the supremum is taken over all 1-Lipschitz functions upon X; that is, $|f(x) - f(y)| \leqslant d(x, y)$ for every $x, y \in X$. The dense set is given by Dirac measures which are the probability measures concentrated at finitely many special points of X. We refer the reader to [65] for further background on computability of measure spaces. Gács [48] initiated a systematic investigation of the effective content of abstract probability and measure spaces in the context of algorithmic randomness.

Perhaps the first known construction of a surjective computable $\Phi : 2^\omega \to \mathcal{P}(2^\omega)$ can be found in Day and Miller [33], but a very closely related argument can be found in the earlier paper [65]. Computable compactness of $\mathcal{P}(2^\omega)$ was later used in [128]. Interestingly, this is a special case of Theorem 3.40 and the lemma below due to Marcone and Valenti [96]. We also note that in the special case of $\mathcal{P}(2^\omega)$ the lemma essentially becomes a triviality; its two-line proof can be extracted from [33].

Lemma 4.50 ([96]). *If X is computably compact then so is $\mathcal{P}(X)$.*

Proof. To cover $\mathcal{P}(X)$, take a finite 2^{-n}-cover of X and let (x_i) be the finitely many centres of the open balls forming the cover. Say, there are N such balls. Take the finite collection of Dirac measures concentrated in the points (x_i) and taking the values of the form $k\dfrac{2^{-n}}{N^2}$, where $k \in \omega$. There are only finitely many such measures, let D be the set of these measures. We claim that balls of radius 2^{-n+2} centred at these points cover the whole space.

Take any other Dirac measure μ concentrated at finitely many points (y_j). We can find, for each j, the least index $i = c(j)$ such that the ball centred at x_i contains y_j. For each i, let

$$C_i = \{y_j : c(j) = i\},$$

and note that $d(x_i, y_j) < 2^{-n}$ for every $y_j \in C_i$. Define

$$\nu(x_i) = \sum_{j \in C_i} \mu(y_j)$$

and let $\rho \in D$ be a measure (from the fixed above finite set) that differs by at most $\dfrac{2^{-n}}{N}$ from ν at every x_i. Fix any 1-Lipschitz function f and assume it takes value 0 at x_0 (recall this means $|f(x) - f(y)| \leqslant d(x, y)$) and assume the diameter of the space X is 1, which makes the absolute value of f also bounded by 1. Then

$$|\int f d\nu - \int f d\rho| = |\sum_{i < N} f(x_i)(\mu(x_i) - \rho(x_i))| \leqslant |\sum_{i < N} (\mu(x_i) - \rho(x_i))| \leqslant N\frac{2^{-n}}{N} = 2^{-n}.$$

On the other hand,

$$| \int f d\mu - \int f d\nu | = | \sum_{i<N} (\sum_{j \in C_i} f(y_j) \mu(y_j) - f(x_i) \nu(x_i)) | = | \sum_{i<N} \sum_{j \in C_i} (f(y_j) - f(x_i)) \mu(y_j) |,$$

and noting that $d(x_i, y_j) < 2^{-n}$ for every $y_j \in C_i$, this is bounded from above by $2^{-n} \sum_j \mu(y_j) = 2^{-n}$. It follows that the distance between ρ and μ is at most 2^{-n+1}. To finish the proof, recall that such Dirac measures are dense in the space, and the choice of μ was arbitrary. □

A few years ago Jason Rute suggested the proof below in a personal communication with Melnikov and Nies. A proof sketch similar to the one that we give below can be found on the logic blog edited by Nies; see [41, Section 17]. Recently this result has been rediscovered in [123]; see [123] for a complete and detailed proof.

Theorem 4.51. *For a compact computable group G, the Haar measure is computable iff G is computably compact.*

Sketch. Suppose G is computably compact. The property of being translation invariant is a Π_1^0 property. So the Haar measure is contained in an effectively closed singleton of the computably compact (by Lemma 4.50) space $\mathcal{P}(G)$. Therefore, the Haar measure is computable by Fact 3.25.

Now take a computable compact group G that has a computable Haar measure. We want to show it is computably compact. Replace $d(x, y)$ with the (integral) average of $d(gx, gy)$, where the average is taken in the Haar measure as g varies across the group. This gives a computable G-invariant metric compatible with the original metric.

Now, to show that G is computably compact (in this new metric), it is enough for each rational k, to effectively find a finite set of points $a_0^k, ..., a_{n-1}^k$ for which every point in G is within distance 2^{-k} of one of these points. Fix k. Using our Haar measure find the measure of a ball of radius $2^{-(k+1)}$. Call this measure δ. (Since the new distance is G-invariant, all balls of the same radius have the same measure.) Find a collection of balls $B_0, ..., B_{n-1}$ with radius $2^{-(k+1)}$ whose union $C = B_0 \cup ... \cup B_{n-1}$ has measure $> 1 - \delta$; recall that the measure is left-c.e. Now, consider any point x not in this union C. It has to be at distance $< 2^{-(k+1)}$ from the union. Otherwise, there would be a ball centred at x with radius $2^{-(k+1)}$, and hence having measure δ, which is disjoint from the union C. But the union C has measure $> 1 - \delta$, so this cannot happen. Therefore all points of G are within distance 2^{-k} of the centres of $B_0, ..., B_{n-1}$. This algorithm shows that the space is computably compact in the new metric. To show it is computably compact in the original metric, for any finite list of rational balls in original metric, convert it to a list of balls in the new metric. Now, if this list of balls covers the space G, by computable compactness, we will eventually find this out. □

It follows from Theorem 4.35 that a profinite group admits a recursive presentation iff it has a computably metrization with computable Haar measure iff it admits a computably compact presentation. (In the latter two cases we need to also assume the operations are computable.) It was established in [99] that recursive profinite groups are exactly the Pontryagin duals of computable torsion groups, and it is not difficult to construct an example of a procyclic, computably metrized group whose dual has no computable copy [99]. (A similar result has recently been established in [123].) In the connected case, one of the proofs of Theorem 4.40 actually builds a computably metrized group with computable operations whose space is not homeomorphic to any computably compact space. We summarise this below:

Corollary 4.52 ([95, 99])**.** *In the classes of connected compact abelian and profinite abelian groups there exist examples of computably metrizable groups no computable metrization of which can compute Haar measure.*

4.11. **Some further open questions.** In Subsection 4.4 we explained why the characterisation problem and the isometric isomorphism problem for compact sets are arithemtical, and also why the homeomorphism problem for compact Polish spaces is Σ_1^1-complete. Similar results for compact Polish groups can be found in [99]. The following related questions are left open:

Question 4.53.

(1) *(Melnikov and Harrison-Trainor[28]) What is the complexity of the isomorphism problem for (not necessarily compact) Polish spaces? (Is the naive upper bound optimal?)*

---

[28]This question and question (2) below have been posed in [60] in a slightly different form.

(2) (Melnikov and Harrison-Trainor) *What is the complexity of the characterisation problem* $\{i : M_i \cong_{hom} S^3\}$
for the 3-sphere S^3? *What about the 2-ball? More generally, is it true that the (topological) characterisation*
problem $\{i : M_i \cong_{hom} S\}$ *for any compact manifold* S *is arithmetical?*

It is known that the characterisation problem or every compact 2-surface (including the 2-sphere, obviously) is
arithmetical [60]. They key step in their proof produces an arithmetical atlas of a given computable surface.

Question 4.54 (Melnikov and Harrison-Trainor [60]). *Suppose a compact computable manifold* S. *Does* S *admit*
an arithmetical atlas?

We have already mentioned above that the index set approach has not yet been applied to the effective enu-
meration of all (partial) computably compact spaces. This seems reasonable assuming the thesis of the article
(that computable compactness is a natural approach to computability in the compact case). Also, there are not
many arithmetical *completeness* index set results in the literature (some can be found in [99]). The approach via
computable compactness seems rather natural for such potential completeness results in the compact case (cf. Re-
mark 4.31).

Question 4.55.
   (1) *Develop the index set approach to classification using the enumeration of all (partial) computably compact*
       spaces.
   (2) *Prove completeness results for the arithmetical index set estimates stated above and also for other results*
       that can be found in, e.g., [40, 60, 95].

In Subsection 4.4 we also explained how to construct a compact computable Polish space not homeomorphic to
any computably compact space; this is Theorem 4.40. We also conjectured that a $0'''$ proof can be used to replace
algebraic topology, but this approach is by no means elementary (if it works). It is rather natural to ask whether
there is a less involved proof of Theorem 4.40. The question below is, of course, loosely stated.

Question 4.56. *Find an elementary (elegant?) proof of Theorem 4.40.*

In Subsection 4.3 we discussed an application of computable compactness to constructing basic sequences in
computable Banach spaces. Bosserhoff [13] constructed a computable Banach space with a Schauder basis and
no computable Schauder basis, and Downey, Long and Greenberg [94] showed that the index set of computable
Banach spaces with computable bases is Σ_3^0 complete. Using the characterisation of Schauder bases together with
computable compactness, it is possible to show [94] that having a basis is a Σ_1^1 property. The following question
seems rather challenging.

Question 4.57. *Is the complexity of the index set* $\{i \mid W_i$ *is a computable Banach space with a Schauder basis*$\}$ Σ_1^1
complete?

There are many interesting open questions in the area of computable Banach spaces; see, e.g., Long [94]. It is
highly plausible that computable compactness can be used to attack some of these questions.

Algebraic topology has played a considerable role in many proofs throughout the paper. We believe that there
is much more to be said about the algorithmic content of algebraic topology, so we state:

Problem 4.58. Develop a general theory of computable algebraic topology and computable homological algebra.

Classically, the class of locally compact spaces is perhaps the narrowest natural class that contains both compact
and discrete spaces. There have been many attempts to define effective local compactness in the literature (e.g.,
[122]). Nonetheless, it seems that there is no commonly accepted and robust notion that would be considered
'standard'.

Problem 4.59. Suggest a robust (and useful) notion of effectively locally compact Polish space.

We of course do not exclude the possibility that some of the known definitions will already be good enough, but
certainly we need to accumulate more results to draw any conclusions.

Our last problem is concerned with primitive recursive analysis. Historically a lot of elementary computable analysis was in fact developed primitively recursively; see, e.g., book [54]. However, gradually, primitive recursiveness was abandoned, perhaps because of technical difficulties that arise while dealing with primitive recursive procedures. Beginning with the 1980-s pretty much all computable analysis has been done using general Turing computability, see [127], [148]. On the other hand, there has been an increasing interest in polynomial-time analysis of continuous functions; see book [83]. Recently in [39] and [132, 133], it has been proposed to revive primitive recursive analysis using modern methods; this program could potentially serve as a link between abstract computable analysis and the more practical polynomial time and computational analysis.

For instance, say that a Polish space is *punctual* is the distances between special points are uniformly primitive recursive *nonzero* reals (to avoid dealing with equality). Some recent results about punctual spaces can be found in [5]. The role of compactness in primitive recursive analysis is very poorly understood. For example, using computable compactness of $[0, 1]$, it is easy to show that every computable continuous function on $[0, 1]$ is effectively uniformly continuous, i.e., has a computable modulus of uniform continuity. The last section of [39] outlines a primitive recursive version of this elementary fact. The proof also uses compactness, but it uses it rather differently from the usual proof. On the other hand, one of the main results in [5] relies on the *standard* computable compactness (in the sense of this paper) to establish a primitive recursive result. So perhaps more insight is needed to attack the following:

Problem 4.60. Give a robust (and useful) definition of a punctually compact space.

We suspect that there are several potentially useful definitions of a punctually compact space that are not equivalent. For instance, we would like to know whether the results discussed in this paper, especially Theorem 1.1, hold primitively recursively.

References

[1] Oliver Aberth. *Computable analysis*. McGraw-Hill International Book Company, New York, 1980.

[2] Paul Alexandroff. Untersuchungen über Gestalt und Lage Abgeschlossener Mengen Beliebiger Dimension. *Ann. of Math. (2)*, 30(1-4):101–187, 1928/29.

[3] C. Ash and J. Knight. *Computable structures and the hyperarithmetical hierarchy*, volume 144 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 2000.

[4] Jeremy Avigad and Vasco Brattka. Computability and analysis: the legacy of Alan Turing. In *Turing's legacy: developments from Turing's ideas in logic*, volume 42 of *Lect. Notes Log.*, pages 1–47. Assoc. Symbol. Logic, La Jolla, CA, 2014.

[5] R. Bagaviev, I. Batyrshin, N. Bazhenov, D. Bushtets, M. Dorzhieva, R. Kornev, A. Melnikov, K.M. Ng, and H.T. Koh. Computably and punctually universal spaces. *Preprint*, 2021.

[6] J. Barrett, R. Downey, and N. Greenberg. Cousin's lemma in second-order arithmetic. *Preprint*.

[7] Nikolay Bazhenov, Rod Downey, Iskander Kalimullin, and Alexander Melnikov. Foundations of online structure theory. *Bull. Symb. Log.*, 25(2):141–181, 2019.

[8] Nikolay Bazhenov, Matthew Harrison-Trainor, and Alexander Melnikov. Computable Stone spaces, 2021.

[9] Itaï Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov. Model theory for metric structures. In *Model theory with applications to algebra and analysis. Vol. 2*, volume 350 of *London Math. Soc. Lecture Note Ser.*, pages 315–427. Cambridge Univ. Press, Cambridge, 2008.

[10] Itaï Ben Yaacov and Arthur Paul Pedersen. A proof of completeness for continuous first-order logic. *J. Symbolic Logic*, 75(1):168–190, 2010.

[11] Errett Bishop. *Foundations of constructive analysis*. McGraw-Hill Book Co., New York-Toronto, Ont.-London, 1967.

[12] Emile Borel. Le calcul des intégrales définies. *Journal de Mathématiques Pures et Appliquées*, 8:159–210, 1912.

[13] Volker Bosserhoff. On the effective existence of Schauder bases. *J.UCS*, 15(6):1145–1161, 2009.

[14] Volker Bosserhoff and Peter Hertling. Effective subsets under homeomorphisms of $\mathbb{R}^n$. *Inform. and Comput.*, 245:197–212, 2015.

[15] Vasco Brattka. Computability of Banach space principles. *FernUniversität Hagen, Fachbereich Informatik, Hagen*, 2001.

[16] Vasco Brattka. Borel complexity and computability of the Hahn-Banach theorem. *Arch. Math. Logic*, 46(7-8):547–564, 2008.

[17] Vasco Brattka. Plottable real number functions and the computable graph theorem. *SIAM J. Comput.*, 38(1):303–328, 2008.

[18] Vasco Brattka, Matthew de Brecht, and Arno Pauly. Closed choice and a uniform low basis theorem. *Ann. Pure Appl. Logic*, 163(8):986–1008, 2012.

[19] Vasco Brattka and Ruth Dillhage. Computability of compact operators on computable Banach spaces with bases. *MLQ Math. Log. Q.*, 53(4-5):345–364, 2007.

[20] Vasco Brattka, Stéphane Le Roux, Joseph S. Miller, and Arno Pauly. Connected choice and the Brouwer fixed point theorem. *J. Math. Log.*, 19(1):1950004, 46, 2019.

[21] Vasco Brattka and Gero Presser. Computability on subsets of metric spaces. *Theoretical Computer Science*, 305(1-3):43–76, 2003.

[22] Mark Braverman and Michael Yampolsky. *Computability of Julia Sets*, volume 23 of *Algorithms and Computation in Mathematics*. Springer, Berlin, 2009.

[23] Konrad Burnik and Zvonko Iljazović. Computability of 1-manifolds. *Log. Methods Comput. Sci.*, 10(2):2:8, 28, 2014.

[24] Caleb Camrud, Isaac Goldbring, and Timothy H. McNicholl. On the complexity of the theory of a computably presented metric structure, arXiv preprint, 2021.

[25] N. L. Carothers. *A short course on Banach space theory*, volume 64 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2005.

[26] G. S. Ceitin. Algorithmic operators in constructive complete separable metric spaces. *Dokl. Akad. Nauk SSSR*, 128:49–52, 1959.

[27] D. Cenzer. Π_1^0 classes in computability theory. In *Handbook of computability theory*, volume 140 of *Stud. Logic Found. Math.*, pages 37–85. North-Holland, Amsterdam, 1999.

[28] D. Cenzer and J. B. Remmel. Π_1^0 classes in mathematics. In *Handbook of recursive mathematics, Vol. 2*, volume 139 of *Stud. Logic Found. Math.*, pages 623–821. North-Holland, Amsterdam, 1998.

[29] D. Cenzer and J. B. Remmel. Π_1^0 classes in mathematics. In *Handbook of recursive mathematics, Vol. 2*, volume 139 of *Stud. Logic Found. Math.*, pages 623–821. North-Holland, Amsterdam, 1998.

[30] Douglas Cenzer. Chapter 2 - Π_1^0 classes in computability theory. In Edward R. Griffor, editor, *Handbook of Computability Theory*, volume 140 of *Studies in Logic and the Foundations of Mathematics*, pages 37–85. Elsevier, 1999.

[31] Peter Cholak, Richard Coles, Rod Downey, and Eberhard Herrmann. Automorphisms of the lattice of Π_1^0 classes: perfect thin classes and anc degrees. *Trans. Amer. Math. Soc.*, 353(12):4899–4924, 2001.

[32] P. J. Couch, B. D. Daniel, and Timothy H. McNicholl. Computing space-filling curves. *Theory Comput. Syst.*, 50(2):370–386, 2012.

[33] Adam R. Day and Joseph S. Miller. Randomness for non-computable measures. *Trans. Amer. Math. Soc.*, 365(7):3575–3591, 2013.

[34] Matthew de Brecht, Arno Pauly, and Matthias Schröder. Overt choice. *Computability*, 9(3-4):169–191, 2020.

[35] David E. Diamondstone, Damir D. Dzhafarov, and Robert I. Soare. Π_1^0 classes, Peano arithmetic, randomness, and computable domination. *Notre Dame J. Form. Log.*, 51(1):127–159, 2010.

[36] R. Downey and D. Hirschfeldt. *Algorithmic randomness and complexity*. Theory and Applications of Computability. Springer, New York, 2010.

[37] R.G. Downey. Abstract dependence, recursion theory, and the lattice of recursively enumerable filters. *Bulletin of the Australian Mathematical Society*, 27(3):461–464, 1983.

[38] Rod Downey and Carl G. Jockusch. Every low Boolean algebra is isomorphic to a recursive one. *Proceedings of the American Mathematical Society*, 122(3):871–880, 1994.

[39] Rod Downey, Alexander Melnikov, and Keng Meng Ng. Foundations of online structure theory II: The operator approach. *Log. Methods Comput. Sci.*, 17(3):Paper No. 6, 35, 2021.

[40] Rodney G. Downey and Alexander G. Melnikov. Computable analysis and classification problems. In Marcella Anselmo, Gianluca Della Vedova, Florin Manea, and Arno Pauly, editors, *Beyond the Horizon of Computability - 16th Conference on Computability in Europe, CiE 2020, Fisciano, Italy, June 29 - July 3, 2020, Proceedings*, volume 12098 of *Lecture Notes in Computer Science*, pages 100–111. Springer, 2020.

[41] A. Nies (editor). Logic Blog 2016. Available at `http://arxiv.org/abs/1703.01573`, 2016.

[42] Per Enflo. A counterexample to the approximation problem in Banach spaces. *Acta Math.*, 130:309–317, 1973.

[43] Y. Ershov and S. Goncharov. *Constructive models.* Siberian School of Algebra and Logic. Consultants Bureau, New York, 2000.

[44] Lawrence Feiner. Hiearchies of Boolean algebras. *J. Symbolic Logic*, 35(3):365–374, 09 1970.

[45] Johanna N. Y. Franklin and Timothy H. McNicholl. Degrees of and lowness for isometric isomorphism. *J. Log. Anal.*, 12:Paper No. 6, 23, 2020.

[46] Johanna N. Y. Franklin and Dan Turetsky. Taking the path computably traveled. *J. Logic Comput.*, 29(6):969–973, 2019.

[47] L. Fuchs. *Infinite abelian groups. Vol. I.* Pure and Applied Mathematics, Vol. 36. Academic Press, New York, 1970.

[48] Peter Gács. Uniform test of algorithmic randomness over a general space. *Theoret. Comput. Sci.*, 341(1-3):91–137, 2005.

[49] Su Gao. *Invariant descriptive set theory*, volume 293 of *Pure and Applied Mathematics (Boca Raton)*. CRC Press, Boca Raton, FL, 2009.

[50] Alex Gavruskin and André Nies. Universality for left-computably enumerable metric spaces. *Lobachevskii Journal of Mathematics*, 35(4):292–294, 2014.

[51] Guido Gherardi and Alberto Marcone. How incomputable is the separable Hahn-Banach theorem? *Notre Dame J. Form. Log.*, 50(4):393–425 (2010), 2009.

[52] S. Goncharov. *Countable Boolean algebras and decidability.* Siberian School of Algebra and Logic. Consultants Bureau, New York, 1997.

[53] S. Goncharov and J. Knight. Computable structure and antistructure theorems. *Algebra Logika*, 41(6):639–681, 757, 2002.

[54] R. Goodstein. *Recursive analysis.* Studies in Logic and the Foundations of Mathematics, North-Holland, Amsterdam, 1961.

[55] Noam Greenberg, Alexander G. Melnikov, Julia F. Knight, and Daniel Turetsky. Uniform procedures in uncountable structures. *J. Symb. Log.*, 83(2):529–550, 2018.

[56] Vassilios Gregoriades, Tamás Kispéter, and Arno Pauly. A comparison of concepts from computable analysis and effective descriptive set theory. *Math. Structures Comput. Sci.*, 27(8):1414–1436, 2017.

[57] A. Grzegorczyk. Computable functionals. *Fund. Math.*, 42:168–202, 1955.

[58] A. Grzegorczyk. On the definition of computable functionals. *Fund. Math.*, 42:232–239, 1955.

[59] A. Grzegorczyk. On the definitions of computable real continuous functions. *Fund. Math.*, 44:61–71, 1957.

[60] M. Harrison-Trainor and A. Melnikov. An arithmetic analysis of closed surfaces. *Preprint*, 2021.

[61] Matthew Harrison-Trainor, Alexander Melnikov, and Keng Meng Ng. Computability of Polish spaces up to homeomorphism. *The Journal of Symbolic Logic*, pages 1–25, 2020.

[62] Edwin Hewitt. The role of compactness in analysis. *The American Mathematical Monthly*, 67(6):499–516, 1960.

[63] Karl H. Hofmann and Sidney A. Morris. *The Structure of Compact Groups.* De Gruyter Studies in Mathematics, 25, Berlin, New York, 1998.

[64] M. Hoyrup, T. Kihara, and V. Selivanov. Degree spectra of homeomorphism types of Polish spaces. *Preprint.*, 2020.

[65] Mathieu Hoyrup and Cristóbal Rojas. Computability of probability measures and Martin-Löf randomness over metric spaces. *Inform. and Comput.*, 207(7):830–847, 2009.

[66] Zvonko Iljazović. Chainable and circularly chainable co-r.e. sets in computable metric spaces. *J.UCS*, 15(6):1206–1235, 2009.

[67] Zvonko Iljazović. Isometries and computability structures. *J.UCS*, 16(18):2569–2596, 2010.

[68] Zvonko Iljazović. Compact manifolds with computable boundaries. *Log. Methods Comput. Sci.*, 9(4):4:19, 22, 2013.

[69] Zvonko Iljazović and Takayuki Kihara. Computability of subsets of metric spaces. In *Handbook of computability and complexity in analysis*, Theory Appl. Comput., pages 29–69. Springer, Cham, [2021] ©2021.

[70] Zvonko Iljazović and Igor Sušić. Semicomputable manifolds in computable topological spaces. *J. Complexity*, 45:83–114, 2018.

[71] Zvonko Iljazović and Lucija Validžić. Effective compactness and orbits of points under the isometry group. *Ann. Pure Appl. Logic*, 174(2):Paper No. 103198, 2023.

[72] Carl G. Jockusch, Jr. and Robert I. Soare. A minimal pair of Π_1^0 classes. *J. Symbolic Logic*, 36:66–78, 1971.

[73] Carl G. Jockusch, Jr. and Robert I. Soare. Degrees of members of Π_1^0 classes. *Pacific J. Math.*, 40:605–616, 1972.

[74] Carl G. Jockusch, Jr. and Robert I. Soare. Π_1^0 classes and degrees of theories. *Trans. Amer. Math. Soc.*, 173:33–56, 1972.

[75] Iraj Kalantari and Galen Weitkamp. Effective topological spaces. I. A definability theory. *Ann. Pure Appl. Logic*, 29(1):1–27, 1985.

[76] Iraj Kalantari and Galen Weitkamp. Effective topological spaces. II. A hierarchy. *Ann. Pure Appl. Logic*, 29(2):207–224, 1985.

[77] Iraj Kalantari and Galen Weitkamp. Effective topological spaces. III. Forcing and definability. *Ann. Pure Appl. Logic*, 36(1):17–27, 1987.

[78] Iraj Kalantari and Larry Welch. On Turing degrees of points in computable topology. *MLQ Math. Log. Q.*, 54(5):470–482, 2008.

[79] H. Kamo. *Computability in some fundamental theorems in functional analysis and general topology*. PhD Thesis, Kyoto University, 2006.

[80] N. Khisamiev. Hierarchies of torsion-free abelian groups. *Algebra i Logika*, 25(2):205–226, 244, 1986.

[81] S. C. Kleene. A note on computable functionals. *Nederl. Akad. Wetensch. Proc. Ser. A.* **59** = *Indag. Math.*, 18:275–280, 1956.

[82] Julia F. Knight and Michael Stob. Computable Boolean algebras. *J. Symbolic Logic*, 65(4):1605–1623, 12 2000.

[83] Ker-I Ko. *Complexity theory of real functions*. Progress in Theoretical Computer Science. Birkhäuser Boston, Inc., Boston, MA, 1991.

[84] Ker-I Ko and Harvey Friedman. Computational complexity of real functions. *Theoret. Comput. Sci.*, 20(3):323–352, 1982.

[85] Margarita Korovina and Oleg Kudinov. Towards computability over effectively enumerable topological spaces. In *Proceedings of the Fifth International Conference on Computability and Complexity in Analysis (CCA 2008)*, volume 221 of *Electron. Notes Theor. Comput. Sci.*, pages 115–125. Elsevier Sci. B. V., Amsterdam, 2008.

[86] Margarita Korovina and Oleg Kudinov. The Rice-Shapiro theorem in computable topology. *Log. Methods Comput. Sci.*, 13(4):Paper No. 30, 13, 2017.

[87] Margarita Korovina and Oleg Kudinov. Highlights of the Rice-Shapiro theorem in computable topology. In *Perspectives of system informatics*, volume 10742 of *Lecture Notes in Comput. Sci.*, pages 241–255. Springer, Cham, 2018.

[88] G. Kreisel. A variant to Hilbert's theory of the foundations of arithmetic. *British J. Philos. Sci.*, 4:107–129 errata and corrigenda, 357 (1954), 1953.

[89] Oleg Kudinov and Victor Selivanov. First order theories of some lattices of open sets. *Log. Methods Comput. Sci.*, 13(3):Paper No. 16, 18, 2017.

[90] Peter La Roche. Effective Galois theory. *J. Symbolic Logic*, 46(2):385–392, 1981.

[91] Peter Edwin La Roche. *Contributions to recursive algebra*. ProQuest LLC, Ann Arbor, MI, 1978. Thesis (Ph.D.)–Cornell University.

[92] Daniel Lacombe. Quelques procédés de définition en topologie recursive. In *Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting)*, Studies in Logic and the Foundations of Mathematics, pages 129–158. North-Holland Publishing Co., Amsterdam, 1959.

[93] Davorin Lešnik. Constructive Urysohn universal metric space. *J.UCS*, 15(6):1236–1263, 2009.

[94] Qian Long. Computability-theoretic complexity of effective Banach spaces, M.Sc. thesis, Victoria University of Wellington, 2021.

[95] M. Lupini, A. Melnikov, and A. Nies. Computable topological abelian groups. *Preprint.*, 2021.

[96] Alberto Marcone and Manlio Valenti. Effective aspects of Hausdorff and Fourier dimension. *Computability*, 11(3-4):299–333, 2022.

[97] A. Melnikov. Enumerations and completely decomposable torsion-free abelian groups. *Theory Comput. Syst.*, 45(4):897–916, 2009.

[98] A. Melnikov. New degree spectra of Polish spaces. *Sibirsk. Mat. Zh.*, (62:5):1091–1108, 2021.

[99] Alexander Melnikov. Computable topological groups and Pontryagin duality. *Trans. Amer. Math. Soc.*, 370(12):8709–8737, 2018.

[100] Alexander Melnikov and Antonio Montalbán. Computable Polish group actions. *J. Symb. Log.*, 83(2):443–460, 2018.

[101] Alexander G. Melnikov. Computably isometric spaces. *J. Symbolic Logic*, 78(4):1055–1085, 2013.

[102] Alexander G. Melnikov and Keng Meng Ng. Computable structures and operations on the space of continuous functions. *Fund. Math.*, 233(2):101–141, 2016.

[103] Alexander G. Melnikov and André Nies. The classification problem for compact computable metric spaces. In *The nature of computation*, volume 7921 of *Lecture Notes in Comput. Sci.*, pages 320–328. Springer, Heidelberg, 2013.

[104] G. Metakides and A. Nerode. Recursively enumerable vector spaces. *Ann. Math. Logic*, 11(2):147–171, 1977.

[105] G. Metakides and A. Nerode. Effective content of field theory. *Ann. Math. Logic*, 17(3):289–320, 1979.

[106] G. Metakides, A. Nerode, and R. A. Shore. Recursive limits on the Hahn-Banach theorem. In *Errett Bishop: reflections on him and his research (San Diego, Calif., 1983)*, volume 39 of *Contemp. Math.*, pages 85–91. Amer. Math. Soc., Providence, RI, 1985.

[107] Joseph S. Miller. Π^0_1 *classes in computable analysis and topology*. PhD thesis, Cornell University, Ithaca, NY, USA, 2002.

[108] Takakazu Mori, Yoshiki Tsujii, and Mariko Yasugi. Computability structures on metric spaces. In Douglas S. Bridges, Cristian S. Calude, Jeremy Gibbons, Steve Reeves, and Ian H. Witten, editors, *Combinatorics, Complexity, and Logic*, volume 1 of *Discrete Mathematics and Theoretical Computer Science*, pages 351–362. Springer, Singapore, 1997.

[109] Y. N. Moschovakis. Recursive metric spaces. *Fund. Math.*, 55:215–238, 1964.

[110] Yiannis N. Moschovakis. *Descriptive set theory*, volume 155 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, second edition, 2009.

[111] Carl Mummert. Reverse mathematics of MF spaces. *J. Math. Log.*, 6(2):203–232, 2006.

[112] James R. Munkres. *Elements of algebraic topology*. Addison-Wesley Publishing Company, Menlo Park, CA, 1984.

[113] J. Myhill. A recursive function, defined on a compact interval and having a continuous derivative that is not recursive. *Michigan Mathematical Journal*, 18(2):97–98, 1971.

[114] Anil Nerode and Wen Qi Huang. An application of pure recursion theory to recursive analysis. *Acta Math. Sinica*, 28(5):625–636, 1985.

[115] Eike Neumann. On the computability of the set of automorphisms of the unit square. *Theoret. Comput. Sci.*, 903:74–83, 2022.

[116] André Nies and Slawomir Solecki. Local compactness for computable Polish metric spaces is Π^1_1 -complete. In *Evolving Computability - 11th Conference on Computability in Europe, CiE 2015, Bucharest, Romania, June 29 - July 3, 2015. Proceedings*, pages 286–290, 2015.

[117] E. Ju. Nogina. Effectively topological spaces. *Dokl. Akad. Nauk SSSR*, 169:28–31, 1966.

[118] E. Ju. Nogina. Correlations between certain classes of effectively topological spaces. *Mat. Zametki*, 5:483–495, 1969.

[119] S. P. Odintsov and V. L. Selivanov. Arithmetic hierarchy and ideals of enumerated Boolean algebras. *Sib. Math. J.*, 30(6):952–960, 1989.

[120] Arno Pauly. On the topological aspects of the theory of represented spaces. *Computability*, 5(2):159–180, 2016.

[121] Arno Pauly. Enumeration degrees and topology. In *Sailing routes in the world of computation*, volume 10936 of *Lecture Notes in Comput. Sci.*, pages 328–337. Springer, Cham, 2018.

[122] Arno Pauly. Effective local compactness and the hyperspace of located sets. *CoRR*, abs/1903.05490, 2019.

[123] Arno Pauly, Dongseong Seon, and Martin Ziegler. Computing Haar Measures. In Maribel Fernández and Anca Muscholl, editors, *28th EACSL Annual Conference on Computer Science Logic (CSL 2020)*, volume 152 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 34:1–34:17, Dagstuhl, Germany, 2020. Schloss Dagstuhl–Leibniz-Zentrum fuer Informatik.

[124] L. S. Pontryagin. *Foundations of algebraic topology*. "Nauka", Moscow, third edition, 1986.

[125] M. B. Pour-El and J. Caldwell. On a simple definition of computable function of a real variable — with applications to functions of a complex variable. *Mathematical Logic Quarterly*, 21(1):1–19, 1975.

[126] M. B. Pour-El and I. Richards. Computability and noncomputability in classical analysis. *Transactions of the American Mathematical Society*, 275(2):539–560, 1983.

[127] Marian B. Pour-El and J. Ian Richards. *Computability in analysis and physics*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1989.

[128] Jan Reimann and Theodore A. Slaman. Measures and their random reals. *Trans. Amer. Math. Soc.*, 367(7):5081–5097, 2015.

[129] Robert Rettinger. Compactness and the effectivity of uniformization. In *How the world computes*, volume 7318 of *Lecture Notes in Comput. Sci.*, pages 626–625. Springer, Heidelberg, 2012.

[130] H. Rogers. *Theory of recursive functions and effective computability*. MIT Press, Cambridge, MA, second edition, 1987.

[131] Hans Sagan. *Space-filling curves*. Universitext. Springer-Verlag, New York, 1994.

[132] Victor Selivanov and Svetlana Selivanova. Primitive recursive ordered fields and some applications, 2021.

[133] V. L. Selivanov and S. Selivanova. Primitive recursive ordered fields and some applications. In F. Boulier, M. England, T. M. Sadykov, and E. V. Vorozhtsov, editors, *Computer Algebra in Scientific Computing, CASC 2021*, volume 12865 of *Lecture Notes in Computer Science*, pages 353–369, Cham, 2021. Springer.

[134] J. R. Shoenfield. Degrees of models. *J. Symbolic Logic*, 25:233–237, 1960.

[135] W. Sierpiński. Sur un espace métrique séparable universel. *Fundamenta Mathematicae*, 33:115–122, 1945.

[136] S. Simpson. *Subsystems of second order arithmetic*. Perspectives in Logic. Cambridge University Press, Cambridge, second edition, 2009.

[137] Rick L. Smith. *The theory of profinite groups with effective presentations*. ProQuest LLC, Ann Arbor, MI, 1979. Thesis (Ph.D.)–The Pennsylvania State University.

[138] Rick L. Smith. Effective aspects of profinite groups. *J. Symbolic Logic*, 46(4):851–863, 1981.

[139] Ernst Specker. Der Satz vom Maximum in der rekursiven Analysis. In *Constructivity in mathematics: Proceedings of the colloquium held at Amsterdam, 1957 (edited by A. Heyting)*, Studies in Logic and the Foundations of Mathematics, pages 254–265. North-Holland Publishing Co., Amsterdam, 1959.

[140] Dieter Spreen. A characterization of effective topological spaces. In *Recursion theory week (Oberwolfach, 1989)*, volume 1432 of *Lecture Notes in Math.*, pages 363–387. Springer, Berlin, 1990.

[141] Dieter Spreen. Representations versus numberings: on the relationship of two computability notions. *Theoret. Comput. Sci.*, 262(1-2):473–499, 2001.

[142] Dieter Spreen and Paul Young. Effective operators in a topological setting. In *Computation and proof theory (Aachen, 1983)*, volume 1104 of *Lecture Notes in Math.*, pages 437–451. Springer, Berlin, 1984.

[143] Ying-Ying Tran. *Computably Enumerable Boolean Algebras*. PhD thesis, 2018.

[144] Alan M. Turing. On computable numbers, with an application to the Entscheidungsproblem. *Proceedings of the London Mathematical Society*, 42:230–265, 1936.

[145] Alan M. Turing. On Computable Numbers, with an Application to the Entscheidungsproblem. A Correction. *Proceedings of the London Mathematical Society*, 43:544–546, 1937.

[146] Matea Čelar and Zvonko Iljazović. Computability of products of chainable continua. *Theory Comput. Syst.*, 65(2):410–427, 2021.

[147] William C. Waterhouse. Profinite groups are Galois groups. *Proc. Amer. Math. Soc.*, 42:639–640, 1974.

[148] Klaus Weihrauch. *Computable analysis*. Texts in Theoretical Computer Science. An EATCS Series. Springer-Verlag, Berlin, 2000. An introduction.

[149] Klaus Weihrauch. Computational complexity on computable metric spaces. *MLQ Math. Log. Q.*, 49(1):3–21, 2003.

[150] Klaus Weihrauch and Tanja Grubba. Elementary computable topology. *J.UCS*, 15(6):1381–1422, 2009.

[151] I. D. Zaslavskiĭ. Some properties of constructive real numbers and constructive functions. *Trudy Mat. Inst. Steklov.*, 67:385–457, 1962.

Victoria University of Wellington
Email address: Rod.Downey@msor.vuw.ac.nz

Victoria University of Wellington, Wellington, New Zealand
Email address: alexander.g.melnikov@gmail.com, sasha.melnikov@vuw.ac.nz