A NOTE ON REALIZATION OF INDEX SETS IN Π^0_1 CLASSES

ROD DOWNEY AND ALEXANDER MELNIKOV

1. Introduction

We assume that the reader is familiar with Π^0_1 classes and index sets as per Cenzer and Jockusch [1], and Soare [4]. The Kriesel Basis Theorem says that each Π^0_1 class has a member of c.e. degree. In [2], Csima, Downey and Ng analysed the problem of determining which sets of c.e. degrees can be realised as members of Π^0_1 classes. Such sets of degrees can be considered as index sets. To wit, we say that e is realized in a Π^0_1 class $\mathcal{C}$ iff there a member P of $\mathcal{C}$ with $\deg_T(W_e) = \deg_T(P)$, and $\mathcal{C}$ is a Π^0_1 class, then $W[\mathcal{C}] = \{e : W_e \text{ is realisable in } \mathcal{C}\}$. Csima, Downey and Ng [2] have recently gaven a precise classification of the index sets which name precisely the c.e. degrees realised in some Π^0_1 class. This involved the following notion. A set S *represents* an index set I iff $I =_{\text{def}} G(S) = \{e : (\exists j \in S)\ W_e \equiv_T W_j\}$.

Theorem 1.1 (Csima, Downey and Ng [2])**.** *An index set I is realisable in a Π^0_1 class iff I has a Σ^0_3 representation iff I has a computable representation.*

Notice that a crude upper bound for the relevant index sets is Σ^0_4, while some Σ^0_4-complete index sets such as $\{e \mid W_e \text{ complete}\}$ have Σ^0_3 representations. (In this last case take the singleton consisting of any index for the halting problem.)

This led to Csima, Downey and Ng trying to ascertain precisely which index sets have Σ^0_3 representations. Classical index set results by Yates [5, 6] show that if A is low$_2$ then $\{e \mid W_e \leq_T A\}$ has a Σ^0_3 representation. Csima, Downey and Ng showed that the collection of superlow c.e. sets have Σ^0_3 representations, as do all upper cones. They asked the following question.

Question 1.2 (Csima, Downey and Ng [2])**.** *Is there some non-low$_2$ c.e. set A such that the c.e. lower cone below A has a Σ^0_3 representation?*

In this note we solve this question verifying a conjecture from [2].

Theorem 1.3. *$\{e \mid W_e \leq_T A\}$ has a Σ^0_3 representation iff A is low$_2$.*

2. The Proof

The proof is not difficult, but involves assembling a number of facts in a new way. First we consider the computable functions f and g defined by the uniform construction which, for each W_k, builds a splitting $W_k = W^1_k \oplus W^2_k \equiv_{\text{def}} W_{f(k)} \oplus W_{g(k)}$, and meets the requirements for $i = 1, 2$,

$$R_{\langle e,i\rangle} : \exists^\infty s(\Phi_e^{W^i_k}(e)\downarrow [s]) \to \Phi_e^{W^i_k}(e)\downarrow .$$

Downey and Melnikov thank the Marsden Fund of New Zealand.

We do this in the usual way: put x entering $W_k[s]$ into the set which does not injure the requirement of highest priority threatened. (This is the standard Sacks' method.)

We could assume that f and g are strictly increasing in their arguments, andnote that their domains do not overlap. In particular, without loss of generality we could assume that:

(1) $f(k) \neq f(j)$, $g(k) \neq g(j)$ whenever $j \neq k$;
(2) $f(k) \neq g(i)$ for any i and j;
(3) the sets $\{f(k) : k \in \omega\}$ and $\{g(k) : k \in \omega\}$ are both computable;

In particular, given $j \in \omega$ we can recognise whether $j = f(k)$ or $j = g(k)$ for some k, and thus compute this k. Also, note that the family of sets $\{W_{g(k)}, W_{f(k)}\}_{k\in\omega}$ is uniformly low.

Assuming $(1) - (3)$ above, the following lemma is immediate:

Lemma 2.1. *Let $S \subset \omega$ and let $\hat{S} = \{f(k) \mid k \in S\} \cup \{g(k) \mid k \in S\}$. Then $S \leq_1 \hat{S}$.*

Proof. Both f and g 1-reduce S to $\hat{S}$. □

Now suppose that A is non-low$_2$. Then $S = \{e \mid W_e \leq_T A\}$ is Σ^0_4 complete by Yates [5, 6]. Suppose that S has a Σ^0_3 representation R. Consider $\hat{S}$.

We claim that $e \in \hat{S}$ if, and only if, either $e = f(k)$ or $e = g(k)$ for some k, and if so then for this k we have

$$(\exists j, i)(W_{f(k)} \equiv_T W_j \,\&\, W_{g(k)} \equiv_T W_i \,\&\, R(j) \,\&\, R(i)).$$

If $e \in \hat{S}$ then e must be either $f(k)$ or $g(k)$ for some k, and since $W_{f(k)}$ and $W_{g(k)}$ split a set below A both halfs must be c.e. sets below A. In particular, their Turing degrees must be listed in the Σ^0_3 representation R of S. Conversely, if both $W_{f(k)}$ and $W_{g(k)}$ are listed in R, up to Turing equivalence, then they must be a split of a set Turing below A.

To produce the upper bound on the syntactical complexity of the definition above, recall that the sequence $\{W_{g(k)}, W_{f(k)}\}_{k\in\omega}$ is uniformly low. In particular, the $\Sigma_3^{W_{g(k)}}$ set

$$\{i : W_{g(k)} \equiv_T W_i\}$$

is Σ^0_3 uniformly in k, and similarly the $\Sigma_3^{W_{f(k)}}$ set

$$\{j : W_{f(k)} \equiv_T W_j\}$$

is Σ^0_3 uniformly in k.

This brings the complexity of the relation $e \in \hat{S}$ down to Σ^0_3. But $S \leq_1 \hat{S}$, contradicting the Σ^0_4-completeness of S. This concludes the proof.

3. Questions

There are a number of quite interesting questions which remain open.

(1) For what intervals of c.e. degrees $[a, b]$ can we realize $\{\mathbf{c} \mid \mathbf{c} \in [a, b]\}$? We know that for any $\mathbf{a}$ and $\mathbf{b} = \mathbf{0}'$, and $\mathbf{a} = \mathbf{0}$ and $\mathbf{b}$ low$_2$. What else?
(2) (Csima, Downey, Ng) What is the situation for *separating classes*? If we insist that the host class is a separating class, what Index Sets can be realized. The only known singleton is $\mathbf{0}'$ as witnessed by, for example, the class of Martin-Löf random reals. It is known by using results of Downey,

Jockusch and Stob [3], no "array computable" singleton is possible. Is any incomplete (nonzero) singleton possible?

(3) What about strong reducibilities? For instance weak truth table reducibility? Again we know that $\emptyset'_{wtt}$ is possible using random reals, but it also seems that some singletons are *not* possible. Of course here the indxex sets will be Σ^0_3 as given.

References

[1] D. Cenzer and C. Jockusch, Π^0_1-classes - Structure and applications, in *Contemporary Mathematics* 257 (2000), 39-59.

[2] B. Csima, R. Downey and K. M. Ng, On the c.e. degrees realizable in Π^0_1 classes, in preparation.

[3] R. Downey, C. Jockusch and M. Stob, Array recursive sets and multiple permitting arguments, in *Proceedings Oberwolfach 1989*, Springer Verlag, *Lecture Notes in Mathematics 1990*, 141-174.

[4] R. I. Soare, *Recursively Enumerable Sets and Degrees*, Springer-Verlag, 1987.

[5] C.E.M. Yates, On the degrees of index sets, *Trans. Amer. Math. Soc.*, 121 (1966), 308-328.

[6] C.E.M. Yates, On the degrees of index sets. II, *Trans. Amer. Math. Soc.*, 135 (1969) 249–266.

School of Mathematics, Statistics and Computer Science, Victoria University of Wellington, PO Box 600, Wellington, New Zealand

Email address: Rod.Downey@ecs.vuw.ac.nz

Mathematics Department, Albany Campus, Massey University

Email address: alexander.g.melnikov@gmail.com