

COMPUTABLY ISOMETRIC SPACES

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ABSTRACT. We say that an uncountable metric space is computably categorical if every two computable structures on this space are equivalent up to a computable isometry. We show that Cantor space, the Urysohn space, and every separable Hilbert space are computably categorical, but the space $\mathcal{C}[0, 1]$ of continuous functions on the unit interval with the supremum metric is not. We also characterize computably categorical subspaces of \mathbb{R}^n , and give a sufficient condition for a space to be computably categorical. Our interest is motivated by classical and recent results in computable (countable) model theory and computable analysis.

Keywords: Computable analysis, metric space theory

1. INTRODUCTION

Constructive mathematics was born quite early in the twentieth century. Its birth pre-dates the formal clarification of what is a computable process. For instance, Brouwer [4], [5] used intuitively effective procedures. Modern computable analysis goes back to at least 1930s to the fundamental papers of Turing [34], [33] and Russian school of constructive mathematics founded by Markov in the late 1940s (see, e.g., Kusner [18]).

In computable analysis one studies algorithmic properties of uncountable classical spaces such as the space of reals \mathbb{R} , the space of continuous functions $\mathcal{C}[0, 1]$, and the space $L_2[0, 1]$. See [37] or [30] for background on computable analysis. There are several ways to define computability in classical spaces; some of them are equivalent, others differ (see [3] for a detailed discussion). Computable analysis is an important basic tool of algorithmic randomness [26, 7], and also has unexpected interactions with advanced topics of algorithmic randomness (see, e.g., [27]).

Given a classical result from analysis, one may ask for its effective versions. For instance, one may prove the effective analogue of the classical Weierstrass theorem for computable functions [30] or study derivatives of computable differentiable functions (see Myhill [25], Pour-El and Richards [29] and Nies [27]). However, results may depend on the initial definition of a computable function (see, e.g., the recent paper [2]). Thus, to develop a meaningful theory we have to choose a definition to work with. We choose an approach which is common for separable metric spaces and, more specifically, for separable Banach spaces [3]:

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Definition 1.1. Let (M, d) be a complete separable metric space, and let $(q_i)_{i \in \omega}$ be a dense sequence without repetitions. The triple $\mathcal{M} = (M, d, (q_i)_{i \in \omega})$ is a *computable metric space* if $d(q_i, q_k)$ is a computable real uniformly in i, k . We say that $(q_i)_{i \in \omega}$ is a *computable structure* on M .

We refer to the elements of the sequence $(q_i)_{i \in \omega}$ as the *special points*. A *Cauchy name* for a point x is a sequence $(q_{f(s)})_{s \in \omega}$ of special points converging to x such that $d(q_{f(s)}, q_{f(t)}) \leq 2^{-s}$ for each $t > s$. An element x of \mathcal{M} is *computable* if there exists a computable function f such that $(q_{f(s)})_{s \in \omega}$ is a Cauchy name for x .

1.1. Computable isometries. A metric space may have more than one computable structure. For example, on the space of continuous functions $\mathcal{C}[0, 1]$, the collection of piecewise linear functions with rational breakpoints is a computable structure, and so is the collection of all polynomials with rational coefficients. These structures are different, but given a Cauchy name of a function f in one structure, one can uniformly pass to a Cauchy name for f in another structure, and vice versa. This leads to the notion of equivalent computable structures which has been intensively studied [3, 30].

Pour-El and Richards [30] were probably the first to observe that in many natural settings the notion of equivalent structures seems too restricted. For instance, consider the reals \mathbb{R} with the usual computable structure given by an effective listing of rationals $(q_i)_{i \in \omega}$. Let γ be a non-computable real. The collection $(q_i + \gamma)_{i \in \omega}$ is a computable structure on \mathbb{R} not equivalent to $(q_i)_{i \in \omega}$. However, the structures $(q_i + \gamma)_{i \in \omega}$ and $(q_i)_{i \in \omega}$ may be represented by the same algorithm computing the distances between the special points. Also, there is an isometry $x \rightarrow x + \gamma$ which preserves computability of points in an algorithmically uniform way. Classically metric spaces are often considered up to surjective isometries. This example suggests to consider *computable* structures up to *computable* surjective isometries:

Definition 1.2. Computable structures $(\alpha_i)_{i \in \omega}$ and $(\beta_i)_{i \in \omega}$ on a complete separable metric space (M, d) are *equivalent up to computable isometry*, or *computably isometric*, if there exists a surjective self-isometry ϕ of M and an effectively uniform algorithm which on input i outputs a Cauchy name for $\phi(\alpha_i)$ in $(\beta_i)_{i \in \omega}$.

Notions similar to Definition 1.2 have already appeared in literature in a different terminology (Pour-El and Richards [30] for Banach spaces, recently and independently Zvonko Iljazović [16] for metric spaces). We may think of the collection of all computable structures as of a category in which computable isometries are the morphisms. The following definition is central to the paper:

Definition 1.3. A metric space (M, d) is *computably categorical* if every two computable structures on M are computably isometric.

Hertling [15] defined a similar (but not equivalent) notion of an effectively categorical space which is based on a generalization of numbering theory [9]. In this paper we address the following question:

Question 1.4. *Which classical metric spaces are computably categorical?*

1.2. Computable categoricity of countable structures. Definition 1.3 is motivated by the classical notion of a computably categorical (autostable) *countable* algebraic structure due to Mal'cev [20] and Rabin [31]. Recall that an infinite *countable* algebraic structure \mathcal{M} is *computable* if the domain of \mathcal{M} can be numbered so

that the operations on \mathcal{M} become computable functions. For example, the group of the rationals $(\mathbb{Q}, +)$ is computable. A *countable* computable algebraic structure \mathcal{M} is *computably categorical* or *autostable* if every computable \mathcal{B} isomorphic to \mathcal{M} is *computably* isomorphic to \mathcal{M} .

Computable categoricity is the central notion of computable model theory and effective algebra (see [14] or [1]). Computably categorical countable structures have been intensively studied. For instance, there are characterizations of computably categorical algebraic structures in the classes of Boolean algebras [12], [19], linear orders [32], torsion-free abelian groups [11], [28], and many other structures [1]. There are notions similar to computable categoricity such as relative computable categoricity [1]. Computable categoricity coincides with relative computable categoricity in many special cases, but not in general [1]. Although computable categoricity has been central to computable model theory for over 50 years, there is still a lot to be understood. For instance, not much is known about computably categorical fields [24]. Also, it is not known if the index set of computably categorical structures is Π_1^1 -hard [38]. For more recent results on computably categorical countable structures see [8].

Computable categoricity is typically dependent on the signature of a given structure. For instance, Mal'cev [20] constructed a computable isomorphic copy of the abelian group $\mathbb{V} = \bigoplus_{i \in \omega} \mathbb{Q}$ in which linear dependence is undecidable. Since there is a computable isomorphic copy of \mathbb{V} in which linear dependence is decidable, Mal'cev concluded that \mathbb{V} is not computably categorical. In contrast, every two computable copies of \mathbb{V} with linear dependence algorithm are computably isomorphic. In other words, \mathbb{V} is computably categorical in the signature of abelian groups augmented by predicates $(P_i)_{i \in \omega}$, where $P_i(x_1, \dots, x_i) = 1$ if, and only if, elements x_1, \dots, x_i are linearly dependent.

Another classical example is the algebraic closure $\mathbb{U}(X)$ of the fraction field $\mathbb{Q}(X)$ over an infinite transcendence base X . It is well-known $\mathbb{U}(X)$ is not computably categorical (Metakides and Nerode [23]). To show $\mathbb{U}(X)$ is not computably categorical, one builds a computable isomorphic copy of $\mathbb{U}(X)$ in which algebraic dependence is undecidable. On the other hand, $\mathbb{U}(X)$ is computably categorical in the signature of fields augmented by algebraic dependence relations (essentially Frölich and Shepherdson [10]). For more examples see [1] and [14].

1.3. Metric spaces associated to Banach spaces. As we have discussed in the previous section, computable categoricity in the countable case depends on the signature of a given structure. This basic idea turns to be useful in the study of *uncountable* metric spaces associated to Banach spaces. For instance, if a metric space \mathcal{M} is associated to a Banach space, than we may ask if the addition operation is computable with respect to every computable structure on \mathcal{M} . We will show that if the answer is “no”, then (under certain extra conditions) it implies \mathcal{M} is not computably categorical. If the answer is “yes”, then it is interesting on its own right. Also, in the case of Hilbert spaces, the positive answer implies computable categoricity (to be shown in Theorem 4.7).

Using this basic idea and the classical theorem of Mazur and Ulam, we prove several technical facts about computably categorical Banach spaces. As a consequence of these facts and a result from [30], the space $l_1 = \{(c_i)_{i \in \omega} : \sum_i |c_i| < \infty\}$ with the metric induced by the l_1 -norm is not computably categorical.

We prove that for every computable structure on a Hilbert space \mathbb{H} , if 0 is computable then the vector space operations are computable as well. Together with the results from [30], it implies:

Theorem. *Every separable Hilbert space is computably categorical.*

We develop the ideas used for Hilbert spaces and characterize computably categorical subsets of \mathbb{R}^n , where $n \in \omega$. We introduce the notion of an intrinsically computable base which is essentially a linearly independent set computable in every computable structure, up to an isometry. We show:

Theorem. *A computable metric space isometric to a subset of \mathbb{R}^n is computably categorical if, and only if, it contains an intrinsically computable base.*

This theorem resembles results on countable Boolean algebras, linear orders and other countable structures mentioned in the previous subsection. We also give an alternative characterization of computably categorical subspaces of \mathbb{R}^n which does not implicitly involve the geometry of \mathbb{R}^n . This leads to a sufficient condition for an arbitrary space to be computably categorical.

In contrast, we prove that the space $\mathcal{C}[0, 1]$ of continuous functions on the unit interval has a computable structure with respect to which the operation $x \rightarrow (1/2)x$ is not computable. As we will show, this implies:

Theorem. *The space $\mathcal{C}[0, 1]$ with the pointwise supremum metric is not computably categorical.*

1.4. Cantor space and the Urysohn space. One uses an effective version of the usual back-and-forth technique to show that the countable dense linear order is computably categorical as a countable algebraic structure (folklore, see also [6]). In the case of uncountable metric spaces the situation is generally more complex. Nonetheless, using a variant of the back-and-forth technique we prove:

Theorem. *Cantor space $\{0, 1\}^\omega$ with the metric $\max\{2^{-n} : f(n) \neq g(n)\}$ is computably categorical.*

Cantor space with the (ultra)metric $\max\{2^{-n} : f(n) \neq g(n)\}$ is central to the modern theory of computably random reals [26], [7].

The Urysohn space [35] is the Fraïssé limit of finite metric spaces. It is the unique ultrahomogeneous universal separable space [17]. It is known that Urysohn space is homeomorphic to a Hilbert space (Uspensky [36]). Remarkably, the original construction of Urysohn [35] was effective. As a consequence, the Urysohn space is computable. We show:

Theorem. *The Urysohn space is computably categorical.*

It is unknown if one can define the Urysohn space “explicitly” without using variations of the Fraïssé construction or a random process. Our theorem essentially shows that the original *effective* construction due to Urysohn is the unique way one can effectively define the Urysohn space.

1.5. The structure of the paper. First, we give formal definitions and prove several not difficult but rather useful facts on computable Banach spaces. Next, we show that Hilbert space is computably categorical. Then we prove that $\mathcal{C}[0, 1]$ is not computably categorical by constructing a computable structure in which 0 is

computable but the operation $x \rightarrow (1/2)x$ is not. We show that Cantor space and the Urysohn space are computably categorical. Next, we characterize computably categorical subsets of \mathbb{R}^n and give a certain general condition for a space to be computably categorical. We conclude the paper by selected open problems.

2. DEFINITIONS

2.1. Definitions and conventions. We give formal definitions of the notions informally used in the introduction. Recall that, given a computable structure $(q_i)_{i \in \omega}$ on a metric space M , an element x of M is *computable* if there exists a computable function f such that $(q_{f(s)})_{s \in \omega}$ is a Cauchy name for x . It is well-known that a point x from $\mathcal{M} = (M, d, (q_i)_{i \in \omega})$ is computable if, and only if, from a positive rational δ one can compute p such that $d(x, q_p) \leq \delta$. Furthermore, we can restrict ourselves to the rationals of the form 2^{-m} . We will use this fact without explicit reference. To emphasize which computable structure on M is considered, we say that x is computable with respect to $(q_i)_{i \in \omega}$ (written w.r.t. $(q_i)_{i \in \omega}$).

We usually identify a special point α_i with its number i and say “find a special point such that ...” in place of “find a number i such that α_i ...”.

Definition 2.1. Let \mathcal{M} and \mathcal{N} be computable metric spaces. A map $F: \mathcal{M} \rightarrow \mathcal{N}$ is *computable* if there is a Turing functional Φ such that, for each x in the domain of F and for every Cauchy name χ for x , the functional Φ enumerates a Cauchy name for $F(x)$ using χ as an oracle¹.

To emphasize which computable structures we consider, we say that a map F is computable with respect to $(\alpha_i)_{i \in \omega}$ and $(\beta_i)_{i \in \omega}$ (written w.r.t. $(\alpha_i)_{i \in \omega}$ and $(\beta_i)_{i \in \omega}$). The composition of two computable maps is computable.

In this paper all metric spaces are complete and separable. For historical reasons, such spaces are often called *Polish*. Also, in the special case of isometric (more generally, bi-Lipschitz) maps, Definition 2.1 is equivalent to saying that for every special point α_i in \mathcal{M} the point $F(\alpha_i)$ is computable uniformly in i . We will use this observation without explicit reference to it. For instance, Definition 1.2 can be equivalently restated as follows:

Definition 2.2. Computable structures $(\alpha_i)_{i \in \omega}$ and $(\beta_i)_{i \in \omega}$ on a Polish space (M, d) are said to be *equivalent up to a computable isometry* or *(computably) isometric*, if there exists a surjective self-isometry U computable w.r.t. $(\alpha_i)_{i \in \omega}$ and $(\beta_i)_{i \in \omega}$.

Note that if U is a computable surjective isometry, then U^{-1} is computable as well. Therefore, equivalence up to a computable isometry induces an equivalence relation on the set of all computable structures on a metric space. Pour-El and Richards [30] used a similar notion restricted to Banach spaces in a different terminology. Their approach is equivalent to the one discussed in the next section.

2.2. Computable spaces with operations. We view a computable Banach space as a computable metric space with distinguished computable operations. An operation is a function which maps tuples of points to points (such as the addition in a Banach space), or tuples of points to reals (such as the inner product in a Hilbert space). Also, we view a distinguished point x as function $T_x: M \rightarrow \{x\}$ such that

¹That is, $(\Phi^X(n))_{n \in \omega}$ is a Cauchy name for $F(x)$.

$T_x(y) = x$, for every y . Thus, distinguished points are operations of a special kind. We could also consider maps from tuples of points *and* reals to points, an example being multiplication by a scalar in \mathbb{R}^2 . Nonetheless, we shall rather substitute such an operation by a uniformly computable list of operations, one for each rational point from \mathbb{R} . For instance, $(r \cdot)_{r \in \mathbb{Q}}$ will be used to represent multiplication by a scalar. One of the reasons is that we will need the operation $(1/2) \cdot$ in our later arguments.

Before we define a computable operation, we need one more definition. In the following, we view a direct power M^k of (M, d) as a metric space with the metric $d_k = \sup_{i \leq k} d(\pi_i x, \pi_i y)$, where π_i is the projection on the i -th component. Let $(\alpha_i)_{i \in \omega}$ be a computable structure on (M, d) . The computable structure $[(\alpha_i)_{i \in \omega}]^k$ on (M^k, d_k) is the effective listing of k -tuples of special points from $(\alpha_i)_{i \in \omega}$.

For convenience, if an operation $X : M^k \rightarrow M$ is computable w.r.t. $[(\alpha_i)_{i \in \omega}]^k$ and $(\alpha_i)_{i \in \omega}$, we simply say that X is computable w.r.t. $(\alpha_i)_{i \in \omega}$. Similarly, instead of saying that an operation $X : M^k \rightarrow \mathbb{R}$ is computable w.r.t. $[(\alpha_i)_{i \in \omega}]^k$ and $(q_i)_{i \in \omega}$, where $(q_i)_{i \in \omega}$ is the usual effective listing of rationals, we say that X is computable w.r.t. $(\alpha_i)_{i \in \omega}$.

Recall that every Turing functional Φ_e can be effectively identified with its computable index e . For instance, we may speak of the index for the distance function d (which depends on the given computable structure). We may also speak of uniformly computable families of maps between computable metric spaces meaning that we can get an index for the functional effectively from the place of the operation on the list.

Definition 2.3. Let $(M, d, (X_j)_{j \in J})$ be a metric space with distinguished operations $(X_j)_{j \in J}$, where J is a computable set. We say that $(\alpha_i)_{i \in \omega}$ is a *computable structure* on $(M, d, (X_j)_{j \in J})$ if $(M, d, (\alpha_i)_{i \in \omega})$ is a computable metric space and the operations $(X_j)_{j \in J}$ are computable w.r.t. $(\alpha_i)_{i \in \omega}$ uniformly in their indices.

We say that an isometry U *respects* an operation $X : M^k \rightarrow M$ if U commutes with this operation: $X \circ U = U \circ X$. An isometry U *respects* an operation $X : M^k \rightarrow \mathbb{R}$ if U preserves the output of the operation: $X \circ U = X$.

Definition 2.4. A space $(M, d, (X_j)_{j \in J})$ is *computably categorical* if every two computable structures $(\alpha_i)_{i \in \omega}$ and $(\beta_i)_{i \in \omega}$ on $(M, d, (X_j)_{j \in J})$ are computably isometric via an isometry which respects X_j for every $j \in J$.

Definition 2.5. We say that operations $(Y_i)_{i \in I}$ *effectively determine* operations $(X_j)_{j \in J}$ on a metric space (M, d) if

1. every isometry of M that respects $(Y_i)_{i \in I}$, respects $(X_j)_{j \in J}$ as well;
2. for any given computable structure $(\alpha_i)_{i \in \omega}$ on (M, d) , the uniform computability of $(Y_i)_{i \in I}$ w.r.t. $(\alpha_i)_{i \in \omega}$ implies the uniform computability of $(X_j)_{j \in J}$ w.r.t. $(\alpha_i)_{i \in \omega}$.

The following consequence of Definition 2.4 and Definition 2.5 is a useful tool.

Fact 2.6. Suppose $(M, d, (Y_i)_{i \in I}, (X_j)_{j \in J})$ is computably categorical, where the operations $(Y_i)_{i \in I}$ effectively determine the operations $(X_j)_{j \in J}$. Then $(M, d, (Y_i)_{i \in I})$ is computably categorical.

3. BANACH SPACES

3.1. Computable Banach spaces. We view a Banach space as a metric space with distinguished operations. Formally, a Banach space \mathbb{B} is a tuple

$$(B, d, 0, +, (r \cdot)_{r \in \mathbb{Q}}),$$

where d is the metric induced by the norm, 0 is the distinguished point for zero, $+$ is the vector summation, and $r \cdot$ is the operator of scalar multiplication by r , for $r \in \mathbb{Q}$ (rational numbers). We shall avoid this complex formal notation if possible. As a special case of Definition 2.3, we have:

Definition 3.1. A collection of points $(\alpha_i)_{i \in \omega}$ is a computable structure on a Banach space \mathbb{B} if $(B, d, (\alpha_i)_{i \in \omega})$ is a computable metric space and $0, +$, and $(r \cdot)_{r \in \mathbb{Q}}$ are uniformly computable operations w.r.t. to $(\alpha_i)_{i \in \omega}$.

Our approach is equivalent to the approach of Brattka, Hertling, and Weihrauch [3, page 466]. It is also equivalent to the existence of an *effectively separable* structure in the sense of Pour-El and Richards [30]. As a special case of Definition 2.4, we have:

Definition 3.2. A Banach space \mathbb{B} is computably categorical if every two computable structures on (B, d) , w.r.t. which the operations $0, +$, and $(r \cdot)_{r \in \mathbb{Q}}$ are uniformly computable, are computably isometric via an isometry which respects $0, +$, and $(r \cdot)_{r \in \mathbb{Q}}$. We also say that \mathbb{B} is computably categorical as a Banach space.

It is not difficult to see that \mathbb{B} is computably categorical as a Banach space if, and only if, every two effectively separable structures on \mathbb{B} are isometric, as defined in Pour-El and Richards [30, Question on page 146].

Remark 3.3. Note that, for a computable structure on a Banach space, the uniform computability of $(r \cdot)_{r \in \mathbb{Q}}$ implies the computability of 0 . By Fact 2.6 we may eliminate 0 from the list of computable operations and obtain equivalent notions of computable Banach space and computably categorical Banach space. However, we may keep 0 for convenience.

If, for a Banach space \mathbb{B} , the associated metric space (B, d) is computably categorical, then we say that \mathbb{B} is *computably categorical as a metric space*.

3.2. Applications of the Mazur-Ulam theorem. The classical theorem of Mazur and Ulam states that every surjective isometry of Banach spaces is affine. In other words, if $U : \mathbb{B}_1 \rightarrow \mathbb{B}_2$ is a surjective isometry of Banach spaces \mathbb{B}_1 and \mathbb{B}_2 , then there exists a linear map $L : \mathbb{B}_1 \rightarrow \mathbb{B}_2$ such that $U(x) = L(x) + U(0)$, for every $x \in \mathbb{B}_1$. We show:

Fact 3.4. If a Banach space \mathbb{B} is computably categorical as a metric space, then it is computably categorical as a Banach space.

Proof. Let $(\alpha_i)_{i \in \omega}$ and $(\beta_i)_{i \in \omega}$ be computable structures on \mathbb{B} w.r.t. which $0, +$, and $(r \cdot)_{r \in \mathbb{Q}}$ are computable. By the assumption, there is a surjective isometry U computable w.r.t. $(\alpha_i)_{i \in \omega}$ and $(\beta_i)_{i \in \omega}$. We need to find a computable surjective isometry which respects $0, +$, and $(r \cdot)_{r \in \mathbb{Q}}$.

By our assumption, the point $U(0)$ is computable w.r.t. $(\beta_i)_{i \in \omega}$. We have $x - y = x + (-1) \cdot y$, showing that the subtraction operation is computable w.r.t. $(\beta_i)_{i \in \omega}$. Thus, the isometry $W(x) = U(x) - U(0)$ is computable w.r.t. $(\alpha_i)_{i \in \omega}$ and $(\beta_i)_{i \in \omega}$. By the Mazur-Ulam theorem, W respects $0, +$, and $(r \cdot)_{r \in \mathbb{Q}}$. \square

Pour-El and Richards [30, page 146] showed that the space l_1 with the usual norm is not computably categorical as a Banach space. As a consequence of their result and Fact 3.4, we have:

Corollary 3.5. *The space l_1 is not computably categorical as a metric space.*

In Theorem 5.2 we will construct a computable structure on the metric space $(\mathcal{C}[0, 1], \text{sup})$ such that 0 is computable w.r.t. this structure, but the operation $(1/2) \cdot$ is not. By Fact 3.6 below, this will imply that $(\mathcal{C}[0, 1], \text{sup})$ is not computably categorical.

Fact 3.6. Let \mathbb{B} be a Banach space. Suppose $(\alpha_i)_{i \in \omega}$ is a computable structure on (B, d) w.r.t. which $+$ and $(r \cdot)_{r \in \mathbb{Q}}$ are uniformly computable, and suppose $(\beta_i)_{i \in \omega}$ is another computable structure on (B, d) w.r.t. which 0 is computable. If $(\beta_i)_{i \in \omega}$ is computably isometric to $(\alpha_i)_{i \in \omega}$, then $+$ and $(r \cdot)_{r \in \mathbb{Q}}$ are uniformly computable w.r.t. $(\beta_i)_{i \in \omega}$.

Proof. Let U be a surjective isometry computable w.r.t. $(\beta_i)_{i \in \omega}$ and $(\alpha_i)_{i \in \omega}$. Recall that U^{-1} is computable w.r.t. $(\alpha_i)_{i \in \omega}$ and $(\beta_i)_{i \in \omega}$. By the theorem of Mazur and Ulam, there exists a linear map $L : X \rightarrow Y$ such that $U(x) = L(x) + U(0)$, for every $x \in B$.

Given $r \in \mathbb{Q}$, we show that the operator $r \cdot$ is computable w.r.t. $(\beta_i)_{i \in \omega}$ uniformly in r , as follows. The operations $+$, $r \cdot$ and $v - w = v + (-1 \cdot w)$ are uniformly computable w.r.t. $(\alpha_i)_{i \in \omega}$. Therefore, the map $x \rightarrow U^{-1}(r(U(x) - U(0)) + U(0))$ is computable w.r.t. $(\beta_i)_{i \in \omega}$. On the other hand,

$$U^{-1}(r(U(x) - U(0)) + U(0)) = U^{-1}(rL(x) + U(0)) = U^{-1}(L(rx) + U(0)) = rx,$$

showing that $r \cdot x$ is a computable w.r.t. $(\beta_i)_{i \in \omega}$ uniformly in r .

The computability of $+$ can be established similarly:

$$U^{-1}(U(\beta) + U(\gamma) - U(0)) = U^{-1}(L(\beta + \gamma) + U(0)) = \beta + \gamma.$$

□

4. HILBERT SPACES

4.1. Operations on a Hilbert space. We view a Hilbert space as a Banach space of a special kind. For instance, for \mathbb{H} a Hilbert space, the associated metric space (H, d) is defined by $d(x, y) = \|x - y\|$. Recall Definition 2.5. We show:

Lemma 4.1. *In the metric space (H, d) associated with a Hilbert space \mathbb{H} , the point 0 effectively determines the operations $+$ and $(r \cdot)_{r \in \mathbb{Q}}$.*

Proof. Suppose $(\alpha_i)_{i \in \omega}$ is a computable structure on (H, d) w.r.t. which 0 is computable. Recall that $d(x, y) = \|x - y\|$. For instance, $\|x\| = d(0, x)$ is computable for every computable point x . It is well-known that the parallelogram identity

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

characterizes Hilbert spaces within the class of Banach spaces. We show:

Claim 4.2. *The operation $+$ is computable w.r.t. $(\alpha_i)_{i \in \omega}$.*

Proof. Given a positive rational $\epsilon < 1$ and Cauchy names for points x and y , find a special point z such that:

$$(1) \quad \left| \|z\|^2 + \|x - y\|^2 - 2\|x\|^2 - 2\|y\|^2 \right| < \delta,$$

$$(2) \quad |||y - z| - |x|| < \delta,$$

$$(3) \quad |||x - z| - |y|| < \delta,$$

where $\delta = \epsilon / (2||x| + 2||y| + 3)$. We may assume that $\delta < 1$. Applying the parallelogram identity, we obtain

$$||z|^2 - |x + y|^2| < \delta.$$

Using the well-known formula for inner product, we get

$$|x + y - z|^2 = |x + y|^2 + |z|^2 - 2\langle x, z \rangle - 2\langle y, z \rangle.$$

Applying this formula again, we obtain

$$|y - z|^2 = |y|^2 + |z|^2 - 2\langle y, z \rangle$$

and

$$|x - z|^2 = |x|^2 + |z|^2 - 2\langle x, z \rangle.$$

We combine the three equations above:

$$|x + y - z|^2 = (|x + y|^2 - |z|^2) + (|x - z|^2 - |y|^2) + (|y - z|^2 - |x|^2).$$

Taking into account $\delta < 1$, observe that

$$\begin{aligned} |||x - z|^2 - |y|^2| &= ||x - z| - |y|| \cdot (|x - z| + |y|) \\ &< \delta(|y| + \delta + |y|) \\ &< \delta(2|y| + 1), \end{aligned}$$

and similarly $||y - z|^2 - |x|^2| < \delta(2|x| + 1)$.

Thus,

$$\begin{aligned} |x + y - z|^2 &\leq ||x + y|^2 - |z|^2| + ||x - z|^2 - |y|^2| + ||y - z|^2 - |x|^2| \\ &< \delta + \delta(2|y| + 1) + \delta(2|x| + 1) \\ &= \delta(2|x| + 2|y| + 3) \\ &= \epsilon. \end{aligned}$$

Thus, we can produce a Cauchy name for $x + y$ uniformly in Cauchy names for x and y , proving the claim. \square

Claim 4.3. *The operations $(r \cdot)_{r \in \mathbb{Q}}$ are uniformly computable w.r.t. $(\alpha_i)_{i \in \omega}$.*

Proof. By Claim 4.2, for every $n \in \omega$ the operation $n \cdot$ is computable uniformly in n . The point 0 is a computable by our assumption, therefore the operation $0 \cdot = 0$ is computable. We show that $(-1) \cdot$ is computable. Given a rational $\eta > 0$ and a Cauchy name for x , find a special point α such that $d(0, x + \alpha) < \eta$. Note that $||(-x) - \alpha| = |x + \alpha| = d(0, x + \alpha) < \eta$. Therefore, we can produce a Cauchy name for $-x$, showing $(-1) \cdot$ is a computable operation.

For every $r = \frac{n}{m} \in \mathbb{Q}$, we have

$$r \cdot x = n \cdot (1/m)x.$$

Note also that the operation $m \cdot$ is bi-Lipschitz with constant m . Therefore, its inverse $(1/m) \cdot$ is computable, uniformly in m . We conclude that $r \cdot$ is computable, uniformly in r . \square

The lemma follows immediately from Claim 4.2 and Claim 4.3. \square

Remark 4.4. The proof of Lemma 4.1 actually shows that the indices for $+$ and $(r \cdot)_{r \in \mathbb{Q}}$ can be obtained effectively from the computable structure and the computable indices for d and 0 .

Remark 4.5. The inner product $\langle \cdot, \cdot \rangle$ is effectively determined by 0 in a Hilbert space. We have $\|x\| = d(0, x)$, $v - w = v + (-1 \cdot w)$, and

$$\langle u, v \rangle = \frac{1}{4}(\|u + v\|^2 - \|u - v\|^2).$$

4.2. Hilbert spaces are computably categorical. Using a different terminology, Pour-El and Richards [30] showed:

Theorem 4.6. *Every separable Hilbert space is computably categorical as a Banach space.*

Proof. The main idea is to use the Gram–Schmidt process. See the discussion on page 146 of Pour-El and Richards [30]. \square

The following consequence of Lemma 4.1 strengthens their result:

Theorem 4.7. *Every separable Hilbert space is computably categorical as a metric space.*

Proof idea. Note that, if 0 was computable w.r.t. every computable structure of a metric space (H, d) associated to a Hilbert space \mathbb{H} , then Fact 2.6 (with 0 in place of $(Y_i)_{i \in I}$), Lemma 4.1 and Theorem 4.6 would imply (H, d) is computably categorical. Unfortunately, 0 does not have to be computable w.r.t. every computable structure on (H, d) . On the other hand, we are given a computable structure on (H, d) , not \mathbb{H} itself, and zero and the operations are not specified in (H, d) . We may pick any special point and declare it to be “zero”. We have to define new vector space operations (to make this element a true zero), and then apply Fact 2.6, Lemma 4.1 and Theorem 4.6.

Proof. Suppose \mathbb{H} is a Hilbert space, and suppose $(\alpha)_{i \in \omega}$ and $(\beta)_{i \in \omega}$ are computable structures on the associated metric space (H, d) . Declare $z_1 = \alpha_0$. Consider the isometry $V(x) = x + z_1$. Clearly, $V(0) = z_1$. Let

$$\|x\|_1 = d(z_1, x).$$

The operation $\|x\|_1$ satisfies the norm axioms with respect to the new vector space operations

$$x +_1 y = x + y - z_1 \text{ and } r \cdot_1 x = r(x - z_1) + z_1,$$

where z_1 plays the role of zero. Furthermore, the norm $\|\cdot\|_1$ satisfies the parallelogram equality, H is complete with respect to $\|\cdot\|_1$, and

$$\|x +_1 (-1) \cdot_1 y\|_1 = d(x, y),$$

for every $x, y \in H$. Thus,

$$\mathbb{H}_1 = (H, d, z_1, +_1, (r \cdot_1)_{r \in \mathbb{Q}})$$

is a Hilbert space.

Similarly, we define a Hilbert space

$$\mathbb{H}_2 = (H, d, z_2, +_2, (r \cdot_2)_{r \in \mathbb{Q}}),$$

where $z_2 = \beta_0$. Note that $\mathbb{H}_2 \cong \mathbb{H}_1 \cong \mathbb{H}$.

By Lemma 4.1, $+_1$ and $(r \cdot)_1$ are uniformly computable w.r.t. $(\alpha)_{i \in \omega}$. Similarly, $+_2$ and $(r \cdot)_2$ are uniformly computable w.r.t. $(\beta)_{i \in \omega}$. Recall that \mathbb{H} is computably categorical as a Banach space (Theorem 4.6). It remains to apply Fact 2.6. \square

We emphasize that the proof of Theorem 4.7 works in the case of any finite dimension:

Corollary 4.8 ([16]). *For every $n \in \mathbb{N}^+$, the metric space \mathbb{R}^n with the usual Euclidean metric is computably categorical.*

5. THE SPACE $\mathcal{C}[0, 1]$

Let $(l_i)_{i \in \omega}$ be the effective list of all continuous piecewise linear functions (written p.l.) on $\mathcal{C}[0, 1]$ which have (finitely many) breakpoints, each breakpoint having rational coordinates in $[0, 1] \times \mathbb{R}$. In the following, we call these functions rational p.l. functions.

Notation 5.1. In this section d stands for the pointwise supremum metric on $\mathcal{C}[0, 1]$:

$$d(f, g) = \sup_{x \in [0, 1]} \{|f(x) - g(x)|\}$$

The sequence $(l_i)_{i \in \omega}$ of rational p.l. functions is a computable structure on $(\mathcal{C}[0, 1], d)$. Furthermore, the operators $+$ and $(r \cdot)_{r \in \mathbb{Q}}$ are uniformly computable w.r.t. $(l_i)_{i \in \omega}$. Thus, $(l_i)_{i \in \omega}$ makes $\mathcal{C}[0, 1]$ a computable Banach space, not merely a computable metric space. Unlike Hilbert spaces, zero does not effectively determine vector space operations on $\mathcal{C}[0, 1]$:

Theorem 5.2. *There is a computable structure on $(\mathcal{C}[0, 1], d)$ in which 0 is a computable point but the operation $(1/2) \cdot$ is not computable.*

Proof idea. We build a computable structure $(f_i)_{i \in \omega}$ on $(\mathcal{C}[0, 1], d)$ which consists of points Δ_2^0 with respect to $(l_i)_{i \in \omega}$. That is, the points are of the form $f_i = \lim_s f_{i,s}$, where $f_{i,s}$ is a computable double sequence of rational p.l. functions, but the computable sequence $(f_{i,s})_{s \in \omega}$ may not have an effective rate of convergence.

We diagonalize against the e 'th Turing functional Ψ_e potentially witnessing the computability of $(1/2) \cdot$, as follows. We choose an interval I_e (which is disjoint from I_j for each $j \neq e$) and a special point f_p not equal to 0 on I_e (a witness). As soon as $\Psi_{e,s}$ on f_p becomes close to our current guess on $f_p/2$ (if ever), we change the approximation of f_p by setting $f_{p,s+1}$ to be far enough from $f_{p,s}$ on I_e . This will make $\Psi_{e,s}$ too far from $(1/2) \cdot f_{p,s+1}$.

Although not every special point from $(f_i)_{i \in \omega}$ will be computable w.r.t. $(l_i)_{i \in \omega}$, we guarantee $\lim_s f_{0,s} = 0$. This makes 0 a special point in the new computable structure we are building. We also make sure $d(f_i, f_j)$ is computable uniformly in i, j by maintaining the equality $d(f_{v,s}, f_{u,s}) = d(f_{v,s+1}, f_{u,s+1})$ at each stage s and for every u and v . This seems to conflict with our attempt to change f_p as described above. However, we are able to make the construction injury-free. Also, some extra work is needed to make $(f_i)_{i \in \omega}$ a dense sequence in $\mathcal{C}[0, 1]$.

Proof. We build a computable double sequence of rational p.l. functions $(f_{i,s})_{i,s \in \omega}$ such that $f_{0,s} = 0$ for all s (thus, 0 is a special point in the new structure). At each stage s of the construction and for every $u, v \in \omega$, we maintain the equality

$$(1) \quad d(f_{v,s}, f_{u,s}) = d(f_{v,s+1}, f_{u,s+1}).$$

To make sure the equality (1) holds for every u, v and s , we will introduce the notion of (J, δ) -variation (Definition 5.3).

At stage s of the construction we will have a finite collection $f_{0,s}, \dots, f_{n(s),s}$ of rational p.l. functions, where $n(s)$ is nondecreasing in s . At the end of stage s we will have another collection of rational p.l. functions $f_{0,s+1}, \dots, f_{n(s+1),s+1}$ so that the equality (1) above holds for every $u, v \leq n(s)$. Note that we will not necessarily have $f_{i,s+1} = f_{i,s}$ for every $i \leq n(s)$. For every i we need to meet the requirement:

$$R_i : \lim_s f_{i,s} \text{ exists.}$$

To meet the requirements $(R_i)_{i \in \omega}$ we will make sure that:

$$(2) \quad (\forall n) (\exists s) (\forall t, z > s) (\forall i) [d(f_{i,t}, f_{i,z}) < 2^{-n}].$$

This will imply R_i is met, for every i . The condition (2) will be satisfied in the construction (to be shown in Claim 5.4). Since the R -requirements will have no conflicts with other requirements (will be clear from the proofs of Claim 5.4 and Claim 5.6), we may assume for notational convenience that $\lim_s f_{i,s}$ exists for every i . We denote $\lim_s f_{i,s}$ by f_i .

For every $j \in \omega$, we need to meet the following requirements:

$$P_j : l_j \text{ belongs to the closure of } (f_i)_{i \in \omega}.$$

Strategy for P_j . If s is a stage such that $s = 2\langle k, j \rangle$ for some integer k , where $\langle \cdot, \cdot \rangle$ is the usual computable bijection of ω^2 onto ω , and if the function l_j is not among $f_{0,s}, \dots, f_{n(s),s}$, then set $f_{n(s)+1,t} = l_j$ for every $t \leq (s+1)$.

End of strategy.

Taking into account (2) one can see that the strategy guarantees there is a sequence of elements in $(f_i)_{i \in \omega}$ converging to l_j (to be shown in Claim 5.5). The most important requirements are:

$$N_e : \Psi_e \text{ does not represent } (1/2)\cdot \text{ in } (f_i)_{i \in \omega}.$$

The *strategy for N_e* is less straightforward and requires some extra work.

Preliminary work towards N_e . First, we simplify the N_e requirements. Note that (p, p, \dots) is a Cauchy name for the special point f_p . There is a primitive recursive function s such that $\Psi_e^{(p,p,\dots)}(n) = \Phi_{s(e)}(p, n)$, where $(\Phi_e)_{e \in \omega}$ is the effective listing of all partial computable functions of two arguments (without an oracle). Thus, it is sufficient to meet the requirements:

$$N'_e : (\exists p)[(\Phi_e(p, n))_{n \in \omega} \text{ is a Cauchy name} \Rightarrow \lim_n \Phi_e(p, n) \neq f_p/2].$$

(Note that if $\Phi_e(p, n) \uparrow$ for some n , then N'_e is met trivially.)

The special element f_p will be the witness for N'_e chosen by the strategy. We need a technical definition which will allow us to make changes to approximations of special points without conflicting the R - and P -requirements:

Definition 5.3. Suppose J is a subinterval of $[0, 1]$, and suppose h_0, \dots, h_k are rational p.l. functions on $[0, 1]$. We say that a finite collection (g_0, \dots, g_k) of rational p.l. functions is a (J, δ) -variation of the collection (h_0, \dots, h_k) if:

$$(a) \quad h_0 = g_0 \text{ and } d(h_i, h_j) = d(g_i, g_j), \text{ for all } i, j \leq k,$$

(b) $h_i = g_i$ on $[0, 1] \setminus J$ and $d(h_i, g_i) \leq \delta$, for every $i \leq k$.

The strategy for N'_e will work within its own interval I_e and have a rational p.l. function w_e with support I_e . The function w_e may eventually become the witness for N'_e . More specifically, fix a computable listing of computable disjoint subintervals $(I_e)_{e \in \omega}$ of $[0, 1]$, where $I_e = [a_e, b_e]$ and $a_e, b_e \in Q$ for every e . We define a rational p.l. function w_e as follows:

$$w_e(x) = \begin{cases} 0, & \text{if } x \notin (a_e, b_e), \\ 2^{-e}, & \text{if } x \in [a_e + (1/4)(b_e - a_e), a_e + (3/4)(b_e - a_e)] \\ \text{linear,} & \text{otherwise.} \end{cases}$$

Strategy for N'_e .

(i) At stage $t = 2 \cdot e + 1$, if w_e is not already among $f_{0,t}, \dots, f_{n(t),t}$, then set $f_{n(t)+1,r} = w_e$ for every $r \leq t$. In the following, we assume that $p \leq n(t) + 1$ is such that $f_{p,t} = w_e$.

(ii) At stage $s > t$ wait for a computation $\Phi_{e,s}(p, -\log_2 \xi_e) \downarrow = h$, where ξ_e is much smaller than 2^{-e-1} (choose $\xi_e = 2^{-2^e-10}$). We have the following possibilities:

- Case 1. The function $f_{h,s}$ has not been defined so far. Then, for each $v \leq s$, set $f_{h,v} = g$ to be a rational p.l. function which is not among $f_{0,s}, \dots, f_{n,s}$ and which satisfies $\sup_{I_e} |g - (1/2)f_{p,s}| > 10$. Stop the strategy.
- Case 2. The function $f_{h,s}$ has already been defined, and $\sup_{I_e} |f_{h,s} - (1/2)f_{p,s}| > \xi_e$. In this case do nothing and stop the strategy.
- Case 3. The function $f_{h,s}$ has already been introduced, and $\sup_{I_e} |f_{h,s} - (1/2)f_{p,s}| \leq \xi_e$. Find a sub-interval J of I_e with rational end-points and a $(J, 2^{-e})$ -variation $(g_0, \dots, g_{n(s)})$ of the collection $(f_{0,s}, \dots, f_{n(s),s})$ such that, for some $y \in J$, we have $f_{p,s}(y) = 2^{-e}$ and $g_p(y) = g_h(y) = f_{h,s}(y)$. We will show in Claim 5.6 that at least one such a $(J, 2^{-e})$ -variation exists and, therefore, can be found effectively. See Figure 1 below for better idea. Set $f_{i,s+1} = g_i$, for all $i \leq n(s)$. Stop the strategy.

End of strategy.

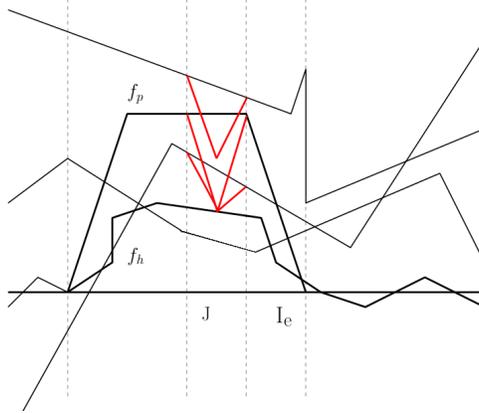


Figure 1. The figure illustrates a $(J, 2^{-e})$ -variation. Within the interval J all the functions we change are linear. The colored lines show the variation.

Comments on the strategy for N'_e . Note that if Φ_e represents $(1/2) \cdot$ in $(f_i)_{i \in \omega}$ then $d(f_h, f_p/2) \leq \xi_e$. In the construction only the N' -strategies will possibly change the approximations of the special points $(f_i)_{i \in \omega}$, and each N'_e -strategy makes changes within its own sub-interval I_e of $[0, 1]$ disjoint from I_j , for $j \neq e$ (see the definitions of I_e and $(J, 2^{-e-1})$ -variation). Thus, in Case 1 and Case 2 we guarantee $d(f_h, f_p/2) > \xi_e$, and Φ_e can not approximate $(1/2) \cdot f_p$ in $(f_i)_{i \in \omega}$.

Observe that if the strategy stops at Case 3, then

$$f_{h,s+1}(y) = f_{h,s}(y) = f_{p,s+1}(y) = f_p(y) \geq 2^{-e-1} - \xi_e.$$

By the choice of y , we have $f_{p,s}(y) = 2^{-e}$. By our assumption, $|f_{h,s}(y) - (1/2)f_{p,s}| \leq \xi_e$. Therefore, by the choice of ξ_e , we obtain

$$\begin{aligned} d(f_h, f_p/2) &\geq |f_{h,s+1}(y) - (1/2)f_{p,s+1}(y)| \\ &\geq |(2^{-e-1} + \xi_e) - (1/2)(2^{-e-1} - \xi_e)| \\ &> \xi_e, \end{aligned}$$

and N'_e is met. We put all the strategies together:

Construction.

At stage 0 of the construction set $f_{0,0} = 0$.

At stage $s > 0$ of the construction let the strategies act according to their instructions. For every $i \leq n(s)$, if $f_{i,s+1}$ have not been defined by the strategies, then set $f_{i,s+1} = f_{i,s}$.

End of construction.

The *verification* is split into several claims.

Claim 5.4. *The requirement R_i is met, for every i .*

Proof. Only the N'_e -requirements may change the approximation $(f_{i,s})_{s \in \omega}$ of a special point f_i , and each N'_e works within its own subinterval I_e . Furthermore, this change (if it is ever done by N'_e) is bounded by 2^{-e} . Therefore, if N'_e never reaches its Case 3, then the condition (2) is satisfied for $n = e$ and $s = 0$. If N'_e reaches its Case 3 at stage s' , then (2) holds for $n = e$ and $s = s'$. Therefore, the condition (2) holds for every n . This implies R_i is met, for every i . \square

Claim 5.5. *The requirement P_j is met, for every j .*

Proof. Let $s(k) = 2\langle k, j \rangle$. The strategy for P_j guarantees that the collection

$$f_{0,s(k)+1}, \dots, f_{n(s(k)+1),s(k)+1}$$

contains $f_{m(k),s(k)}$ which is equal to l_j at that stage² Suppose also that that $s(k)$ is so large that for every $j \leq e$ the strategy for N_j never reaches its Case 3 after stage $s(k)$. Then $d(f_{m(k)}, l_j) \leq 2^{-e}$. Therefore, $(f_{m(k)})_{k \in \omega}$ converges to l_j . \square

Claim 5.6. *The requirement N'_e is met, for every e .*

²Notice that l_j is uniquely determined by its index. Also note that $f_{m(k),s(k)}$ equals to l_j only at stage $s(k)$, at a later stage t we may have $f_{m(k),t} \neq f_{m(k),s(k)}$. In this case P_j , when it is active again, will add a function which currently will be equal to l_j , etc.

Proof. For functions f and h and a set $X \subseteq [0, 1]$, write $f \leq_X g$ if $f(x) \leq g(x)$ for every $x \in X$. Define $<_X$ and $=_X$ similarly.

From now on we use notations from the strategy for N'_e . Assume we are at Case 3 of the strategy. We need to find an interval $J \subseteq I_e$ and a $(J, 2^{-e})$ -variation $(g_0, \dots, g_{n(s)})$ of $(f_{0,s}, \dots, f_{n(s),s})$ such that, for some $y \in J$, we have $f_{p,s}(y) = 2^{-e}$ and $g_p(y) = g_h(y) = f_{h,s}(y)$. Find a rational point y and subinterval $J = [c, d] \subseteq I_e$ containing y , where $c, d \in \mathbb{Q}$, such that:

- (1) $f_{p,s}(y) = 2^{-e}$ (notice that this implies $f_{p,s}(y) \neq f_{h,s}(y)$, by the choice of $f_{p,s}$ and $\xi_e \ll 2^{-e}$);
- (2) $f_{v,s}$ is linear on J , for each $v \leq n(s)$
- (3) for every $v, m \leq n(s)$ either $f_{m,s} <_J f_{v,s}$, or $f_{v,s} <_J f_{m,s}$ or $f_{v,s} =_J f_{m,s}$.

Recall that $f_{p,s}$ is equal to w_e on I_e , and there is a subinterval of I_e such that $f_{p,s}$ is equal to 2^{-e} when restricted to this subinterval. Note that the functions $f_{0,s}, \dots, f_{n(s),s}$ have finitely many breakpoints, and so do the functions $\{f_{k,s} - f_{m,s}\}_{k,m \leq n(s)}$. It is sufficient to choose $J = [c, d]$ so that $f_{p,s} =_J 2^{-e}$ and J does not contain any of these points, and so that either the functions do not intersect within the interval, or are equal everywhere within this interval. Let y be any rational point from J .

Denote $f_{i,s}$ restricted to $J = [c, d]$ by F_i . For every i , the function F_i is linear. Without loss of generality, we may assume

$$F_0 <_J \dots <_J F_h < \dots <_J F_p <_J \dots <_J F_{n(s)}.$$

Note that, by the choice of J , we must have $p < h$ in this list. (Note that we possibly have to change indexing and identify functions equal under $=_J$. We, however, slightly abuse our notation and assume that p and h remain untouched.) Let $k = p - h$:

$$F_0 <_J \dots F_h <_J \dots <_J F_{h+k} \dots <_J F_{n(s)}.$$

Given $i \in \{1, \dots, k\}$, define $\delta_i = |F_{h+i}(y) - F_h(y)|$. For each $j \leq n(s)$, define a new p.l. function G_j to be equal to F_j on the end-points of $J = [c, d]$, set

$$G_j(y) = \begin{cases} F_j(y) - \delta_k, & \text{if } j \geq h + k, \\ F_j(y) - \delta_{j-h}, & \text{if } h < j < h + k, \\ F_j(y), & \text{otherwise,} \end{cases}$$

and make it linear on $x \in J \setminus \{y\}$ (see Figure 1).

Recall that F_h is ξ_e -close to $F_p/2$, where ξ_e is much smaller than 2^{-e} . Recall also that, by the choice of y , we have $F_p(y) = 2^{-e}$. Therefore,

$$|F_p(y) - F_h(y)| < 2^{-e} \text{ and } \delta_i \leq 2^{-e},$$

for every $i \in \{1, \dots, k\}$. This implies that, for every $j \leq n(s)$,

$$\sup_J |G_j - F_j| \leq 2^{-e}.$$

Note that $G_i(c) = F_i(c)$ and $G_i(d) = F_i(d)$. Also,

$$|G_i(x) - G_j(x)| \leq |F_i(x) - F_j(x)|$$

for every $x \in (c, d)$, by the definition of G_i and G_j . Also, the functions $(F_i)_{i \leq n(s)}$ are linear, and $\sup_J |F_i - F_j| = \sup_{[c,d]} |F_i - F_j|$ realizes on c or d . We conclude that

$$\sup_J |G_i - G_j| = \sup_J |F_i - F_j|,$$

for every $i, j \leq n(s)$.

For every $j \leq n(s)$, define

$$g_j(x) = \begin{cases} G_j(x), & \text{if } x \in [c, d], \\ f_{j,s}(x), & \text{otherwise.} \end{cases}$$

We have $f_{p,s} \geq_{I_e} 0$ and $f_h \geq_{I_e} 0$. Thus, the definition ensures $g_0 = f_{0,s}$. It follows that (g_0, \dots, g_n) is a $(J, 2^{-e})$ -variation of $(f_{0,s}, \dots, f_{n(s),s})$. By its definition, we have $f_{p,s}(y) = 2^{-e}$ and $g_p(y) = g_h(y) = f_{h,s}(y)$. We proved the claim in the case when all the inequalities are strict.

The general case is done by a simple inductive argument. Suppose, say, $F_1 = F_2$, and suppose there is a $(J, 2^{-e})$ -variation (g_0, g_2, \dots, g_n) of $(f_{0,s}, f_{2,s}, \dots, f_{n(s),s})$ with the needed properties. Define

$$g_1(x) = \begin{cases} f_{1,s}(x), & \text{if } x \notin J, \\ g_2(x), & \text{if } x \in J. \end{cases}$$

The collection (g_0, g_1, \dots, g_n) is the needed $(J, 2^{-e})$ -variation of $(f_{0,s}, f_{1,s}, \dots, f_{n(s),s})$. \square

It remains to observe that all the stages are effective, because at each stage we have a collection of rational p.l. functions, and all the questions we ask about these collections are effectively decidable. \square

Theorem 5.7. *The space $\mathcal{C}[0, 1]$ is not computably categorical.*

Proof. This follows from Theorem 5.2 and Fact 3.6. \square

6. CANTOR SPACE

Recall that Cantor space is the set of infinite strings of 0's and 1's. We show:

Theorem 6.1. *Cantor space $\{0, 1\}^\omega$ with the metric $d(\xi, \phi) = \max\{2^{-n} : \xi(n) \neq \phi(n)\}$ is computably categorical.*

Proof idea. Let $(\alpha_i)_{i \in \omega}$ and $(\beta_i)_{i \in \omega}$ be computable structures on Cantor space. The computable structures are rational-valued metric spaces. Furthermore, every surjective isometry of these rational-valued subspaces is uniquely expandable to a surjective isometry of their closures. Therefore, it is sufficient to build a computable bijection $f : \omega \rightarrow \omega$ such that $d(\alpha_i, \alpha_j) = d(\beta_{f(i)}, \beta_{f(j)})$, for every $i, j \in \omega$.

Without loss of generality, we may assume $(\alpha_i)_{i \in \omega}$ is the usual computable structure on Cantor space given by the collection of infinite strings that are eventually 0. To define $f(i)$, we find the least j such that $\{\alpha_k\}_{k \leq i}$ and $\{\beta_{f(k)}\}_{k < i} \cup \beta_j$ are isometric via $\alpha_k \rightarrow \beta_{f(k)}$ and $\alpha_i \rightarrow \beta_j$, and set $f(i) = j$. Since the distances are rational-valued, the definition of f is effective.

Proof. Let $(\alpha_i)_{i \in \omega}$ be the usual computable structure on Cantor space given by the infinite strings that are eventually 0, and let $(\beta_i)_{i \in \omega}$ be another computable structure on Cantor space. If we think of the Cantor space as of a binary tree, the special points in $(\alpha_i)_{i \in \omega}$ are enumerated level-by-level, excluding repetitions:

$$\emptyset, 1, 01, 11, 001 \dots$$

We define f in the following procedure:

Construction.

At stage 0, set $f(0) = 0$.

At stage $i > 0$, we assume that $f(k)$ has already been defined for every $k < i$. Say that j is *good for extension* if the isometry $\alpha_k \rightarrow \beta_{f(k)}$ of $\{\alpha_k\}_{k < i}$ onto $\{\beta_{f(k)}\}_{k < i}$ can be extended to an isometry of $\{\alpha_k\}_{k \leq i}$ onto $\{\beta_{f(k)}\}_{k < i} \cup \{\beta_j\}$. Find least j good for extension and set $f(i) = j$.

End of construction.

Verification. By the density of $(\beta_i)_{i \in \omega}$, at every stage of the construction there exists at least one j good for extension. In fact, every special point β_j in a certain open ball in 2^ω is good for extension. The formal argument is not difficult and can be left to the reader. Thus, f is total. We show:

Claim 6.2. f is computable.

Proof. Recall that, for each v and w in Cantor space, the distance $d(v, w)$ is of the form 2^{-n} , for some n . Suppose we have a computable structure on Cantor space. Given unequal computable points v and w , we can effectively find n such that $d(v, w) = 2^{-n}$. We conclude that the construction is effective and, consequently, the function f is computable. \square

Claim 6.3. f is an isometry.

Proof. Note that any isometric embedding of Cantor space into itself is onto, by compactness. Thus, it is sufficient to show that at every stage there is a point good for extension.

The usual visualization of Cantor set by the complete binary tree may help in understanding of the proof below. Recall that special points from $(\alpha_i)_{i \in \omega}$ are enumerated “level-by-level”. The structure $(\beta_i)_{i \in \omega}$ can be visualized as a set of infinite paths through the complete binary tree. If at stage s we have to choose an extension of f to another point of $(\alpha_i)_{i \in \omega}$, we can pick any point in $(\beta_j)_{j \in \omega}$ within an open ball uniquely determined by the collection of distances between points in $\text{dom}(f_s)$ and $\text{range}(f_s)$. To see that the ball is uniquely determined by the distances, observe that the finite collection of distances are completely determined by finite initial segments of the special points $\text{dom}_s(f)$ and $\text{range}_s(f)$. An inductive argument shows that the way we list elements of $(\alpha_i)_{i \in \omega}$ ensures that every open ball of the form $\sigma 2^\omega$ will be among these “determined” balls. Furthermore, the balls 02^ω and 12^ω will correspond (in some order) to stages 0 and 1, the balls $\sigma 2^\omega$, where $\text{lgth}(\sigma) = n$, will appear (in some order) at stages $\sum_{k < n} 2^k \leq s < \sum_{k \leq n} 2^k$. \square

Define a surjective self-isometry U of Cantor space:

$$\text{if } (\alpha_{g(i)})_{i \in \omega} \text{ is a Cauchy name then } U(\lim_i \alpha_{g(i)}) = \lim_i \beta_{fg(i)}.$$

By Claims 6.3 and 6.2, the isometry U witnesses computable categoricity of Cantor space. \square

Remark 6.4. Note that the operator U witnessing computable categoricity of $\{0, 1\}^\omega$ may be obtained uniformly from the given computable structures on $\{0, 1\}^\omega$.

Remark 6.5. Our construction additionally guarantees that f is a permutation of ω . We pick $v \in \omega$ and show that β_v is in the range of f . Suppose f has already been defined for all $u < v$, and s is the least stage when that happened. Let n be largest having the property $\sum_{k < n} 2^k \leq s$. There must be a stage $t > \sum_{k < n} 2^k$ such that the open ball $\tau 2^\omega$, where $\tau \subset \beta_v$, is the ball determined by the distances in $\text{dom}(f_t)$ and $\text{range}(f_t)$. The construction ensures that the least special point from the ball must be put into the range of f . If it has not already happened to β_v , then β_v must be good for extension at stage t .

Remark 6.6. The same proof would work if we replaced the standard distance on 2^ω by $\max\{2^{-n}\gamma: \xi(n) \neq \phi(n)\}$, where γ is a computable real.

7. THE URYSOHN SPACE

The rational Urysohn space $\mathbb{Q}\mathbb{U}$ is the Fraïssé limit of finite rational-valued metric spaces. The Urysohn space \mathbb{U} is the completion of $\mathbb{Q}\mathbb{U}$. We can effectively list all finite rational-valued metric spaces. Therefore, the points in $\mathbb{Q}\mathbb{U}$ form a computable structure of \mathbb{U} . We will need the following definitions which can be found in [17] and [21].

Definition 7.1. Let X be a metric space. A map $f : X \rightarrow \mathbb{R}^+$ is a *Katetov map* if for $z \notin X$ setting $d(x, z) = f(x)$ defines a metric space on $X \cup \{z\}$ which extends X .

A map f is Katetov if, and only if, $(\forall x, y \in X) |f(x) - f(y)| \leq d(x, y) \leq f(x) + f(y)$. (The collection $E(X)$ of all Katetov maps together with the metric $\sup_{x \in X} |f(x) - g(x)|$ is a metric space.) Informally, Katetov maps reflect all one-point metric extensions of a given metric space.

Definition 7.2. A space X has the *approximate extension property* if for every finite subset A of X , for every $f \in E(A)$, and every $\epsilon > 0$ there exists a point $z \in X$ such that

$$(\forall a \in A) |d(z, a) - f(a)| \leq \epsilon.$$

It is known that a Polish metric space has the approximate extension property if, and only if, it is isometric to the Urysohn space ([21]). The approximate extension property is equivalent to the extension property which is the approximate extension property with $\epsilon = 0$. The approximate extension property is central to the proof of the theorem below.

Theorem 7.3. *The Urysohn space is computably categorical.*

Proof idea. Suppose we are given two computable structures, we need to build a computable isometry between their completions. Without loss of generality, we may assume that one of these computable structures is $\mathbb{Q}\mathbb{U}$, the rational-valued Urysohn space (this is not important for the construction though).

The proof is essentially an effective version of the usual back-and-forth argument. The problem is that we can not define the image of a point in one step. The approximate extension property allows us to run the argument on the special points with an “arbitrarily good precision”. This property allows us to search for a point in a given computable structure $(\alpha_i)_{i \in \omega}$ which is “approximately” an image of a special point r from $\mathbb{Q}\mathbb{U}$. As was mentioned above, a Polish metric space has the approximate extension property if, and only if, it is isometric to the Urysohn space. The proof of this fact (see, e.g., [21]) guarantees that it is possible to define a

sequence of “approximate” images of r so that it is a Cauchy sequence of points from $(\alpha_i)_{i \in \omega}$. As is described in detail in the formal proof below, one can use the approximate extension property to define an injection of \mathbb{QU} into the closure of $(\alpha_i)_{i \in \omega}$.

However, it does not guarantee the map is surjective. We should force every special point from $(\alpha_i)_{i \in \omega}$ to be in the closure of the image of the embedding of \mathbb{QU} . Clearly, it can not be implemented as it was done for Cantor space, because we don’t know the precise distances between special points in $(\alpha_i)_{i \in \omega}$ at a stage. We need to make sure that, for every ϵ and n , the special element α_n is in the ϵ -neighbourhood of the image of \mathbb{QU} .

At a stage s , the element α_n may be within the 2^{-m} -neighbourhood of the image of (a finite part of) \mathbb{QU} . At a later stage we discover it is outside the 2^{-m-3} -neighbourhood of the current image. In this case we need to put a new special point r from \mathbb{QU} into the domain of our map so that α_n belongs to the 2^{-2m} -neighbourhood of the image of r .

For technical reasons, we use markers on the (numbers of) special elements from \mathbb{QU} to implement this idea. The markers allow us to label elements for which the map has already been defined. The formal details can be found in the proof below.

Proof. Let $(r_i)_{i \in \omega}$ be the computable structure on \mathbb{U} given by an effective listing of the points in \mathbb{QU} , and let $(\alpha_i)_{i \in \omega}$ be another computable structure on \mathbb{U} .

First, we define a computable double sequence $(f_{j,s})_{s,j \in \omega}$ of special points in $(\alpha_i)_{i \in \omega}$ such that $\lim_s f_{j,s}$ exists with the rate of convergence computable uniformly in j , and $d(r_i, r_j) = d(\lim_s f_{i,s}, \lim_s f_{j,s})$ for each $i, j \in \omega$. In the construction for every k we will have a strategy L_k which defines the computable sequence $(f_{k,s})_{s \in \omega}$ of special points in $(\alpha_i)_{i \in \omega}$.

Strategy L_0 . Set $f_{0,s} = \alpha_0$ for every s .

Given $k > 0$, suppose that for every $i < k$ the (strategies L_i enumerating) computable sequences $(f_{i,s})_{s \in \omega}$ have already been defined, and for every $i < k$ the point $g_i = \lim_s f_{i,s}$ is computable uniformly in i . We have:

Strategy L_k . Suppose we need to define $f_{k,s}$, and either $s = 0$ or $f_{k,s-1}$ has already been defined. Find a special point α in $(\alpha_i)_{i \in \omega}$ such that for every $i < k$

$$|d(r_i, r_k) - d(g_i, \alpha)| < 2^{-s},$$

and $d(f_{k,s-1}, \alpha) < 2^{2-s}$ if $s > 0$. Set $f_{k,s} = \alpha$.

Clearly, if we can show that for every k, s at least one point α with the needed properties exists (to be shown), then L_k eventually finds α (or any other point satisfying these properties would work). If so, then $g_k = \lim_s f_{k,s}$ exists and is a computable point. Furthermore, the computable index for the Cauchy name of g_k can be obtained uniformly from the computable indices for the Cauchy names $(f_{i,s})_{s \in \omega}$ of $(g_i)_{i < k}$. Thus, the map $F : r_j \rightarrow g_j$ is an isometry which is uniquely expandable to a computable isometric embedding of \mathbb{U} into itself. To define a surjective embedding we need extra requirements (to be introduced later) and movable markers.

Movable markers. Recall that we identify special elements and their numbers. At each stage of the construction we will have markers $(m_k)_{k \in \omega}$ on the special elements from \mathbb{QU} . If a special point r at stage s carries the marker m_k , then we write

$\boxed{\Gamma}_k^s$ or $m_{k,s} = r$. As usual for movable markers arguments, we will describe how to effectively move all the markers at once instead of dealing with finitely many markers at every stage of the construction. At the end of every stage s , for every k there will be exactly one r such that $m_{k,s} = r$, and every r will carry a marker. We will show that for every k there exists a stage t such that $m_{k,s} = m_{k,t}$ for every $s \geq t$, and for every r there exists k such that $r = \lim_s m_{k,s}$.

We need the following important modifications of L_k :

- Replace every instance of r_k and r_i in the strategy by $m_{k,t}$ and $m_{i,t}$ respectively, where t is the stage of the construction at which L_k defines $f_{k,s}$.
- If L_k defines $f_{k,0}$ at stage t , declare $m_{k,t} = \lim_s m_{k,s} = m_k$ and say that m_k has *settled*.

The isometric map will be defined by the rule $\lim_s m_{k,s} \rightarrow g_k$ for every k .

The requirements for surjectivity. It is sufficient to meet, for every $v, n \in \omega$, the requirements:

$$P_{v,n} : (\exists i) d(\alpha_v, g_i) < 2^{-n}.$$

The strategy for $P_{v,n}$. Suppose we are at stage t of the construction, and $m_{0,t}, \dots, m_{j,t}$ are all the markers which have already settled. Equivalently, $(j+1)$ is least such that $f_{j+1,0}$ has not been defined yet. Wait until one of the two possibilities is effectively recognized:

- (1) $d(\alpha_v, g_k) < 2^{-n}$ holds, for some $k \leq j$. In this case $P_{v,n}$ is met. Stop the strategy.
- (2) $d(\alpha_v, g_k) > 2^{-n-1}$ holds for every $k \leq j$. Find $m_{h,t}$ with $h > j$ such that for every $k \leq j$

$$|d(m_{k,t}, m_{h,t}) - d(g_k, \alpha_v)| < 2^{-n-2}.$$

Define $m_{j+1,t+1} = m_{h,t}$. Declare m_{j+1} settled. (Note that temporarily one special point carries no marker.)

For every $i > (j+1)$, in increasing order, find the least x such that r_x does not carry a marker m_l with $l < i$ and set $m_{i,t+1} = r_x$. (In other words, move all the non-settled markers one step left, avoiding the settled markers.)

Then set $f_{j+1,s} = \alpha_v$, for every $s \leq n+2$. Stop the strategy. Note that the modified strategy L_{j+1} guarantees $d(g_{j+1}, \alpha_v) = d(g_{j+1}, f_{j+1,n+2}) < 2^{2-(n+2)} = 2^{-n}$, and $P_{v,n}$ is met.

Construction.

At stage t , let k be least such that $m_{k,t}$ has not settled yet. For every $j \leq k$ let the modified L_j define $f_{j,s}$ for at least every $s \leq t$. Then let $P_{v,n}$ with $\langle v, n \rangle = t$ act according to its instructions, where $\langle \cdot, \cdot \rangle$ is the usual computable bijection of ω^2 onto ω .

Verification. We split the verification into claims.

Claim 7.4. *For every k , the strategy L_k defines a computable infinite sequence $(f_{k,s})_{s \in \omega}$ of special points.*

Proof. The statement is clear for $k = 0$. Suppose for every $i \leq k$ the strategy L_k defines a computable infinite sequence $(f_{i,s})_{s \in \omega}$ of special points, and either $s = 0$ or $f_{k,s-1}$ has already been defined. Note that, by inductive assumption, we may assume that we have indices for the Cauchy names of $(g_i)_{i < k}$ (observe that the values of $(f_{i,s})_{s \in \omega}$ depend on the values of $(f_{j,s})_{s \in \omega, j < i}$ only, and we can speed-up the enumerations $(f_{j,s})_{s \in \omega}$ and postpone the definition of $f_{i,s}$ if needed). Therefore, if a special point α satisfies the conditions $|d(r_i, r_k) - d(g_i, \alpha)| < 2^{-s}$ and $d(f_{k,s-1}, \alpha) < 2^{2-s}$, then we will eventually see that it indeed satisfies these conditions. We need only to show that at least one such α exists.

Recall that we need to find a special point α in \mathbb{QU} such that for every $i < k$ $|d(r_i, r_k) - d(g_i, \alpha)| < 2^{-s}$, and $d(f_{k,s-1}, \alpha) < 2^{2-s}$ if $s > 0$.

Recall that a Polish metric space has the approximate extension property if, and only if, it has the extension property. The proof of this fact goes as follows. We suppose that a space has the approximate extension property. We take finitely many points and a Katetov map h on these points (equivalently, a potential 1-point extension of the finite metric space on these points) and wish to find a sequence which converges to some point in the space which realizes this map. We take $\epsilon = 1/2$ and take x_1 which realizes the Katetov map with precision ϵ . We can set $\epsilon = 1/4$ and define x_2 similarly, but there is no guarantee that $d(x_1, x_2)$ is small. A more careful analysis of the situation which makes use of the distance between Katetov maps shows that it can be done (see, e.g., [21], Exercise 3 and the proof of Theorem 3.4). It is not important for us how exactly it is done.

Now we take the Katetov map which is given by $d(g_i, U(r_k))$, where U is *some* self-isometry of the Urysohn space, and let $f_{k,s}$ play the role of x_s in the explanation above. The ultra homogeneity of the Urysohn space implies that the choice of U does not matter, and the density of $(\alpha_i)_{i \in \omega}$ in \mathbb{U} implies that we can set $f_{k,s} = \alpha$ for some special point α in $(\alpha_i)_{i \in \omega}$. \square

Claim 7.5. *The requirement $P_{v,n}$ is met for every $v, n \in \omega$.*

Proof. Note that at least one of the two possibilities in the strategy for $P_{v,n}$ will be eventually effectively recognized. The effectiveness follows from the fact that the indices for $(f_{i,s})_{s \in \omega}$ are given ahead of time, as it is explained in the proof of Claim 7.4. The rest follows from the density of $(r_i)_{i \in \omega}$ and $(\alpha_i)_{i \in \omega}$ and the strategy for L_k . \square

Claim 7.6. *For every k , the movable marker m_k settles, and every special point eventually carries a settled marker.*

Proof. At some stage t , the marker $m_{k,t}$ will be either used by an L - or P -strategy to define $f_{k,0}$. In both cases it will be declared settled. The construction is organized so that at the beginning of stage t the least j such that r_j does not carry a settled marker will be occupied by $m_{k,t}$, where k is least such that $m_{k,t}$ has not settled yet. The marker will be declared settled by L_k at stage t . \square

Denote the element which eventually carries the marker m_k by c_k . Define F by setting $F(c_k) = \lim_s f_{k,s}$, for every s , and then extending F to the whole \mathbb{U} . The map F is a surjective self-isometry computable w.r.t. $(r_i)_{i \in \omega}$ and $(\alpha_i)_{i \in \omega}$. \square

8. SUBSPACES OF \mathbb{R}^n .

In this section d stands for the Euclidean metric. We denote the affine span of $M \subseteq \mathbb{R}^n$ over \mathbb{R} by $\langle M \rangle_{\mathbb{R}}$ (alternatively, we may assume that 0 belongs to M). If $M = \mathbb{R}^n$, then one may use the Gram–Schmidt process to show \mathbb{R}^n is computably categorical (Corollary 4.8). In the general case $M \subseteq \mathbb{R}^n$, there are two difficulties. First, M may not be a Banach space. Second, even if we isometrically and computably embed M into \mathbb{R}^n , we may not be able to run Gram - Schmidt within M . The definition below is central to this section:

Definition 8.1. Let $M \subseteq (\mathbb{R}^n, d)$ be closed such that $\langle M \rangle_{\mathbb{R}}$ has dimension $m \leq n$, and assume (M, d) possesses a computable structure. We say that points $x_0, \dots, x_m \in M$ form an *intrinsically computable base* of (M, d) if:

1. the vectors $x_0 - x_1, \dots, x_0 - x_m$ are linearly independent in \mathbb{R}^n ,
2. for every computable structure $(\alpha_i)_{i \in \omega}$ on (M, d) there is a surjective self-isometry W of (M, d) such that $W(x_0), \dots, W(x_m)$ are computable in $(\alpha_i)_{i \in \omega}$.

We call m from the definition above *the dimension of M* . We show:

Theorem 8.2. *Let M be a closed subspace of (\mathbb{R}^n, d) having dimension $m \leq n$ which possesses a computable structure. The following are equivalent:*

- (1) (M, d) is computably categorical;
- (2) M has an intrinsically computable base x_0, \dots, x_m .

Proof idea. The proof of (1) \Rightarrow (2) is rather straightforward, we briefly outline (2) \Rightarrow (1). Suppose we are given two computable structures on M . Using an intrinsically computable base, we embed M into \mathbb{R}^n and define new computable structures on \mathbb{R}^n . Then we observe that these new structures are computably isometric via a surjective isometry which maps M onto itself. Then we show that the restriction of this self-isometry to M is computable w.r.t. the given computable structures on M , and conclude that M is computably categorical.

Proof. In the proof below, we consider the linear span of M within \mathbb{R}^n and, without loss of generality, set $m = n$.

We prove (1) \Rightarrow (2). Suppose $(\alpha_i)_{i \in \omega}$ is a computable structure on (M, d) . By the density of $(\alpha_i)_{i \in \omega}$ and the choice of M and n , we can choose special points x_0, \dots, x_n such that $x_0 - x_1, \dots, x_0 - x_n$ are linearly independent. By our assumption, for every computable structure $(\beta_i)_{i \in \omega}$ on (M, d) there exists a computable isometry

$$U : (M, d, (\alpha_i)_{i \in \omega}) \rightarrow (M, d, (\beta_i)_{i \in \omega}).$$

The points $U(x_0), \dots, U(x_n)$ are computable in $(\beta_i)_{i \in \omega}$.

We show (2) \Rightarrow (1). We need the following fact. Although the fact is intuitively clear, we give a proof of it which uses elementary affine geometry.

Fact 8.3. *Every surjective self-isometry W of M can be uniquely extended to a surjective self-isometry \bar{W} of \mathbb{R}^n . Furthermore, both W and its extension are completely determined by the images of x_0, \dots, x_n .*

Proof. Note that the points x_0, \dots, x_n are affine independent which means that the smallest convex set containing the points has non-zero volume in \mathbb{R}^n (recall that here $m = n$). The volume is determined by the value of the Cayley-Maneger matrix which involves only $d(x_i, x_j)$, for $i, j \leq n$. The isometry W preserves the values of $d(x_i, x_j)$ and, consequently, the images of x_0, \dots, x_n are affine independent. Thus,

$W(x_0) - W(x_1), \dots, W(x_0) - W(x_n)$ are linearly independent. The vectors $x_0 - x_i$ are linearly independent. Thus, every point $z \in M \subseteq \mathbb{R}^n$ is uniquely determined by $d(z, x_i), i \leq n$. On the other hand, every point y from \mathbb{R}^n , and from M in particular, is uniquely determined by the distances $d(y, W(x_i)), i \leq n$. Notice that these distances are preserved under W . Since $x_0 - x_1, \dots, x_0 - x_n$ are linearly independent, every element z of \mathbb{R}^n can be uniquely written as $z = \sum_{0 < i \leq n} f_i(x_0 - x_i)$, where f_i are reals. Now it is clear that the map

$$\bar{W} : x_0 + \sum_{0 < i \leq n} f_i(x_0 - x_i) \rightarrow W(x_0) + \sum_{0 < i \leq n} f_i(W(x_0) - W(x_i))$$

is the needed extension. \square

By Fact 8.3, we may assume that $W(M)$ is a subset of an isometric copy of \mathbb{R}^n . Note that the proof of Fact 8.3 implies that $W(x_0) - W(x_1), \dots, W(x_0) - W(x_n)$ are linearly independent within this copy.

Claim 8.4. *Let v_1, \dots, v_n be linearly independent vectors in \mathbb{R}^n with $\|v_i\|$ and $\|v_i - v_j\|$ computable. Then $(\sum_{1 \leq i \leq n} r_i v_i)_{(r_1, \dots, r_n) \in \mathbb{Q}^n}$ is a computable structure on (\mathbb{R}^n, d) , where d is the Euclidean metric.*

Proof. The collection of points $(\sum_{1 \leq i \leq n} r_i v_i)_{(r_1, \dots, r_n) \in \mathbb{Q}^n}$ is clearly dense in \mathbb{R}^n . We have

$$\begin{aligned} d^2(r_1 v_1 + \dots + r_n v_n, q_1 v_1 + \dots + q_n v_n) &= \|(r_1 - q_1)v_1 + \dots + (r_n - q_n)v_n\|^2 \\ &= \sum_{0 < i \leq m} (r_i - q_i)^2 \|v_i\|^2 + \sum_{0 < i < j \leq m} 2(r_i - q_i)(r_j - q_j) \langle v_i, v_j \rangle. \end{aligned}$$

By the assumption, for every $i, j \in \{1, \dots, n\}$ the reals $\|v_i - v_j\|$ and $\|v_i\|$ are computable. Thus, the real $\langle v_i, v_j \rangle = \frac{1}{2}(\|v_i\|^2 + \|v_j\|^2 - \|v_i - v_j\|^2)$ is computable, for every $i, j \in \{1, \dots, n\}$. Therefore, the distances between the special points in $(r_1 v_1 + \dots + r_n v_n)_{(r_1, \dots, r_n) \in \mathbb{Q}^n}$ are uniformly computable. \square

Let $(\alpha_i)_{i \in \omega}$ and $(\beta_i)_{i \in \omega}$ be computable structures on M . Let W_1 and W_2 be self-isometries of M such that $W_1(x_0), \dots, W_1(x_n)$ are computable in $(\alpha_i)_{i \in \omega}$ and $W_2(x_0), \dots, W_2(x_n)$ are computable in $(\beta_i)_{i \in \omega}$, respectively. Let $v_i = W_1(x_i) - W_1(x_0)$ and $w_i = W_2(x_i) - W_2(x_0)$, for $i \in \{1, \dots, n\}$. By the choice of x_0, \dots, x_n , these points as well as their W -images are computable. Thus, the norms $\|v_i - v_j\|$, $\|w_i - w_j\|$, $\|w_i\|$ and $\|v_i\|$ are computable reals, for every $i, j \in \{1, \dots, n\}$. Fixing some effective enumeration of all rational n -tuples, denote $(\sum_{1 \leq i \leq n} r_i v_i)_{(r_1, \dots, r_n) \in \mathbb{Q}^n}$ by $(\gamma_i)_{i \in \omega}$, and $(\sum_{1 \leq i \leq n} r_i w_i)_{(r_1, \dots, r_n) \in \mathbb{Q}^n}$ by $(\theta_i)_{i \in \omega}$.

Claim 8.5. *The point α_k is computable w.r.t. $(\gamma_i)_{i \in \omega}$ uniformly in k , for every $k \in \omega$.*

Proof. We have $\alpha_k = W_1(x_0) + \sum_{0 < i \leq n} f_i v_i$ for reals f_1, \dots, f_n . Given $\delta > 0$ and a tuple of rationals (q_1, \dots, q_n) such that $|f_i - q_i| < \delta$ for every $i \in \{1, \dots, n\}$, we obtain

$$\begin{aligned} \left\| \sum_{0 < i \leq n} (f_i - q_i) v_i \right\|^2 &= \sum_{0 < i \leq n} (f_i - q_i)^2 \|v_i\|^2 + \sum_{0 < i < j \leq n} 2(f_i - q_i)(f_j - q_j) \langle v_i, v_j \rangle \\ &\leq \delta^2 \left(\sum_{0 < i \leq n} \|v_i\|^2 + \sum_{0 < i < j \leq n} 2|\langle v_i, v_j \rangle| \right). \end{aligned}$$

Thus, it is sufficient to show that f_i is computable, for every $i \in \{1, \dots, n\}$.

By the choice of W_1 , the real $d(\alpha_k, W_1(x_i))$ is computable, for every $i \leq n$. Therefore, the real

$$B_0 = d^2(\alpha_k, W_1(x_0)) = \left\| \sum_{0 < i \leq n} f_i v_i \right\|^2$$

is computable, and so is

$$B_i = d^2(\alpha_k, W_1(x_i)) = \left\| \left(\sum_{0 < i \leq n} f_i v_i \right) - v_i \right\|^2,$$

for every $i \in \{1, \dots, n\}$.

We express B_i using the inner products $\langle v_i, v_j \rangle$ and the norms $\|v_i\|^2$, for $i, j \leq n$. After a simplification, we get the system of equations

$$\frac{B_0 - B_i + \|v_i\|^2}{2} = \sum_{0 < j \leq n} \langle v_i, v_j \rangle f_j.$$

The set $\{v_1, \dots, v_n\}$ is a basis of \mathbb{R}^n , which implies that the matrix corresponding to this system is invertible. Furthermore, it has computable coefficients. Thus, for every $i \in \{1, \dots, n\}$ the real f_i is computable. \square

Claim 8.6. *The point β_k is computable w.r.t. $(\theta_i)_{i \in \omega}$ uniformly in k , for every $k \in \omega$.*

Proof. Similar to the proof of Claim 8.5. \square

By Claim 8.4, $(\gamma_i)_{i \in \omega}$ and $(\theta_i)_{i \in \omega}$ are computable structures on (\mathbb{R}^n, d) . Taking into account Fact 8.3, observe that these structures are computably isometric via $W(r_1 v_1 + \dots + r_n v_n) = r_1 w_1 + \dots + r_n w_n$. By Claim 8.5, the inclusion i of M into \mathbb{R}^n is computable w.r.t. $(\alpha_i)_{i \in \omega}$ and $(\gamma_i)_{i \in \omega}$. By Claim 8.6, i is computable w.r.t. $(\beta_i)_{i \in \omega}$ and $(\theta_i)_{i \in \omega}$. Note that M is closed, and i^{-1} is computable on its domain w.r.t. $(\gamma_i)_{i \in \omega}$ and $(\beta_i)_{i \in \omega}$. Similarly, i^{-1} is computable w.r.t. $(\theta_i)_{i \in \omega}$ and $(\alpha_i)_{i \in \omega}$. The following diagram commutes, where U is W restricted to M :

$$\begin{array}{ccc} (M, d, (\alpha_i)_{i \in \omega}) & \xrightarrow{U} & (M, d, (\beta_i)_{i \in \omega}) \\ \downarrow i & & \downarrow i \\ (\mathbb{R}^n, d, (\gamma_i)_{i \in \omega}) & \xrightarrow{W} & (\mathbb{R}^n, d, (\theta_i)_{i \in \omega}) \end{array}$$

This shows that U is computable w.r.t. $(\alpha_i)_{i \in \omega}$ and $(\beta_i)_{i \in \omega}$, proving the theorem. \square

The usual computable structure on \mathbb{R}^n is given by the tuples of rationals which are coordinates in an orthonormal base. The proof of Theorem 8.2 leads to a fact which resembles similar results on computable closures in computable algebra:

Fact 8.7. Let $M \subseteq (\mathbb{R}^n, d)$ be a closed subspace. For every computable structure $(\alpha_i)_{i \in \omega}$ on $\mathcal{M} = (M, d)$ there exists an isometric embedding $U : \mathcal{M} \rightarrow (\mathbb{R}^n, d)$ which is computable with respect to $(\alpha_i)_{i \in \omega}$ and the usual computable structure on \mathbb{R}^n . Furthermore, its inverse is computable on its domain.

Proof. Since for every $1 \leq m < n$ there is a natural computable inclusion of the standard computable structure on \mathbb{R}^m into the standard computable structure \mathbb{R}^n , we may assume that $\langle M \rangle_{\mathbb{R}}$ has dimension n . By the choice of n and the density of $(\alpha_i)_{i \in \omega}$ in \mathcal{M} , there exist special points $\gamma_0, \dots, \gamma_n$ in $(\alpha_i)_{i \in \omega}$ such that $\{\gamma_0 - \gamma_1, \dots, \gamma_0 - \gamma_n\}$ is linearly independent in \mathbb{R}^n . Denote $\gamma_0 - \gamma_k$ by v_k , for every $k \in \{1, \dots, n\}$. By Claim 8.4, the collection of points $(\sum_{1 \leq i \leq n} r_i v_i)_{(r_1, \dots, r_n) \in \mathbb{Q}^n}$ is a computable structure on \mathbb{R}^n . By Claim 8.5, the inclusion I of \mathcal{M} into \mathbb{R}^n is computable with respect to $(\alpha_i)_{i \in \omega}$ and $(\sum_{1 \leq i \leq n} r_i v_i)_{(r_1, \dots, r_n) \in \mathbb{Q}^n}$. By Theorem 4.7, the computable structure

$$\left(\sum_{1 \leq i \leq n} r_i v_i \right)_{(r_1, \dots, r_n) \in \mathbb{Q}^n}$$

is computably isometric to the usual structure on \mathbb{R}^n via a computable isometry U . The composition $U \circ I : \mathcal{M} \rightarrow \mathbb{R}^n$ is the needed isometry. \square

As a consequence of Theorem 8.2, many common computable compact subsets of (\mathbb{R}^n, d) with the inherited metric are computably categorical. In the following, $B_n(r)$ denotes the n -dimensional ball of radius r , and $\text{Cube}_n(r)$ stands for the n -dimensional cube (with its inside) with edge of length r . In particular, $\text{Cube}_1(r)$ is isometric to the interval $[0, r]$. The metrics on $B_n(r)$ and $\text{Cube}_n(r)$ are Euclidean. The following fact is rather straightforward:

Fact 8.8. *Let $M_n(r)$ be either the ball $B_n(r)$ or the cube $\text{Cube}_n(r)$. The space $M_n(r)$ possesses a computable structure if, and only if, r is left-c.e.*

Proof. Suppose r is left-c.e. and $r = \sup_s q_s$ for a computable sequence of positive rationals $(q_s)_{s \in \omega}$. If $M_n(r) = B_n(r)$ is a ball, then define a computable structure starting from the geometrical center of $M_n(r)$ and expanding the structure on the stages s such that $q_s > q_{s-1}$. More formally, define a computable sequence of finite rational-valued metric subspaces X_n such that for every m we have $X_{\langle k, m \rangle} \subseteq B_n(q_m)$ and, furthermore, $\bigcup_k X_{\langle k, m \rangle}$ is a computable structure on $B_n(q_m)$. The sequence can be organized so that $d(x, y)$ is computable for every $x, y \in \bigcup_{k, m \in \omega} X_{\langle k, m \rangle}$. The desired structure is $\bigcup_{k, m \in \omega} X_{\langle k, m \rangle}$. The case when $M_n(r) = \text{Cube}_n(r)$ can be done similarly.

Suppose $(\alpha_i)_{i \in \omega}$ is a computable structure on $M_n(r)$. The real $\mu = \sup\{d(\alpha_i, \alpha_j) : i, j \in \omega\}$ is clearly left-c.e. Note that $\mu = 2r$ if $M_n(r) = B_n(r)$, and $\mu = r\sqrt{n}$ if $M_n(r) = \text{Cube}_n(r)$. \square

We have:

Corollary 8.9 ([16]). *For every $n \in \omega$ and every computable real r , the cube $\text{Cube}_n(r)$ (the ball $B_n(r)$) is computably categorical.*

Proof. We prove the corollary for $\text{Cube}_n(r)$ with $n = r = 1$, the general case is not significantly different from the case $n = r = 1$. We show that $\{0, 1\}$ is an intrinsically computable base of $[0, 1]$.

Let $(\alpha_i)_{i \in \omega}$ be a computable structure on $\text{Cube}_1(1) = [0, 1]$. Define two uniformly computable sequences, $(\theta_k)_{k \in \omega}$ and $(i_k)_{k \in \omega}$ of special points of $(\alpha_i)_{i \in \omega}$ by recursion. The sequence $(\theta_k)_{k \in \omega}$ will be a Cauchy name for 0, and the sequence $(i_k)_{k \in \omega}$ will be a Cauchy name for 1. Let v and w be special points such that $d(v, w) > 3/4$ and $d(v, 0) < d(w, 0)$. Set $\theta_0 = v$ and $i_0 = w$. We use these points as non-uniform parameters. For $k > 1$, search in $(\alpha_i)_{i \in \omega}$ for special points x and y such

that $d(x, y) > 1 - 2^{-k-2}$ and $d(x, \theta_{k-1}) < d(x, i_{k-1})$. Set $\theta_k = x$ and $i_k = y$. This completes the definition of $(\theta_k)_{k \in \omega}$ and $(i_k)_{k \in \omega}$. Clearly, $d(\theta_k, 0) \leq 2^{-k}$ and $d(i_k, 1) \leq 2^{-k}$, for every $k \geq 0$.

We also outline the proof for $B_2(r)$. Pick a triple $x, y, z \in B_2(r)$ such that $d(x, y) = 2r$ and $d(x, z) = d(y, z) = \sqrt{2}r$. We claim that $\{x, y, z\}$ is an intrinsically computable base of $B_2(r)$. Given a computable structure $(\alpha_i)_{i \in \omega}$, we find a pair of special points (a_1, b_1) such that $d(x_0, y_0) > 2r - \epsilon$, where ϵ is a small positive rational. At the next stage, we need to find another pair (x_2, y_2) such that $d(x_2, y_2) \geq 2r - \epsilon/2$ so that x_2 is close to x_1 , and y_2 is close to x_2 . More specifically, we additionally require $d(x_2, x_1) < 3\sqrt{r}\epsilon$ and $d(y_2, y_1) < 3\sqrt{r}\epsilon$. To see why such x_2 and y_2 exist, we could use elementary geometry³. The computable sequences $(x_i)_{i \in \omega}$ and $(y_i)_{i \in \omega}$ witness computability of $\{x, y\}$ up to a self-isometry of $B_2(r)$. Suppose $a = \lim_i x_i$ and $b = \lim_i y_i$. Given $(x_i)_{i \in \omega}$ and $(y_i)_{i \in \omega}$, we can produce a computable sequence $(z_i)_{i \in \omega}$ converging to a point c such that $d(c, a) = d(c, b) = \sqrt{2}r$. We use elementary geometry to obtain such a sequence similarly to how it was done for $(x_i)_{i \in \omega}$ and $(y_i)_{i \in \omega}$. \square

Remark 8.10. If in Definition 8.1 we replace “there exists a surjective self-isometry W ” by “for every surjective self-isometry W ” then the corresponding analog of the preceding theorem will state that every isometry from a space having such a base is computable w.r.t. to any given computable structures. The proof needs only minor adjustments.

Fact 8.11. *Every self-isometry of $[0, 1]$ with the usual metric is computable w.r.t. $(\alpha_i)_{i \in \omega}$ and $(\beta_j)_{j \in \omega}$, for each computable structures $(\alpha_i)_{i \in \omega}$ and $(\beta_j)_{j \in \omega}$ on $[0, 1]$.*

Proof. A straightforward modification of the proof of the preceding corollary shows that $\{0, 1\}$ is an intrinsically computable base of $[0, 1]$ having the stronger property from Remark 8.10. \square

As a consequence, every two computable structures on $[0, 1]$ are equivalent (that is, the identity map is computable w.r.t. to these structures). What is the number of non-isometric computable structures on an interval of length r , where r is left-c.e.? We show:

Theorem 8.12. *Let n be a positive natural and r a left-c.e. real. The following are equivalent:*

- (1) r is computable,
- (2) the interval $[0, r]$ is computably categorical,
- (3) there exists only finitely many computable structures on $[0, r]$ which are pairwise not computably isometric.

Proof. The implication (1) \rightarrow (2) is given by Corollary 8.9, and the implication (2) \rightarrow (3) is trivial. We show $\neg(1) \rightarrow \neg(3)$.

³Hint: Consider a triangle $\triangle abc$ such that $a, b, c \in \partial B_2(r)$, $[x_1, y_1] \subset [a, b]$, and the geometric center of $B_2(r)$ belongs to $[c, b]$. Then a is at most ϵ far from x_1 or y_1 , say from x_1 . We have $d(y_1, b) < \epsilon < 3\sqrt{r}\epsilon$. The non-trivial situation is when the geometric center of $B_2(r)$ is not within $[a, b]$. Since $d(a, b) \geq d(x_1, y_1) > 2r - \epsilon$, we have $d^2(c, a) < (2r)^2 - (2r - \epsilon)^2 = 4r\epsilon - \epsilon^2 < 4r\epsilon$. Since $d(a, x_1) \leq \epsilon$, we have $d^2(c, x_1) = d^2(c, a) + d^2(a, x_1) \leq 4r\epsilon + \epsilon^2 < 9r\epsilon$. Thus, (c, b) could serve as (x_2, y_2) . By the density of the given computable structure, x_2 and y_2 can be chosen special points.

Suppose r is a non-computable left-c.e. real. We define infinitely many structures on $[0, r]$, as follows. For a natural $m > 0$, let $\alpha(m)_i$ and $\beta(m)_i$ be the i 'th rational in any fixed enumeration of positive rationals from the left cuts of $2^{-m}r$ and $(1 - 2^{-m})r$, respectively. Clearly, $(\alpha(m)_i)_{i \in \omega}$ is a computable structure on $[0, 2^{-m}r]$, and $\beta(m)_i$ is a computable structure on $[0, (1 - 2^{-m})r]$. Note that 0 does not belong to either $(\alpha(m)_i)_{i \in \omega}$ or $(\beta(m)_i)_{i \in \omega}$.

Define a computable structure $(h(m)_i)_{i \in \omega}$ on $[0, r]$, as follows. Let $h(m)_0 = 2^{-m}r$. Given $i > 0$, let

$$h(m)_i = \begin{cases} 2^{-m}r + \beta(m)_k, & \text{if } i = 2k, \\ 2^{-m}r - \alpha(m)_k, & \text{if } i = 2k - 1. \end{cases}$$

Observe that the distance between any two points from $(h(m)_i)_{i \in \omega}$ is uniformly computable, and the sequence $(h(m)_i)_{i \in \omega}$ is dense in $[0, r]$.

We show that $(h(m)_i)_{i \in \omega}$ is not computably isometric to $(h(n)_i)_{i \in \omega}$ if $m > n$. Pick a surjective isometry U of $[0, r]$ and assume (towards a contradiction) that U is computable with respect to $(h(m)_i)_{i \in \omega}$ and $(h(n)_i)_{i \in \omega}$. The point $U(h(m)_0)$ has to be computable in $(h(n)_i)_{i \in \omega}$. Note that $U(h(m)_0)$ is either $2^{-n}r$ or $(1 - 2^{-n})r$. In either case, $d(U(h(m)_0), h(n)_0)$ is not computable as it is a rational multiple of r , contradicting the choice of U . \square

9. A SUFFICIENT CONDITION

We say that a collection B of points of a metric space $\mathcal{M} = (M, d)$ is an *automorphism base* of \mathcal{M} if and only if any nontrivial surjective self-isometry of \mathcal{M} necessarily moves at least one of its elements – or, equivalently, the global action of any such surjective self-isometry is completely determined by that on B . We need an effective version of this notion:

Definition 9.1. We say that a finite automorphism base $B = \{b_1, \dots, b_k\}$ of a Polish space $\mathcal{M} = (M, d)$ is an *effective automorphism base* of \mathcal{M} if:

1. for every computable structure $(\alpha_i)_{i \in \omega}$ on \mathcal{M} there is a surjective self-isometry U of \mathcal{M} such that $U(b_1), \dots, U(b_k)$ are computable in $(\alpha_i)_{i \in \omega}$;
2. for every rational $\epsilon > 0$ we can compute a rational $\delta \in (0, 1)$ such that for every $x, y \in \mathcal{M}$ the inequality $\sup_j |d(b_j, x) - d(b_j, y)| < \delta / C_{x,y}$ implies $d(x, y) < \epsilon$, where $C_{x,y} = 1 + \sup_i d(b_i, x) + \sup_i d(b_i, y)$.

Thus, points in \mathcal{M} are effectively determined by their distances to points in an effective automorphism base. Clearly, if the diameter of \mathcal{M} is finite we can eliminate $(1 + \sup_i d(b_i, x) + \sup_i d(b_i, y))$ from (2.) in Definition 9.1. If the diameter of \mathcal{M} is infinite, then $C_{x,y}$ is needed to make Theorem 9.2 work for subspaces of \mathbb{R}^n . We show:

Theorem 9.2. *Suppose a Polish space \mathcal{M} possesses a computable structure. If \mathcal{M} has an effective automorphism base then \mathcal{M} is computably categorical.*

Proof. Suppose $\{b_1, \dots, b_k\}$ is an effective automorphism base for \mathcal{M} . Let $(\alpha_s)_{s \in \omega}$ and $(\beta_s)_{s \in \omega}$ be two computable structures on \mathcal{M} . Let W_1 and W_2 be surjective self-isometries such that $W_1(b_1), \dots, W_1(b_k)$ are computable w.r.t. $(\alpha_s)_{s \in \omega}$ and $W_2(b_1), \dots, W_2(b_k)$ are computable w.r.t. $(\beta_s)_{s \in \omega}$. Let $U = W_2 \circ W_1^{-1}$. We show that U is computable with respect to $(\alpha_s)_{s \in \omega}$ and $(\beta_s)_{s \in \omega}$.

Note that $W_1(b_1), \dots, W_1(b_k)$ is an effective automorphism base. Therefore, without loss of generality, we may assume $W_1 = \text{Id}$. Thus, we assume the points $U(b_0), \dots, U(b_k)$ are computable w.r.t. $(\beta_s)_{s \in \omega}$, and the points b_0, \dots, b_k are computable w.r.t. $(\alpha_s)_{s \in \omega}$.

Given $\epsilon > 0$ and a special point α_s , compute the rational $\delta < 1$ corresponding to ϵ . Find a special point β_v such that

$$\sup_j |d(b_j, \alpha_s) - d(U(b_j), \beta_v)| < \frac{\delta}{2 + 2 \sup_j d(b_j, \alpha_s)}.$$

By the density of $(\beta_s)_{s \in \omega}$, such a special point can be effectively found. We have $d(b_j, \alpha_s) = d(U(b_j), U(\alpha_s))$ and $d(b_j, U^{-1}(\beta_v)) = d(U(b_j), \beta_v)$. Note that $\delta < 1$ implies $\sup_j |d(U(b_j), U(\alpha_s)) - d(U(b_j), \beta_v)| < 1$. Consequently,

$$2 + 2 \sup_j d(b_j, \alpha_s) > 1 + \sup_j d(U(b_j), \beta_v) + \sup_j d(U(b_j), U(\alpha_s))$$

and

$$|d(U(b_j), U(\alpha_s)) - d(U(b_j), \beta_v)| < \frac{\delta}{1 + \sup_j d(U(b_j), \beta_v) + \sup_j d(U(b_j), U(\alpha_s))}.$$

By the choice of δ , we obtain $d(U(\alpha_s), \beta_v) < \epsilon$, showing that $U(\alpha_s)$ is computable w.r.t. $(\beta_s)_{s \in \omega}$. Thus, U is a computable map. \square

Corollary 9.3. *The unit circle S^1 with the distance given by the shortest arc between points is computably categorical.*

Proof. Let (α_i) be a computable structure on S^1 . This structure contains special points α_0, α_1 such that $d(\alpha_0, \alpha_1) < 1/2$. It is straightforward to check that α_0, α_1 form an effective automorphism base of S^1 . \square

Effective automorphism bases yield an alternative characterization of computable categoricity of subspaces of \mathbb{R}^n :

Theorem 9.4. *A closed subspace \mathcal{M} of \mathbb{R}^n which possesses a computable structure is computably categorical if, and only if, \mathcal{M} has an effective automorphism base.*

Proof. By Theorem 9.2, it is sufficient to show that \mathcal{M} contains an effective automorphism base provided that \mathcal{M} is computably categorical. Let $(\alpha_i)_{i \in \omega}$ be a computable structure on \mathcal{M} . Let n be least such that \mathcal{M} isometrically embeds into \mathbb{R}^n . By Theorem 8.2, the space \mathcal{M} contains an intrinsically computable base b_0, \dots, b_n . Without loss of generality, we may assume that b_0, \dots, b_n are computable w.r.t. $(\alpha_i)_{i \in \omega}$. By Fact 8.7, we may assume that \mathcal{M} is a subspace of \mathbb{R}^n such that each α_i is computable in the usual structure on \mathbb{R}^n uniformly in i . As a consequence, the points b_0, \dots, b_n are computable in the usual computable structure on \mathbb{R}^n . We may further suppose that b_0 is 0 in \mathbb{R}^n , re-defining operations on the standard computable structure if necessary. By the definition of an intrinsically computable base, the elements b_0, \dots, b_n satisfy (1.) of Definition 9.1. It remains to check (2.) of Definition 9.1 for b_0, \dots, b_n .

Let $\epsilon < 1$, and suppose $x, y \in \mathcal{M}$ are such that

$$|d(b_j, x) - d(b_j, y)| < \delta / (1 + \sup_i d(b_i, x) + \sup_i d(b_i, y))$$

for every $j \in \{1, \dots, n\}$. We need to choose a value for δ which will depend on ϵ . We will choose δ later, based on the considerations below.

Let $v_j = b_j - b_0$ for $j \in \{1, \dots, n\}$. There are uniquely defined tuples of reals $\bar{f} = (f_1, \dots, f_n)$ and $\bar{h} = (h_1, \dots, h_n)$ such that

$$y = \sum_{1 \leq j \leq n} f_j v_j \text{ and } x = \sum_{1 \leq j \leq n} h_j v_j.$$

Define

$$\begin{aligned} \bar{u} &= (d^2(b_0, y) - d^2(b_1, y) + d^2(b_0, b_1), \dots, d^2(b_0, y) - d^2(b_n, y) + d^2(b_0, b_n)); \\ \bar{w} &= (d^2(b_0, x) - d^2(b_1, x) + d^2(b_0, b_1), \dots, d^2(b_0, x) - d^2(b_n, x) + d^2(b_0, b_n)). \end{aligned}$$

As we have seen in the proof of Claim 8.5 in Theorem 8.2, the computability of b_0, \dots, b_n implies that there is a computable matrix B with a computable inverse such that $\bar{f} = B \cdot \bar{u}$ and $\bar{h} = B \cdot \bar{w}$. Let D be the matrix which corresponds to the Gram-Schmidt orthogonalization of v_1, \dots, v_n . By the choice of v_1, \dots, v_n , the matrix D and its inverse are computable. Let $\|\cdot\|$ stand for the usual norm in the space \mathbb{R}^n of n -tuples of reals. We have

$$d^2(y, x) = \|DB(\bar{u}) - DB(\bar{w})\|^2 \leq \|DB\|^2 \|\bar{u} - \bar{w}\|^2,$$

where $\|DB\| > 0$ is a computable real. Let $\delta = \min\{\epsilon \frac{1}{2\|DB\|\sqrt{n}}, \epsilon\}$ and

$$\delta_1 = \frac{\delta}{1 + \sup_j d(b_j, x) + \sup_j d(b_j, y)}.$$

We obtain

$$\begin{aligned} \|\bar{u} - \bar{w}\|^2 &= \sum_{1 \leq j \leq n} (d^2(b_0, y) - d^2(b_j, y) - d^2(b_0, x) + d^2(b_j, x))^2 \\ &< \sum_{1 \leq j \leq n} (2\delta_1 \sup_i (d(b_i, y) + d(b_i, x)))^2 \\ &\leq 4n(\delta_1(1 + \sup_j d(b_j, x) + \sup_j d(b_j, y)))^2 \\ &= 4n\delta^2. \end{aligned}$$

Therefore, $d(y, x) < \|DB\| \cdot 2\sqrt{n}\delta \leq \epsilon$. This finishes the proof. \square

Thus, Theorem 9.2 is a generalization of (2) \rightarrow (1) part of Theorem 8.2 to metric spaces which are not subspaces of \mathbb{R}^n . We obtain:

Corollary 9.5. *For a closed subspace \mathcal{M} of \mathbb{R}^n having a computable structure, the following are equivalent:*

1. \mathcal{M} is computably categorical;
2. \mathcal{M} has an intrinsically computable base;
3. \mathcal{M} has an effective automorphism base.

Proof. By Theorem 8.2 and Theorem 9.4. \square

10. FURTHER PROBLEMS

As we have seen, l_2 is computably categorical, but l_1 is not.

Problem 10.1. Is l_p computably categorical if, and only if, $p = 2$?

All (ultra)homogeneous spaces that we studied in the paper happen to be computably categorical. It is not obvious why it should be the case for *every* such space:

Problem 10.2. Is every computable ultra-homogeneous space computably categorical?

Keng Meng Ng and the author [22] have recently showed that $C[0,1]$ is not computably categorical as a Banach space and even as a Banach algebra.

Problem 10.3. Find a Banach space which is computably categorical (as Banach space) but is not computably categorical as a metric space.

There has been a lot of work on the number of non-equivalent computable copies of countable algebras. For instance, Goncharov’s Branching Theorem [13] gives a sufficient condition for a countable structure to have infinitely many non-equivalent computable copies. It would be nice to have a similar condition for Polish spaces. Khoussainov and Melnikov (unpublished) recently constructed a Polish space having exactly two non-isometric computable structures. This space is rather artificial from the analytical point of view. Thus:

Problem 10.4. Find an example of classical space (such as l_p) that is not c.c. and has a finite number non-isometric computable structures.

Also, is it true that a subspace of \mathbb{R}^n has either one or infinitely many non-isometric computable structures?

As we mentioned in the introduction, in computable model theory there are notions similar to computable categoricity. It would be interesting to introduce and study similar notions for computable metric spaces. More generally, we hope that it is possible to develop computable continuous model theory and apply these results to classical objects of computable analysis.

Problem 10.5. Extend the results of computable model theory to computable uncountable metric spaces.

Also, it would be interesting to study computable bi-Lipschitz and quasi-isometric maps between computable metric spaces.

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