NEW DEGREE SPECTRA OF POLISH SPACES

ALEXANDER G. MELNIKOV

Abstract. The main result is as follows. Fix an arbitrary prime q. A q-divisible torsion-free (discrete, countable) abelian G has a Δ^0_2-presentation if, and only if, its connected Pontryagin – van Kampen Polish dual $\widehat{G}$ admits a computable complete metrization (in which we do not require the operations to be computable).

We use this jump-inversion/duality theorem to transfer results on degree spectra of torsion-free abelian groups to results about degree spectra of Polish spaces up to homeomorphism. For instance, it follows that for every computable ordinal $\alpha > 1$ and each $\mathbf{a} > \mathbf{0}^{(\alpha)}$ there is a connected compact Polish space having proper α^{th} jump degree $\mathbf{a}$ (up to homeomorphism). Also, for every computable ordinal β of the form $1 + \delta + 2n + 1$, where δ is zero or is a limit ordinal and $n \in \omega$, there exists a connected Polish space having an X-computable copy if, and only if, X is *non-low*$_\beta$. In particular, there is a connected Polish space having exactly the *non-low*$_2$ complete metrizations. The case when $\beta = 2$ is an unexpected consequence of the main result of the author's M.Sc. thesis written under the supervision of Sergey S. Goncharov.

In the present paper we establish a new algorithmic version of Pontryagin – van Kampen duality between compact and discrete groups and apply it to derive new results on degree spectra of Polish spaces up to homeomorphism. To motivate such investigations, we note that the paper contributes to a new theme in effective mathematics that combines methods of computable algebra [1–3] with tools of computable analysis [4–6] to advance both subjects. The main tools of such studies are the notions of computability of algebraic and topological structures; Turing [7, 8], Banacha and Mazur [9], Fröhlich and Shepherdson [10], Maltsev [11], Rabin [12] and others suggested various notions of computability for infinite mathematical structures and spaces. Historically, the study of computable processes in separable uncountable structures [5, 6] and in countable discrete algebraic structures [1, 2] have been rather independent, even though both theories share essentially the same motivation. In particular, one of the central problems in both themes has traditionally been:

Describe computably presentable mathematical structures.

Beginning with [13], there has been a line of investigation that aims to unite these two subjects into one general theory; we cite [14–16] for several recent results in this direction, and we also cite [17] for a detailed exposition of this new general theory. The main result of the present paper completely reduces the problem of computable metrizability of a broad class of compact Polish spaces to the problem of Δ^0_2-presentability of countable torsion-free abelian groups. This result provides an explicit link between computable analysis and effective algebra.

Fix a prime q. Say that an (additive) abelian group A is q-divisible if, for every $n \in \mathbb{N}$ and every $a \in A$, $q^n x = a$ has a solution in A.

Theorem 0.1. *For a (discrete, countable) q-divisible torsion-free abelian group H, the following are equivalent:*

(1) H has a Δ^0_2 presentation.
(2) The connected compact Pontryagin – van Kampen dual $\widehat{H}$ of H admits a computable (compatible, complete) metrization.

We will clarify the terminology in due course, but we note that, similarly to Stone duality for Boolean algebras and Stone spaces, Pontryagin – van Kampen duality is injective on the isomorphism types of discrete and compact Polish abelian groups, and indeed $G \cong \widehat{\widehat{G}}$ for any (Hausdorff) locally compact abelian group G. In a related work [18], Lupini, Melniikov and Nies have recently established a version of Pontryagin – van Kampen duality between computable torsion-free abelian H and effectively compact $\widehat{H}$. Although the proof of (2) $\implies$ (1) uses

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the apparatus developed in [18], the proof of (1) $\implies$ (2) relies on some new technical ideas. Indeed, the jump inversion technique that we introduce to prove (1) $\implies$ (2) seems to have no direct analogy in computable analysis.

We emphasise that in (2) of Theorem 0.1 we do not require the group operations of $\widehat{H}$ to be computable with respect to the metric. However, the proof of (1) $\implies$ (2) gives a computable metrization of $\widehat{H}$ in which the operations are computable. It follows that, for a q-divisible torsion-free abelian H, if $\widehat{H}$ is computably metrizable, then there is also a metrization of $\widehat{H}$ that makes the operations computable. We also (strongly) conjecture that the proof of (1) $\implies$ (2) produces a primitive recursive ("punctual") metrization of the connected compact $\widehat{H}$ (which is likely polynomial-time). Compare this to (3) of Theorem 1.5 in [19] saying that every computable torsion-free abelian group has a punctual presentation. We refer the reader to [20, 21] for the foundations of punctual structure theory and to [22] to the connections of this theory with computable analysis. We note however that this conjectured consequence is perhaps not that exciting because we still do not know if there is a computable Polish space not homeomorphic to a primitive recursive one[1]. (This is an open question recently raised by Ng.) Further consequences of our result are concerned with spaces that are *not* computably presentable. We discuss this in detail below.

0.1. **Degree spectra of countable algebras.** In computable algebra, one seeks to find a computably-theoretic sufficient condition for a structure to be computably presentable. For example, every low Boolean algebra[2] is isomorphic to a computable one [24], and indeed every low$_4$ Boolean algebra also has this property [25]. More generally, we follow Richter [26] and define the *degree spectrum* of a countable algebraic structure A to be the set of Turing degrees that compute a presentation of the structure:

$$DSp(A) = \{\mathbf{a} : \mathbf{a} \text{ computes } B \cong A\},$$

where $\cong$ stands for algebraic isomorphism. So, for a Boolean algebra A, if a low degree is in its degree spectrum then A has a computable presentation. Therefore, the study of degree spectra is directly related to the problem of computable presentability of structures. Much work has been done on degree spectra in common algebraic classes [27–30] as well as for structures in general [26, 31–34].

One naturally seeks to understand what kind of collections of Turing degrees can be realized as a degree spectrum of some structure. One of the most notable and counterintuitive results of this sort was obtained by Wehner [32] and, independently, Slaman [33]: There exists a structure whose degree spectrum contains exactly the non-computable degrees. (We call the collection of all non-zero Turing degrees the *Slaman-Wehner spectrum.*) The result answered a question of Lempp in negative. One standard notion that can be used to measure the complexity of a spectrum is as follows. If A is a countable structure, α is a computable ordinal, and $\mathbf{a} \geq \mathbf{0}^{(\alpha)}$ is a Turing degree, then A has α^{th} *jump degree* $\mathbf{a}$ if the set

$$\{\mathbf{d}^{(\alpha)} : \mathbf{d} \in DSp(A)\}$$

has $\mathbf{a}$ as its least element. In this case, the structure A is said to have α^{th} *jump degree*. A structure A has *proper* α^{th} *jump degree* $\mathbf{a}$ if A has α^{th} jump degree $\mathbf{a}$ but not β^{th} jump degree for any $\beta < \alpha$. In this case, the structure A is said to have *proper* α^{th} *jump degree*. We cite [35–39] for various results on jump degrees of structures.

Usually, if a structure has α^{th} jump degree this means that one can code a set of natural numbers into its Σ^c_α diagram (in the sense of the infinitary computable language $\mathcal{L}^c_{\omega_1\omega}$ [2]), while the standard way of getting a structure with the non-low$_n$-spectrum is to code a family of sets into the Σ^c_{n+1}-diagram of the structure.

0.2. **Degree spectra of Polish spaces.** While Banach spaces are usually considered up to linear isometries, Polish spaces and Polish groups are often studied up to homeomorphism. Given a Polish space, one way to define its computable presentation is to say that there exists a complete compatible metric d and a dense countable sequence (x_i) such that $d(x_i, x_j)$ is a computable real uniformly in i, j. In this case we also say that the space admits a (compatible, complete) computable metrization.

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[1] By this we mean the metric completion of a countable metric space upon the domain of ω in which the predicates $D_<(i,j,k) \iff d(i,j) < 2^{-k}$ and $D_>(i,j,k) \iff d(i,j) > 2^{-k}$ are uniformly primitive recursive.

[2] Recall that an oracle X is low if the Halting problem X' for Turing machines with oracle X can be computed using $\emptyset'$, the usual Halting problem (with no oracles. The class low$_4$ is defined similarly, but using $X^{(4)}$ and $\emptyset^{(4)}$. This is a degree-invariant property. We cite [23] for further background.

While the study of degree spectra of countable algebraic structures is a well-developed theory, essentially nothing was known about the degree spectra of Polish spaces up to homeomorphism until the publication of the three recent papers [40–42]. We follow [40] and define the degree spectrum of a Polish space M *up to homeomorphism* as follows:

$$DSp_{hom}(M) = \{\mathbf{a} : \mathbf{a} \text{ computes } N \cong_{hom} M\},$$

where N is a separable metrized space with a distinguished dense set (x_i) for which $\mathbf{a}$ can uniformly compute $d(x_i, x_j)$.

We discuss the little that is known about degree spectra of spaces up to homeomorphism. Results from [42] and the unpublished manuscript [43] provide direct ways to code a set into the space using 4 and 3 jumps, respectively, giving examples of spaces having proper 4th and 3rd jump degree. In [42, 43] two computable versions of Stone duality have been established. As a consequence of the version contained in [42], degree spectra of Boolean algebras are in a 1-1 correspondence with the degree spectra of the respective Stone spaces. We also cite [44] for a further detailed computability-theoretic analysis of Stone duality, and for an application of Stone spaces to degree spectra of Banach spaces up to linear isometry. However, degree spectra of Boolean algebras are not particularly rich. For instance, the above-mentioned result about low$_4$ Boolean algebras implies that the Slaman-Wehner spectrum (indeed, the non-low$_n$ spectrum for $n < 4$) cannot be realized for Stone spaces. Also, the main result of [39] says that no Boolean algebra has nth jump degree $\mathbf{d} > 0^{(n)}$. Thus, to get interesting examples of degree spectra of Polish spaces, one needs to consider spaces that are not totally disconnected compact.

A version of the main result of the present paper for effectively compact presentations can be found in the recent work [18]. Using this version of Pontryagin – van Kampen duality from [18], one can get corollaries similar to the ones we will give below shortly, but for effectively compact presentations. We will not define what it means to be effectively compact here, but we do say that there is no direct way to connect the above-mentioned consequences of [18] with degree spectra up to homeomorphism in our sense. This is because there exist computably metrized spaces that have no effectively compact presentation [43], and indeed this in particular is true for the spaces considered in [18] (as explained in [18]).

We now give several consequences of our main result.

Corollary 0.2 (Follows from [45, 46])**.** For every computable ordinal β of the form $1 + \delta + 2n + 1$, where δ is zero or is a limit ordinal and $n \in \omega$, there exists a a connected compact Polish space having an X-computable copy if, and only if, X is *non-low*$_\beta$.

As an immediate consequence of the above, we obtain:

Corollary 0.3. For every β as in Corollary 0.2, there exists a connected compact Polish space having $0^{(\beta+1)}$ as its proper $(\beta + 1)^{th}$-jump degree.

Corollary 0.4 (Follows from [36, 37, 46, 47])**.** For every computable ordinal $\alpha > 1$ and degree $\mathbf{a} > \mathbf{0}^{(\alpha)}$, there is a connected compact Polish space having proper α^{th} jump degree $\mathbf{a}$.

We now discuss how these corollaries follow from our main theorem and the cited results from the literature. Recall that (2) of Theorem 0.1 does not assume that the group operations are computable. However, the theorem does assume that the discrete group is q-divisible, and there is no such assumption in the cited results. Luckily, all these constructions of interesting degree spectra of (discrete) torsion-free abelian groups in the literature share the same feature, namely, they remain true if one also additionally makes the groups q-divisible for some prime q (that is not used anywhere in the definition of the group otherwise). For instance, to see that this is indeed the case for groups constructed in [36, 45] one has to get accustomed with the heavy machinery used in these papers. We refer the reader to, e.g., Lemma 4.4 of [36] for a sample technical result of this sort where this stability under q-divisibility is explicitly stated and verified. Discussing this machinery in any reasonable detail is outside the scope of this paper; we cite the survey [48] for an informal explanation, and we also cite [49] for a more detailed technical exposition of this machinery. When $\beta = 2$ in Corollary 0.2, no such complex machinery is required, and indeed the space admits a relatively nice description. We give a complete explanation of this case in the last section.

We leave open whether Corollary 0.4 holds for $\alpha = 0$, i.e. when $\mathbf{a}$ is just the degree of M. Also, we do not know if Corollaries 0.2 and 0.3 hold for $\alpha = 0, 1$. In particular, we do not know if there is a Polish space with the Slaman-Wehner spectrum, up to homeomorphism. Such examples are not known for Polish spaces in general (up to homeomorphism), let alone compact or connected compact Polish spaces. Finally, we leave open whether the main result holds without any extra assumption on the isomorphism type of the torsion-free group.

1. Pontryagin - van Kampen duality

All groups in this section are separable and Hausdorff. Given a topological abelian group G, the character group $\widehat{G}$ of G is the collection of all continuous homomorphisms from G to the unit circle group $\mathbb{R}/\mathbb{Z}$ under the compact-open topology (the topology of uniform convergence on compact sets), with pointwise addition. Note that $\widehat{G}$ is abelian as well. To avoid repetitions, henceforth *we assume that all our groups are Polish and abelian.* Pontryagin - van Kampen duality states that, if G is locally compact, then $\widehat{G}$ is also locally compact; furthermore $\widehat{\widehat{G}}$ is topologically isomorphic to G via the map sending $g \in G$ to the evaluation map $\phi \mapsto \phi(g)$ [50]. Similarly to Stone duality in the case of Boolean algebras, the character group $\widehat{G}$ contains all the information about G. For a locally compact abelian group G, the character group $\widehat{G}$ is usually called the Pontryagin - van Kampen dual of G, the Pontryagin dual of G, or simply the dual of G if there is no danger of confusion. We refer the reader to the books [51, 52] for more on this subject. We will need that G is discrete countable torsion-free iff $\widehat{G}$ is compact connected Polish.

We explain why the dual of a discrete group can be viewed as a closed subgroup of $\mathbb{A} = \mathbb{T}^\omega$, where $\mathbb{T}$ is the unit circle group. Let $\mathbb{T}$ be the group $\mathbb{R}/\mathbb{Z}$, which is isomorphic as topological group to the multiplicative group of complex numbers having norm 1. We say that a point $x \in \mathbb{T}$ is *rational* if the respective point of the unit interval is a rational number. Then $\mathbb{T}$ equipped with rational points is a computable Polish group. The direct product

$$\mathbb{A} = \prod_{i \in \mathbb{N}} \mathbb{T}_i,$$

of infinitely many identical copies $\mathbb{T}_i$ of $\mathbb{T}$ carries the natural product-metric

$$D(\chi, \rho) = \sum_{i=0}^{\infty} \frac{1}{2^{-i-1}} d_i(\chi_i, \rho_i),$$

where each of the d_i stands for the shortest arc metric on $\mathbb{T}_i$. Under this metric and the component-wise operation $\mathbb{A}$ is a computably metrized Polish abelian group. The dense sets are given by sequences (a_i), where a_i is a rational point in $\mathbb{T}_i$, and almost all a_i are equal to zero. The basic open sets in $\prod_{i \in \mathbb{N}} \mathbb{T}_i$ are direct products of intervals with rational end-points such that a.e. interval in the product is equal to the respective $\mathbb{T}_i$. Clearly, we can effectively list all such open sets. (The exact choice of this basic system of balls is not crucial, but it will be convenient to assume that the end-points of the intervals are rational.) Every compact abelian group can be realised as a closed subgroup of $\mathbb{A}$, as explained below.

Suppose $G = \{g_0 = 0, g_1, g_2, \ldots\}$ is a countably infinite discrete group. Let $\mathrm{Hom(G, \mathbb{T})}$ be the subset of $\mathbb{A} = \prod_{i \in \mathbb{N}} \mathbb{T}_i$, (each $\mathbb{T}_i$ is a copy of $\mathbb{T}$) consisting of tuples $\chi = (\chi_0, \chi_1, \ldots)$, where each such tuple represents a group-homomorphism $\chi : G \to \mathbb{T}$ such that $\chi(g_i) = \chi_i \in \mathbb{T}_i$. Since G is discrete, every group homomorphism $\chi : G \to \mathbb{T}$ is necessarily continuous. Thus, $\widehat{G} \cong \mathrm{Hom(G, \mathbb{T})}$. Since being a group-homomorphism is a universal property, $\mathrm{Hom(G, \mathbb{T})}$ is a closed subspace of $\mathbb{A}$. Pontryagin - van Kampen duality implies that every separable compact abelian group is homeomorphic to a closed subgroup of $\mathbb{A}$.

2. Proof of Theorem 0.1

2.1. **Turning a computable Polish space into a Δ^0_2 discrete group.** Lupini, Melnikov and Nies [18] proved that, for a connected compact Polish abelian group G, the following are equivalent:

(1) G has an effectively compact presentation (as a Polish space);
(2) $\widehat{G}$ has a computable presentation.

In (1) there is no assumption about the computability of the group operations. A computably metrized compact space is effectively compact if for every n we can uniformly computably list all of its open 2^{-n}-covers. Since every computably metrized Polish space is $0'$-effectively compact, we obtain that for every computably metrized connected Polish abelian G, its dual admits a Δ^0_2-presentation. The effectively compact case requires extra care because the non-emptiness of the intersection of basic open balls can be an undecidable property. In [18] this is circumvented using a new constructive version of a metric nerve to replace the usual nerve of a cover. However, if our goal is to show that $\widehat{G}$ admits a Δ^0_2-presentation for a computably metrized G, then we do not need this new notion, and therefore the proof becomes much more direct and transparent. We explain this below.

Our goal is to prove a computable version of the remarkable result stated below.

Theorem 2.1 (See Part 5 of Chapter 8 of [53])**.** *For a compact connected Polish abelian group G,*

$$H^1(G) \cong \widehat{G},$$

where as usual $\widehat{G}$ denotes the Pontryagin – van Kampen dual of G and $H^1(G)$ stands for the first Čech cohomology group of the underlying metric space.

One does not need the operation of G to define $H^1(G)$; therefore homeomorphic connected compact abelian groups are necessarily isomorphic as topological groups. See, e.g., [54] for a detailed exposition of cohomology theory for compact abelian groups. So, our goal is to prove the following.

Theorem 2.2. *Let M be a compact, computably metrized Polish space. Then, for each i, its ith Čech cohomology group $H^i(G)$ admits a Δ^0_2-presentation.*

2.1.1. *The necessary tools from algebraic topology.* Given a compact M, let $\mathcal{N}$ be the directed set of all its finite open covers (under refinement). Since the covers by basic ϵ-balls, where ϵ ranges over positive rationals, are cofinal among all covers, without loss of generality we can restrict ourselves only to covers by basic open balls with rational radii. For each member C of $\mathcal{N}$, define its nerve $N(C)$ to be the collection of all sets in the cover that intersect non-trivially. One can view $N(C)$ as a (finite) simplicial complex in which the n-dimensional faces are exactly the n-element subsets X of $N(C)$ such that $\bigcap\{Y : Y \in X\}$ is a non-empty set. For these finite simplicial complexes we can define their cohomology groups $H^*(N(C))$ (with coefficients in $\mathbb{Z}$).

We follow §73 of [55] and define the Čech cohomology group of a compact metrized space as follows. For a fixed finite set of basic open balls $C \in \mathcal{N}$ and the respective metric simplex $N(C)$, define the simplicial chain complex as usual:

$$\ldots \to_{\delta_3} A_2 \to_{\delta_2} A_1 \to_{\delta_1} A_0$$

where A_i are finitely generated free abelian groups and δ_i are boundary homomorphisms, and then define the associated cochain complex $A^i = Hom(A_i, \mathbb{Z})$ and define $d_i : A^i \to A^{i-1}$ to be the dual homomorphism of δ_{i+1}. Then $H^i(N(C)) = Ker(d_i)/Im(d_{i-1})$ is the ith cohomology group of the simplex $N(C)$ which is a finitely generated abelian group which can be thought of as given by finitely many generators and relations. Let $H^*(M)$ be the direct limit of $H^*(N(C))$ induced by the inverse system $\mathcal{N}$ under the refinement maps.

Chapter 1 (§11) of [55] contains a careful verification of the computability of the homology groups for finite simplicial complexes, in the following sense. Given a (strong index of a) simplicial complex, one can uniformly compute its i-th homology group represented as $\bigoplus_{i \leq k} \langle a_i \rangle$, where $a_0, \ldots, a_k$ are the generators of the group such that the orders of the cyclic $\langle a_i \rangle$ are also uniformly computable. Since $A^i = Hom(A_i, \mathbb{Z})$, one can easily observe that that respective cohomology groups are also computable in this strong sense. We need a similar result for the Čech cohomology groups, but these groups are no longer finitely generated.

2.1.2. *Computability of the Čech cohomology.*

Proof of Theorem 2.2. A version of this proof for effectively compact spaces can be found in [18]. We say that a sequence of finitely generated uniformly computable abelian groups (B_j) is *strongly completely decomposable* if each B_i uniformly splits into a direct sum of its cyclic subgroups, and furthermore the sets of generators of the cyclic summands are given by their strong indices.

Claim 1. The groups $H^i(N(C))$ are strongly completely decomposable (uniformly in C and i).

Proof. A close examination of the definitions shows that, given C (as a finite set of parameters) and i, we can compute the generators of $A^i = Hom(A_i, \mathbb{Z})$ and compute d_i. We will need the fact below which is well-known; see [56] for a proof.

Fact 2.3. Let $G \leq F$ be free abelian groups. There exist generating sets $g_1, \ldots, g_k$ and $f_1, \ldots f_m$ $(k \leq m)$ of G and F, respectively, and integers $n_1, \ldots, n_k$ such that for each $i \leq k$, we have $g_i = n_i f_i$.

We can computably find the set of generators (a_j) of $Ker(d_i)$ and a set of generators (b_s) of $Im(d_{i-1})$ such that for each s there is an integer m and an index i such that $ma_i = b_s$; we know that such generators exist so we just search for the first found ones. It follows that the factor $H^i(N(C)) = Ker(d_i)/Im(d_{i-1})$ is strongly completely decomposable with all possible uniformity. ☐

Recall that a group admits a Σ^0_2 presentation if it is isomorphic to a factor of a Δ^0_2 group by a Σ^0_2 subgroup.

Claim 2. The direct limit $\lim_{C\in\mathcal{N}} H^i(N(C))$ admits a Σ^0_2 presentation.

Proof. Note that $0'$ can list all open ϵ-covers and decide whether two given basic open balls intersect. A refinement map between two covers $C \leq C'$ in $\mathcal{N}$ induces a simplicial map between the respective nerves $N(C)$ and $N(C')$, and this induces a homomorphism between the respective cohomology groups $H^i(N(C)) \to H^i(N(C'))$. By Claim 1, these finitely generated abelian groups are $0'$-effectively completely decomposable uniformly in C and i. Note that $Im\,\phi$ is generated in $H^i(N(C'))$ by the images of the generators of $H^i(N(C))$. Similarly to the proof of Claim 1, choose new generators of $H^i(N(C'))$ and $Im\,\phi$ so that the latter are integer multiples of the former. In particular, it is easy to see that $Im\,\phi$ is a Δ^0_2 subgroup of $H^i(N(C'))$. This means that we can augment $Im\,\phi$ with extra generators in a $0'$-computable way to expand it to $H^i(N(C'))$. It follows that $\lim_{C\in\mathcal{N}} H^i(N(C)) = H^i(G)$ can be consistently defined as the "union" of the $H^i(N(C)), C \in \mathcal{N}$, to obtain a group in which the operations are $0'$-computable and the equality is Σ^0_2. (The equality is merely Σ^0_2 since an element $a \in H^i(N(C))$ can be mapped to 0 in some $H^i(N(C''))$ which appears arbitrarily late in the directed system.) ☐

Since $\widehat{G}$ is torsion-free, to finish the proof it is sufficient to apply the result below relativized to $0'$.

Proposition 2.4 (Khisamiev [57])**.** *Every c.e.-presented torsion-free abelian group A has a computable presentation.*

It follows that, for a computably metrized connected abelian G, its torsion-free discrete dual is Δ^0_2-presentable. ☐

2.2. **Turning a Δ^0_2 discrete group into a computable Polish space.** The goal of this section is to prove the lemma below which is a partial converse of Theorem 2.2 sufficient for our goals. Fix a prime number q. We say that H is q-divisible if $q^\infty | h$ for every $h \in H$.

Lemma 2.5. *There is a uniform procedure which, given any non-zero q-divisible Δ^0_2-group H outputs a computably metrized presentation of $\widehat{H}$.*

Proof. The proof uses ideas and techniques from [18, 58], however, new tools are necessary to handle the case of an arbitrary q-divisible Δ^0_2 group.

2.2.1. *The setup.* We build $G \leq \mathbb{A} = \mathbb{T}^\omega$, where $\mathbb{T}$ is the "natural" computably metrized presentation of the unit circle group. Our goal is to make sure $G \cong \widehat{H}$. We define G to be a closed subgroup of the product; indeed, it is sufficient to specify a dense subset of it.

First, imagine that H is computable. We use the well-known result of Dobrica [59] and fix a presentation of H with a computable basis B which may or may not be finite. (We cite [58] for a modern proof of the result of Dobrica that uses the meta-theorem from [60], and we also cite [18] fora modern proof of a more general result that implies the theorem of Dobrica.) The group is generated by elements that satisfy relations of the form $ng = \sum_{b\in B} m_b b$, where almost all coefficients are zero, and we can list such relations effectively. We reserve a special copy $\mathbb{T}_g$ of $\mathbb{T}$ for every such generator g, including $\mathbb{T}_b$ for each $b \in B$. In this notation, the elements of G are sequences of the form $(\chi_h)_{h\in H}$ going though the computably metrized $\mathbb{A} = \prod_g \mathbb{T}_g$.

We declare $n\chi_g = \sum_{b\in B} m_b \chi_b$ for all $\chi \in \mathbb{T}_g \cap G$ whenever we have a relation $ng = \sum_{b\in B} m_b b$. We will proceed in this way and will end up with a computably metrized presentation G of $\widehat{H}$ [58]. The issue is, of course, that H is merely Δ^0_2.

2.2.2. *An informal outline of the basic strategy.* We fix a Δ^0_2 presentation with a Δ^0_2 basis. In fact, taking the retract of the domain under a suitable Δ^0_2 map, we can make sure that the indices of the basic elements form a computable set. So the only difference with the computable case is that the relations are no longer computable but are merely Δ^0_2, in the sense that, for a fixed g, $ng = \sum_{b\in B} m_b b$ can change to some other $n'g = \sum_{b'\in B} m'_b b$ finitely many times before it settles.

The idea is to ensure that $n\chi_g = \sum_{b\in B} m_b\chi_b$ holds up to (say) error 2^{-s} at stage s, and therefore it can be corrected later if necessary. If we decide that $n\chi_g = \sum_{b\in B} m_b\chi_b$ no longer holds for any g, we will be able to turn it into a relation of the form $q^k g = \sum_{b\in B} m_b b$ (where q is the fixed prime and k is very large).

We give more details. At every stage we will put only finitely many rational points into each circle. Suppose we initially had

$$n\chi_g = \sum_{b\in B} m_b\chi_b, \tag{†}$$

that is witnessed by finitely many intervals in the respective copies of the unit circle that approximate this relation up to error 2^{-s}, where s is the stage. If this relation will never change for g, we will keep making these intervals smaller, and will end up with a closed set for which the relation does indeed hold. The shrinking intervals will allow us to approximate a dense computable subset of the closed set similarly to how one lists a dense set for (say) the Baire space; we omit the standard details.

Now suppose the relation $m\chi_g = \sum_{b\in B} m_b\chi_b$ has to be replaced with some other relation at stage s. Note that at the stage we have only finitely many intervals approximating the relation with precision 2^{-s}. Take k so large that each of these finitely many intervals of $\mathbb{T}_g$ contains at least one point of the form $\dfrac{r}{q^k - m}$, where $r \leq (q^n - m)$. Under $x \to q^k x$, the point $\dfrac{r}{q^k - m}$ on the unit circle is mapped to

$$\frac{rq^k}{q^k - m} = \frac{r(q^k - m + m)}{q^k - m} = r + \frac{rm}{q^k - m} = \frac{rm}{q^k - m}. \tag{‡}$$

In other words, $x \to mx$ and $x \to q^k x$ agree on all points of this form. Consequently, for these points

$$q^k\chi_g = \sum_{b\in B} m_b\chi_b \tag{*}$$

will also hold "up to 2^{-s}", and this corresponds to

$$\exists g\, q^k g = \sum_{b\in B} m_b b$$

in H that vacuously holds since the group is q-divisible. We can therefore consistently *switch* the approximation of our closed set locally perhaps making further necessary adjustments.

Remark 2.6. For instance, if we are dealing with $mg = b$ and changing it to $q^k g = b$ then we need to introduce 2^{-s}-approximations to many new pre-images of b under the (adjusted) map. For example, there are only two preimages under $x \to 2x$, but there are four under $x \to 4x$. Such adjustments make the closed set that we build not effectively closed in general.

Then we introduce a fresh circle corresponding to the relation $ng = \sum_{b\in B} m_b b$, and repeat.

To make our proof a bit more transparent, we will actually split $m\chi_g = \sum_{b\in B} m_b\chi_b$ into $\chi_R = \sum_{b\in B} m_b\chi_b$ and $m\chi_g = \chi_R$ and work only with $m\chi_g = \chi_R$. This of course is merely a notational convenience.

2.2.3. *Putting the strategies together.* At this stage the reader hopefully sees how to organise the construction using, say, movable markers. Note there is essentially no interaction between strategies working with different generators. There are only two further subtleties that needs to be addressed.

Issue 1 Among other things, we need to make sure that the group is isomorphic to the dual of H. In particular, one potential difficulty is that at some stage the (approximations to) relations imply that the group is torsion; for example, we could potentially have that $mx = b$ and $nx = b$ for $m \neq n$. These potentially bad cases can be easily excluded. Indeed, all groups will be finitely generated abelian, where all such properties are easily decidable (see, e.g., the analysis of f.g abelian groups contained in the previous subsection). We therefore do not believe a relation unless, together with the previous relations, it gives us a finitely generated torsion-free group.

Issue 2 The relations of q^n-divisibility play a special role in the construction; in particular, we must make sure that all elements are q^∞-divisible, but we also have to avoid problems similar to Issue 1 restricted to relations of the form $q^n x = \sum_b m_b b$. This can be done as follows. In the construction, we implement the basic strategies only for generators that have their relations of the form $ng = \sum_{b \in B} m_b b$, where n and q are co-prime. Meanwhile, we manually introduce more and more new circle-components for relations of the form $q^k g = b$, for every b, to make sure that no such relations are missing.

2.2.4. *Formal proof.* By the Dobrica's result [59], we can assume that H has a Δ^0_2 basis. We can view H as a Σ^0_2 subgroup of $V^{|B|} = \bigoplus_{i \leq card(B)} Q$ so that the basis B is equal to the standard basis of $V^{|B|}$. In particular, we can indeed assume that B is a computable set. To list H as a subgroup of $V^{|B|}$, it is sufficient to list elements $g \in H$ that satisfy relations of the form

$$p^v g = \sum_{b \in B} m_b b,$$

where p ranges over primes and v over positive integers. Note that if such a g exists then it is unique. We computably guess whether the relation holds in the group in the spirit of the limit lemma. The relations holds in the group iff

$$\frac{\sum_{b \in B} m_b b}{p^v} \in H.$$

Since H is a Σ^0_2 subgroup in general, the relation holds iff we eventually never see it to fail. However, if the relation fails we can have infinitely many stages at which we believe that the relation might hold.

We first give a "macro-construction" that manipulates with copies of the unit circle and rules of the form $n\chi_i = \sum_{b \in B} m_b \chi_b$. Then we give the full construction that turns the macro-construction into a construction of a computable Polish group G isomorphic to $\widehat{H}$.

2.2.5. *The macro-construction.* For every $b \in B$, reserve a copy $\mathbb{T}_b$ of the unit circle group. At stage s of the macro-construction, introduce one more $\mathbb{T}_b$ and monitor one more relation of the form $p^v g = \sum_{b \in B} m_b b$. For every such relation R that had not previously been considered, do the following:

(a.1) Introduce a copy of the unit circle group $\mathbb{T}_R$ and declare

$$\chi_R = \sum_{b \in B} m_b \chi_b.$$

(a.2) Introduce a new copy of the unit circle group $\mathbb{T}_{R,0}$ and declare it active.
(a.3) Declare $\chi_R = p^v \chi_{R,0}$.

For every relation R' that has already been considered, check if R' still holds according to the Σ^0_2-approximation that we fixed above.

If it does not hold, then let u be largest such that $\mathbb{T}_{R',u}$ is currently active, and so that $\chi_{R'} = p^w \chi_{R',u}$ was declared at the stage when $\mathbb{T}_{R',u}$ was introduced.

(b.1) Declare $\mathbb{T}_{R',u}$ inactive and declare $\chi_{R'} = p^v \chi_{R',u}$ *dismissed.*
(b.2) Choose k very large (to be clarified) and declare $\chi_{R'} = q^k \chi_{R',u}$.
(b.3) Introduce a new copy $\mathbb{T}_{R',u+1}$ of the unit circle group and declare it active.
(b.4) Declare $\chi_{R'} = p^v \chi_{R',u+1}$.

2.2.6. *The definition of G.* The macro-construction builds a closed subgroup of $\mathbb{T}^\omega$. We now explain how the macro-construction can be computably approximated, and then we explain how this approximation allows us to computably list a dense subset of G.

Definition 2.7. For a finite equation of the form $ng = \sum_{b \in B} m_b b$, its ϵ-name is a finite collection of open basic intervals $U^i_g \in \mathbb{T}_g$ and $U^j_b \in \mathbb{T}_b$ of diameter ϵ with the following properties:

(1) U^j_b cover $\mathbb{T}_b$;
(2) U^i_g cover $\mathbb{T}_g$;

(3) If $nx = \sum_{b\in B} m_b y_b$ holds for $x \in \mathbb{T}_g$ and $y_b \in \mathbb{T}_b$ then for some intervals $U_g^i \ni x$ and $U_b^{j_b} \ni y_b$,

$$nU_g^i = \sum_{b\in B} m_b U_b^{j_b}$$

Such an approximation is possible due to the especially nice properties of the usual operations and the geometry of the unit circle. (We do not really need our approximations to be that nice though.)

To define G, we monitor the macro-construction. At every stage s, we do the following:

(1) Enumerate more points into each copy of the unit circle that is currently introduced by the macro-construction.
(2) For every declared relation which have not yet been dismissed, with the exception of those declared in $(b.4)$ of the current stage, refine its 2^{-s+1}-name to a 2^{-s} name.
(3) For every relation of the form $\chi_R = q^k \chi_{R,u}$ that has been introduced at a (b.2)-type substage of the current stage for some R, replace the 2^{-s+1}-name of the dismissed $\chi_R = p^v \chi_{R,u}$ with a 2^{-s}-name of $\chi_R = q^k \chi_{R,u}$.

In particular, in (b.2) we choose k so large that substep (3) can be performed according to (‡) and the discussion preceding (‡). (Note that there is no circularity here.)

2.2.7. *The verification.* We first explain how to list a dense subset of G. At every stage we list only finite tuples of special points in the constructed copy of $\mathbb{T}^\omega$ that satisfy the currently declared 2^{-s}-names rules up to 2^{-s}. During the construction, we will keep refining and extending each such finite tuple, so that the result will converge uniformly to an infinite tuple of points going through the whole of $\mathbb{T}^\omega$. This is done arbitrarily, say, by choosing the smallest available index among the special points that obey the current 2^{-s}-names.

Remark 2.8. One can argue that the closed set defined in the construction is effectively overt, that is, the set of basic open balls intersecting the set is c.e. It is well-known that such closed sets admit a computable dense sequence (e.g., [61]). We provide a detailed (but informal) explanation that avoids the use of effective overtness.

Suppose at stage s the rule is $p\chi_1 = \chi_0$. Note that there are p pre-images of $x \in \mathbb{T}_0$ in $\mathbb{T}_1$ under $x \to px$. We put finitely many rational points $r^s_{0,1}, \ldots, r^s_{0,s}$ in $\mathbb{T}_0$ and also use the 2^{-s}-name of $p\chi_1 = \chi_0$ to find for each i all points $r^s_{1,j}$ such that $pr^s_{1,j} = r^s_{0,i}$. (We actually do not really need the 2^{-s} name as this can be done directly.)

If there is a rule relating $\mathbb{T}_2$ to either $\mathbb{T}_0$ or $\mathbb{T}_1$, then there will will also be at most finitely many points $r^s_{2,k}$ suitable for the third coordinate in

$$(r^s_{0,i}, r^s_{1,j}, r^s_{2,k}, \ldots).$$

If there is no such rule, then at the stage there will be finitely many points in $\mathbb{T}_2$ so we can pick the third coordinate arbitrarily using any of these points. According to the standard metric on $\mathbb{T}^\omega$, the circles get "smaller" as i gets larger. So we continues at this manner until we get to an index u such that the diameter of $\prod_{v>u} \mathbb{T}_v$ is smaller than 2^{-s}. Then $(r^s_{0,i}, r^s_{1,j}, r^s_{2,k}, \ldots, r^s_{u,d}, 0, 0, \ldots)$ can be declared 2^{-s}-approximation of a special point in G. Note that at stage s there are only finitely many such approximations.

At the next stage, we need to refine it to $(r^{s+1}_{0,i'}, r^{s+1}_{1,j'}, r^{s+1}_{2,k'}, \ldots, r^{s+1}_{u,d'}, 0, 0, \ldots)$. For that, we will choose the (lexicographically) smallest-index finite tuple that is within 2^{-s} of the previous tuple and furthermore satisfies the (perhaps, updated) rules with a better precision.

It is crucial that the rules are updated so that (say) $r^s_{1,j}$, can be replaced with $r^{s+1}_{1,j'}$ that is arbitrarily close to $r^s_{1,j}$ in $\mathbb{T}_1$. Also, as we already noted above once, a new rule implies that there are more preimages that need to be listed. For instance, if $pr^s_{1,j} = r^s_{0,i}$ needs to be replaced with $q^k r^{s+1}_{1,j} = r^{s+1}_{0,i}$, where k is very large, then there will be $q^k > p$ points that satisfy the new equation for $r^{s+1}_{0,i}$. Only p-many of them will be used as better approximations of the previously introduced (procedures approximating) points. The new ones will have to be listed into new tuples that will be (approximations to) new special points. As we have already mentioned, this is where we might lose effective compactness in the definition of G.

We keep making better and better approximations using the smallest available index principle every time we have a choice, and this way we will build a dense subset of G.

It should be clear that G is equal to the completion of the uniformly computable list of points that we build according to the procedure described above. To finish the verification, note that G is topologically isomorphic to the character group of H, and therefore is a computable presentation of $\widehat{H}$ under the operations inherited from $\mathbb{A} = \mathbb{T}^\omega$. □

3. Completely decomposable groups and solenoid spaces

In this section we explain in detail the special case of Corollary 0.2 when $\beta = 2$. Thus, our goal is to construct a compact space with non-low$_2$ degree spectrum.

Assuming that the reader is willing to see what the isomorphism type of the space actually is, we give details below. (If the reader does not need to see this, we just say that there is a completely decomposable group with exactly the non-low copies [46]. It is easy to see that it can be made 2-divisible, and the jump inversion from groups to *spaces* (in view of Theorem 2.1) makes it non-low$_2$, as usual.)

A set W is the enumeration of a family U if $\{W^{[e]} : e \in \omega\} = U$, where $W^{[e]} = \{x : \langle x, e\rangle \in W\}$. Note that repetitions do not matter here. We also say that U admits a Σ^X_{n+1} enumeration if there is a set W with the above property such that which is c.e. relative to $X^{(n)}$. Recall that a group is completely decomposable if it is isomorphic to a direct sum of additive subgroups of the rationals. The following proposition is a variation of a result from [46].

Proposition 3.1. Let U be an infinite family of finite sets that contains the empty set. There exists a completely decomposable 2-divisible group $G(U)$ such that the following conditions are equivalent.

(1) $G(U)$ has an X-computable presentation.
(2) U admits a Σ^X_2-enumeration.

Proof sketch. Let $(p_i)_{i\in\omega}$ be the standard enumerations of all primes; in particular, $p_0 = 2$. Let A_S be the subgroup of $\mathbb{Q}$ that contains 1 and such that $p_{i+1} \not| 1$ if $D_i \subseteq S$ and $p^\infty_{i+1}|1$ otherwise. Also, declare $2^\infty|1$. Define

$$G_U = \bigoplus_{i\in\omega}\left(\bigoplus_{S\in U} A_S\right).$$

It is not hard to show that G_U has the desired property. □

A space is a *solenoid space* if it is equal to the inverse limit of the unit circles under multiplicative maps of the form $x \to mx$, where m is a positive integer:

$$\mathbb{T} \leftarrow_{m_0} \mathbb{T} \leftarrow_{m_1} \mathbb{T} \leftarrow_{m_2} \ldots,$$

where $(m_i)_{i\in\omega}$ is allowed to have repetitions. It is not hard to see that such as space is homeomorphic to the domain of the dual of the group $\langle\{\frac{1}{\prod_{j\leq i} m_i} : i \in \omega\}\rangle \leq (Q, +)$. Conversely, the dual of every subgroup of the rationals is a solenoid group. Since $\widehat{\bigoplus_{i\in\omega} H_i} \cong \prod_{i\in\omega} \widehat{H_i}$, completely decomposable groups correspond to direct products of solenoid groups under duality.

It follows from Theorem 0.1 that, for $H(U) = \widehat{G(U)}$, U admits a Σ^0_3-enumeration iff $H(U)$ has a computable metrization. Relativise the result of Wehner [32] to $0''$ to obtain a family U of finite sets which is has no Σ^0_3-enumeration but is Σ^X_3-enumerable for any X such that $X'' > 0''$. Then $H(U)$ admits a Y-computable metrization iff $\Sigma^Y_3 \neq \Sigma^0_3$ iff $Y^{(3)} \not\leq_1 0^{(3)}$ iff $Y^{(2)} \not\leq_T 0^{(2)}$ which is the same as to say that Y is not low$_2$.

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School of Mathematics and Statistics, Victoria University of Wellington, New Zealand and Sobolev Institute of Mathematics, Novosibirsk, Russia

Email address: alexander.g.melnikov@gmail.com