Ulm type, and coding rank-homogeneous trees in other structures

E. Fokina, J. F. Knight, C. Maher, A. Melnikov, and S. M. Quinn*

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Abstract

The first main result isolates some conditions which fail for the class of graphs and hold for the class of Abelian p-groups, the class of Abelian torsion groups, and the special class of "rank-homogeneous" trees. We consider these conditions as a possible definition of what it means for a class of structures to have "Ulm type". The result says that there can be no Turing computable embedding of a class not of Ulm type into one of Ulm type. We apply this result to show that there is no Turing computable embedding of the class of graphs into the class of "rank-homogeneous" trees. The second main result says that there is a Turing computable embedding of the class of rank-homogeneous trees into the class of torsion-free Abelian groups. The third main result says that there is a "rank-preserving" Turing computable embedding of the class of rank-homogeneous trees into the class of Boolean algebras. Using this result, we show that there is a computable Boolean algebra of Scott rank ω_1^{CK}.

1 Introduction

There are many known transformations between classes of structures, used in different ways. Mal'cev [19] considered the transformation taking rings to their Heisenberg groups. He showed that there is a copy of the input ring, defined with parameters, in the output group. Mal'cev used this idea to obtain, from the ring of integers, a group whose elementary first order theory is hereditarily undecidable. Hirschfeldt, Khoussainov, Shore, and Slinko [13] used the Mal'tsev transformation for results on computable dimension.

There is quite a lot of work comparing classes of countable structures and saying, in various concrete ways, that the "classification problem" for one class is more difficult than that for the other class. We think of the classification

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problem as the problem of describing the members of the class, up to isomorphism. There are different approaches. One involves cardinality. If the class K has only countably isomorphism types of countable structures, while the class K' has uncountably many, then the classification problem for K' is clearly more difficult. There are other approaches which allow finer distinctions. Friedman and Stanley [9], introduced an approach involving "Borel embeddings" and "Borel cardinality". Borel embeddability gives a pre-ordering $\leq_B$ on classes of structures. Friedman and Stanley located various familiar classes on top under $\leq_B$—graphs, fields of any desired characteristic, groups, trees, linear orderings. They located other familiar classes below the top under $\leq_B$—fields of finite transcendence degree, Abelian p-groups, Abelian torsion groups. Camerlo and Gao [6] showed that the class of Boolean algebras lies on top under $\leq_B$. Friedman and Stanley left open the question of whether the class of torsion-free Abelian groups lies on top. This has stimulated quite a lot of work, by Hjorth [12] and others.

In [3], there is an effective version of the Friedman and Stanley approach, involving "Turing computable embeddings" and "effective cardinality". Some of the known Borel embeddings are in fact computable. Some of the embeddings also turn out to preserve more. "Scott rank" is a measure of internal model theoretic complexity. Computable structures have Scott rank at most $\omega_1^{CK}+1$. Examples of computable structures with various computable ranks, and of rank $\omega_1^{CK}+1$ have been known for some time. In [5], there is an example of a computable tree with Scott rank ω_1^{CK}. In [4], further examples (an undirected graph, a field, a linear ordering) are obtained from the tree, using "rank preserving" Turing computable embeddings.

In the remainder of the introduction, we discuss Borel cardinality and effective cardinality. We also describe the class of rank-homogeneous trees. In Section 2, we give some background from infinitary logic, and we also say a little about Scott rank. In Section 3, we propose a definition of "Ulm type", and we prove our general result saying that there is no Turing computable embedding of a class not of Ulm type into a class of Ulm type. We show that the class of rank-homogeneous trees has Ulm type. Borrowing ideas from Friedman and Stanley, we show that the class of graphs is not of Ulm type. Hence, there is no Turing computable embedding of the class of graphs into the class of rank-homogeneous trees.

In Section 4, we give a Turing computable embedding of the class of rank-homogeneous trees into the class of torsion-free Abelian groups. We use a transformation defined by Hjorth [12] and also used by Downey and Montalbán. It is not clear that the transformation on the full class of trees is $1-1$ on isomorphism types, but we can show this for the class of rank-homogeneous trees. In Section 5, we give a Turing computable embedding of the class of rank-homogeneous trees into the class of Boolean algebras, and we show that this embedding has a property of "rank preservation". As a corollary, we obtain a computable Boolean algebra of Scott rank ω_1^{CK}—adding one more to the list of familiar classes known to contain such structures.

1.1 Borel cardinality

We are ready to describe in a precise way the approach of Friedman and Stanley [9]. Let L be a countable language, and let $Mod(L)$ be the class of all L-structures with universe ω. There is a natural topology on $Mod(L)$, with basic open neighborhoods of the form $Mod(\sigma)$, where σ is a finitary quantifier-free $(L \cup \omega)$-sentence. Closing under countable unions and complements, we obtain Borel subsets of $Mod(L)$. Using the product topology and again closing under countable unions and complements, we obtain Borel subsets of $Mod(L) \times Mod(L')$. Friedman and Stanley [9] considered classes K such that for some L, $K \subseteq Mod(L)$, and K is closed under isomorphism. In this subsection, all of our classes are assumed to satisfy these conventions.

Definition 1 (Friedman-Stanley)**.** *A* Borel embedding *of K into K' is a Borel transformation $\Phi : K \to K'$ such that for $\mathcal{A}, \mathcal{A}' \in K$,*

$$\mathcal{A} \cong \mathcal{A}' \text{ iff } \Phi(\mathcal{A}) \cong \Phi(\mathcal{A}') \ .$$

We write $K \leq_B K'$ if there is such a transformation. We write $K \equiv_B K'$ if $K \leq_B K'$ and $K' \leq_B K$. We write $K <_B K'$ if $K \leq_B K'$ and not $K' \leq_B K$.

The relation $\leq_B$ is a pre-ordering on classes. Two classes have the same *Borel cardinality* if they are $\equiv_B$-equivalent. If we have a Borel embedding Φ of K into K', then we can describe a member of K by finding its Φ-image and describing *that.* All classes with $\aleph_0$ isomorphism types are $\equiv_B$-equivalent. Among classes with $2^{\aleph_0}$ isomorphism types, the notion of Borel cardinality makes some important distinctions.

There are well-known transformations showing that the class UG of undirected graphs lies "on top" under the relation $\leq_B$ (see [20] or [22]). Friedman and Stanley gave transformations showing that the class of trees, the class of linear orderings and the class of fields of any fixed characteristic are also on top. The class of 2-step nilpotent groups is on top—one way to see this uses the Mal'cev transformation. Camerlo and Gao [6] showed that the class BA of Boolean algebras lies on top. In fact, for any completion T of the theory of Boolean algebras, the class $Mod(T)$ lies on top.

Friedman and Stanley showed that classes for which the isomorphism relation is Borel lie strictly below the top. This is true, in particular, for the class of fields of finite transcendence degree, and for the class of equivalence structures. They also showed that the class of Abelian p-groups, and the broader class of Abelian torsion groups, do not lie on top under the pre-ordering $\leq_B$. The reason is different. The idea behind the argument that Friedman and Stanley give is that we may code a family of sets in a graph, while for Abelian p-groups, and Abelian torsion groups, the invariants are essentially sets of countable ordinals.

Friedman and Stanley asked whether the class TFA of torsion-free Abelian groups lies on top. This problem remains open. Hjorth [14] showed that the isomorphism relation on TFA is not Borel. Downey and Montalbán [8] showed that the isomorphism relation is complete analytic. A torsion-free Abelian group is essentially a subgroup of a $\mathbb{Q}$-vector space, and the *rank* is the least dimension of a vector space in which the group can be embedded. Let TFA_n be the class of torsion-free Abelian groups of rank n. With great effort, Hjorth and Thomas [15], [24] showed that for all n, $TFA_n <_B TFA_{n+1}$.

1.2 Effective cardinality

In [3], there is an effective analogue of Borel embedding. We modify the conventions slightly. Languages are computable, not just countable. The universe of each structure is a subset of ω, not necessarily all of ω. As above, a class K consists of structures for a fixed language, and it is closed under isomorphism (modulo the restriction on the universe).

Definition 2.

1. *A* Turing computable transformation *from K to K' is a computable operator $\Phi = \varphi_e$ such that for each $\mathcal{A} \in K$, there exists $\mathcal{B} \in K'$ with $\varphi_e^{D(\mathcal{A})} = \chi_{D(\mathcal{B})}$. We write $\Phi(\mathcal{A})$ for $\mathcal{B}$.*
2. *A* Turing computable embedding *of K into K' is a Turing computable transformation Φ such that for $\mathcal{A}, \mathcal{A}' \in K$, $\mathcal{A} \cong \mathcal{A}'$ iff $\Phi(\mathcal{A}) \cong \Phi(\mathcal{A}')$. We write $K \leq_{tc} K'$ if there is such a Φ. We write $K \equiv_{tc} K'$ if $K \leq_{tc} K'$ and $K' \leq_{tc} K$. We write $K <_{tc} K'$ if $K \leq_{tc} K'$ and not $K' \leq_{tc} K$.*

Again we have a pre-ordering on classes. Two classes have the same effective cardinality if they are $\equiv_{tc}$-equivalent. Many Borel embeddings are actually Turing computable. However, in some cases, we make finer distinctions using Turing computable embeddings. For example, if NF is the class of number fields and FVS is the class of $\mathbb{Q}$-vector spaces of finite dimension, then $NF \equiv_B FVS$, but $NF <_{tc} FVS$ [11]. For a completion T of PA, the class $Mod(T)$ of models of T lies on top under $\leq_B$, but not under $\leq_{tc}$ [7].

As in the Borel setting, if $K \leq_{tc} K'$ via Φ, then we can describe $\mathcal{A} \in K$ by computing $\Phi(\mathcal{A})$ and describing *that*. It is more satisfying to describe $\mathcal{A}$ in its own language. The following result is from [17].

Theorem 1.1 (Pullback Theorem)**.** *Suppose $K \leq_{tc} K'$ via Φ. Then for any computable infinitary sentence φ in the language of K', we can effectively find a computable infinitary sentence φ^* in the language of K such that for $\mathcal{A} \in K$,*

$$\mathcal{A} \models \varphi^* \text{ iff } \Phi(\mathcal{A}) \models \varphi$$

Moreover, if φ is computable Σ_α, or computable Π_α, then so is φ^*.

1.3 Rank-homogeneous trees

We begin by recalling some definitions and basic facts about rank-homogeneous trees (from [5]).

Definition 3 (tree rank)**.** *Let T be a subtree of $\omega^{<\omega}$. We define the* tree rank *of $x \in T$, denoted by $tr(x)$, by induction.*

1. *$tr(x) = 0$ if x has no successor,*

2. *for $\alpha > 0$, $tr(x) = \alpha$ if α is the least ordinal greater than $tr(y)$ for all successors y of x,*

3. *$tr(x) = \infty$ if x does not have ordinal tree rank.*

Tree rank is sometimes called *foundation rank.* Note that $tr(x) = \infty$ if and only if x extends to a path.

Definition 4 (rank-homogeneous tree)**.** *A tree $T \subseteq \omega^{<\omega}$ is* rank-homogeneous *provided that for all x at level n,*

1. *if $tr(x)$ is an ordinal, then for all y at level $n+1$ such that $tr(y) < tr(x)$, x has infinitely many successors z such that $tr(z) = tr(y)$,*

2. *if $tr(x) = \infty$, then for all y at level $n+1$, x has infinitely many successors z such that $tr(z) = tr(y)$.*

For a rank-homogeneous tree T, let $R(T)$ be the set of pairs (n, α) such that there is an element at level n of tree rank α (where α is an ordinal, not ∞). Note that the top node in T has rank ∞ just in case $R(T)$ has no pair of the form $(0, \alpha)$. Also note if T has a node of rank ∞, then the top node must have rank ∞, and if the top node has rank ∞, then there are nodes of rank ∞ at all levels. Thus, from the set of pairs $R(T)$ in which the second components are ordinals, we can deduce all of the information that would be given if we included pairs with second component ∞.

Proposition 1.2. *Suppose T, T' are rank-homogeneous trees. Then $T \cong T'$ iff $R(T) = R(T')$.*

Proof. Clearly, if $T \cong T'$, then $R(T) = R(T')$. Suppose $R(T) = R(T')$. To see that there is an isomorphism, we show that the set of finite partial rank-preserving isomorphisms between subtrees of T and T' has the back-and-forth property. The subtrees must be closed under predecessor in the large trees, and the finite partial isomorphisms must preserve all ranks, both ordinals and ∞. Given a finite subtree of one of the large trees, we can reach any further node by a finite sequence of steps in which the node being added is a successor of one already included. Therefore, it is enough to prove the following.

Claim: Let p be a rank-preserving isomorphism from the finite subtree τ of T onto the finite subtree τ' of T', and let $a \in T - \tau$ be a successor of $b \in \tau$. Suppose $b' = p(b)$. Then there exists a', a successor of b' in T', not already in $ran(p)$, such that a' and a have the same rank.

The rank of $p(b)$ is the same as that of b. If a has rank ∞, then b and b' also have rank ∞, and b' has infinitely many successors of rank ∞. If a has ordinal rank α, then b and b' have rank either ∞ or some $\beta > \alpha$. In either case, b' has infinitely many successors of rank α. We choose a' to be a successor of b', of the proper rank, not already in $ran(p)$.

□

2 Background on infinitary logic

In this section, we give some background on infinitary logic and Scott rank.

2.1 Infinitary logic

For a countable language L, the $L_{\omega_1\omega}$ formulas allow countable conjunctions and disjunctions. We consider countable fragments, so that we can apply a version of the Henkin method to construct models. While the usual Compactness Theorem fails for $L_{\omega_1\omega}$, Barwise [2] proved a limited version of Compactness for admissible fragments. Ressayre [23] considered saturation properties related to Barwise Compactness. We state special cases of these results, involving "computable" infinitary sentences. For more about computable infinitary formulas, see [1].

The *computable infinitary formulas* are formulas of $L_{\omega_1\omega}$ in which the infinite disjunctions and conjunctions are over c.e. sets. The computable infinitary formulas are essentially the formulas in the least admissible fragment. We consider formulas in "normal" form, with the negations brought inside. These are classified as computable Σ_α or computable Π_α, where α is a computable ordinal. For any computable infinitary formula φ, we have a formula $neg(\varphi)$ such that $neg(\varphi)$ is logically equivalent to the negation of φ.

Definition 5.

1. *if φ is finitary quantifier-free, then $neg(\varphi) = \neg\varphi$,*
2. *if φ is computable Σ_α, of the form $\bigvee\!\!\!\!\bigvee_i (\exists\overline{u}_i)\,\psi_i(\overline{u}_i)$, then $neg(\varphi)$ is computable Π_α, of the form $\bigwedge\!\!\!\!\bigwedge_i (\forall\overline{u}_i)\,neg(\psi_i(\overline{u}_i))$,*
3. *if φ is computable Π_α, of the form $\bigwedge\!\!\!\!\bigwedge_i (\forall\overline{u}_i)\,\psi_i(\overline{u}_i)$, then $neg(\varphi)$ is computable Σ_α, of the form $\bigvee\!\!\!\!\bigvee_i (\exists\overline{u}_i)\,neg(\psi_i(\overline{u}_i))$.*

Note: If φ is computable Σ_α, then $neg(\varphi)$ is computable Π_α and vice versa.

Lopez-Escobar showed that for a class $K \subseteq Mod(L)$ (closed under isomorphism), K is Borel if and only if it is axiomatized by a sentence of $L_{\omega_1\omega}$. Vaught showed that K is $\mathbf{\Sigma}^0_\alpha$ in the Borel hierarchy if and only if it is axiomatized by a Σ_α sentence. Vanden Boom [25] showed that K is Σ^0_α in the effective Borel hierarchy if and only if it is axiomatized by a computable Σ_α sentence.

Theorem 2.1 (Barwise-Kreisel Compactness). *If Γ is a Π^1_1 set of computable infinitary sentences, and every Δ^1_1 set $\Gamma' \subseteq \Gamma$ has a model, then Γ has a model.*

Ressayre [23] showed the following.

Theorem 2.2 (Ressayre). *Suppose Γ is a Π^1_1 set of computable infinitary sentences, where every Δ^1_1 subset has a model. Then Γ has a model $\mathcal{A}$ such that $\omega_1^{\mathcal{A}} = \omega_1^{CK}$.*

Remark: The condition $\omega_1^{\mathcal{A}} = \omega_1^{CK}$ holds iff $\mathcal{A}$ is contained in a fattening of the least admissible set.

Theorem 2.2 is a special case of the following result on expansions [23].

Theorem 2.3 (Ressayre)**.** *Suppose $\omega_1^{\mathcal{A}} = \omega_1^{CK}$. Let Γ be a Π_1^1 set of computable infinitary sentences involving symbols from the language of $\mathcal{A}$, possibly a finite tuple of parameters from $\mathcal{A}$, and finitely many further relation or constant symbols. Suppose that for every Δ_1^1 set $\Gamma' \subseteq \Gamma$, there is an expansion of $\mathcal{A}$ satisfying Γ'. Then $\mathcal{A}$ has an expansion $\mathcal{A}'$ such that $\omega_1^{\mathcal{A}'} = \omega_1^{CK}$ and $\mathcal{A}'$ satisfies all of Γ.*

2.2 Scott rank and rank preservation

Scott showed that for any countable structure $\mathcal{A}$ for a countable language L, there is an $L_{\omega_1\omega}$- sentence whose countable models are just the isomorphic copies of $\mathcal{A}$. In the proof, Scott assigned ordinals to tuples in the structure, and to the structure itself. Scott rank is a measure of model theoretic complexity of countable structures. In general, countable structures may have arbitrarily large countable ordinal ranks. Nadel [21] showed that if $\mathcal{A}$ is a computable, or hyperarithmetical, structure, then $SR(\mathcal{A}) \leq \omega_1^{CK}+1$. More generally, $SR(\mathcal{A}) \leq \omega_1^{\mathcal{A}} + 1$, where $\omega_1^{\mathcal{A}}$ is the first ordinal not computable relative to $\mathcal{A}$.

There are several different ways to define Scott rank. Different definitions of Scott rank may assign different ordinal ranks to some structures. However, the definitions currently in use all agree on which computable structures are assigned Scott rank ω_1^{CK} and which are assigned Scott rank $\omega_1^{CK} + 1$. We are primarily interested in Scott ranks for computable structures. Instead of choosing a particular definition of Scott rank, we give, in Proposition 2.4 below, conditions which, for computable, or hyperarithmetical, structures, may be used as a definition of these special Scott ranks.

There are familiar examples of computable structures having various computable Scott ranks. The Harrison ordering has Scott rank $\omega_1^{CK} + 1$ [12]. It took longer to find examples of computable structures of Scott rank ω_1^{CK}. For some classes, such as Abelian p-groups, rank ω_1^{CK} does not occur.

Makkai [18] produced an arithmetical structure of Scott rank ω_1^{CK}. In [16], Makkai's example is made computable. In [5], it is shown that there is a computable tree T of Scott rank ω_1^{CK}. This tree has a special property of "rank-homogeneity" (defined in the next subsection). In [4], there are further examples of computable structures of Scott rank ω_1^{CK} in the following classes: graphs, linear orderings, and fields of any desired characteristic. These examples are obtained from the tree by applying familiar effective transformations which have been shown to "preserve" Scott rank in the sense that either the input and output structures have the same Scott rank, or else both Scott ranks are computable relative to the input structure.

Proposition 2.4 (Scott ranks for computable structures)**.** *If $\mathcal{A}$ is a computable or hyperarithmetical structure, then*

1. *$SR(\mathcal{A}) < \omega_1^{CK}$ if there is some $\alpha < \omega_1^{CK}$ such that the orbits of all tuples are all Δ^0_α,*

2. *$SR(\mathcal{A}) = \omega_1^{CK}$ if the orbits of all tuples are hyperarithmetical, but there is no computable ordinal as in 1,*

3. *$SR(\mathcal{A}) = \omega_1^{CK} + 1$ if there is some tuple whose orbit is not hyperarithmetical.*

This result is given in [16]. Related results are given in [1] and [5]. It is not difficult to show that for a hyperarithmetical structure $\mathcal{A}$, $SR(\mathcal{A}) < \omega_1^{CK}$ if the orbits of all tuples are definable by computable infinitary formulas of bounded complexity, $SR(\mathcal{A}) = \omega_1^{CK}$ if the orbits are all definable by computable infinitary formulas, but there is no bound on the complexity, and $SR(\mathcal{A}) = \omega_1^{CK}+1$ if there is a tuple whose orbit is not definable by any computable infinitary formula. In [10], it is shown that for a structure $\mathcal{A}$ that is computable, or hyperarithmetical, the orbit of a tuple is hyperarithmetical if and only if it is definable in $\mathcal{A}$ by a computable infinitary formula.

2.3 Scott ranks of rank-homogeneous trees

Proposition 2.5. *Let T be a rank-homogeneous tree. If $\omega_1^T = \omega_1^{CK}$, then the ordinals that occur as tree ranks are all computable.*

Proof of Proposition 2.5. Supposing that $tr(x) \geq \alpha$ for all computable ordinals α, we can show that $tr(x) = \infty$. It is enough to show that any node with tree rank $\geq \alpha$ for all computable ordinals α has a successor with the same feature. We use Theorem 2.3. Let a be a node at level n such that $tr(a) \geq \alpha$ for all computable ordinals α. Let e be a new constant, and let $\Gamma(a, e)$ say that e is at level $n + 1$, e is a successor of a, and $tr(e) \geq \alpha$ for all computable ordinals α. For any Δ^1_1 subset Γ' of $\Gamma(a, e)$, there is a computable bound α^* on the ordinals mentioned. We satisfy Γ' by letting e be a successor of a such that $tr(a) \geq \alpha^*$. $\square$

While T may live in a proper fattening of the least admissible set, computable infinitary formulas suffice to describe the tree ranks of the elements of T. More generally, if $\omega_1^T = \alpha$, then the ordinals that occur as tree ranks are all less than α. While T itself may not be in the admissible set L_α, the natural formulas saying that $tr(x) = \beta$, for $\beta < \alpha$ are in L_α.

Proposition 2.6. *Let T be a rank-homogeneous tree such that $\omega_1^T = \omega_1^{CK}$. Then*

1. *$SR(T) < \omega_1^{CK}$ if there is a computable bound α on the ordinals that occur as tree ranks,*

2. *$SR(T) = \omega_1^{CK}$ if for each n, there is a computable bound α_n on the ordinals that occur as tree ranks of nodes at level n, but there is no bound as in 1,*

3. $SR(T) = \omega_1^{CK}$ *if there is some* n *such that there is no computable bound on the ordinals that occur as tree ranks at level* n.

In a rank-homogeneous tree T, the orbit of a tuple is determined by the tree ranks of the elements in the finite subtree generated by the tuple, under predecessor. If, for each of the finitely many levels represented, there is a computable bound on the ordinal ranks, then we have a computable infinitary formula defining the orbit (see [5]).

3 First main result

Let UG be the class of undirected graphs. Friedman and Stanley [9] showed that there is no Borel embedding of UG in the class of Abelian p-groups, or the class of Abelian torsion groups. If there were a Turing computable embedding taking graphs with universe ω to Abelian p-groups, or Abelian torsion groups, with universe ω, then this would be a Borel embedding. So, the result of Friedman and Stanley implies that there is no Turing computable embedding of this kind. Below, we prove a general result that yields, as corollaries, the fact that there is no Turing computable embedding of UG into these classes, or into the class RHT of rank-homogeneous trees.

Theorem 3.1. *Let* K, K' *be classes of structures, closed under isomorphism. Suppose* K *contains a pair of structures* $\mathcal{A}, \mathcal{A}'$ *such that* $\omega_1^{\mathcal{A}} = \omega_1^{\mathcal{A}'} = \omega_1^{CK}$*,* $\mathcal{A}, \mathcal{A}'$ *satisfy the same computable infinitary sentences, and* $\mathcal{A} \not\cong \mathcal{A}'$*, while* K' *contains no such pair. Then* $K \not\leq_{tc} K'$.

Before giving the proof of Theorem 3.1, we say something about the significance of the statement. The condition on the class K' is, it seems to us, a possible definition of what it means for a class to have "Ulm type".

Definition 6. *A class has* Ulm type *provided that for any structures* $\mathcal{A}, \mathcal{A}'$ *in the class, if there are no non-computable ordinals computable relative to* $\mathcal{A}$ *or* $\mathcal{A}'$*, and* $\mathcal{A}$ *and* $\mathcal{A}'$ *satisfy the same computable infinitary sentences, then they are isomorphic.*

Equivalently, we may say that a class has *Ulm type* provided that for any $\mathcal{A}$ in the class such that $\mathcal{A}$ lives in a fattening of the least admissible set, the computable infinitary sentences true of $\mathcal{A}$ are enough to distinguish it from other members of the class that live in fattenings of the least admissible set. We note that the computable infinitary sentences live in the least admissible set.

Theorem 3.1 says that if K does not have Ulm type, while K' does have Ulm type, then there is no Turing computable embedding of K into K'.

Proof of Theorem 3.1. We shall use the Pull-back Theorem (Theorem 1.1). Suppose $K \leq_{tc} K'$ via Φ. Let $\Phi(\mathcal{A}) = \mathcal{B}$ and let $\Phi(\mathcal{A}') = \mathcal{B}'$. Since Φ is Turing computable, we have $\mathcal{B} \leq_T \mathcal{A}$ and $\mathcal{B}' \leq_T \mathcal{A}'$. Therefore, if $\omega_1^{\mathcal{A}} = \omega_1^{\mathcal{A}'} = \omega_1^{CK}$, then $\omega_1^{\mathcal{B}} = \omega_1^{\mathcal{B}'} = \omega_1^{CK}$. Since $\mathcal{A} \not\cong \mathcal{A}'$, we must have $\mathcal{B} \not\cong \mathcal{B}'$. There is a computable infinitary sentence φ that is true in $\mathcal{B}$ and not $\mathcal{B}'$. If φ^* is the pullback of φ, then φ^* should be true in $\mathcal{A}$ and not $\mathcal{A}'$. This is a contradiction. $\square$

Theorem 3.2. *Let $\mathcal{A}$ and $\mathcal{A}'$ be rank-homogeneous trees. If $\omega_1^{\mathcal{A}} = \omega_1^{\mathcal{A}'} = \omega_1^{CK}$, and $\mathcal{A}, \mathcal{A}'$ satisfy the same computable infinitary sentences, then $\mathcal{A} \cong \mathcal{A}'$.*

Proof. Suppose $\mathcal{A}$ is a rank-homogeneous tree. If $tr(\emptyset) = \infty$ in $\mathcal{A}$, then $R(\mathcal{A})$ has no pair of the form $(0, \alpha)$. By Proposition 2.5, if $\omega_1^{\mathcal{A}} = \omega_1^{CK}$, then the computable ordinals are the only possible ordinal tree ranks. For any computable ordinal α, and any n, we have a computable infinitary sentence saying that there is a node of tree rank α at level n. Therefore, the computable infinitary sentences determine $R(\mathcal{A})$. $\square$

Remarks: More generally, if $\mathcal{A}, \mathcal{A}'$ are rank-homogeneous trees, each living in an admissible set with ordinal α, and $\mathcal{A}, \mathcal{A}'$ satisfy the same $L_{\omega_1\omega}$-sentences in L_α, then $\mathcal{A} \cong \mathcal{A}'$. The same is true for Abelian p-groups and for Abelian torsion groups.

Theorem 3.3. *There exist non-isomorphic undirected graphs $\mathcal{A}$ and $\mathcal{A}'$ such that $\omega_1^{\mathcal{A}} = \omega_1^{\mathcal{A}'} = \omega_1^{CK}$, and $\mathcal{A}$ and $\mathcal{A}'$ satisfy the same computable infinitary sentences.*

Proof. We consider graphs of a special form, coding a countable family of sets. Let S be a countable family of sets. The graph $G(S)$ will have one connected component for each $A \in S$. The connected component for A is a "daisy", with a "center" c, and for each n, a "petal", which is a cycle, containing c, and having length $2n+2$, if $n \in A$, or $2n+3$ if $n \notin A$. The cycles are disjoint except for the common center c.

We consider "generic" graphs of the form $G(S)$. For simplicity, we first say how to produce one such graph. The universe will be ω. The forcing conditions are finite graphs p—subgraphs of a graph of the proper form. Let F be the set of forcing conditions. We write $p \subseteq q$ if p is a subgraph of q. The forcing language consists of sentences $\varphi(\overline{a})$, where $\varphi(\overline{x})$ is a computable infinitary formula in the language of graphs, and $\overline{a}$ is a tuple of constants from ω. We let $\mathcal{S}$ be the set of these sentences. We define the relation $p \Vdash \varphi$ (p forces φ), for $p \in F$ and $\varphi \in S$.

Definition 7 (Forcing)**.**

1. *If φ is finitary quantifier-free, then $p \Vdash \varphi$ if p contains all of the constants appearing in φ and $p \models \varphi$.*
2. *If φ is computable Σ_α, of the form $\bigvee\!\!\!\bigvee_i (\exists \overline{u}_i)\, \psi_i(\overline{u}_i)$, then $p \Vdash \varphi$ if for some i and some $\overline{a}_i$, $p \Vdash \psi_i(\overline{a}_i)$.*
3. *If φ is computable Π_α, of the form $\bigwedge\!\!\!\bigwedge_i (\forall \overline{u}_i)\, \psi_i(\overline{u}_i)$, then $p \Vdash \varphi$ if for all i, all $\overline{a}_i$, and all $q \supseteq p$, some $r \supseteq q$ forces $\psi_i(\overline{a}_i)$.*

We have the usual forcing lemmas.

Lemma 3.4 (Extension)**.** *If $p \Vdash \varphi$ and $q \supseteq p$, then $q \Vdash \varphi$.*

Proof. We proceed by induction on sentences φ in $\mathcal{S}$. □

Lemma 3.5 (Consistency)**.** *We cannot have $p \Vdash \varphi$ and also $p \Vdash neg(\varphi)$.*

Proof. We proceed by induction on sentences φ in $\mathcal{S}$. □

Lemma 3.6 (Density)**.** *For any p and φ, there exists $q \supseteq p$ such that $q \Vdash \varphi$ or $q \Vdash neg(\varphi)$.*

Proof. Again we proceed by induction on sentences φ in $\mathcal{S}$. □

We say that p *decides* φ if $p \Vdash \varphi$ or $p \Vdash neg(\varphi)$.

Definition 8. *A* complete forcing sequence (c.f.s.) *is a sequence $(p_n)_{n\in\omega}$ such that*

1. *for each sentence φ in $\mathcal{S}$, there is some n such that $p_n \Vdash \varphi$ or $p_n \Vdash neg(\varphi)$.*
2. *$p_n \subseteq p_{n+1}$.*

Using Density, we obtain the existence of a c.f.s. $(p_n)_{n\in\omega}$. Then we obtain a graph $G = \cup_n p_n$. The definition of forcing assures that G will have universe ω—we must decide the sentences $a = a$, and we do this by putting a into the universe of some p_n.

Definition 9. *We say that G is* generic *if it is the union of a complete forcing sequence $(p_n)_{n\in\omega}$.*

Lemma 3.7 (Truth and Forcing)**.** *For any sentence φ of the forcing language, the following are equivalent:*

1. *$G \models \varphi$,*
2. *for some n, $p_n \Vdash \varphi$,*
3. *for some finite subgraph q of G, $q \Vdash \varphi$.*

It is not difficult to see that if G is generic, then G has the form $G(S)$, where S is a countably infinite family of subsets of ω. There are infinitely many centers, and for each center, for each n, there is a petal of length $2n + 2$ or $2n + 3$, and not both. For each element a, either a is a center or else a belongs to a petal connected by a finite chain to a center. Thus, G consists of infinitely many daisies, each coding a set.

Proposition 3.8. *Suppose G is generic. Let a and b be centers in G, and let S_a and S_b be the sets represented by the corresponding daisies. Then $S_a \neq S_b$.*

Proof. There is a sentence in the forcing language saying that $S_a = S_b$—the conjunction over $k > 1$ of sentences saying that there is a chain of length k through a if and only if there is one through b. This sentence cannot be forced, since for any p, there exists n and $q \supseteq p$ such that q has a chain of length $2n+2$ about one of a, b and one of length $2n+3$ about the other. $\square$

There is a great deal of freedom in producing a generic graph. A surprising fact is that for all generic graphs G, the computable infinitary theory is the same. Essentially, this is because the forcing conditions satisfy the joint embedding property.

Lemma 3.9. *Suppose $p \Vdash \varphi$. For any permutation f of ω, if p' is the forcing condition isomorphic to p under f, and φ' is the sentence obtained by applying f to the constants in φ, then $p' \Vdash \varphi'$.*

Proof. We proceed by induction on φ. If φ is finitary quantifier-free, then $p \models \varphi$ and $p' \models \varphi'$, so $p' \Vdash \varphi'$. Consider $\varphi = \bigvee\!\!\!\bigvee_i (\exists \overline{u}_i)\, \psi_i(\overline{u}_i)$. We have $p \Vdash \psi_i(\overline{a}_i)$, for some i and $\overline{a}_i$. By the Induction Hypothesis, we have $p' \Vdash \psi_i(f(\overline{a}_i))$, so $p' \Vdash \varphi'$. Finally, consider $\varphi = \bigwedge\!\!\!\bigwedge_i (\forall \overline{u}_i)\, \psi_i(\overline{u}_i)$. For all i and $\overline{a}_i$, and for all $q \supseteq p$, there exists $r \supseteq p$ such that $r \Vdash \psi_i(\overline{a}_i)$. Fix i and $\overline{a}'_i$. Say $q' \supseteq p'$. Let $\overline{a}_i = f^{-1}(\overline{a}'_i)$ and let q be the extension of p isomorphic to q' under f^{-1}. We have $r \supseteq q$ such that $r \Vdash \psi_i(\overline{a}_i)$. Let r' be the extension of q' isomorphic to r under f. Then by the Induction Hypothesis, we have $r' \Vdash \psi_i(\overline{a}'_i)$. Therefore, $p' \Vdash \varphi'$. $\square$

Lemma 3.9 implies that if φ is a sentence with no constants, and p, p' are isomorphic forcing conditions, then $p \Vdash \varphi$ if and only if $p' \Vdash \varphi$.

Lemma 3.10. *Suppose G and G' are both generic. Then for any computable infinitary sentence φ with no constants, $G \models \varphi$ iff $G' \models \varphi$.*

Proof. If not, we would have p, q such that $p \Vdash \varphi$ and $q \Vdash neg(\varphi)$. There is a forcing condition r extending disjoint copies of both p and q. Then r must force both φ and $neg(\varphi)$, a contradiction. $\square$

The next lemma gives an alternative definition of forcing for computable Π_α sentences.

Lemma 3.11. *$p \Vdash \bigwedge\!\!\!\bigwedge_i (\forall \overline{u}_i)\, \psi_i(\overline{u}_i)$ iff for all i and $\overline{a}_i$ and all $q \supseteq p$, it is not the case that $q \Vdash neg(\psi_i(\overline{a}_i))$.*

Proof. Suppose $p \Vdash \bigwedge\!\!\!\bigwedge_i (\forall \overline{u}_i)\, \psi_i(\overline{u}_i)$. If $q \supseteq p$, there exists $r \supseteq q$ such that $r \Vdash \psi_i(\overline{a}_i)$. By Extension and Consistency, we cannot have $q \Vdash neg(\psi_i(\overline{a}_i))$. Now, suppose that for all $q \supseteq p$, it is not the case that $q \Vdash \psi_i(\overline{a}_i)$. Then for each q, there is some $r \supseteq q$ such that $r \Vdash \psi_i(\overline{a}_i)$. $\square$

We calculate the complexity of the relations $p \Vdash \varphi$.

Lemma 3.12. *If φ is finitary quantifier-free, then the relation $q \Vdash \varphi$ is computable. For $\alpha > 0$, if φ is computable Σ_α, the relation $q \Vdash \varphi$ is Σ^0_α, and if φ is computable Π_α, the relation $q \Vdash \varphi$ is Π^0_α, all uniformly.*

Proof. If φ is finitary quantifier-free, the statement is clear. Suppose $\alpha > 0$, and the statement holds for $\beta < \alpha$. The statement is clear in the case where φ is computable Σ_α. Suppose φ is computable Π_α, say $\varphi = \bigwedge\!\!\!\bigwedge_i (\forall \overline{u}_i)\, \psi_i(\overline{u}_i)$. Then the statement is clear from Lemma 3.11. $\square$

Lemma 3.13. *For each formula $\varphi(\overline{x})$, and any tuple $\overline{u}$, we can find a computable infinitary formula $Force_{\varphi(\overline{x}),\overline{u}}(\overline{x},\overline{u})$, giving a disjunction of possible equality relations on $\overline{x},\overline{u}$ and graph structures on $\overline{u}$, such that if $\overline{a},\overline{b}$ satisfies the formula and p is the graph on $\overline{b}$ then $p \Vdash \varphi(\overline{a})$.*

Sketch of proof. We use Lemma 3.9. For any formula $\varphi(\overline{x})$ and any tuple $\overline{u}$, our preliminary version of $Force_{\varphi(\overline{x}),\overline{u}}(\overline{x},\overline{u})$ is the disjunction of the finitary quantifier-free formulas $\delta(\overline{x},\overline{u})$ such that $\delta(\overline{x},\overline{u})$ describes an equality relation on $\overline{x},\overline{u}$ and a graph structure on $\overline{u}$, and if $\overline{a},\overline{b}$ satisfies $\delta(\overline{x},\overline{u})$ and p is the graph on $\overline{b}$, then $p \Vdash \varphi(\overline{a})$. If $\varphi(\overline{x})$ is computable Σ_α, then the preliminary version of $Force_{\varphi(\overline{x}),\overline{u}}(\overline{x},\overline{u})$ is the disjunction of a Σ^0_α set of finitary quantifier-free formulas. By results in [1], we can effectively find a logically equivalent formula that is computable Σ_α. If $\varphi(\overline{x})$ is computable Π_α, then the preliminary version of $Force_{\varphi(\overline{x}),\overline{u}}(\overline{x},\overline{u})$ is the disjunction of a Π^0_α set of finitary quantifier-free formulas, and by the results in [1], we can effectively find a logically equivalent formula that is computable $\Sigma_{\alpha+1}$. $\square$

Lemma 3.14. *There exists a generic G such that $\omega_1^G = \omega_1^{CK}$.*

Proof. We use the result of Ressayre (Theorem 2.3). We have a Π^1_1 set Γ of computable infinitary sentences describing generic graphs G. For each computable infinitary formula $\varphi(\overline{x})$, we include the sentence F_φ saying

$$(\forall \overline{x}) \bigvee\!\!\!\bigvee_{\overline{u}} (\exists \overline{u})\, Force_{\varphi(\overline{x}),\overline{u}}(\overline{x},\overline{u}) \ .$$

If G is a generic graph, then it is a model of Γ. Conversely, if G is a model of Γ with universe ω, then we can produce a c.f.s. with union G, so G is generic.

For any Δ^1_1 $\Gamma' \subseteq \Gamma$, there is a computable bound α on the complexity of the formulas φ such that $F_\varphi \in \Gamma'$. For any computable ordinal α, there is a hyperarithmetical c.f.s. deciding all computable Σ_α sentences in the forcing language. The union of this c.f.s. is a hyperarithmetical graph $G' = (\omega, R')$ satisfying the sentences of Γ'. We are in a position to apply Theorem 2.3. We get $G = (\omega, R)$ such that $\omega_1^G = \omega_1^{CK}$ and G satisfies all of Γ. Then G is generic. $\square$

We want two non-isomorphic graphs G_1, G_2 such that $\omega_1^{G_i} = \omega_1^{CK}$, and G_1, G_2 are both generic. We modify the forcing argument above. Our new forcing conditions are pairs (p, q), where p, q are finite subgraphs of graphs of the form $G(S)$. We may suppose that p, q have the same universe. We let $(p, q) \subseteq (p', q')$ if $p \subseteq p'$ and $q \subseteq q'$. In our forcing language, we use two different binary relation symbols, R_1 and R_2, one for each graph. We consider computable infinitary sentences $\varphi(\overline{a})$ which result from substituting a tuple of constants $\overline{a}$ for the free variables in some computable infinitary formula $\varphi(\overline{x})$. The formulas may describe both graphs. Given a c.f.s. $(p_n, q_n)_{n\in\omega}$, we obtain two graphs $G_1 = \cup_n p_n$ and $G_2 = \cup_n q_n$, each with universe ω. Both are generic in the old sense.

Lemma 3.15. *If a is a center in G_1 and b is a center in G_2, the sets represented by the daisies about a and b cannot be equal.*

Proof. Suppose (p, q) forces the sentence saying that the sets are equal; i.e., for all k, G_1 has a petal of length k about a if and only if G_2 has a petal of length k about b. Take m such that there is no petal of either length $2m+2$ or $2m+3$ about a in p or about b in q. Take $(p', q') \supseteq (p, q)$ with a petal of length $2m+2$ about a in p' and with a petal of length $2m+3$ about b in q'. $\square$

We have formulas $Force_{\varphi(\overline{x}),\overline{u}}(\overline{x}, \overline{u})$ obtained as the disjunction of quantifier-free formulas $\delta(\overline{x}, \overline{u})$, specifying an equality relation on $\overline{x}, \overline{u}$ and two graph relations on $\overline{u}$, such that if $\overline{a}, \overline{b}$ satisfies $\delta(\overline{x}, \overline{u})$ and (p, q) is the forcing condition given by the two graph relations on $\overline{b}$, then $(p, q) \Vdash \varphi(\overline{a})$. We have a Π^1_1 set Γ with a sentence F_φ, for each computable infinitary formula $\varphi(\overline{x})$, saying that for all $\overline{x}$, $\varphi(\overline{x})$ holds if and only if there is some $\overline{u}$ such that $Force_{\varphi(\overline{x}),\overline{u}}(\overline{x}, \overline{u})$.

For each computable ordinal α, there is a hyperarithmetical model satisfying F_φ for all computable Σ_α formulas φ. It follows that for any Δ^1_1 set $\Gamma' \subseteq \Gamma$, Γ' has a model. By Ressayre's Theorem, Γ has a model $G = (\omega, R_1, R_2)$ such that $\omega_1^G = \omega_1^{CK}$. We get a pair $G_1 = (\omega, R_1)$, $G_2 = (\omega, R_2)$ such that G_1, G_2 are mutually generic, and both live in the same fattening of the least admissible set. By Lemma 3.15, the two graphs are not isomorphic. We have completed the proof of Theorem 3.3. $\square$

We are in a position to apply Theorem 3.1 to get the following.

Corollary 3.16. *$UG \not\leq_{tc} RHT$.*

We may improve Theorem 3.3 as follows.

Theorem 3.17. *There is a family of $2^{\aleph_0}$ non-isomorphic graphs $(G_i)_{i\in I}$, all satisfying the same computable infinitary sentences, and with the feature that $\omega_1^{G_i} = \omega_1^{CK}$.*

Sketch of proof. Let Γ be the Π^1_1 set of sentences describing a generic special graph. We have seen that the models of Γ all satisfy the same computable infinitary sentences with no constants. Take a constant a and let $\varphi(a)$ be an existential sentence saying that a is connected to at least three other points—this says that a is the center of a daisy. For each n, let $\psi_n(a)$ be an existential sentence saying that there is a petal of length $2n+2$ attached to a. Similarly, let $\psi'_n(a)$ say that there is a petal of length $2n+3$ attached to a.

In Ressayre's construction of a model G of Γ such that $\omega_1^G = \omega_1^{CK}$, at each step, we have a Π^1_1 set of computable infinitary sentences $\Gamma \cup \Lambda(\overline{a})$, mentioning finitely many constants. The set is consistent in the sense that the consequences (in the language with just equality) are all true of the distinct elements $\overline{a}$ in ω.

Let S be the set of n such that $\Gamma \cup \Lambda(\overline{a}), \varphi(a) \vdash \psi_n(a)$, and let S' be the set of n such that $\Gamma \cup \Lambda(\overline{a}), \varphi(a) \vdash \psi'_n(a)$. Both S and S' are Π^1_1. If $S \cup S' = \omega$, then S would be Δ^1_1. However, the sets coded in a generic graph cannot be hyperarithmetical. In particular, for any hyperarithmetical set A, we have a sentence in the forcing language saying that $S_a = A$. No forcing condition p can force this sentence, for we could take n not in either S nor S', and we may consistently add either $\psi_n(a)$ or $\psi'_a(n)$ and then continue with the next step in Ressayre's construction. We obtain models G of Γ with $2^{\aleph_0}$ different sets S_a. These models G all have the feature that $\omega_1^G = \omega_1^{CK}$. $\square$

4 Second main result—torsion-free Abelian groups

Torsion-free Abelian groups are subgroups of $\mathbb{Q}$-vector spaces. Hjorth [14] gave a transformation from trees to torsion-free Abelian groups which enabled him to show that the isomorphism relation on these groups is not Borel. Downey and Montalbán [8] built on Hjorth's ideas to show that the isomorphism relation on these groups is analytic complete. The transformation from [14] and [8] is described below.

We consider the elements of $\omega^{<\omega}$ as a basis for a $\mathbb{Q}$-vector space V^*. Let T be a subtree of $\omega^{<\omega}$, and let V be the subspace of V^* with basis T. Let T_n be the set of elements at level n of T. If u is at level $n > 0$, let u^- be the predecessor of u. Let $(p_n)_{n\in\omega}$ be the standard computable list of primes, in increasing order. We let $G(T)$ be the subgroup of V generated by the vector space elements of the following forms:

1. $\frac{v}{(p_{2n})^k}$, where $v \in T_n$, and $k \in \omega$,
2. $\frac{v+v'}{(p_{2n+1})^k}$, where $v \in T_n$, v' is a successor of v, and $k \in \omega$.

If P is a finite set of prime numbers, we let $\mathbb{Q}_P$ be the set of rationals of the form $\frac{k}{m}$, where $k \in \mathbb{Z}$ and m is a product of powers of elements of P.

Facts.

1. $\mathbb{Q}_\emptyset = \mathbb{Z}$
2. $\mathbb{Q}_P \cap \mathbb{Q}_R = \mathbb{Q}_{P\cap R}$
3. $\mathbb{Q}_P + \mathbb{Q}_R$ (the set of sums of elements of $\mathbb{Q}_P$ and $\mathbb{Q}_R$) is $\mathbb{Q}_{P\cup R}$

Note that each element of $G(T)$ can be expressed in the form

$$h = \Sigma_{v\in V} a_v v + \Sigma_{u\in U} b_u(u^- + u)$$

where

1. U, V are finite subsets of T, $\emptyset \notin U$,

2. if $v \in V \cap T_n$, then $a_v \in \mathbb{Q}_{\{p_{2n}\}}$,

3. if $u \in U \cap T_{n+1}$, then $b_u \in \mathbb{Q}_{\{p_{2n+1}\}}$.

The transformation described above takes the full class of trees to the class TFA of torsion-free Abelian groups. It is not clear that the transformation is $1-1$ on isomorphism types. Our goal is to show that the restriction of the transformation to the class RHT of rank-homogeneous trees is $1-1$ on isomorphism types. We use the definitions and techniques from [14], [8]. We do not distinguish between elements of the tree T and the corresponding elements of $G(T)$, which we call *vertex elements*. We will describe elements of $G(T)$ that resemble vertex elements. We will also describe a relation on these elements that resembles the successor relation. From this, we obtain a notion of rank. For each $n \in \omega$ and each countable ordinal α, we will have a sentence of $L_{\omega_1,\omega}$ that is true in $G(T)$ if and only if T has a node at level n of tree rank α. From this, it follows that rank-homogeneous trees that give rise to isomorphic groups must be isomorphic.

The results in [8] use only a few simple facts, which they extract from the proofs in [14]. We begin with these same facts, but we shall need more. Recall that $\emptyset$ is the top node in the tree T. We write $p^\infty|h$ if h is divisible by all powers of p.

Lemma 4.1. *Let $h \in G(T)$, say $h = \sum_{v\in V} q_v v$, where V is a finite set of vertex elements and $q_v \in \mathbb{Q} - \{0\}$. If p is a prime and $p^\infty|h$, then there is some $g \in G(T)$ such that $g = \sum_{v\in V} r_v v$, where $p^\infty|g$, and for all $v \in V$, $r_v \in \mathbb{Q}_{\{p\}} - \mathbb{Z}$.*

Proof. We multiply h by an appropriate integer and then divide by a power of p. □

The next two lemmas are given explicitly in [8].

Lemma 4.2. *Let h be an element of $G(T)$, say $h = \sum_{v\in V} r_v v$, where V is a finite subset of T and $r_v \in \mathbb{Q} - \{0\}$. If $(p_{2n})^\infty|h$, then for all $v \in V$, v has length n.*

Proof. We take $g = \sum_{v\in V} r'_v v$ as in Lemma 4.1. For $v \in V$, the coefficient r'_v has the form $a_v + (\sum_{u\in U_v} b_u) + b_v$, where U_v consists of successors of v, if v has length m, then $a_v \in \mathbb{Q}_{\{p_{2m}\}}$, if $u \in U_v$, then $b_u \in \mathbb{Q}_{\{p_{2m+1}\}}$, and $b_v \in \mathbb{Q}_{\{p_{2m-1}\}}$. Since $r'_v \in \mathbb{Q}_{\{p_{2n}\}} - \mathbb{Z}$, we must have $m = n$ and $a_v \neq 0$. Note that $(\sum_{u\in V_v} b_u) + b_v$ must be in $\mathbb{Z}$. □

Lemma 4.3. *Let h be an element of $G(T)$, say $h = \sum_{v\in V} r_v v$, where V is a finite subset of T and $r_v \in \mathbb{Q} - \{0\}$. If $(p_{2n+1})^\infty|h$, then for all v of length n in V, there is a successor u.*

Proof. Again take $g = \sum_{v\in V} r'_v v$ as in Lemma 4.1. For $v \in V$, of length m, r'_v has the form $a_v + (\sum_{u\in U_v} b_u) + b_v$, where U_v consists of successors of v, $a_v \in \mathbb{Q}_{\{p_{2m}\}}$, $b_u \in \mathbb{Q}_{\{p_{2m+1}\}}$, and $b_v \in \mathbb{Q}_{\{p_{2m-1}\}}$. Since $r'_v \in \mathbb{Q}_{\{p_{2n+1}\}} - \mathbb{Z}$, we must have $m = n$, and there must exist $u \in U_v$ with $b_u \notin \mathbb{Z}$. □

It is useful to keep in mind the following example.

Example: Let $h = u - u' = (v + u) - (v + u')$, where $v \in T_n$ and u, u' are successors of v in T_{n+1}. Then $p_{2n+1}^{\infty}|h$, although in our expression for h, the coefficient of v is 0.

The following is taken from Hjorth [14] (Propositions 2.2 and 2.5).

Proposition 4.4. *Let φ be a homomorphism from $G(T)$ to $\mathbb{Q}$ such that $\varphi(v) = 1$ for $v \in T_n$ and $\varphi(v) = -1$ for $v \in T_{n+1}$. Let $h = \sum_{v\in V} c_v v + \sum_{u\in U} a_u u$, where $V \subseteq T_n$ and $U \subseteq T_{n+1}$. If $(p_{2n+1})^{\infty}|h$, then $\varphi(h) = 0$. Moreover, for each $v \in V$, if $h_v = c_v v + \sum_{u\in U_v} a_u u$, then $(p_{2n+1})^{\infty}|h_v$, and $\varphi(h_v) = 0$.*

Using Proposition 4.4, we obtain the following.

Lemma 4.5.

1. *Suppose $h = a_{\emptyset}\emptyset + \sum_{u\in U} a_u u$, where $U \subseteq T_1$. If $(p_1)^{\infty}|h$, then $a_{\emptyset} = \sum_{u\in U} a_u$.*
2. *Suppose $h = \sum_{v\in V} a_v v + \sum_{u\in U} b_u u$, where $U \subseteq T_{n+1}$, and V is the set of predecessors of these elements. For $v \in V$, let U_v be the set of successors of v. If $(p_{2n+1})^{\infty}|h$, then for each $v \in V$, $a_v = \sum_{u\in U_v} b_u$.*

Proof. For 1, we consider a homomorphism φ taking $\emptyset$ to 1 and taking elements at level 1 to -1. We have $\varphi(h) = 0 = a_{\emptyset} - \sum_{u\in U} a_u$. By Proposition 4.4, $a_{\emptyset} = \sum_{u\in U} a_u$. For 2, we consider a homomorphism φ taking all elements of V to 1 and all elements of U to -1. By Proposition 4.4, for each $v \in V$, $\varphi(a_v v + \sum_{u\in U_v} b_u) = 0 = a_v - \sum_{u\in U_v} b_u$. Therefore, $a_v = \sum_{u\in U_v} b_u$. □

Note that in Lemma 4.5, in Case 1, we may have $a_{\emptyset} = 0$ and $\sum_{u\in U} a_u = 0$, and in Case 2, we may have $a_v = 0$, and $\sum_{u\in U_v} b_u = 0$. We need a refinement of Lemma 4.2.

Lemma 4.6. *Suppose $(p_{2n})^{\infty}|h$.*

1. *If $n > 0$, then h can be expressed in the form $\sum_{v\in V} r_v v$, where $V \subseteq T_n$ and r_v is in $\mathbb{Q}_{\{p_{2n}, p_{2n-1}\}}$.*
2. *If $n = 0$, then h has the form $r\emptyset$, where $r \in \mathbb{Q}_{\{p_0\}}$.*

Proof. We consider the two cases separately.

Case 1: Suppose $n > 0$. By Lemma 4.2, h can be expressed in the form $\sum_{v\in V} r_v v$, where $V \subseteq T_n$, and $r_v \in \mathbb{Q}$. Just because $h \in G(T)$, we have $h = \sum_{u\in U} a_u u + \sum_{u\in W} b_u(u + u^-)$, where if $u \in T_k$, then $a_u \in \mathbb{Q}_{\{p_{2k}\}}$, and $b_u \in \mathbb{Q}_{\{p_{2k-1}\}}$. For u at level $k \neq n$, the coefficient of u in the expression for h must be 0. This coefficient has the form $a_u + (\sum_{w^-=u} b_w) + b_u$.

Claim: For all $k > n$, for u at level k (appearing in our decomposition), a_u and b_u are integers.

Proof of Claim. We work our way back from the largest $k > n$ with some u at level k that appears. For the greatest k, if u is at level k, and u appears, then no successor of u appears. We have $0 = a_u + b_u$, where $a_u \in \mathbb{Q}_{\{p_{2k}\}}$ and $b_u \in \mathbb{Q}_{\{p_{2k-1}\}}$. Then both a_u and b_u must be integers. Supposing that the claim holds for $k' > k$, where $k > n$, let u be an element at level k that appears. We have $0 = a_u + (\sum_{w^-=u} b_w) + b_u$, where $a_u \in \mathbb{Q}_{\{p_{2k-1}\}}$, $b_u \in \mathbb{Q}_{\{p_{2k-1}\}}$, and $\sum_{w^-=u} b_w \in \mathbb{Z}$. Again a_u and b_u must be integers. □

Using the Claim, we can complete the proof for Case 1. For v at level n, the coefficient is $r_v = a_v + (\sum_{w^-=v} b_w) + b_v$, where $\sum_{w^-=v} b_w \in \mathbb{Z}$, $a_v \in \mathbb{Q}_{\{p_{2n}\}}$ and $b_v \in \mathbb{Q}_{\{p_{2n-1}\}}$. Therefore, $r_v \in \mathbb{Q}_{\{p_{2n},p_{2n-1}\}}$.

Case 2: Suppose $n = 0$. Then the only possible v is $\emptyset$, so $h = r\emptyset$. Since there is no $\emptyset^-$, we have $r = a_\emptyset + \sum_{w^-=\emptyset} b_w$. By the argument above, $\sum_{w^-=\emptyset} b_w \in \mathbb{Z}$. Since $a_\emptyset$ is in $\mathbb{Q}_{\{p_0\}}$, r is also. □

A node in T_n has the feature that there is a successor chain of length n leading from $\emptyset$ to it. We try to describe this in the group $G(T)$. We define first the *pseudo-vertex-like* elements at level n, and then the *vertex-like* elements at level n.

Definition 10 (pseudo-vertex-like)**.** *An element $h \in G(T)$ is* pseudo-vertex-like*, or* p.v.l.*, at level n, if one of the following holds:*

1. *$n = 0$ and $p_0^\infty | h$ or*
2. *$n > 0$ and*
   (a) *$p_{2n}^\infty | h$,*
   (b) *there exists a sequence $g_0, g_1, \ldots, g_n = h$, such that g_0 satisfies the formula $\Theta(x)$ from Lemma 4.7, and for all $i < n$, we have $p_{2i}^\infty | g_i$ and $p_{2i+1}^\infty | (g_i + g_{i+1})$.*

It is easy to see that all vertex elements are pseudo-vertex-like. For each n, we have a computable infinitary formula that defines the set of p.v.l. elements of $G(T)$. The formula is independent of T.

Next, we give a notion of successor on the p.v.l. elements.

Definition 11 (pseudo-successor)**.** *Suppose g is p.v.l. at level n and let h be p.v.l. at level $n+1$. We say that h is a* pseudo-successor *of g if $(p_{2n+1})^\infty | (g+h)$.*

Lemma 4.7. *There is a computable infinitary formula $\Theta(x)$ such that for all $T \in RHT$ with $T_1 \neq \emptyset$, $\Theta(x)$ is satisfied just by $\emptyset$ and $-\emptyset$.*

Proof. We let $\Theta(x)$ say the following:

1. $(p_0)^\infty | x$,

2. for primes $q \neq p_0$, $q \not| x$,

3. x has a pseudo-successor,

4. $\frac{1}{p_0}x$ has no pseudo-successor.

It is not difficult to see that $\emptyset$ and $-\emptyset$ satisfy $\Theta(x)$. We must show that other elements do not. If x satisfies Condition 1, we can apply Part 2 of Lemma 4.6, to see that x has the form $r\emptyset$, where $r \in \mathbb{Q}_{\{p_0\}}$. Then r has the form $\frac{z}{(p_0)^m}$, where $z \in \mathbb{Z}$. Condition 2 implies that z is not divisible by any primes other than p_0. Therefore, x has the form $\pm p^k\emptyset$. Condition 3 says that x has a successor. Using this, we show that $k \geq 0$. Take y such that $(p_2)^\infty | y$. By Part 1 of Lemma 4.6, $y = \sum_{v \in V} s_v v$, where $V \subseteq T_1$ and $s_v \in \mathbb{Q}_{\{p_2,p_1\}}$. If $(p_1)^\infty|(x+y)$, then by Lemma 4.5, $\pm(p_0)^k = \sum_{v \in V} s_v$. This implies that the right-hand side is an integer, and then the left-hand side is as well. Therefore, $x = \pm p_0^k$, where $k \geq 0$. Finally, we show that if x satisfies Condition 4, then k cannot be positive. If $k > 0$, then $\frac{1}{p_0}x = p_0^{k-1}\emptyset$. This satisfies Conditions 1 and 2. Moreover, if $v \in T_1$, then $p_0^{k-1}v$ is a successor of $\frac{1}{p_0}x$, contradicting Condition 4. Therefore, x must have the form $\pm\emptyset$. $\square$

Remark. For each n, we have a computable infinitary formula defining in $G(T)$ the set of pairs (g, h) such that g is p.v.l. at level n and h is a pseudo-successor of g. The formula is independent of T.

We define rank for p.v.l. elements by analogy with tree rank. We write $rk(h)$ for the rank of h in the group $G(T)$, and $tr(v)$ for the tree rank of v in the tree T.

Definition 12 (rank). *Let h be p.v.l. at level n.*

1. *$rk(h) = 0$ if h has no pseudo-successors,*

2. *for $\alpha > 0$, $rk(h) = \alpha$ if all pseudo-successors of h have ordinal rank, and α is the least ordinal greater than these ranks,*

3. *$rk(h) = \infty$ if h does not have ordinal rank.*

We note that $rk(h) = \infty$ if and only if there is an infinite sequence $(g_i)_{i \in \omega}$ such that each g_i is p.v.l., $g_0 = h$ and g_{i+1} is a pseudo-successor of g_i.

Lemma 4.8. *Suppose h is p.v.l at level n, expressed in the form $\sum_{v \in V} r_v v$, where V is a finite subset of T_n and $r_v \neq 0$. Then for all v, $tr(v) \geq rk(h)$.*

Proof. We show by induction on α that if $rk(h) > \alpha$, then for all $v \in V$, $tr(v) \neq \alpha$. (We allow the possibility that $rk(h) = \infty$.) Let $rk(h) > 0$. Let g be a p.v.l. pseudo-successor for h. Then $(p_{2n+1})^\infty|(h+g)$. Say $g = \sum_{u \in U} s_u u$, where U is a set of vertex elements at level $n+1$ and $s_u \neq 0$. By Lemma 4.3, for each $v \in V$, there is some $u \in U$ such that u is a successor of v. Therefore, $tr(v) \neq 0$.

Consider $\alpha > 0$, where the statement holds for $\beta < \alpha$. Suppose $rk(h) > \alpha$. Let g be a p.v.l. pseudo-successor of h such that $rk(g) \geq \alpha$. Say $g = \sum_{u \in U} s_u u$, where U is a set of vertex elements at level $n+1$ and $s_u \neq 0$. By the Induction Hypothesis, $tr(u) \neq \beta$ for any $\beta < \alpha$, so $tr(u) \geq \alpha$. By Lemma 4.3, some $u \in U$ is a successor of v. Then $tr(v) \neq \alpha$. Finally, we show that if $rk(h) = \infty$, then for all $v \in V$, $tr(v) = \infty$. There must be an infinite sequence of p.v.l. elements $(g_k)_{k \in \omega}$ such that $g_0 = h$ and g_{k+1} is a pseudo-successor of g_k. We have $g_k = \sum_{u \in U_k} s_u u$, where U_k is a set of vertex elements at level $n+k$, and $s_u \neq 0$. For each element of U_k, there is a successor in U_{k+1}. We obtain a chain of successors, starting with $v = v_0 \in U_0$, and choosing v_{k+1} a successor of v_k in U_{k+1}. Therefore, $tr(v) = \infty$. $\square$

Remark. For each n and α, we have a formula of $L_{\omega_1,\omega}$ defining in $G(T)$ the set of p.v.l. elements at level n of rank α. The formula is independent of T. Moreover, it lies in the least admissible set containing the ordinal α.

It is again helpful to consider an example.

Example: Let $v \in T_1$ and let u and u' be successors of v in T_2. Suppose that both u and u' have successors in T_3. Let $g = \frac{1}{11}u + \frac{10}{11}u'$. Since $p_4^\infty | u, u'$, we have $p_4^\infty | g$. Since $v + g = \frac{1}{11}(v+u) + \frac{10}{11}(v+u')$, we see that $p_5^\infty | (v+g)$. Therefore, g is p.v.l. and it is a pseudo-successor of v. We can show that g has no pseudo-successor, even though we have expressed it in terms of u and u', both of which have successors in T_4. Suppose that h is a pseudo-successor at level 4. Then $h = \sum_{w \in W} r_w w$, where $W \subseteq T_4$ and r_w. By Lemma 4.6, we must have $r_w \in \mathbb{Q}_{7,8}$. We must have $p_9^\infty | (g+h)$. By Lemma 4.5, if $W_u, W_{u'}$ are, respectively, the sets of successors of u, u' in W, then $\sum_{w \in W_u} r_w = \frac{1}{11}$, and $\sum_{w \in W_{u'}} r_w = \frac{10}{11}$. This is a contradiction.

We strengthen the definition of p.v.l. element in order to rule out examples like the one above, in which g has no successor, but it has a decomposition in terms of elements all having successors.

Definition 13 (vertex-like)**.** *Let $g \in G(T)$. We say that g is* vertex-like*, or* v.l.*, if*

1. *g is p.v.l. at some level n, and*
2. *either*
   (a) *$rk(g) > 0$, or*
   (b) *$rk(g) = 0$ and for any decomposition $g = \sum_j r_j g_j$ such that all g_j are p.v.l. at level n, there exists j such that $rk(g_j) = 0$.*

Lemma 4.9. *If v is a vertex element, then it is vertex-like.*

Proof. We already noted that a vertex element is p.v.l. Suppose v is at level n, and $rk(v) = 0$. Then v has no successors. We must show that if $v = \sum_j g_j$, where each g_j is p.v.l. at level n, then for some j, $rk(g_i) = 0$. Suppose that for all j, $rk(g_j) \neq 0$. Say h_j is a p.v.l. pseudo-successor of g_j at level $n+1$. By Lemma 4.2, each g_j has a decomposition in terms of tree elements at level n. Since $v = \sum_j g_j$, v must appear with non-zero coefficient in the decomposition of some g_j. Then by Lemma 4.5, the corresponding h_j has a decomposition that involves successors of v with non-zero coefficients. This is a contradiction. □

We would like to show that if g is v.l. at level n, expressed in the form $\sum_{v \in V} r_v v$, where $V \subseteq T_n$ and $r_v \neq 0$, then $rk(g)$ is the minimum of $tr(v)$, for $v \in V$.

Lemma 4.10. *Suppose g is v.l. at level n. Say $g = \sum_{v \in V} r_v v$, where $V \subseteq T_n$. Then $rk(g) = 0$ iff there exists $v \in V$ such that $tr(v) = 0$.*

Proof. First, suppose there exists $v \in V$ such that $tr(v) = 0$. By Lemma 4.8, $tr(v) \geq rk(g)$, so $rk(g) = 0$. Next, suppose $rk(g) = 0$. The elements of V are p.v.l. and one of the decompositions of g is $\sum_{v \in V} r_v v$. By the definition of vertex-like, there is some v such that $rk(v) = 0$. Then v has no pseudo-successors, so v has no successors in T. Therefore, $tr(v) = 0$. □

Lemma 4.11. *If g is v.l. at level n and $rk(g) > 0$, then g has a decomposition $\sum_{v \in V} m_v v$ where all coefficients m_v are integers.*

Proof. By Lemma 4.6, g can be expressed in the form $\sum_{v \in V} r_v v$, where $V \subseteq T_n$ and $r_v \in \mathbb{Q}_{\{p_{2n}, p_{2n-1}\}}$. Since $rk(g) > 0$, we have a p.v.l. pseudo-successor g', expressed in the form $\sum_{u \in U} s_u u$, where $U \subseteq T_{n+1}$ and $s_u \in \mathbb{Q}_{\{p_{2n+2}, p_{2n+1}\}}$. Consider $h = g + g'$. Since $(p_{2n+1})^\infty | h$, we can apply Lemma 4.5. For each $v \in V$, let U_v be the set of successors of v in U. We have $r_v = \sum_{u \in U_v} s_u$. It follows that $\sum_{u \in U_v} s_u$ and r_v are integers. □

Suppose g is a v.l. element at level n. Recall that the definition of v.l. has two condidions, with the second split into two cases. If Condition 2 (a) holds for g, then Lemma 4.10 says that g can be expressed as a sum of vertex elements on level n with integer coefficients. If Condition $rk(g) = 0$, then the decomposition of g involves some terminal vertex element.

Lemma 4.12. *Let g be v.l. at level n, with a decomposition $\sum_{v \in V} r_v v$, where $V \subseteq T_n$ and all coefficients r_v are non-zero. Then $rk(g) = min_{v \in V} tr(v)$.*

Proof. By Lemma 4.8, $tr(v) \geq rk(g)$ for all $v \in V$. We show by induction on α that if $tr(v) \geq \alpha$ for all $v \in V$, then $rk(g) \geq \alpha$. For $\alpha = 0$, the statement is trivially true. Suppose $\alpha > 0$, where the statement holds for all $\beta < \alpha$. If g satisfies Condition 2 (b) from the definition of v.l., then by Lemma 4.11, there is some $v \in V$ such that $tr(v) = 0$. Suppose $rk(g) = \beta$, where $0 < \beta < \alpha$. For all $v \in V$, $tr(v) > \beta$, so v has a successor u_v with $tr(u_v) \geq \beta$. By Lemma 4.11, we may suppose that all r_v are integers. We have a successor h of g, of the form $\sum_{v \in V} r_v u_v$. This h is vertex-like at level $n+1$, and by the Induction Hypothesis,

$rk(h) \geq \beta$. Then $rk(g)$ cannot be β after all. Finally, suppose $tr(v) = \infty$ for all $v \in V$. For each v, there is an infinite successor chain, and we can use these to form an infinite chain of successors of g, so $rk(g) = \infty$. □

Recall that for a tree T, $R(T)$ is the set of pairs (n, α) such that there is some $v \in T$ at level n with $tr(v) = \alpha$. Proposition 1.2 says that for rank-homogeneous trees T, T', $T \cong T'$ if and only if $R(T) = R(T')$.

We give one last example.

Example: Let $g = \sum_{v \in V} r_v v$, where $V \subseteq T_n$ and $r_v \neq 0$. Let $h = \sum_{w \in W} s_w w$ be a successor of g, where $W \subseteq T_{n+1}$ and $s_w \neq 0$. For each $v \in V$, we must have successors $w \in W$, but there may be some $w \in W$ such that $w^- \notin V$. If this happens, we call h a "bad" successor. Let W' be the set of such w. Then $h' = \sum_{w \in W - W'} s_w w$ is a "good" successor of g. We can show that $rk(h') \leq rk(h)$. If $rk(h')$ and $rk(h)$ are both non-zero, then h and h' are both v.l., and by Lemma 4.12, $rk(h') \geq rk(h)$. If $rk(h) = 0$, then $rk(h') \geq rk(h)$. Finally, suppose $rk(h') = 0$ and $rk(h) > 0$. If k is a successor of h, then we obtain a successor for h' by removing from the decomposition the terms corresponding to successors of elements of W'. When we compute $rk(g)$, we may ignore any "bad" successors such as h and consider only good successors.

Theorem 4.13. *For $T, T' \in RHT$, $T \cong T'$ iff $G(T) \cong G(T')$.*

Proof. We let $R(G(T))$ be the set of pairs (n, α) such that there is a v.l. $g \in G(T)$ at level n with $rk(g) = \alpha$. We can show that $R(T) = R(G(T))$. If $(n, \alpha) \in R(T)$, then v is v.l. in $G(T)$, expressed simply as $1v$, and $tr(v) = rk(v)$. Therefore, $(n, \alpha) \in R(G(T))$. If $(n, \alpha) \in R(G(T))$, witnessed by $g = \sum_{v \in V} r_v v$, then by Lemma 4.12, there is some vertex element $v \in V_n$ such that $tr(v) = \alpha$. Therefore, $(n, \alpha) \in R(T)$. □

This completes the proof that Φ is $1 - 1$ on isomorphism types.

5 Third main result—Boolean algebras

In this section, our first goal is to define a Turing computable transformation from RHT to BA that is $1 - 1$ on isomorphism types. After this, we show that the transformation preserves Scott rank.

Theorem 5.1. *$RHT \leq_{tc} BA$*

Proof. Let $(A_n)_{n \in \omega}$ be an effective partition of ω into disjoint infinite sets. Let $(L_n)_{n \in \omega}$ be a uniformly computable sequence of orderings, where L_n has order type $\omega^{n+1} + \eta + 1$. For an input tree T, let $S(T)$ consist of the finite sequences of the form $r_0 q_1 r_1 \ldots q_n r_n x$ satisfying the following conditions:

- for some tree element $a_1 \ldots a_n$, $q_i \in A_{a_i}$, for $i < n$,

- $r_i \in A_0$,
- $r_n \in A_1$, and
- $x \in L_n$, where x is not last in L_n.

From the tree element $\emptyset$, we obtain sequences of the form $r_0 x$, where $x \in L_0$, not last.

For $\rho = r_0 q_1 \ldots r_{n-1} q_n$ of length $2n$ with extensions in $S(T)$, let N_ρ be the set of these extensions. For $\sigma = r_0 q_1 \ldots q_n r_n$ of length $2n+1$, with extensions in $S(T)$ and with $r_n \in A_1$, and for I a half open interval in L_n, let $M_{\sigma,I}$ be the set of extensions of σ in $S(T)$—this looks like the interval I in L_n. Note that $N_\emptyset = S(T)$. Let $B(T)$ be the set algebra generated by the special sets N_ρ and $M_{\sigma,I}$. The Turing computable transformation that we want to consider is the one that takes T to $B(T)$. Note that if T and T' are isomorphic trees, then $B(T) \cong B(T')$.

Remark: The transformation that we have described makes sense for arbitrary trees. However, it is not $1-1$ on isomorphism types. For example, let $T = 2^{<\omega}$, and let $T' = 3^{<\omega}$. Then $T \not\cong T$, but $B(T) \cong B(T')$.

We show that the restriction of the transformation to rank-homogeneous trees is $1-1$ on isomorphism types.

Definition 14. *For $\rho = r_0 q_1 \ldots r_{n-1} q_n$, where $q_i \in A_{a_i}$, we say that N_ρ* represents *the sequence $s = a_1 \ldots a_n$.*

Each tree element s is represented by many elements N_ρ. For each n, we have a computable infinitary formula $\lambda_n(x)$ describing the elements N_ρ which represent elements at level n of T. The formula (which is independent of our choice of T) says the following:

1. x bounds infinitely many copies of $I(L_n)$ and no copies of $I(L_k)$, for $k < n$.
2. if $y \le x$, then only one of y or $x - y$ bounds infinitely many copies of $I(L_n)$.

For a given n, we have an ideal in $B(T)$ generated by the elements of the form N_ρ, for ρ representing a sequence of length $> n$, and elements of the form $M_{\sigma,I}$, where I is isomorphic to a sub-interval of L_m for some $m \ge n$. We say that $x \sim_n y$ if $x \Delta y$ is in this ideal. If a satisfies $\lambda_n(x)$, then there is a unique ρ of length $2n$ such that $a \sim_n N_\rho$. If a satisfies $\lambda_n(x)$, we say that a is at *level n*.

Suppose a is at level n, b is at level $n+1$, and $b \subseteq a$. Let ρ, ρ' be the unique sequences such that $a \sim_n N_\rho, b \sim_{n+1} N_{\rho'}$, and let s, s' be the sequences in T represented by $N_\rho, N_{\rho'}$. Then s' is a successor of s. We say that b is a *successor* of a. Again we have a computable infinitary formula expressing this. Note that a given b may be a successor of many different a.

Lemma 5.2. *Suppose $\omega_1^T = \omega_1^{CK}$. For each $n \in \omega$ and each computable ordinal β, there is a computable infinitary formula $\nu_{n,\beta}(x)$ such that $B(T) \models \nu_{n,\beta}(a)$ iff a is at level n and for the unique ρ such that $a \sim_n N_\rho$, N_ρ represents $s \in T$ such that $tr(s) \geq \beta$.*

Proof of Lemma. We define the formulas $\nu_{n,\beta}(x)$ inductively as follows.

1. $\nu_{n,0}(x)$ says that x is at level n.
2. $\nu_{n,\beta+1}(x)$ says
   (a) x is at level n,
   (b) x has infinitely many successors satisfying $\nu_{n+1,\beta}(x)$,
3. for limit β, $\nu_{n,\beta}(x)$ is the conjunction of $\nu_{n,\gamma}(x)$, for $\gamma < \beta$.

□

We may generalize this as follows.

Lemma 5.3. *Suppose $\omega_1^T \leq \alpha$, where α is admissible. For each $n \in \omega$ and each $\beta < \alpha$, there is a formula $\nu_{n,\beta}(x)$, in the fragment $L_{\alpha,\omega} = L_{\omega_1\omega} \cap L_\alpha$, such that $B(T) \models \nu_{n,\beta}(a)$ iff a is at level n and for the unique ρ such that $a \sim_n N_\rho$, N_ρ represents $s \in T$ such that $tr(s) \geq \beta$.*

We have formulas saying in $B(T)$ that x is at level n and $x \sim N_\rho$, where N_ρ represents $s \in T$ of tree rank *at least* β. We also have formulas saying that x is at level n and $x \sim N_\rho$, where N_ρ represents $s \in T$ of tree rank *exactly* β. We let $\tau_{n,\beta}(x)$ say that $\nu_{n,\beta}(x)$ holds and $\nu_{n,\beta+1}(x)$ does not hold.

Lemma 5.4.

1. *If $\omega_1^T = \omega_1^{CK}$, then for each $n \in \omega$ and each computable ordinal β, there is a computable infinitary formula $\tau_{n,\beta}(x)$ such that $B(T) \models \tau_{n,\beta}(a)$ iff a is at level n and for the unique ρ such that $a \sim N_\rho$, N_ρ represents $s \in T$ such that $tr(s) = \beta$.*
2. *If $\omega_1^T \leq \alpha$, where α is admissible, then for each $n \in \omega$ and each $\beta < \alpha$, there is a formula $\tau_{n,\beta}(x)$ in $L_{\alpha,\omega}$ such that $B(T) \models \tau_{n,\beta}(a)$ iff a is at level n and for the unique ρ such that $a \sim N_\rho$, N_ρ represents $s \in T$ such that $tr(s) = \beta$.*

Remark: For $\tau_{n,0}(x)$, it may be tempting to say simply that x is at level n, and x has no successor; i.e., there is no $y \subseteq x$ such that y is at level $n+1$. This does not work. To see the difficulty, consider the element $a = N_\rho \cup N'_\rho$, where N_ρ represents s at level n, and $N_{\rho'}$ represents s', at level $n+1$. Now, $N_{\rho'}$ is a successor of a, but $a \sim_n N_\rho$.

We are ready to complete the proof of the theorem. We consider the sentences of the form $(\exists x)\,\tau_{n,\beta}(x)$. We have $B(T) \models (\exists x)\,\tau_{n,\beta}(x)$ if and only if there exists $a \in B(T)$ at level n, with a unique ρ such that $a \sim N_\rho$. This N_ρ represents a unique $s \in T$ such that $tr(s) = \beta$. Then $B(T) \models (\exists x)\, \tau_{n,\beta}(x)$ if and only if $(n, \beta) \in R(T)$. If $B(T) \cong B(T')$, then the two Boolean algebras satisfy the same infinitary sentences. It follows that $R(T) = R(T')$, so $T \cong T'$. $\square$

5.1 Preservation of Scott rank

Definition 15. *Let Φ be a Turing computable transformation from K to K'. We say that Φ has the* rank preservation property *if for all $\mathcal{A} \in K$, either $SR(\mathcal{A}) = SR(\Phi(\mathcal{A}))$ or else $SR(\mathcal{A}), SR(\Phi(\mathcal{A})) < \omega_1^{\mathcal{A}}$.*

In [4], there are some transfer theorems, giving conditions sufficient to guarantee preservation of Scott rank. However, we cannot apply any of these transfer theorems here. We note that if N_ρ represents the sequence s in T, then the orbit of N_ρ in $B(T)$ depends only on the tree rank of s, while the orbit of s in T depends on the tree rank of s, plus the tree ranks of the other elements generated by s under the predecessor function.

Theorem 5.5. *Let $\Phi : RHT \to BA$ be the transformation taking T to $B(T)$. Then Φ has the rank preservation property.*

Proof. We already understand the orbits of tuples in T in terms of tree ranks. We need to understand the orbits of tuples in $B(T)$. For a tuple $\bar{b}$, we consider the finite algebra generated by the elements of $\bar{b}$. We look at the atoms of this algebra. Say these are $b_1, \ldots, b_n$. The orbit of $\bar{b}$ is determined by the isomorphism types of b_i, for $i = 1, \ldots, n$, considered as sub-algebras of $B(T)$.

We focus on the isomorphism type of a single $b = b_i$. We may suppose that b is a finite disjoint sum $d_1 \oplus \ldots \oplus d_k$, where each d_j is a finite intersection of generating elements and their complements. We can see that each d_j is isomorphic to a single N_ρ, or $M_{\sigma,I}$. Then b is isomorphic to a finite sum of these. Just as we can collapse atoms into 1-atoms, we may be able to collapse certain d_j into others.

It is easy to describe the isomorphism type of $M_{\sigma,I}$—it is a familiar interval algebra. We must describe the isomorphism type of an element N_ρ. Suppose N_ρ represents some $s \in T$ at level n, where $tr(s) = \beta$. Suppose $N_{\rho'}$ also represents some $s' \in T$ at level n, with $tr(s') = \beta$. Then we have $N_\rho \cong N_{\rho'}$. If a is in the orbit of N_ρ, then we must have $a \sim N_{\rho'}$ for such a ρ'. However, this is not enough. If $x \leq a$, there must be some $x' \leq N_\rho$ such that $x \cong x'$. We can give a formula describing the orbit of N_ρ. Even in the case where $\omega_1^T = \omega_1^{CK}$. The formula that we give may not be computable infinitary, but it is T-computable infinitary.

Lemma 5.6. *Suppose $\omega_1^T = \alpha$.*

1. *For each n and $\beta < \alpha$, we have a T-computable infinitary formula $\varphi_{n,\beta}(x)$ such that $B(T) \models \varphi_{n,\beta}(a)$ iff $a \cong N_\rho$, where N_ρ represents a sequence $s \in T$, at level n, with $tr(s) = \beta$.*

2. *Suppose that for each n, there is a bound $\beta_n < \alpha$ on the tree ranks at level n. Then we have a T-computable infinitary formula $\varphi_{n,\infty}(x)$ such that $B(T) \models \varphi_{n,\infty}(a)$ iff a is isomorphic to some N_ρ representing a sequence $s \in T$, at level n, with $tr(s) = \infty$.*

Proof. For 1, we proceed by induction. We let $\varphi_{n,0}(x)$ say that x is at level n with no successors. For $\beta > 0$, let $S_\beta = \{\gamma < \beta : (n+1, \gamma) \in R(T)\}$. We let $\varphi_{n,\beta}(x)$ say that for all $\gamma \in S_\beta$, x has infinitely many successors y such that $\varphi_{n+1,\gamma}(y)$ holds, and for each successor y of x, there is some $\gamma \in S_\beta$ such that $\varphi_{n+1,\gamma}(y)$ holds.

For 2, we let $\varphi_{n,\infty}(x)$ say that x is at level n, and it does not satisfy any of the formulas $\tau_{n,\beta}$, for $(n, \beta) \in R(T)$. If a satisfies this formula, then it is almost equal to N_ρ, for ρ representing some sequence $s \in T$ at level n. This s does not have tree rank β for any $\beta < \beta_n$ such that $(n, \beta) \in R(T)$. Then $tr(s)$ must be ∞. This means that everything that can occur at levels $m \geq n$ occurs in the subtree below s. □

We are prepared to complete the proof of rank preservation. We suppose that $\omega_1^T = \alpha$.

Case 1: Suppose $SR(T) < \alpha$. There is a bound $\beta < \alpha$ on the ordinal tree ranks that occur. Then there is some $\beta' < \alpha$ such that for all tuples $\bar{b}$ in $B(T)$, there is a T-computable $\Sigma_{\beta'}$ formula defining the orbit. Therefore, $SR(B(T)) < \alpha$.

Case 2: Suppose $SR(T) = \alpha$. For level n of T, there is a bound $\beta_n < \alpha$ on the ordinal tree ranks, but there is no bound over all. For all tuples $\bar{b}$ in $B(T)$, there is a T-computable infinitary formula defining the orbit, but there is no over-all bound on the complexity of the formulas.

Case 3: Suppose $SR(T) = \alpha + 1$. For some level n, there is no bound $\beta < \alpha$ on the tree ranks at level n. It follows that T has paths. Let s be a node at level n of rank ∞ and let p be the predecessor of s. The set of successors of p of tree rank ∞ is not defined by any formula in $L_{\alpha,\omega}$. We have a T-computable function h mapping each successor s' of p to some N_ρ representing it. Suppose the orbit of $h(s)$ is defined by a T-computable infinitary formula $\psi(x)$. We get a contradiction by showing that there is some bound $\beta < \alpha$ on the tree ranks of successors of p. If not, then we can use Theorem 2.3 to get a successor s' of p such that $tr(s') = \infty$ and $h(s')$ does not satisfy $\psi(x)$. Therefore, if T has Scott rank $\alpha + 1$, so does $B(T)$. □

In [5], there is a computable rank-homogeneous tree T of Scott rank ω_1^{CK}. Moreover, the tree T shares with the Harrison ordering the following feature.

Definition 16. *A computable structure $\mathcal{A}$, of non-computable Scott rank, is* strongly computably approximable *if for any Σ_1^1 set S, there is a uniformly computable sequence $(\mathcal{A}_n)_{n\in\omega}$ such that $\mathcal{A}_n \cong \mathcal{A}$ if $n \in S$, and $SR(\mathcal{A}_n) < \omega_1^{CK}$, otherwise.*

Our rank-preserving computable embedding yields a computable Boolean algebra $\mathcal{B} = B(T)$ such that $SR(\mathcal{B}) = \omega_1^{CK}$, and $\mathcal{B}$ is strongly computably approximable.

6 Conclusion

Camerlo and Gao [6] showed that BA lies on top under $\leq_B$. It is not clear that their transformation is effective. The transformation that we gave is effective, but it does not answer the following.

Problem 1. *Does BA lie on top under $\leq_{tc}$?*

There are computable structures of Scott rank ω_1^{CK} in a number of familiar classes, obtained by applying rank-preserving transformations to a rank homogeneous tree.

Problem 2. *Let $\Phi : RHT \rightarrow TFA$ be the transformation of Hjorth and Downey-Montalbán, taking T to $G(T)$. Does Φ have the rank preservation property?*

To show that the transformation preserves Scott rank, we would need to understand the orbits of tuples in $G(T)$ and show that they are simply described in terms of orbits in T.

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