Comparing classes of finite sums

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Abstract

The notion of *Turing computable embedding* [4] is a computable analog of Borel embedding. It provides a way to compare classes of countable structures, effectively reducing the classification problem for one class to that for the other. Most of the known results on non-existence of Turing computable embeddings reflect differences in the complexity of the sentences needed to distinguish among non-isomorphic members of the two classes. Here we consider structures obtained as sums. We show that the n-fold sums of members of certain classes lie strictly below the $(n+1)$-fold sums. The differences reflect model theoretic considerations related to Morley degree, not differences in the complexity of the sentences that describe the structures. We consider three different kinds of sum structures: cardinal sums, in which the components are named by predicates; equivalence sums, in which the components are equivalence classes under an equivalence relation; and direct sums of certain groups.

1 Introduction

"Borel embeddings" were introduced by Friedman and Stanley [8], as a way of comparing the classification problems for classes of countable structures. "Turing computable embeddings", introduced in [4], are an effective analogue of Borel embeddings. These allow some finer distinctions. We consider classes K consisting of structures all having the same computable language, and with universe a subset of ω. Our classes are closed under isomorphism. We say that a class is *nice* if it is axiomatized by a computable infinitary sentence.

We identify each structure $\mathcal{A}$ with its atomic diagram $D(\mathcal{A})$, and we identify this, via Gödel numbering, with a subset of ω. For our purposes, a *Turing operator* is a partial function from sets to sets, computed by an oracle Turing machine. For a Turing operator $\Phi = \varphi_e$ given by the oracle machine with

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index e, we write $\Phi(A) = B$ if $\varphi_e^A = \chi_B$. The following is the main definition from [4].

Definition 1. *A* Turing computable embedding *of K into K' is a Turing operator $\Phi = \varphi_e$ such that the following hold:*

1. *For each $\mathcal{A} \in K$, there is some $\mathcal{B} \in K'$ such that $\varphi_e^{D(\mathcal{A})} = \chi_{D(\mathcal{B})}$. Identifying the structures with their atomic diagrams, we write $\Phi(\mathcal{A}) = \mathcal{B}$.*
2. *For $\mathcal{A}, \mathcal{A}' \in K$, $\mathcal{A} \cong \mathcal{A}'$ iff $\Phi(\mathcal{A}) \cong \Phi(\mathcal{A}')$.*

We write $K \leq_{tc} K'$ if there is a a Turing computable embedding of K into K'. The relation $\leq_{tc}$ is a pre-ordering on classes of structures. Many of the results for Borel embeddings carry over to Turing computable embeddings. For both kinds of emberddings, the class LO of linear orderings lies on top, along with the class UG of undirected graphs. For both kinds of embeddings, the class of fields of finite transcendence degree does not lie on top, and neither does the class ApG of Abelian p-groups. Details of the proof for ApG may be found in [7]. Using Turing computable embeddings, we can make some finer distinctions. Under Borel embeddings, all classes with $\aleph_0$ isomorphism types are equivalent. Under Turing computable embeddings, we have

$$NF <_{tc} V$$

where NF is the class of number fields, and V is the class of non-trivial $\mathcal{Q}$-vector spaces. We note that the class V is tc-equivalent to the class of free groups.

Suppose Φ is a Turing computable embedding (or tc-embedding, for short) from K to K' reduces the classification problem for K to the classification problem for K'. We do not say exactly what a classification is, but it should involve describing the (countable) members of K in a way that lets us tell non-isomorphic members apart. Assuming that we know how to describe the structures in K', we can describe a structure $\mathcal{A} \in K$ by computing the Φ-image $\mathcal{B} \in K'$ and giving the description of $\mathcal{B}$.

The single main tool in showing non-tc-embeddability of a class K into K' has been to consider the complexity of sentences needed to distinguish the different members of the classes, and apply the "Pullback Theorem" from [11].

Theorem 1.1 (Pullback Theorem)**.** *Suppose $K \leq_{tc} K'$ via Φ. Then for any computable infinitary sentence φ in the language of K', we can find a sentence φ^* in the language of K such that for $\mathcal{A} \in K$, $\mathcal{A} \models \varphi^*$ iff $\Phi(\mathcal{A}) \models \varphi$. Moreover, if φ is computable Σ_α, where α is a computable ordinal ≥ 1, then φ^* is also computable Σ_α.*

As an example of the use of the Pullback Theorem, we show that $V \not\leq_{tc} NF$ (see [12]).

Sample proof. Suppose $V \leq_{tc} NF$ via Φ, expecting a contradiction. Let $\mathcal{A}, \mathcal{A}'$ be non-isomorphic $\mathbb{Q}$-vector spaces. Then $\Phi(\mathcal{A}), \Phi(\mathcal{A}')$ are non-isomorphic number fields. The number fields must differ on some existential sentence φ. The

Pullback Theorem gives a computable Σ_1 sentence φ^* on which $\mathcal{A}, \mathcal{A}'$ should differ. However, all elements of V satisfy the same computable Σ_1 sentences [11]. $\square$

In [7], the Pullback Theorem is used to prove the following.

Proposition 1.2. *Suppose K and K' are classes of structures such that K has a non-isomorphic pair of structures satisfying the same computable infinitary sentences, while K' has no such pair. Then $K \not\leq_{tc} K'$.*

Proposition 1.2 implies that $UG \not\leq_{tc} ApG$. There are non-isomorphic undirected graphs G_1, G_2 such that $\omega_1^{G_i} = \omega_1^{CK}$ and G_1, G_2 satisfy the same computable infinitary sentences, but non-isomorphic Abelian p-groups that compute no non-computable ordinals must differ on some computable infinitary sentence.

In the current paper, we give prove some non-embeddability results using the model-theoretic idea of "Morley degree". We consider classes K^s (K^{s*}) in which each element is an ordered (unordered) collection of s members of the class K. We show that $K^s <_{tc} K^{s+1}$ under some conditions on K, which cover many cases of interest. We apply this result to $K = S$, the class of structures in the pure language, $K = V$, the class of infinite $\mathbb{Q}$-vector spaces, and others.

The motivating example was V. We observe that there is a single "most complicated" structure in V; namely, the saturated model. The elementary first order theory of V is strongly minimal, so the formula $x = x$ has Morley rank 1. For the elementary first order theory of V^s, the formula $x = x$ has Morley rank 1, but Morley degree s. If $s > 1$, there are structures in V^s with different numbers of saturated components, and the number increases with s. The intuition behind the proof that $V^{s+1} \not\leq_{tc} V^s$ is that there ought not to be enough "room" in V^s to code all of the structures in V^{s+1} with at least one saturated component.

1.1 $\leq_{FF}$ and $\leq_{FF^*}$

Friedman and Fokina [5] defined a reducibility for comparing Σ^1_1 equivalence relations on ω.

Definition 2. *Let E, E' be equivalence relations on ω. We write $E \leq_{FF} E'$ if there is a total computable function $f : \omega \to \omega$ such that mEn iff $f(m)E'f(n)$.*

Recall that a structure $\mathcal{A}$ is *computable* if the atomic diagram $D(\mathcal{A})$ is computable; i.e., there is some e such that $\varphi_e = \chi_{D(\mathcal{A})}$. We refer to e as an index for $\mathcal{A}$. There may be different computable copies of a particular abstract structure, and each of these will have infinitely many different indices. For an abstract structure $\mathcal{M}$, we write $I(\mathcal{M})$ for the set of indices for computable copies of $\mathcal{M}$. For a class K, closed under isomorphism, we write $I(K)$ for the set of indices for computable members of K. We write $E(K)$ for the set of pairs (a, b) of indices for isomorphic computable members of K.

A class of structures K, closed under isomorphism, is said to be *nice* if it is axiomatized by a computable infinitary sentence. For a class K, let $E^*(K)$ be the set of pairs $(a, b) \in \omega^2$ such that either a and b are indices for isomorphic computable members of K, or else a and b are both not in $I(K)$. Thus we introduce an extra equivalence class for the numbers not in $I(K)$. If K is a nice class, then $E^*(K)$ is a Σ^1_1 equivalence relation on ω. In [6], it is shown that isomorphism on computable trees lies on top under $\leq_{FF}$. The same is true for isomorphism of computable torsion-free Abelian groups and even Abelian p-groups. The proofs of these results make no use of the extra equivalence class. In particular, for an arbitrary Σ^1_1 equivalence relation E on ω, we produce a uniformly computable sequence of trees $(T_n)_{n\in\omega}$ such that mEn iff $T_m \cong T_n$. With this in mind, we introduce another reducibility $\leq_{FF^*}$, which is a natural adjustment of $\leq_{FF}$ for comparing classes of computable structures.

Definition 3. *Let K and K' be two classes of structures, closed under isomorphism. We write $K \leq_{FF*} K'$ if there is a partial computable function f satisfying the following conditions.*

- *For each $a \in I(K)$, $f(a)$ is defined, with value $b \in I(K')$.*
- *For $a, a' \in I(K)$, indices for structures $\mathcal{A}, \mathcal{A}'$, corresponding to structures $\mathcal{B}$ with index $f(a)$ and $\mathcal{B}'$ with index $f(b')$, $\mathcal{A} \cong \mathcal{A}'$ iff $\mathcal{B} \cong \mathcal{B}'$.*

It is easy to see that $\leq_{tc}$ reducibility implies $\leq_{FF*}$ reducibility. Suppose $K \leq_{tc} K'$ via $\Phi = \varphi_e$. Given an index $a \in I(K)$ for a computable structure $\mathcal{A}$ in K, we know a procedure for computing the diagram of $\Phi(\mathcal{A}_e)$—we take $\varphi_e^{D(\mathcal{A}_a)}$. We let $f(a)$ be the resulting index.

1.2 New results

Let K be a class of L-structures, closed under isomorphism. For convenience, we suppose that L is a relational language. We can treat functions and constants as relations, so this is no real restriction on our classes.

Definition 4.

1. *K^n is the class of* cardinal sums *of n elements of K. The language of these structures consists of the symbols of L, plus new unary predicates $U_1, \ldots, U_n$. For each $\mathcal{A} \in K^n$, we have n structures $\mathcal{A}_1, \ldots, \mathcal{A}_n \in K$, with disjoint universes, the universe of $\mathcal{A}_i$ is $U_i^{\mathcal{A}}$, and for $R \in L$, $R^{\mathcal{A}} = \cup_{1\leq i\leq n} R^{\mathcal{A}_i}$.*
2. *K^{n*} is the class of* equivalence sums *of n elements of K. The language of these structures consists of the symbols of L, plus a new binary relation symbol $\sim$. For each $\mathcal{A} \in K^{n*}$, $\sim^{\mathcal{A}}$ is an equivalence relation with n equivalence classes. We have n structures $\mathcal{A}_1, \ldots, \mathcal{A}_n \in K$ whose universes are the $\sim$-equivalence classes, and for $R \in L$, $R^{\mathcal{A}} = \cup_{1\leq i\leq n} R^{\mathcal{A}_i}$.*

For any class K with just one computable member, up to isomorphism, whose index set is maximally complicated, we show that $K^n <_{tc} K^{n+1}$. This implies the following.

Theorem 1.3. *For each of the following classes K, we have $K^n <_{tc} K^{n+1}$ and $K^n <_{FF^*} K^{n+1}$:*

1. *S—the class of structures in the pure language of equality.*
2. *V—the class of $\mathbb{Q}$-vector spaces*
3. *computable well orderings, plus orderings of Harrison type,*
4. *computable rank-saturated trees.*

We then show that for each n, and for each of the classes S and V, the two kinds of n-fold sums, cardinal sums and equivalence sums, are tc-equivalent.

Theorem 1.4. *$S^n \equiv_{tc} S^{n*}$ and $V^n \equiv_{tc} V^{n*}$;*
thus, $S^{n*} <_{tc} S^{(n+1)*}$ and $V^{n*} <_{tc} V^{(n+1)*}$.

Finally, we consider direct products of certain groups. Let P^n be the class of direct products of n groups of the form $\mathbb{Z}_{p^m}$ ($m \geq 1$) or $\mathbb{Z}_{p^\infty}$ (the Prüfer group).

Theorem 1.5. *$P^n \equiv_{tc} S^n$: thus, $P^n <_{tc} P^{n+1}$.*

The results on cardinal sums are in Section 3. In Section 2, we give some background on index sets. The results on equivalence sums are in Section 4. The results on direct products of groups are in Section 5.

2 Index sets

In this section we give some further background on index sets. Recall from the introduction that a class K, closed under isomorphism, is *nice* if it is axiomatized by a computable infinitary sentence.

Note: If K is a nice class, then $I(K)$ is hyperarithmetical.

Definition 5 (Γ-hard, Γ-complete)**.** *Let Γ be a complexity class (such as Π^0_n or d-Σ_n or Δ^1_1), and let $A \subseteq \omega$.*

1. *A is* Γ-hard *if for every $S \in \Gamma$, $S \leq_m A$,*
2. *A is* m-complete Γ *if $A \in \Gamma$, and A is Γ-hard.*

Sometimes, we want to describe a specific structure $\mathcal{A}$ so as to differentiate it from other members of a class K. The description of K may be complicated, more complicated than the description of $\mathcal{A}$ within K. The following definitions are from [2].

Definition 6 (Γ within, Γ-hard within, Γ-complete within)**.** *Let $A, B \subseteq \omega$, where $A \subseteq B$.*

1. *A is* Γ within B *if there is some $C \in \Gamma$ such that $A = C \cap B$.*
2. *A is* Γ-hard within B *if for each $S \in \Gamma$, there is a computable function f such that for all n, $f(n) \in B$, and $f(n) \in A$ iff $n \in S$.*
3. *A is m-complete* Γ within B *if it is Γ within B, and Γ-hard within B.*

Note that if A is Γ and $A \subseteq B$, then A is trivially Γ within B. The following index set calculations are given in [3].

Example 1. *Recall that V is the class of non-trivial $\mathbb{Q}$-vector spaces. Let $\mathcal{A} \in V$.*

1. *If $\mathcal{A}$ has infinite dimension, then $I(\mathcal{A})$ is m-complete Π^0_3. It is Π^0_3 hard within $I(V)$.*
2. *If $\mathcal{A}$ has finite dimension, then $I(\mathcal{A})$ is m-complete d-Σ^0_2. It is d-Σ^0_2-hard within $I(V)$.*

3 Cardinal sums

In this section, we consider cardinal sums of pure sets, cardinal sums of vector spaces, and cardinal sums of structures from some further classes that are not nice. In each case, we consider sums of structures from a class K with one computable member that is harder to describe than the others. We consider complexity classes Γ of the forms Π^0_n, d-Σ^0_n $(n \geq 1)$ or Δ^1_1. Recall that a set is d-Σ_n if it is a difference of Σ^0_n sets. This is the same as the intersection of a Σ^0_n set and one that is Π^0_n. These classes Γ have the following features.

1. Γ is closed under finite intersection. This implies that if $u \in K^s$ is s-fold sum of structures with index sets that are Γ within $I(K)$, then $I(u)$ is Γ within $I(K^s)$.
2. If $B \in \Gamma$ and $A \leq_m B$, then $A \in \Gamma$. More is true. Suppose $A \subseteq C$ and $B \subseteq D$, and f is a partial computable function from C to D such that for $x \in C$, we have $x \in A$ iff $f(x) \in B$—we might say that f reduces A within C to B within D. If B is Γ within D, then A is Γ within C. This implies that if $\mathcal{A} \in K$ and $\mathcal{B} \in K'$ are computable structures, and f is a partial computable function taking $I(\mathcal{A})$ to $I(\mathcal{B})$ and $I(K) - I(\mathcal{A})$ to $I(K') - I(\mathcal{B})$, and $I(\mathcal{B})$ is Γ within $I(K')$, then $I(\mathcal{A})$ is Γ within $I(K)$.

Theorem 3.1. *Let Γ be one of the complexity classes Π^0_n, d-Σ^0_n, or Δ^1_1. Suppose K is a class with (up to isomorphism) exactly one computable member $\mathcal{A}$ such that $I(\mathcal{A})$ is not Γ within $I(K)$. Then $K^n <_{tc} K^{n+1}$.*

Proof. There is an obvious embedding witnessing that $K^n \leq_{tc} K^{n+1}$. We copy the first n components of the input structure, and we fix the last component, giving it type $\mathcal{A}$. To show that $K^{n+1} \not\leq_{tc} K^n$, it is enough to prove the following.

Lemma 3.2. *If $K^s \leq_{tc} K^t$ via Φ, then Φ maps computable structures with at least r components of type $\mathcal{A}$ to structures with at least r components of type $\mathcal{A}$.*

Proof. We proceed by induction on r. For $r = 0$, the result is trivial. Assuming that the result holds for r, we prove it for $r+1$. Let u be a computable member of K^s with $r+1$ components of isomorphism type $\mathcal{A}$, and let v be a computable member of K^t with just r components of type $\mathcal{A}$. Suppose $\Phi(u) = v$, expecting a contradiction. Let L^s consist of the elements of K^s with at least r components of type $\mathcal{A}$, and let L^t consist of the elements of K^t with at least r components of type $\mathcal{A}$. From Φ we get a partial computable function f from $I(K^s)$ to $I(K^t)$, taking $I(u)$ to $I(v)$ and taking $I(L^s)$ to $I(L^t)$.

Claim 1: $I(v)$ is Γ within $I(L^t)$. We describe v within L^t by describing within K the $t - r$ components not of type $\mathcal{A}$.

Claim 2: $I(u)$ is not Γ within $I(L^s)$. Let Ψ be a Turing operator taking K to K^s such that in the output structure, a copy of the input structure fills the last place filled by $\mathcal{A}$ in u, and the other components of the output are the same as for u. This gives a partial computable function g taking $I(K)$ to $I(L^s)$ such that $e \in I(\mathcal{A})$ iff $g(e) \in I(u)$. If $I(u)$ were Γ within $I(L^s)$, say $C \cap I(L^s)$, then $I(\mathcal{A})$ would be Γ within K, as $I(\mathcal{A}) = g^{-1}(C) \cap K$. It follows that $I(u)$ is not Γ within $I(L^s)$.

From Φ, we get a partial computable function f taking $I(K^s)$ to $I(K^t)$, such that f takes $I(L^s)$ to $I(L^t)$, and $e \in I(u)$ iff $f(e) \in I(v)$. Therefore, $I(u)$ is Γ within $I(L^s)$, contradicting Claim 2 above. □

The theorem is proved. □

We have the corresponding result for FF^*-embeddings.

Theorem 3.3. *Let Γ be one of the complexity classes Π^0_n, d-Σ^0_n, or Δ^1_1. Suppose K is a class with exactly one computable member $\mathcal{A}$ (up to isomorphism) such that $I(\mathcal{A})$ is not Γ within K. Then $K^n <_{FF*} K^{n+1}$.*

Proof. The Turing computable embedding of K^n into K^{n+1} gives an FF^*-embedding. Suppose $K^{n+1} \leq_{FF*} K^n$ via f. Let u be a computable element of K^{n+1} with all components of type $\mathcal{A}$. Then f gives an m-reduction of $I(u)$ to $I(v)$ for some computable structure $v \in K^n$. The proof above shows that if $K^s \leq_{FF*} K^t$ via f, then f maps indices for computable structures with at least r components of type $\mathcal{A}$ to indices for structures with at least r components of type $\mathcal{A}$. □

3.1 Sums of sets

Let S be the class of structures in the empty language. Then S^n is the class of structures $\mathcal{A}$ in the language $\{U_i \mid 1 \leq i \leq n\}$, where each U_i is a unary relation symbol, and the universe of $\mathcal{A}$ is the disjoint union of the sets $U_i^{\mathcal{A}}$.

Proposition 3.4. $S^n <_{tc} S^{n+1}$

Proof. Let $\mathcal{A}$ be the countably infinite structure in the empty language. Then $I(\mathcal{A})$ is Π^0_2-complete within $I(S)$, whereas for any finite structure $\mathcal{B}$ in the empty language, $I(\mathcal{B})$ is d-Σ_1. Taking Γ to be the complexity class d-Σ_1, we are in a position to apply Theorem 3.1. □

3.2 Sums of vector spaces

Recall that V is the class of non-trivial $\mathbb{Q}$-vector spaces. Then V^n is the class of cardinal sums of n vector spaces.

Proposition 3.5. $V^n <_{tc} V^{n+1}$.

Proof. Let $\mathcal{A}$ be the infinite-dimensional $\mathbb{Q}$-vector space. Then $I(\mathcal{A})$ is Π^0_3-complete within $I(V)$ [3]. If $\mathcal{B}$ is a finite-dimensional vector space, then $I(\mathcal{B})$ is d-Σ_2. Taking Γ to be the complexity class d-Σ^0_2, we are in a position to apply Theorem 3.1. □

3.3 Sums of ordinals and rank-saturated trees

We consider some examples of classes K with, up to isomorphism, a single computable member $\mathcal{A}$ of K such that $I(\mathcal{A})$ is non-hyperarithmetical within $I(K)$, and we apply Theorem 3.1 to these classes.

Example 2. *Let K be the class of orderings that are either well orderings or have the form $\alpha(1+\eta)$, for an admissible ordinal α. The single computable member of K whose index set is non-hyperarithmetical within $I(K)$ is the Harrison ordering, which has order type $\omega_1^{CK}(1+\eta)$. Letting Γ be the complexity class Δ^1_1 (hyperarithmetical), we can apply Theorem 3.1 to show that $K^n <_{tc} K^{n+1}$.*

The next example is the class of *rank-saturated* subtrees of $\omega^{<\omega}$. For a tree $T \subseteq \omega^{<\omega}$, the tree rank of the nodes is defined as follows. We write $tr(\sigma)$ for the *tree rank*, or *foundation rank* of the node σ. We have $tr(\sigma) = 0$ if σ has no successors. For $\alpha > 0$, $tr(\sigma) = \alpha$ if all successors of σ have ordinal ranks, and α is the least ordinal greater than all of these. We let $tr(\sigma) = \infty$ if σ does not have ordinal rank.

Definition 7 (Rank-saturated tree)**.** *A tree $T \subseteq \omega^{<\omega}$ is* rank-saturated *if, for the least ordinal α that is the ordinal of an admissible set containing a copy of T, for all $\sigma \in T$, either*

1. *$tr(\sigma)$ is an ordinal less than α and for all $\beta < tr(\sigma)$, σ has infinitely many successors σ' with $tr(\sigma') = \beta$, or*

2. $tr(\sigma) = \infty$ *and for all* $\beta < \alpha$, σ *has infinitely many successors* σ' *with* $tr(\sigma') = \beta$, *plus infinitely many successors* σ' *with* $tr(\sigma') = \infty$.

For a computable rank-saturated tree, the ordinal α must be ω_1^{CK}.

Example 3. *Let* K *be the class of rank-saturated trees* $T \subseteq \omega^{<\omega}$. *These trees are characterized by the rank of the root node. If* $\mathcal{A}$ *is the computable rank-saturated tree with rank* ∞, *then* $I(\mathcal{A})$ *is not hyperarithmetical within* K. *The other computable members of* K *all have hyperarithmetical index sets. Again, letting* Γ *be the class* Δ^1_1, *we can apply Theorem 3.1 to show that* $K^n <_{tc} K^{n+1}$.

The two classes K above are not nice. If K is a nice class with just one computable member $\mathcal{A}$ (up to isomorphism) such that $I(\mathcal{A})$ is not hyperarithmetical, of course, Theorem 3.1 would apply. We do not actually know of a class like this. Assuming that Vaught's Conjecture fails, we get a relativized version. Becker [1] and Montalbán [13], showed that if there is a counterexample to Vaught's Conjecture, then there is an $L_{\omega_1\omega}$ sentence φ such that for a cone of sets X, φ has, up to isomorphism, a unique X-computable model $\mathcal{A}$ for which the set of X-computable indices is not X-hyperarithmetical; i.e., it is not in the least admissible set containing X. The two proofs were independent, with Becker's just a little earlier. We may suppose that φ is X-computable. Then $Mod(\varphi)$ is a "nice" class relative to X, and $\mathcal{A}$ is (up to isomorphism), the unique X-computable member such that $I(\mathcal{A})$ is not X-hyperarithmetical, so it is not X-hyperarithmetical within $I(K)$.

4 Equivalence sums

Recall that for a class K, K^{n*} consists of structures with an equivalence relation that partitions the universe into n equivalence classes, with a structure from K on each equivalence class. Recall also that S is the class of sets (structures for the empty language), and V is the class of non-trivial $\mathbb{Q}$-vector spaces.

Proposition 4.1.

1. $S^{n*} <_{tc} S^{(n+1)*}$
2. $V^{n*} <_{tc} V^{(n+1)*}$

Proof. We know that $S^n <_{tc} S^{n+1}$ and $V^n <_{tc} V^{n+1}$. Therefore, to prove Proposition 4.1, it is enough to prove that $S^n \equiv_{tc} S^{n*}$ and $V^n \equiv_{tc} V^{n*}$.

Lemma 4.2. $S^n \equiv_{tc} S^{n*}$

Proof. We first show that $S^n \leq_{tc} S^{n*}$. Given an input structure $\mathcal{A} \in S^n$, we produce an output structure $\mathcal{B} \in S^{n*}$ that codes which components of $\mathcal{A}$ have which size. When we are building $\mathcal{B}$, there is an ordering on the classes, corresponding to that in $\mathcal{A}$. If at stage s, we have seen m elements in the k^{th} component of $\mathcal{A}$, then the k^{th} class in $\mathcal{B}$ has size $\langle k, m\rangle$. From the isomorphism type of the output $\mathcal{B}$, we can recover the set of pairs (k, m) such that the k^{th} component of the input $\mathcal{A}$ has size m. For $1 \leq k \leq n$, if there is no such pair (k, m), then the k^{th} component of the input is infinite.

Next, we show that $S^{n*} \leq_{tc} S^n$. Given an input structure $\mathcal{A} \in S^{n*}$, we produce an output structure $\mathcal{B} \in S^n$ that codes the number of components of each size. At stage s, we produce a sequence of numbers $k_1^s, \ldots, k_n^s$, representing the current sizes of the components of $\mathcal{A}$, arranged in non-decreasing order. We give the i^{th} component of $\mathcal{B}$ size k_i^s. $\square$

Lemma 4.3. $V^n \equiv_{tc} V^{n*}$

Proof. We first show that $V^n \leq_{tc} V^{n*}$. Recall the standard computable guessing function $d(k, s)$ such that $\liminf_s d(k, s)$ is the dimension of the k^{th} component—we are guessing at basis elements. For an input $\mathcal{A} \in V^n$, we build an output $\mathcal{B} \in V^{n*}$. At stage s, $d(k, s)$ is our guess at the dimension of the k^{th} component in $\mathcal{B}$. We give the k^{th} component in $\mathcal{B}$ (the ordering of the components is not part of the structure) dimension $\langle k, d(k, s)\rangle$. We designate the basis elements. When the dimension of a component increases, we add new basis elements, and when the dimension decreases, we keep the first ones, and remove the last ones. From $\mathcal{B}$, we can recover the set of pairs (k, m) such that $m = \liminf_s d(k, s)$. For $1 \leq k \leq n$ such that $\liminf_s d(k, s) = \infty$, the k^{th} component of the input has infinite dimension.

Next, we show that $V^{n*} \leq_{tc} V^n$. For an input $\mathcal{A} \in V^{n*}$, we build an output $\mathcal{B} \in V^n$ with the same dimensions, but arranged in a non-decreasing sequence. We order the equivalence classes of $\mathcal{A}$ by the first element, so we have components $V_1, \ldots, V_n$. There is a standard procedure for guessing the dimension of a vector space, so that the dimension of V is the $\liminf$ of the stage s guesses, and the dimension is infinite if the $\liminf$ is ∞ (does not exist). If $n = 2$, and both components have dimension 2, we may always guess that one has dimension at least 3. We must think of a scheme that yields $\mathcal{B}$ with the correct dimensions.

Using the jump of the atomic diagram of $\mathcal{A}$, we can determine whether a given tuple of elements is linearly independent in V_i. Then we can compute a basis for each component V_i. We look at the elements, in order, and add an element to our basis if it is independent of the elements we have already included. For each s, we count the number of basis elements among the first s elements of V_i, and we arrange these numbers in a non-decreasing sequence $f(s) = (d_1^s, \ldots, d_n^s)$. Obviously, for each s, d_i^s is non-decreasing in i. We can see that for each i, d_i^s is non-decreasing in s. If some V_i has finite dimension d, and all other V_j have dimension at least d, then for all sufficiently large s, $d_1^s = d$. More generally, if $(d_1, \ldots, d_k)$ is the sequence of dimensions for the components of finite dimension, arranged in non-decreasing order, then $(d_1, \ldots, d_k)$ is an initial segment of $f(s)$ for all sufficiently large s.

Keep in mind that the function f is $\Delta_2^0(\mathcal{A})$. Computably in $\mathcal{A}$, we guess the values so that we are eventually correct on each initial segment. At stage t, we have what we believe to be $f \upharpoonright s$, for some s. At stage $t + 1$, we either extend, adding one more value $f(s)$ at the end of our stage t sequence, or drop back and change the first value that no longer seems correct. The guesses are sensible; for our stage t guess $f \upharpoonright s$, where for all $k < s$, $f(k) = (d_1^{k,t}, \ldots, d_n^{k,t})$, we have that for each fixed i, $d_i^{k,t}$ is non-decreasing in k. Of course, for each fixed k, $d_i^{k,t}$ is non-decreasing in i.

We explain how to construct $\mathcal{B}$, based on the guesses at f. At stage t, suppose the last term of our guess is $f(s-1) = (d_1, \ldots, d_n)$ $(d_i = d_i^{s-1,t})$. We picture $\mathcal{B}$ with components $U_1, \ldots, U_n$ such that U_i has dimension d_i. Of course, we enumerate only finitely much of the diagram, but we mention the basis elements. Suppose at stage $t+1$, we have $f(0), \ldots, f(s), f(s+1)$, where $f(s+1) = (d'_1, \ldots, d'_n)$ $(d'_i = d_i^{s+1,t+1})$. We must have $d'_i \geq d_i$. We add basis elements, as needed, to the components U_i. We keep track of which were the basis at stage s, for $f(s-1)$, and which are the new ones, for $f(s)$. Now, suppose at stage $t+1$, our guesses at $f(j)$, for $j < k$ are unchanged, but we have a new guess at $f(k)$, say $(d_1^*, \ldots, d_n^*)$, where $d_i^* = d_i^{k,t+1})$. We may have $d_i^* < d_i$, for some i. For such i, we introduce dependencies to collapse any extra basis elements in U_i. We keep as part of our basis for U_i the elements created for $f(k-1)$.

If for some i, there is a limiting value for d_i^s, then U_i will eventually have d basis elements which are never collapsed, and any further basis elements that we add temporarily will later be collapsed. If there is no limiting value for d_i^s, then for each s, there is some stage t after which our guess at $f(s) = (d_1^s, \ldots, d_n^s)$ will never change, and if U_i will eventually have d_i^s basis elements that will never be collapsed. In the end, the dimension is infinite. $\square$

$\square$

5 Abelian groups

In this section, we consider the class P^n of Abelian p-groups that are the direct product of n groups, where each component is either $\mathbb{Z}_{p^m}$ for some $m \geq 1$ or $\mathbb{Z}_{p^\infty}$ (the Prüfer p-group), all for some fixed prime p. The Prüfer p-group $\mathbb{Z}_{p^\infty}$ is the additive Abelian group that can be described as $\langle a_i : i \in \omega | pa_0 = 0, pa_i = a_{i-1} : i > 0 \rangle$. The Prüfer p-group $\mathbb{Z}_{p^\infty}$ can also be viewed as the direct limit of $(\mathbb{Z}_{p^k})_{k \in \omega}$ under the natural inclusion.

We will need some elementary facts about Abelian p-groups. A standard reference is [10]. All groups in this subsection are countable, commutative, and additive. Abelian groups are naturally $\mathbb{Z}$-modules: for an element a of the group and an integer n, we set $na = \mathrm{sign}(n) \cdot \sum_{i \in |n|} a$ if $n \neq 0$, and we define $0a = 0$. Given a prime p, an Abelian group A and $k \geq 1$, we write $p^k A$ to denote the subgroup $\{p^k x | x \in A\} \leqq A$, and we write $A[p]$ for $\{g \in A : pg = 0\}$. Note that $A[p]$ is naturally a $\mathbb{Z}_p$-vector space, its $\mathbb{Z}_p$-dimension is denoted by $\dim_{\mathbb{Z}_p} A[p]$.

Recall that an Abelian group A is a direct sum of its subgroups X and Y if $A = X + Y$ and $X \cap Y = \{0\}$. This direct sum is isomorphic to the direct product of X and Y, and the same holds for any finite number of direct summands. Thus, we can use direct sums instead of direct products. Recall also that every finite Abelian p-group is isomorphic to a direct sum of cyclic groups of the form $\mathbb{Z}_{p^k}$ $(k \geq 1)$; such a *full* decomposition is also unique in the usual sense (i.e., the orders and the number of the cyclic summands are invariants of the group). Finally, it is well-known that every element in P^n is uniquely described by the number and the orders of cyclic and Prüfer summands in its (full) direct decomposition. We prove:

Theorem 5.1. $P^n <_{tc} P^{n+1}$.

Proof. By Proposition 3.4, $S^n <_{tc} S^{n+1}$. Hence, it is enough to show that $P^n \equiv_{tc} S^n$. One of the embeddings is easy.

Lemma 5.2. $S^n \leq_{tc} P^n$.

Proof. Given $V \in S^n$, produce $A \in P^n$ such that if the k^{th} component of V has size α, then the k^{th} direct summand of A is $\mathbb{Z}_{p^\alpha}$. □

Showing that $P^n \leq_{tc} S^n$ will require more work.

Lemma 5.3. $P^n \leq_{tc} S^n$.

Proof. We need a claim.

Claim 5.4. *Suppose $A \in P^n$ and $B \leqq A$, and suppose $\infty \geq r_1 \geq \ldots \geq r_k \geq 1$ and $\infty \geq s_1 \geq \ldots \geq s_m \geq 1$ are such that $A \cong \mathbb{Z}_{p^{r_1}} \oplus \ldots \oplus \mathbb{Z}_{p^{r_k}}$ and $B \cong \mathbb{Z}_{p^{s_1}} \oplus \ldots \oplus \mathbb{Z}_{p^{s_m}}$. Then $k \geq m$ and $r_i \geq s_i$ for $i = 1, \ldots, m$.*

Proof. Recall that for $r \in \omega$, $p^r A = \{p^r x | x \in A\} \leqq A$. Before we prove the claim, we establish several basic properties of the functor $A \to p^r A$.

Note that if $U = X \oplus Y$ then $p^r U = p^r X \oplus p^r Y$ (to be explained shortly). Indeed, $p^r X \leqq p^r U$ and $p^r Y \leqq p^r U$. Furthermore, for every $w \in p^r U$ we have $w = p^r v$ for some $v \in U$. Since $U = X \oplus Y$, there exist $x \in X$ and $y \in Y$ such that $v = x + y$. Thus, $w = p^r(x + y) = p^r x + p^r y \in p^r X + p^r Y$ and

$$p^r U = p^r X + p^r Y.$$

But $p^r X \leqq X$ and $p^r Y \leqq Y$ imply that $p^r X \cap p^r Y = \{0\}$, therefore the above sum is direct.

Observe also that, for any $\alpha \in \omega \cup \{\infty\}$ and $r \in \omega$, we have

$$p^r \mathbb{Z}_{p^\alpha} \cong \mathbb{Z}_{p^{\alpha - r}},$$

where $\infty - r = \infty$ and $\alpha - r = 0$ for $r > \alpha$. (In fact, $p^r \mathbb{Z}_{p^\infty}$ is *equal* to $\mathbb{Z}_{p^\infty}$.)

We now prove the claim. Clearly, $k = \dim_{\mathbb{Z}_p}(A[p]) \geq \dim_{\mathbb{Z}_p}(B[p]) = m$. Suppose $r_t \geq s_t$ for $t = 1, \ldots, i-1$, but $r_i < s_i$. Note that the assumption implies r_i is finite and is smaller than r_j, s_j for each $j < i$. Then $p^{r_i} A$ is isomorphic to $\mathbb{Z}_{p^{r_1 - r_i}} \oplus \ldots \oplus \mathbb{Z}_{p^{r_{i-1} - r_i}}$ and $p^{r_i} B$ is isomorphic to $\mathbb{Z}_{p^{s_1 - r_i}} \oplus \ldots \oplus \mathbb{Z}_{p^{s_i - r_i}} \cdots$, where $\infty - r_i = \infty$. Thus

$$\dim_{\mathbb{Z}_p}((p^{r_i} B)\,[p]) \geq i > (i-1) = \dim_{\mathbb{Z}_p}((p^{r_i} A)\,[p]),$$

which contradicts $(p^{r_i} B)\,[p] \subseteq (p^{r_i} A)\,[p]$. □

Note that that the claim above fully describes finite subgroups of $A \cong \mathbb{Z}_{p^{r_1}} \oplus \ldots \oplus \mathbb{Z}_{p^{r_k}}$ up to isomorphism[1].

We now define a computable reduction witnessing $P^n \leq_{tc} S^n$. We identify a structure with its open diagram and we use Claim 5.4 throughout. Using $A \in P^n$ as an oracle, we can effectively define a *nested* approximation of A by its finite subgroups, i.e.

$$A = \bigcup_s A_s,$$

where $(A_s)_{s\in\omega}$ is a computably enumerable sequence in which $A_s \leqq A$ and $A_s \subseteq A_{s+1}$ (and thus $A_s \leqq A_{s+1}$). Since every A_s is finite, it splits into a direct sum $A_s = \bigoplus_{1\leq i\leq l[s]} \mathbb{Z}_{p^{t_i[s]}}$, where the finite sequence $(t_i[s])_{i\leq l[s]}$ is non-decreasing in i and $l[s] \leq n$. Suppose

$$A \cong \mathbb{Z}_{p^{r_1}} \oplus \ldots \oplus \mathbb{Z}_{p^{r_n}}.$$

Since $A_s \leqq A_{s+1} \leqq A$, we have $l[s] \leq l[s+1] \leq n$ and furthermore, by Claim 5.4, $t_i[s] \leq t_i[s+1] \leq r_i$ for every $i \leq l[s]$.

We define a computable enumeration $(V_s)_{s\in\omega}$ of finite cardinal sums of sets that are nested under componentwise inclusion, as follows. Declare V_s equal to the cardinal sum of sets $X_1[s], \ldots, X_{l[s]}[s]$ having sizes $t_1[s], \ldots, t_{l[s]}[s]$, respectively. Notice that the properties of $(A_s)_{s\in\omega}$ described above guarantee that we can make V_s a substructure of V_{s+1} under the natural componentwise inclusion. Thus, we can define $V = \bigcup_s V_s$ consistently and furthermore effectively and uniformly in A. The structure V is the intended effective image of A in S^n.

It remains to show that the effective transformation $P^n \to S^n$ described above is injective on isomorphism types. Recall that

$$A \cong \mathbb{Z}_{p^{r_1}} \oplus \ldots \oplus \mathbb{Z}_{p^{r_n}},$$

where the first $i \geq 0$ summands are $\mathbb{Z}_{p^\infty}$ and the rest summands are finite. Since $A = \bigcup_s A_s$, there exists an s such that $G \leqq A_s$, where

$$G \cong (\mathbb{Z}_{p^{r_{i+1}+1}} \oplus \ldots \oplus \mathbb{Z}_{p^{r_{i+1}+1}}) \oplus \mathbb{Z}_{p^{r_{i+1}}} \oplus \ldots \oplus \mathbb{Z}_{p^{r_n}} \leqq A.$$

By Claim 5.4, for every $v > s$ we have

$$A_v \cong \mathbb{Z}_{p^{t_1[v]}} \oplus \ldots \oplus \mathbb{Z}_{p^{t_i[v]}} \oplus \mathbb{Z}_{p^{r_{i+1}}} \oplus \ldots \oplus \mathbb{Z}_{p^{r_n}},$$

where $t_j[v] \geq r_{i+1} + 1$ for $j \leq i$. Therefore $\lim_s l[s] = n$ and $\lim_s t_j[s] = r_j$ for $j > i$. Note that for any choice of ('arbitrarily large') cyclic p-groups $C_1, \ldots, C_i$, the group

$$C_1 \oplus \ldots \oplus C_i \oplus \mathbb{Z}_{p^{r_{i+1}}} \oplus \ldots \oplus \mathbb{Z}_{p^{r_n}}$$

[1]Indeed, given any finite p-group B, find its full decomposition $B \cong \mathbb{Z}_{p^{s_1}} \oplus \ldots \oplus \mathbb{Z}_{p^{s_m}}$. If we have $k \geq m$ and $r_i \geq s_i$ for $i = 1, \ldots, m$ then we can embed B into A. Conversely, if $B \leqq A$ is finite then (using the same notation) we get $k \geq m$ and $r_i \geq s_i$ for $i = 1, \ldots, m$ by the claim. We also note that the claim in fact describes the isomorphism types of arbitrary subgroups of $A \in P^n$, but this fact has no use for us.

is embeddable into A. Thus, the same argument as we had for G illustrates $\lim_s t_j[s] = \infty,\ \ j = 1, \ldots, i$.

We conclude that

$$\lim_s t_j[s] = r_j,\ \ j = 1, \ldots, n,$$

where each sequence $(t_j[s])_s$ is non-decreasing (Claim 5.4). Hence, the output V will be the cardinal sum of n sets X_j whose cardinalities match with the respective $\lim_s t_j[s]$ (under the identification $\omega = \infty$). It remains to recall that the tuple $(r_1, \ldots, r_n)$, where $\infty \geq r_1 \geq \ldots \geq r_n \geq 1$, is a full and injective isomorphism invariant of both A and the corresponding V. $\square$

We note that proving $P^n <_{tc} P^{n+1}$ directly (i.e., without any reference to S^n) would require more work. $\square$

References

[1] Becker, H., "Isomorphism of computable structures and Vaught's Conjecture", *J. Sym. Logic*, vol. 78(2013), p. 1328-1344.

[2] W. Calvert, "The isomorphism problem for computable Abelian p-groups of bounded length", *J. Symb. Logic*, vol. 70(2005), pp. 331-345.

[3] W. Calvert, V. S. Harizanov, J. F. Knight, S. Miller, "Index sets of computable structures", *Algebra and Logic*, vol. 45(2006), pp. 306-325.

[4] W. Calvert, D. Cummins, J. F. Knight, and S. Miller, "Comparing classes of finite structures", *Algebra and Logic*, vol. 43(2004), pp. 666-701.

[5] E. B. Fokina, S-D. Friedman, "On Σ^1_1 equivalence relations over the natural numbers", *Math. Log. Quarterly*, vol. 58(2012), pp. 113-124.

[6] E. B. Fokina, S-D. Friedman, V. S. Harizanov, J. F. Knight, C. McCoy, and A. Montalbán, "Isomorphism relations on computable structures", *J. Symb. Logic*, vol. 77(2012), pp. 122-132.

[7] E. B. Fokina, J. F. Knight, A. Melnikov, S. M. Quinn, and C. M. Safranski, "Classes of Ulm type and coding rank-homogeneous trees in other structures", *J. Symb. Logic*, vol. 76(2011), pp. 846-869.

[8] H. Friedman and L. Stanley, "A Borel reducibility theory for classes of countable structures", *J. Symb. Logic*, vol. 54(1989), pp. 894-914.

[9] S. S. Goncharov and J. F. Knight, "Computable structure and non-structure theorems", *Algebra and Logic*, vol. 41(2002), pp. 351-373.

[10] I. Kaplansky, *Infinite Abelian Groups*, Univ. of Michigan Press, 1954.

[11] J. F. Knight, S. Miller, and M. Vanden Boom, "Turing computable embeddings", *J. Symb. Logic*, vol. 72(2007), pp. 901-918.

[12] Christina Maher (now Safranski), *On Embeddings of Computable Structures, Classes of Structures, and Computable Isomorphism*, PhD dissertation, University of Notre Dame, 2009.

[13] A. Montalb'an, "A computability theoretic equivalent to Vaught's Conjecture", pre-print.