COMPUTABLE TOPOLOGICAL GROUPS AND PONTRYAGIN DUALITY

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ABSTRACT. The well-known Pontryagin Duality (classically) reduces the study of compact abelian groups to the algebraic theory of discrete abelian groups. At first glance, Pontryagin Duality seems to be "algorithmic" in nature. Quite unexpectedly, the situation is more intricate. Nonetheless, using methods of computable analysis [Wei00] and modern techniques of computable algebra (e.g., the recent metatheorem [HTMM15]), we establish a partial algorithmic analogy of Pontryagin Dulity and use it to derive a handful of applications. We believe that most of these applications are fundamental to the emerging systematic theory of computable Polish groups. We also apply our techniques to measure the complexity of the classification problem for profinite and connected compact Polish groups.

1. INTRODUCTION

Which topological groups admit algorithmic presentations? Can we classify common subclasses of compact topological groups – such as profinite abelian groups – using reasonable invariants? Can we characterise profinite groups that admit a unique algorithmic presentation? In this paper we combine methods of topological group theory, computable analysis, and computable structure theory to answer questions of this sort.

Metakides and Nerode [MN79] initiated the study of algorithms in topological groups ([GR93, LR81, Smi81b, MM]), and Mal'cev [Mal62, Mal61] laid the foundations of constructive (discrete) group theory ([EG00, AK00, Khi98]). In this paper we establish a close technical connection between these two subjects. Thus, the paper contributes to the general program proposed in [Mel13] which is focused on applications of effective algebra to computable analysis (see also [MN16, MN13, MM, GMKT, McN15, MS]).

1.1. **Algorithms in group theory.** Algorithmic problems have been central to group theory for many decades, see e.g., Novikov [Nov55], Boone [Boo59], Higman [Hig61], and book [LS01]. In the 1960's Mal'cev [Mal62, Mal61] initiated the systematic study of infinitely generated groups with solvable Word Problem.

Definition 1.1 (Mal'cev). A countably infinite group $(A, +)$ is *constructive* if there exists an injective numbering of its universe $A = \{a_0, a_1, a_2, \ldots\}$ by natural numbers, and a Turing machine T such that $a_i + a_j = a_{T(i,j)}$, for every i, j.

A countably generated group is constructive if, and only if, it has a group-presentation

$$\{x_1, x_2, \ldots \mid r_1, r_2, \ldots\}$$

in which the set of generators and relations are computably enumerable and the Word Problem is algorithmically decidable. If we do not require the Word Problem to be solvable in the presentation, then we say that the group admits a *computably enumerable* (c.e.) presentation.

The theory of infinitely generated constructive groups, and especially of constructive abelian groups, has been very popular within the former USSR [EG00, Ers, Gon81, Khi98, Dob81, Nur74], but some of the key results were independently discovered by experts in the USA and Australia [Lin81, Smi81a, Bar95, Dow97, DK86]. Over the past 60 years the subject has accumulated many results, see books [AK00, EG00] and surveys [Khi98, Mel14]. Of course, such studies have not

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been restricted to groups. There has been a lot of work on constructive fields, Boolean algebras, linear orders and other structures [Gon97, Dow98, AK00, EG00] (we omit the general definition).

The development of constructive field theory led to investigations into algorithmic aspects of profinite groups. Based on Metakides and Nerode [MN79], La Roche [LR81] and Smith [Smi81b] defined and studied *recursive* and *co-recursively enumerable* (co-r.e.) profinite groups.

Definition 1.2. We say that a profinite group P is *recursive* if there exists a uniformly computable (strong) array of finite groups $(F_i)_{i \in \mathbb{N}}$ and surjective homomorphisms $(\phi_i)_{i \in \mathbb{N}}$ such that the inverse system $0 \leftarrow_{\phi_0} F_0 \leftarrow_{\phi_1} F_1 \leftarrow_{\phi_2} \ldots$ has limit P.

If we do not require the ϕ_i to be onto, we get the (provably) weaker notion of a *co-r.e.* profinite group [LR81]. We note that recursive (co-r.e.) profinite groups are exactly the Galois groups of constructive (resp., c.e.-presented) Galois field extensions [LR81].

What if a topological group is not profinite? The general theory of computable topological groups is still to be developed, but a lot of work has been done on computable Banach spaces and computable Polish spaces [PER89, Wei00, BHW08]. It is natural to extend the classical notions of effectiveness from Polish spaces to Polish groups. Using approximations by points in a computable dense set, we can define the notion of a computable function between Polish spaces (to be clarified in the preliminaries). In particular, it gives the notion of a computable operation upon a computable Polish space, and thus of a computable Polish group:

Definition 1.3 ([MM]). A *computable Polish group* is a computable Polish space equipped with computable group operations ($^{-1}$ and $\cdot$).

We briefly discuss the relationship between Def. 1.3 and Def. 1.2. Every recursive profinite group is clearly computable Polish. As a consequence of our main results, every computable Polish profinite group has $0'$-recursive presentation (Cor. 1.8), and there exists a computable Polish profinite group with no recursive presentation (Cor. 1.6). (We believe that these facts are new.)

1.2. **A computable version of Pontryagin Duality.** Pontryagin Duality is one of the main tools of abstract harmonic analysis (see textbook [Fol16]), and it will also be central to this paper.

Let $\mathbb{T} = (\mathbb{R}, +)/\mathbb{Z}$. For any topological group G, form its *dual group*

$$\widehat{G} = \{\chi \mid \chi \text{ is a continuous group homomorphism from } G \text{ to } \mathbb{T}\}.$$

It is easily seen that $\widehat{G}$ is itself a topological group under the operation $(\chi + \xi)(a) = \chi(a) + \xi(a)$ and the topology of uniform convergence. A locally compact abelian G is discrete iff $\widehat{G}$ is compact abelian, and G compact Polish abelian iff $\widehat{G}$ is countable discrete [Pon66, Mor77]. Pontryagin Duality states that, if G is compact abelian then $G \cong \widehat{\widehat{G}}$ [Pon66, Mor77]. This means that the discrete dual $\widehat{G}$ of a compact abelian G contains all the information about G. Thus, the Duality essentially reduces the study of compact abelian groups to the algebraic theory of abelian groups, see e.g. book [Lot98]. We note that van Kampen extended the Duality to arbitrary locally compact abelian groups [Pon66, Mor77], but we will focus on the compact Polish/discrete countable case.

Our initial (naive) hope was that Pontryagin Duality behaves well with respect to computable Polish and constructive presentations. If that was the case then we would be able to reduce the study of computable Polish compact abelian groups to the theory of constructive (discrete) abelian groups [Khi98, Mel14]. Although classical proofs tend to be non-effective, at first glance the duality seems fully uniformly computable. Quite unexpectedly, the situation is a lot more complicated.

Theorem 1.4.
   (1) Let G be a constructive abelian group. Then $\widehat{G}$ is a computable Polish group.
   (2) There exists a computable Polish compact abelian group W whose (discrete) dual $\widehat{W}$ has no constructivisation.

Our proof of part (1) is not uniform and uses a variety of techniques. We will clarify the difficulty in Proposition 3.3. To circumvent this difficulty, we introduce a new notion of a tractable constructivisation (Def. 3.4). Remarkably, in Lemma 3.5 we prove that this new purely technical property is

equivalent to the existence of a computable Prüfer basis. Dobrica [Dob83] showed that every constructive abelian group has a constructivisation with a computable Prüfer basis. Dobrica's original proof relies on clever combinatorics such as the factorial trick, see [Mel14] for a sketch of the torsion-free case. Unfortunately, the proof in [Dob83] is quite compressed and incomplete (we will discuss this further in Remark 3.9). To clarify the situation, in Proposition 3.10 we give a new and detailed proof of Dobrica's theorem that replaces the combinatorics with abelian group theory and elements of model theory.

In contrast with part (1) of Theorem 1.4, the proof of the "non-effective" part (2) of Theorem 1.4 is more straightforward, but it does require a new idea. The proof of (2) also has two important consequences. Recall that a compact group possesses a unique left-invariant probability measure, the Haar measure. Willem Fouche has asked whether every computable compact Polish group has computable Haar measure. We will show that the group W from the proof of Theorem 1.4(2) witnesses:

Corollary 1.5. *There is a computable compact Polish abelian group in which the Haar measure is not computable.*

Furthermore, we will see that W is the first known example separating definitions 1.3 and 1.2 for profinite groups.

Corollary 1.6. *There exists a computable Polish profinite group that has no recursive presentation.*

Our proofs of the two corollaries above use the third main result of the paper (Theorem 1.9, to be stated). See Section 5 for the proofs. Also, see the Logic Blog (edited by Nies) for a direct proof of Cor. 1.5 that does not use Theorems 1.4 and 1.9.

There are several further questions related to the effective content of Pontryagin Duality that we leave open (to be discussed in Section 6). In this paper we concentrate on further applications of Theorem 1.4.

1.3. **Applications to classification problems.** The computable enumeration of all Turing machines leads to an effective listing of all (partially) computable Polish groups:

$$G_0, G_1, G_2, \dots.$$

Let K be a class of Polish groups. We follow the general framework [GK02] and define the index set of K:

$$I(K) = \{i : G_i \in K\},$$

and the *topological isomorphism problem* for K:

$$E(K) = \{(i, j) : G_i, G_j \in K \text{ and } G_i \cong_{hom} G_j\},$$

where $\cong_{hom}$ stands for topological group-isomorphism (i.e., homeomorphic isomorphism). The more complicated $I(K)$ and $E(K)$ are, the harder it is to classify members of the class K. We will use Arithmetical and Analytical hierarchies [Rog87] to measure the complexity of $I(K)$ and $E(K)$. Our results will be stated in terms of computable groups, but they can be relativised to any given oracle. Therefore, the results measure the complexity of the classification problem in general, not the complexity of only computable objects in the class. See [GK02, DM08] for further background on applications of recursion theory to classification problems.

It takes a bit of work to show that the index set

$$CPGr = \{i : G_i \text{ is a compact Polish group}\}$$

is Arithmetical, i.e. it is definable by a first-order formula in $(\mathbb{N}, 0, +, \times)$ (it is Π_3^0-complete, see Prop. 4.1). Since being a compact Polish group is itself an Arithmetical property, it is natural to investigate the complexity of index sets for various natural subclasses of compact Polish groups[1].

Recall that the connected component of the identity element C forms a normal subgroup, and thus every compact group is an extension of a profinite group by a connected group. We prove:

Theorem 1.7.

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[1] We conjecture that in absence of compactness the index set $\{i : G_i \text{ is a Polish group}\}$ becomes Π_1^1-complete.

(1) *The index sets of profinite and of connected compact Polish groups are both Arithmetical (Π_2^0-and Π_3^0-complete within $CPGr$, respectively).*

(2) *The topological isomorphism problems for profinite abelian groups and for connected compact abelian groups are both Σ_1^1-complete.*

The first part of Theorem 1.7 says that both profinite and connected compact groups admit very low complexity listings among all computable Polish groups, but these lists contain repetitions of isomorphism types. The proof of part (1) relies on a careful definability analysis. Since the index sets are complete in their classes, it follows that the produced formal definitions are optimal. We will use our definability techniques to establish the mentioned above:

Corollary 1.8. *Every computable Polish profinite group has a $0'$-recursive presentation.*

In Section 4 we prove a slightly stronger uniform version of this corollary (see Cor. 4.8). By Corollary 1.6, "$0'$-recursive" cannot be improved to "recursive".

The second part of Theorem 1.7 illustrates that removing repetitions from the listings produced in Theorem 1.7(1) is as hard as it could be. It follows that deciding whether two profinite (or connected compact) abelian groups are topologically isomorphic is as hard as just saying that there exists a topological group isomorphism between them, which itself is a Σ_1^1-statement (Fact 4.9). Note the Σ_1^1-completeness is witnessed by abelian groups, but as far as we know Theorem 1.7(2) is new (but far more expected) for compact groups that are not necessarily abelian.

1.4. **Profinite abelian groups.** Recall that a constructive A is computably categorical (autostable) iff it has a unique constructivization, up to computable isomorphism. Note that a finitely generated constructive algebraic structure is necessarily computably categorical. Thus, computable categoricity can be viewed as a computability-theoretic generalisation of being finitely generated, and there is enough evidence supporting this intuition, see books [AK00, EG00]. The study of computably categorical algebras is central to effective algebra (see books [AK00, EG00]), and there has been an increasing interest in computably categorical Polish and Banach spaces [GMKT, McN15, MS, Mel13].

If a group is profinite, it is more natural to use recursive presentations (Def. 1.2) rather than computable Polish presentations (Def. 1.3). We say that a profinite group is *computably categorical* if it has a unique recursive presentation, up to computable topological group-isomorphism. Can we describe those Polish abelian groups that are both profinite and computably categorical? The reader might expect that such groups should be rather simple, but surprisingly the situation is more complicated. First, we prove:

Theorem 1.9. *Let P be a profinite abelian group.*

(1) *P is recursive iff $\widehat{P}$ is constructive.*

(2) *P is co-r.e. iff $\widehat{P}$ is c.e.-presented.*

(3) *P is computably categorical iff $\widehat{P}$ is computably categorical.*

The proofs of (1) and (2) of Theorem 1.9 are quite straightforward, but part (3) requires some care. The main strength of Theorem 1.9 is in its applications.

It is not hard to see that the duals of profinite abelian groups are torsion [Pon66]. The recent work of M. and Ng [MN] on constructive torsion abelian groups and Theorem 1.9(3) imply:

Corollary 1.10. *The index set of computably categorical profinite abelian groups is Π_4^0-complete.*

Corollary 1.10 implies that one should not expect a *purely algebraic* description of computably categorical profinite abelian groups since categoricity in this class is not relativizable, see [MN] for a discussion. On the other hand, these groups admit a rather syntactically and algorithmically simple listing (without repetition). The main result in [MN] has an involved proof. Proving Corollary 1.10 directly with profinite groups, i.e. without Theorem 1.9, would be an even more challenging task.

In contrast, computably categorical pro-p abelian groups do admit an algebraic description. Theorem 1.9 and the well-known result of Gocharov [Gon80] (and, independently, Smith [Smi81a]) imply:

Corollary 1.11. *Let p be a prime. A pro-p abelian group P is computably categorical iff it is home-omorphic to a (topological) direct product of cyclic p-groups and the group of p-adic integers in which a.e. factor is equal to some fixed cyclic or p-adic group.*

Theorem 1.9 has several further consequences whose statements we delay until Section 5; some of its corollaries have already been stated above (e.g., Cor. 1.6 and Cor. 1.5).

2. Elements of computable analysis

The standard references for computable analysis are [PER89, Wei00, BHW08]. We will also use elements of abelian group theory [Fuc70, Fuc73] and topological group theory [Pon66, Mor77]. This section contains definitions and basic facts that are necessary for the rest of the paper, further definitions will be given when necessary.

2.1. Computable maps between computable Polish spaces.
Recall that a real α is *computable* (Turing [Tur36, Tur37]) if there exists a Turing machine that, given $n \in \mathbb{N}$, outputs a rational r within 2^{-n} of α. A Polish space (M, d) is *computable* if there exists a sequence $(\alpha_i)_{i \in \mathbb{N}}$ of M-points which is dense in M and such that, for every $i, j \in \mathbb{N}$, the distance $d(\alpha_i, \alpha_j)$ is a computable real, uniformly in i and j [Wei00].

Definition 2.1. Let f be a continuous function between Polish metric spaces M and N. A *name* of f the any collection of pairs of basic open balls (B, C) such that $f(B) \subseteq C$, and for every $x \in M$ and every $\epsilon > 0$ there exists $(B, C) \in \Psi$ such that $B \ni x$ and $r(C) < \epsilon$.

It is clear that every continuous function has a name.

Definition 2.2. A function $f : M \to N$ between computable Polish spaces M, N is *computable* if it possesses a c.e. name.

A function is continuous iff it has an X-c.e. name, where X is some oracle.

In a metric space, we say that a Cauchy sequence (x_i) is *fast* if $d(x_i, x_{i+1}) < 2^{-i-1}$. The above definition of a computable map is equivalent to saying that f is represented by a Turing functional that maps fast Cauchy sequences to fast Cauchy sequences (folklore).

Definition 2.3. A computable Polish group is a triple (G, Φ, Ψ), where G is a computable Polish space and Φ and Ψ are (indices for) c.e. names of group-operations $\cdot$ and $^{-1}$ upon G.

The definition below is a variation of similar notions that can be found in [Wei01, MM].

Definition 2.4. Let B, B' be (closed or open) balls in a Polish space M. We say that B is formally included in B' and write $B \subset_{from} B'$ if there exits a rational $\epsilon > 0$ such that $d(cntr(B), cntr(B')) + r(B) + \epsilon < r(B')$.

We write $B \subseteq_{form} B'$ if $B \subset_{form} B'$ or $B = B'$, where the latter means that B and B' have the same description as basic open balls (not that they are equal as sets). Note that $B \subset_{form} B'$ implies $cl(B) \subseteq B'$. It is evident that formal inclusion of basic balls is Σ_1^0.

2.2. The universal compact Polish abelian group.
Let $\mathbb{T}$ be the circle of perimeter 1 equipped with the shortest arc metric. Equivalently, $\mathbb{T}$ is the unit interval in $\mathbb{R}$ in which the end-points are identified. We say that a point $x \in \mathbb{T}$ is *rational* if the respective point of the unit interval is a rational number. Then $\mathbb{T}$ equipped with rational points is a computable Polish group. The direct product

$$\mathbb{A} = \prod_{i \in \mathbb{N}} \mathbb{T}_i,$$

of infinitely many identical copies $\mathbb{T}_i$ of $\mathbb{T}$ carries the natural product-metric

$$D(\chi, \rho) = \sum_{i=1}^{\infty} \frac{1}{2^{-i}} d_i(\chi_i, \rho_i),$$

where each of the d_i stands for the shortest arc metric on $\mathbb{T}_i$. Under this metric and the component-wise operation $\mathbb{A}$ is a computable compact Polish abelian group. The basic open sets in $\prod_{i \in \mathbb{N}} \mathbb{T}_i$ are direct products of intervals with rational end-points such that a.e. interval in the product is equal to the respective $\mathbb{T}_i$. Clearly, we can effectively list all such open sets. (The exact choice of this basic system of balls is not crucial, but it will be convenient to assume that the end-points are rational.) Every compact abelian group can be realised as a closed subgroup of $\mathbb{A}$, this is explained in the next subsection.

2.3. **The group Hom$(\mathbf{G}, \mathbb{T})$.** Suppose $G = \{g_0 = 0, g_1, g_2, \ldots\}$ is a countably infinite discrete group. Let $\mathrm{Hom}(\mathbf{G}, \mathbb{T})$ be the subset of $\mathbb{A} = \prod_{i \in \mathbb{N}} \mathbb{T}_i$, (each $\mathbb{T}_i$ is a copy of $\mathbb{T}$) consisting of tuples $\chi = (\chi_i, \chi_2, \ldots)$, where each such tuple represents a group-homomorphism $\chi : G \to \mathbb{T}$ such that $\chi(g_i) = \chi_i \in \mathbb{T}_i \equiv \mathbb{T}$. Since G is discrete, every group homomorphism $\chi : G \to \mathbb{T}$ is necessarily continuous. Thus, $\widehat{G} \cong \mathrm{Hom}(\mathbf{G}, \mathbb{T})$. Since being a group-homomorphism is a universal property, $\mathrm{Hom}(\mathbf{G}, \mathbb{T})$ is a closed subspace of $\mathbb{A}$. Pontryagin Duality implies that every compact abelian group is homeomorphic to a closed subspace of $\mathbb{A}$. In our later proofs we will need a more detailed understanding of the dual of a discrete G within $\mathbb{A} = \prod_{i \in \mathbb{N}} \mathbb{T}_i$. See [Pon66] and Chapter VIII of [Fuc70] for more information on the duals of elementary groups.

We represent G as the union of its finitely generated subgroups:

$$\{0\} \subseteq \langle g_1 \rangle \subseteq \langle g_1, g_2 \rangle \subseteq \ldots \subseteq \langle g_1, g_2, \ldots, g_i \rangle \subseteq \ldots.$$

It is well-known that the character group $\mathrm{Hom}(\mathbf{G}, \mathbb{T}) \cong \widehat{G}$ is homeomorphic the inverse limit of the system

$$\{0\} \leftarrow \widehat{\langle g_1 \rangle} \leftarrow \widehat{\langle g_1, g_2 \rangle} \leftarrow \ldots \leftarrow \ldots,$$

where embeddings are defined in a certain canonical way. More specifically, if $f : A \to B$ then define $\hat{f} : \widehat{B} \to \widehat{A}$ by the rule $f(\gamma)(a) = \gamma f(a)$, for every $a \in A$ and each $\gamma \in \widehat{B}$. Recall also that for a cyclic group H, $\widehat{H}$ is either $\mathbb{T}$ (if $H \cong \mathbb{Z}$) or a discrete subgroup of $\mathbb{T}$ (if H is finite). We thus can "build" a closed subgroup of $\mathbb{A}$ homeomorphic to $\widehat{G}$. A reader not familiar with such constructions should look at the example below.

Example 2.5. Recall, $\mathbb{A} = \prod_{i \in \mathbb{N}} \mathbb{T}_i$ where each $\mathbb{T}_i$ is a copy of $\mathbb{T}$. Let π_i be the projection of $\mathrm{Hom}(\mathbf{G}, \mathbb{T})$ onto $\mathbb{T}_i$. Clearly, $\chi(0) = 0$ for any character χ of G, thus $\pi_0(\mathrm{Hom}(\mathbf{G}, \mathbb{T})) = 0$. Suppose $\langle g_1 \rangle \cong \mathbb{Z}$, and therefore its dual $\widehat{\langle g_1 \rangle}$ is homeomorphic to $\mathbb{T}$. We then declare $\pi_1(\mathrm{Hom}(\mathbf{G}, \mathbb{T})) = \mathbb{T}_1$.

Now consider $\langle g_1, g_2 \rangle$ and suppose $2g_2 = 0$. Then the possibilities for $\chi(g_2)$ are exhausted by: $\chi(g_2) = 1/2$ or $\chi(g_2) = 0$. We have $\widehat{\langle g_2 \rangle} \cong \mathbb{Z}_2$. If we were to stop here then our dual would consist of arbitrary paris (x, a), where $x \in \mathbb{T}$ and $a \in \{0, 1/2\}$, under the product topology. The respective (finite) inverse sequence would be $0 \leftarrow \mathbb{T} \leftarrow \mathbb{T} \oplus \mathbb{Z}_2$, where the second map is the natural projection onto the first coordinate. We thus declare $\pi_2(\mathrm{Hom}(\mathbf{G}, \mathbb{T})) = \{0, 1/2\} \subseteq \mathbb{T}_2$ without any further restriction whatsoever.

If g_3 does not generate a new fresh direct summand, then the situation becomes slightly more complicated. Assume that (say) $2g_3 = g_1$. Then the order of g_3 is infinite, and thus $\widehat{\langle g_3 \rangle} \cong \mathbb{T}$. For any $x \in \mathbb{T}$, $\chi(g_3) = x$ implies $\chi(g_1) = 2x$. We then declare $\pi_3(\mathrm{Hom}(\mathbf{G}, \mathbb{T})) = \mathbb{T}_3$, but we also require $2\chi_3 = \chi_2$ for each character $\chi = (\chi_0, \chi_1, \ldots)$ of G. We can proceed in this manner to define the closed set representing $\widehat{G}$ within $\mathbb{A}$.

We note that the decision procedure for the projections π_i described in the example above is not necessarily computable (to be illustrated in Proposition 3.3).

3. The effective content of Pontryagin Duality

3.1. **Computably enumerable closed sets.** We say that a closed subset C of a computable Polish metric space M is *computably enumerable* (c.e.) if

$$\{i : B_i \text{ basic open in } M \text{ and } B_i \cap C \neq \emptyset\}$$

is a computably enumerable set. The fact below will be useful in constructing a computable copy of $\widehat{G}$ in Theorem 1.4(1).

Fact 3.1. A closed subset C of a computable Polish space M is computably enumerable if, and only if, C possesses a uniformly computable (in M) dense sequence points.

Note that that the desnse sequence makes C a computable Polish space under the induced metric.

Proof. Suppose C possesses such a computable sequence $(\alpha_i)_{i\in\mathbb{N}}$. Then the density of the sequence in C implies that $B_i \cap C \neq \emptyset$ iff $\exists j \alpha_j \in B_i$, which is a uniformly Σ_1^0 statement.

Now suppose C is a computably enumerable closed subset of M. Our goal is to construct a uniformly computable sequence of points $(\alpha_i)_{i\in\mathbb{N}}$ that is dense in C. Without loss of generality, we assume C is infinite. (If C is finite, then it clearly contains only computable points.) We uniformly approximate the sequence by stages. Before we describe stage s, we give a technical definition.

Definition 3.2. *Two basic open balls U and W are formally s-disjoint if $r(U) + r(W) < d(cntr(U), cntr(W))$ and this can be seen after calculating the radii and the distance with precision 2^{-s}. We say that U and W are formally disjoint if the are formally s-disjoint for some s.*

At stage 0 fix one basic ball $B_{0,0}$ of radius < 1 such that $B_{0,0} \cap C \neq \emptyset$. At stage $s > 1$ first check whether there exists a basic open ball with index $< s$ which is formally s-disjoint from $B_{0,s-1}, \ldots, B_{s-1,s-1}$. If such a basic open B exists, then choose the first fund $B_{s,s} \subseteq_{form} B$ and $B_{i,s} \subseteq_{form} B_{i,s-1}$, $i < s$ such that $B_{j,s} \cap C \neq \emptyset$, the $B_{j,s}$ are pairwise formally disjoint and $r(B_{j,s}) < 2^{-s}$, $j = 0, \ldots, s$. Otherwise, of no such B exists, fix the first found pairwise formally disjoint $B_{0,s}, \ldots, B_{s,s}$ that intersect C, have radii $< 2^{-s}$, and such that $B_{i,s} \subseteq_{form} B_{i,s-1}$ for $i < s$ (note there is no further restriction on $B_{s,s}$). This ends the construction.

Note that all the conditions (that we check at every stage) are computable. Since C is infinite, at no stage we are stack. We then let α_i be the unique point of the Polish space such that $\{\alpha_i\} = \bigcap_{j\geq i} B_{i,j}$. Since the construction is uniform and the radii of balls are rapidly shrinking, the points α_i form a uniformly computable sequence. Since each of the $B_{i,j}$ $(j = i, i+1, \ldots)$ intersects C and C is closed, each $\alpha_i \in C$. It remains to check that $\overline{(\alpha_i)}_{i\in\mathbb{N}}$ is dense in C.

Suppose $c \in C$. We claim that $c \in \overline{(\alpha_i)}_{i\in\mathbb{N}}$. Assume $c \notin \overline{(\alpha_i)}_{i\in\mathbb{N}}$, and there is a ball U centred in c which is outside $\overline{(\alpha_i)}_{i\in\mathbb{N}}$. There will be a basic open ball $B' \ni c$ of radius at most 2^{-n} and which is formally contained in U with precision 2^{-n}:

$$d(cntr(U), cntr(B')) + r(B') < r(U) + 2^{-n}.$$

Then at every stage $s > n + 4$ the balls $B_{i,s-1}$, $i = 0, \ldots, s-1$ will be formally s-disjoint from B, as will be readily witnessed by the metric. At some late enough stage s' we will get a confirmation that $B \cap C \neq \emptyset$. There exist only finitely many basic balls that have their index smaller than the index of B. Therefore, eventually B will be used to define $B_{t,t} \subseteq_{form} B$, contradicting the assumption that $U \cap \overline{(\alpha_i)}_{i\in\mathbb{N}} = \emptyset$. $\square$

Recall that $\mathbb{A} = \prod_{i\in\omega} \mathbb{T}_i$ carries a natural computability structure that makes it a computable compact Polish group. Suppose G is a constructive discrete group. It is reasonable to ask whether the set $\mathrm{Hom}(G, \mathbb{T})$ – considered as a closed subset of $\mathbb{A}$ – is necessarily a computably enumerable closed subset of $\mathbb{A}$. If the answer was positive then Fact 3.1 would allow us to build a computable copy of $\mathrm{Hom}(G, \mathbb{T}) \cong \widehat{G}$. The negative result below justifies the use of Dobrica's result in the proof of Theorem 1.4, but it also has some independent interest.

Proposition 3.3. *There exists a constructivisation of the free abelian group F of rank ω such that $\mathrm{Hom}(F, \mathbb{T})$ is not a c.e. subset of $\prod_{i\in\mathbb{N}} \mathbb{T}_i$.*

Proof. The constructvisation of F will be constructed by stages. In the construction, for every index e we will reserve two witnesses a_e and b_e in F, and we will initially keep them in the generating basis of F. Let $F = \{f_0 = 0, f_1, \ldots\}$, and assume $a_e = f_{u(e)}$ and $b_e = f_{v(e)}$. For each e we will also fix two basic open sets A_e and B_e uniquely determined by intervals $(7/16, 9/16)$ and $(3/16, 5/16)$ in (respectively) $\mathbb{T}_{u(e)}$ and $\mathbb{T}_{v(e)}$:

$$A_e = \mathbb{T}_0 \times \mathbb{T}_1 \times \ldots \times \mathbb{T}_{u(e)-1} \times (7/16, 9/16) \times \mathbb{T}_{u(e)+1} \times \ldots$$

and

$$B_e = \mathbb{T}_0 \times \mathbb{T}_1 \times \ldots \times \mathbb{T}_{v(e)-1} \times (3/16, 5/16) \times \mathbb{T}_{v(e)+1} \times \ldots,$$

and let

$$C_e = A_e \cap B_e.$$

Recall that $\mathbb{T}_{u(e)}$ and $\mathbb{T}_{v(e)}$ are both copies of $\mathbb{T}$. It will be convenient to identify these two copies with $\mathbb{T}$. In this case we write X' and Y' to denote the respective natural images of $X \subset \mathbb{T}_{u(e)}$ and $Y \subset \mathbb{T}_{v(e)}$ in $\mathbb{T}$.

If a_e and b_e were indeed elements of the generating base of F, then would have continuum elements in $C_e \cap \mathrm{Hom}(\mathrm{F}, \mathbb{T})$. We could however declare a_e and b_e linearly dependent arbitrarily late in the construction, and furthermore we could do it so that the new relation that we impose contradicts $\frac{m}{p} A'_e \cap B'_e \neq \emptyset$. This idea leads to a straightforward diagonalisation, as described below.

To diagonalise against the e'th potential decision procedure for the closed set $\mathrm{Hom}(\mathrm{F}, \mathbb{T})$, wait until the procedure declares $C_e \cap \mathrm{Hom}(\mathrm{F}, \mathbb{T}) \neq \emptyset$. If this never happens then we keep a_e and b_e in the generating basis of F and do nothing. In this case we will have diagonalised since there must be uncountably many elements in $C_e \cap \mathrm{Hom}(\mathrm{F}, \mathbb{T})$. If at some stage s we see $C_e \cap \mathrm{Hom}(\mathrm{F}, \mathbb{T}) \neq \emptyset$, then find a prime p and integer $m > 0$ that satisfy $\frac{p}{4} > m > s$ and declare $ma_e = pb_e$. In this case $\frac{1}{4} > \frac{m}{p} > 0$, and therefore $\frac{m}{p} A'_e \cap B'_e = \emptyset$. It will lead to a contradiction since the relation on the group must imply $\frac{m}{p} \chi_{u(e)} = \chi_{v(e)}$ for any character $\chi \in \mathrm{Hom}(\mathrm{F}, \mathbb{T})$. However, every character in $C_e = A_e \cap B_e$ must fail this property, as witnessed by $\frac{m}{p} A'_e \cap B'_e = \emptyset$.

It remains to note that m and p such that $\frac{p}{4} > m > s$ are necessarily co-prime. Therefore, the new relation $ma_e = pb_e$ that we introduce to F at stage s will not imply any relation involving a_e and b_e with smaller coefficients. The only relations upon a_e and b_e implied by $ma_e = pb_e$ are $nma_e = npb_e$, $n \in \mathbb{Z}$. Thus, there will be no contradiction with the piece of F we will have constructed before stage s. $\qquad\square$

3.2. Proof of Theorem 1.4(1).

Let G be a constructive (discrete) abelian group. Our goal is to show that $\widehat{G}$ is homeomorphic to a computable Polish group.

3.2.1. Tractable and effectively predictable presentations.

Note that any constructive abelian group G can be represented as the union of a computable ascending chain of finitely generated groups G_s. Each of the finitely generated groups in the sequence possesses a direct decomposition into cyclic summands, but these decompositions – as well as the isomorphism types of their summands – do not have to be uniformly computable. We exploited this feature in Proposition 3.3 to construct a counter-example. We specifically designed the definition below to avoid this bad scenario.

Definition 3.4. We say that G has a *tractable constructivisation* if there exists a uniformly computable ascending sequence of f.g. abelian groups $(F_i)_{i \in \mathbb{N}}$ with the following properties:

(1) $G = \bigcup_{i \in \mathbb{N}} F_i$.
(2) $0 \subseteq F \subseteq F_2 \subseteq F_3 \subseteq \ldots$ is a uniformly computable sequence in which the natural embeddings are also computable.
(3) For every F_i we can uniformly compute a finite set of elements $h_0, \ldots, h_{k(i)}$ such that the orders of $h_0, \ldots, h_{k(i)}$ are (uniformly) computable and $F_i = \langle h_0 \rangle \oplus \langle h_1 \rangle \oplus \ldots \langle h_{k(i)} \rangle$.

Lemma 3.5. Suppose $(F_i)_{i \in \mathbb{N}}$ is a tractable constructivisation of a (discrete and countable) abelian group G, and let $H = \bigcup_i F_i$. Then $\mathrm{Hom}(\mathrm{H}, \mathbb{T})$ is a c.e. closed subset of $\mathbb{A} = \prod_{i \in \mathbb{N}} \mathbb{T}$.

Proof of Lemma 3.5. We need to argue that if $(F_i)_{i \in \mathbb{N}}$ is tractable then the respective inverse limit (see Example 2.5) can be realized as a computable closed subgroup of $\prod_{i \in \mathbb{N}} \mathbb{T}_i$. By Proposition 3.3, the proof must make an essential use of tractability of the constructivization of G. The plan is as follows. We first introduce a certain special kind of closed subgroups of $\prod_{i \in \mathbb{N}} \mathbb{T}_i$, then we prove that each such closed subgroup forms a c.e. closed subset of $\mathbb{A}$, and then we argue that $\mathrm{Hom}(\bigcup_{i \in \mathbb{N}} \mathrm{F}_i, \mathbb{T}) \cong \widehat{\mathrm{G}}$ is a closed subgroup of this form.

Suppose $\prod_{i\in\mathbb{N}} C_i \subseteq \prod_{i\in\mathbb{N}} \mathbb{T}_i$ is a closed subgroup (and thus so is each of the $C_i \subseteq \mathbb{T}_i$). Assume further that either $C_i = \mathbb{T}_i$ or C_i is finite, and that $C_0 = \{0\}$. We say that C_i is *defined by a primitive relation* if one of the following possibilities is realized:

(1) there is a $j < i$ and a positive integer k such that every $\chi \in \prod_{i\in\mathbb{N}} C_i$ satisfies $\chi_i = k\chi_j$,
(2) there is a $j < i$ and a non-negatiove integer k such that every $\chi \in \prod_{i\in\mathbb{N}} C_i$ satisfies $k\chi_i = \chi_j$,
(3) there exist a $u, v < i$ such that every $\chi \in \prod_{i\in\mathbb{N}} C_i$ satisfies $\chi_i = \chi_u - \chi_v$,
(4) there exist a $u, v < i$ such that every $\chi \in \prod_{i\in\mathbb{N}} C_i$ satisfies $\chi_i = \chi_u + \chi_v$.

In each of the four cases we assume that every finite sequence $(\chi_0, \ldots, \chi_i)$ such that $\chi_j \in C_j$ $(j \leq i)$ satisfy the respective linear conditions can be extended to an infinite sequence $\chi \in C$. (Note if $k = 0$ in (2) then essentially there is no restriction on χ_i.)

We also say that $\prod_{i\in\mathbb{N}} C_i \subseteq \prod_{i\in\mathbb{N}} \mathbb{T}_i$ is *effectively predictable* if there exists a uniform algorithm that decides whether $C_i \cong \mathbb{T}$ or not, and if not then it also outputs the finite tuple of rationals $r_0, \ldots, r_k$ such that $C_i = \{r_i : i \leq k\} \subseteq \mathbb{T}_i$. We say that C is *fully effectively predictable* if furthermore every C_i is defined by a primitive relation that can be computed uniformly in i.

Claim 3.6. *Every fully effectively predictable (thus, closed) subgroup of $\prod_{i\in\mathbb{N}} \mathbb{T}_i$ is a c.e. closed subset of $\prod_{i\in\mathbb{N}} \mathbb{T}_i$.*

Proof of Claim. Suppose we are given a basic open set $W = A_0 \times A_1 \times A_2 \times \ldots$ in which $A_i = \mathbb{T}_i$ for every $i > n$. Now compute the isomorphism types of $C_0, \ldots, C_n$ and the respective primitive relations that define C_i. Recall the end-points used in the definitions of the basic intervals A_i were rational, and thus we can effectively decide which (rational) points are in $A_i \cap C_i$ for each $i \leq n$. Note that we can also check whether at least one rational tuple $(\chi_1, \ldots, \chi_n)$ in W satisfies the first n primitive relations. From this we can decide whether the basic open set W intersects the subgroup. $\square$

Claim 3.7. *Suppose $(F_i)_{i\in\mathbb{N}}$ is a tractable constructivisation of a (discrete and countable) abelian group $H = \bigcup_i F_i$. Then $\mathrm{Hom}(H, \mathbb{T})$ is fully effectively predictable.*

Proof of Claim. Suppose $H = \{h_0 = 0, h_1, \ldots\}$. Our goal is to define $\prod_i C_i \subseteq \prod_i \mathbb{T}_i$ such that we can effectively decide the isomorphism type of C_i and also list primitive relations defining each of the $C_i \subseteq \mathbb{T}_i$ in terms of some C_k, $k < i$.

We can effectively refine the sequence $(F_i)_{i\in\mathbb{N}}$ witnessing Def. 3.4 and assume that for every i, there exists an $a \in F_{i+1}$ such that either $F_{i+1} = \langle a \rangle \oplus F_i$ or $F_{i+1} = \langle F_i, a \rangle$ and for some m we have $ma \in F_{i+1}$. In the former case the order of a is infinite (otherwise we are in the second case). We can effectively find full decompositions of F_i and F_{i+1}, and thus we can effectively figure out which of the two possibilities is realised, and in particular what the order of a is.

In the former case, assuming $a = h_j \in H = \bigcup_i F_i$, we set $C_i = \mathbb{T}_i$ and do not put any restriction (formally, we declare that $0 \cdot \chi_i = \chi_0 = 0$). For $h_k = na = nh_j$, we will set $C_k = \mathbb{T}_k$ and declare $\chi_k = n\chi_j$ as the respective primitive relation. Otherwise, if we have $ma \in F_i$ and $a = h_j \in H$, then we first see what C_k is for $h_k = mh_j \in F_i$, $k < j$. If $C_k = \mathbb{T}_k$ then we declare $C_j = \mathbb{T}_j$ and declare that $\chi_k = m\chi_j$. Otherwise, if C_k is a finite closed subset (in which case it must form a discrete cyclic group), we set $C_j \subset \mathbb{T}_j$ to be the pre-image of C_k along $\phi : \mathbb{T}_i \to \mathbb{T}_k$ that takes $x \in \mathbb{T}_i \cong \mathbb{T}$ and outputs $mx \in \mathbb{T}_k \cong \mathbb{T}$. We also impose the primitive relation $\chi_k = m\chi_j$ on C_j. If $h_s = na = nh_j$ for some $s > j$, then we use a similar procedure to decide what C_i is, and we declare $\chi_s = n\chi_j$ for C_s.

The procedure described above witnesses that $\prod_i C_i = \mathrm{Hom}(H, \mathbb{T}) \cong \widehat{G}$ is fully effectively predictable. $\square$

Lemma 3.5 follows from the two claims above. $\square$

3.2.2. *Tractable groups are exactly the groups with dependence algorithm.* Our goal is to show that all constructive abelian groups admit tractable constructivisations. Recall that elements $g_1, \ldots, g_k$ of an abelian group are *Prüfer independent* (or linearly independent) if, for any integer coefficients $m_1, \ldots, m_k$,

$$\sum_{i \leq k} m_i g_i = 0 \text{ implies } m_1 = \ldots = m_k = 0.$$

According to this definition, every torsion element is dependent on itself. We will write $Span(B)$ for the collection of all elements of the group dependent on B. It should not be confused with the group-theoretic span $\langle B \rangle$ of B. Thus, in particular, $Span(\emptyset) = T(G)$ which is the torsion subgroup of G. Therefore, we are really looking at a basis in the torsion-free $G/T(G)$. (See Fuchs [Fuc70] for an alternate approach related to p-basic subgroups.) Recall that a subgroup H of an abelian A is pure (in A) if, for any $h \in H$ and each positive integer k, $\exists a \in A \ ka = h$ implies $\exists u \in H \ ku = h$. It is well-known that a pure f.g. subgroup of an abelian group A detaches in A (i.e., forms a direct summand of A).

Lemma 3.8. *Let G be a constructive (discrete) abelian group. The following are equivalent:*

   (1) *G is tractable.*
   (2) *G has a computable basis.*

We will be using only (2) $\Rightarrow$ (1) and therefore our proof of (2) $\Rightarrow$ (1) will be somewhat compressed. Nonetheless, the reader should agree that (1) $\iff$ (2) is worth noting.

Proof of Lemma. Suppose G has a computable basis $B = \{b_1, b_2 \ldots\}$ (we include the possibility of B being finite or empty). We can effectively list the torsion subgroup $T(G)$ of G. At every stage we will be enumerating more of B and more of $T(G)$. Suppose at a stage we have effectively defined a f.g. partial subgroup

$$G_s = \langle h_1, \ldots, h_k; t_1, \ldots, t_m \rangle \restriction_t,$$

where k, t and m will depend on the stage, the h_i are linearly independent in $G/T(G)$, and the t_j are torsion elements. We also assume that we can effectively see this information about the isomorphism types of the summands (this is a recursive argument).

We may assume that $Span(h_1, \ldots, h_k) = Span(b_1, \ldots, b_k)$ in $G/T(G)$. Furthermore, w.l.o.g. each t_j generates a (finite and primary) cyclic summand of G_s, and each h_i generates a finite initial segment of the infinite cyclic group:

$$G_s = \langle h_1 \rangle \restriction_t \oplus \ldots \oplus \langle h_k \rangle \restriction_t \oplus \langle t_1 \rangle \oplus \ldots \oplus \langle t_m \rangle.$$

Suppose a new element h enters the enumeration of the group. We will then keep adjoining elements $b_{k+1}, b_{k+2}, \ldots$ from the basis B to $h_1, \ldots h_k$ (note that $\{h_1, \ldots, h_k, b_{k+1}, b_{k+2}, \ldots\}$ forms a basis of G), and we will keep adjoining new elements from $T(G)$ to G_s. At a later stage we will have a f.g. partial group

$$G'_u = \langle h_1 \rangle \restriction_u \oplus \ldots \oplus \langle h_k \rangle \restriction_u \oplus \langle b_{k+1} \rangle \restriction_u \oplus \ldots \oplus \langle b_{k'} \rangle \restriction_u \oplus T(G'_u)$$

containing G_s. We keep doing so until we find a linear combination

$$mh = \sum_{i \leq k} n_i h_i + \sum_{k \leq j \leq k'} n'_j b_j + t,$$

where $t \in T(G'_u)$. We now are ready to define G_{s+1}. It will be a large enough finite partial subgroup approximating the $\langle h, G'_u \rangle$. Note that $\langle h \rangle$ is pure in $\langle h, G'_u \rangle$, and therefore it detaches as a direct summand,

$$\langle h, G'_u \rangle = \langle u \rangle \oplus H.$$

We choose any direct decomposition of H into cyclic summands, and we wait until a late enough stage v such that the generators of all these summands appear in the enumeration of G at stage v. Then we set $G_{s+1} = \langle h, G_u \rangle$ restricted to this stage. We also put the respective generators of G_{s+1}, the information about their orders, and the natural embedding of G_s into G_{s+1}, as the additional information about G_{s+1}. (Note that the orders of these generators of G_{s+1} can be found uniformly effectively using G_s and its decomposition.) Clearly, the Prüfer span of the generators of the infinite cyclic summands will be equal to $Span(b_1, \ldots, b_{k'})$, which is the part of the basis we've used so far.

Strictly speaking, $(G_s)_{s \in \mathbb{N}}$ is a sequence of finite partial subgroups, not a sequence of subgroups of G (as required). Otherwise, all the other properties that we need are satisfied by the sequence $(G_s)_{s \in \mathbb{N}}$. But note that $G = \bigcup_{i \in \mathbb{N}} \langle G_s \rangle$. Based on this observation, we claim that $\langle G_s \rangle$ is a uniformly computable subgroup of G. Indeed, for any $g \in G$ wait for $g \in G_v$ ($s \leq v$), and then use the information about G_v, its generators, and how G_s is embedded into G_v to see if $g \in \langle G_v \rangle$. Finally, observe that all

the embedding of $\langle G_s \rangle$ into $\langle G_s \rangle$ is completely described by the embedding of G_s into G_{s+1}. Thus, $G = \bigcup_{i \in \mathbb{N}} \langle G_s \rangle$ witnesses that G is tractable.

Conversely, suppose G is a tractable constructive group, and let $(F_i)_{i \in \mathbb{N}}$ be an ascending sequence of its subgroups witnessing its tractability. Suppose we have $g_1, \ldots, g_k \in G$. Then for some large enough m we must have $g_1, \ldots, g_k \in F_m$. Since we can compute a full decomposition of F_k, we can decide whether $g_1, \ldots, g_k$ are independent just by analysing the matrix of the projections of $g_1, \ldots, g_k$ onto the infinite cyclic summands in the decomposition. (Attempt to transform the matrix into a diagonal one using integer elementary operations; then see whether the result has zero entrees along the main diagonal.) We can use the dependence algorithm to produce a computable basis of the group in the usual way (see, e.g., [Mel14]). □

3.2.3. *A new proof of Dobrica's theorem.* Recall that tractability is equivalent to having a linear dependence algorithm. Thus, to show that every constructive abelian group has a tractable copy, it is sufficient to use the result of Dobrica [Dob83] saying that every constructive abelian group has a copy with computable basis. As we promised in the introduction, in this subsection we give a new and very detailed proof of the result of Dobrica.

Remark 3.9. We also discuss the problem with the proof in [Dob83]. The proof does not explain why partial embeddings at intermediate stages preserve the partial diagram of the group built so far. This is the key subtlety in the proof, but it is not even briefly mentioned. Dobrica uses a rather elaborate choice of coefficients at intermediate stages (e.g., $k_i \dfrac{mt!}{k}$), but this particular choice is never clarified. We believe that Dobrica's sketch can be adjusted and extended to a complete and correct proof, but we chose to give a new proof that avoids dealing with complicated coefficients.

Recall that the cardinality of any basis is called the rank of an abelian G. The rank of a subset X of G is the rank of the subgroup generated by this set. If the rank of G is finite, then it has a computable basis.

Proposition 3.10 (Dobrica)**.** *There exists a uniform procedure which transforms any constructive abelian G of infinite rank into a constructive $H \cong G$ that has a computable Prüfer basis.*

Proof of Proposition 3.10. Clearly, the linear span (defined above) is a closure operator and therefore it induces a r.i.c.e. pre-geometry, see [HTMM15] or book [Mar02] for definitions. For the rest of the proof it is not really necessary to know what a r.i.c.e. pregeometry is. We will only use that x, y are independent over a subset X if $x \notin Span(X \cup \{y\})$ and $y \notin Span(X \cup \{x\})$. We also say that such x, y are interchangeable over X if $Span(X \cup \{y\}) = Span(X \cup \{x\})$.

The *independence diagram* $\mathcal{I}(\bar{c})$ of $\bar{c}$ in G is the collection of all existential formulas true of tuples independent over $\bar{c}$. We say that independent tuples in G are *locally indistinguishable* if for every tuple $\bar{c}$ in G, any $\bar{u}, \bar{v}$ independent tuples over $\bar{c}$, and each existential formula ϕ such that $G \models \phi(\bar{c}, \bar{u})$ there exists a tuple $\bar{w}$ that is independent over $\bar{c}$, has $G \models \phi(\bar{c}, \bar{w})$, and (with $\bar{w} = (w_1, ..., w_n)$ and $\bar{v} = (v_1, ..., v_n)$) we have $w_i \in Span(\bar{c}, v_1, ..., v_i)$. According to the meta-theorem from [HTMM15], it is sufficient to check that tuples in G are locally indistinguishable and that for each tuple $\bar{c}$ its independence diagram is c.e. uniformly in $\bar{c}$.

The intuition is that $\bar{c}$ is the part of the basis we've built at a stage, and ϕ is the part of the open diagram we've listed. Later we may discover that $\bar{u}$ is dependent over $\bar{c}$, and we will need to correct our embedding. To make sure that the embedding can always be corrected, we always check whether facts we wish to enumerate in the open diagram are consistent with $\mathcal{I}(\bar{c})$. The proof can be viewed as a finite injury construction, and the meta-theorem allows us to separate algebra from recursion theory. See [HTMM15] for more detail. In [HTMM15] we also required $rank(\bar{c}) \geq 2$ in the definition of locally indistinguishable tuples. This was necessary to cover ordered abelian groups, this assumption can be ignored in our case. Without this extra assumption, the meta-theorem from [HTMM15] becomes uniform.

Lemma 3.11. *Prüfer independent tuples are locally indistinguishable in G.*

Proof. Suppose $\bar{u} = (u_1, ..., u_n)$ is independent over $\bar{c}$, and assume $G \models \exists \bar{x} \phi(\bar{c}, \bar{u}, \bar{x})$. Fix $\bar{b}$ in G witnesses the existential quantifier. Let $\bar{v} = (v_1, ..., v_n)$ be any other tuple independent over $\bar{c}$. We must find a tuple $\bar{w} = (w_1, ..., w_n)$ independent over $\bar{c}$ such that $G \models \exists \bar{x} \phi(\bar{c}, \bar{w}, \bar{x})$ and $w_i \in Span(\bar{c}, v_1, ..., v_i)$, $i = 1, \ldots, n$. We give a very detailed proof.

Let $X = \langle \bar{c}, \bar{u}, \bar{b} \rangle \leqq G$. Suppose $\bar{c} = \bar{c}'\bar{c}''$, where $\bar{c}'$ consists of independent elements and $\bar{c}''$ belongs to the (linear) span of $\bar{c}'$. Note that $\langle \bar{c}', \bar{u} \rangle$ forms a free subgroup of $\langle \bar{c}, \bar{u}, \bar{b} \rangle$ of rank $|\overline{c'u}|$. It well-known that for any f.g. abelian groups $U \leqq V$ there exist full decompositions $\langle u_1 \rangle \oplus \langle u_2 \rangle \oplus \ldots$ and $\langle v_1 \rangle \oplus \langle v_2 \rangle \oplus \ldots$ of U and V into cyclic summands such that for some choice of integers $m_1, m_2, \ldots$ we have $u_i = m_i v_i$, $i = 1, 2, \ldots$ [Fuc70]. We apply this fact to $\langle \bar{c}, \bar{u} \rangle \leqq \langle \bar{c}, \bar{u}, \bar{b} \rangle$ to obtain a tuple $\bar{h} = \bar{h}'\bar{h}''$ in $\langle \bar{c}, \bar{u}, \bar{b} \rangle$ such that:

(1) $\bar{h}'$ has the same length (and rank) as the tuple $\overline{c'u}$,

(2) $\langle \bar{c}, \bar{u} \rangle \leqq \langle \bar{h} \rangle$ (but not equal, in general),

(3) $\langle \bar{h} \rangle$ is a direct summand of $\langle \bar{c}, \bar{u}, \bar{b} \rangle$, say $\langle \bar{c}, \bar{u}, \bar{b} \rangle = \langle \bar{h} \rangle \oplus W$,

(4) $\langle h \rangle = \langle h' \rangle \oplus \langle h'' \rangle$.

In particular, (1) implies that $\langle \bar{h}' \rangle$ is free abelian. Within $\langle \bar{h}' \rangle$, it makes sense to define the least pure subgroup $\langle \bar{c}' \rangle^*$ of $\langle \bar{h}' \rangle$ containing $\bar{c}'$. It detaches in $\langle \bar{h}' \rangle$, say $\langle \bar{h}' \rangle = \langle \bar{c}' \rangle^* \oplus U$. Every u_i in $\bar{u}$ is (linearly) interchangeable with its projection onto U (over $\bar{c}'$ and equivalently over $\bar{c}$). Let $u_1', u_2', \ldots \in U$ be these projections. We would like to come up with a linearly independent set of generators $\bar{z} = (z_1, z_2, \ldots)$ of U such that z_i and u_i' are linearly interchangeable in U over $Span(\bar{c}, u_1', \ldots, u_{i-1}')$. This is done as follows. First, consider the least pure subgroup H_1 of U that contains u_1'. It must be cyclic and infinite, suppose it is generated by z_1. Clearly, z_1 and u_1' are interchangeable, and $U = H_1 \oplus K_1$. Take the K_1-projections $u_2'', u_3'', \ldots$ of the remaining elements of $\bar{u}'$. Observe that $u_2'', u_3'', \ldots$ are interchangeable with $u_2', u_3', \ldots$ over $Span(u_1') = Span(z_1)$. We repeat the process with u_2'' and K_1 playing the roles of u_1' and U respectively, to define H_2, K_2 and $u_3''', u_4''', \ldots$, etc. At every step we will have a further direct decomposition and a new tuple interchangeable with the original one over the respective initial segment of $\bar{u}'$. We will arrive at a generating set $z_1, z_2, \ldots$ of U such that the z_i and u_i' are interchangeable over $Span(u_1', \ldots, u_{i-1}')$. Since u_i and u_i' were interchangeable over $\bar{c}'$ (equivalently, over $\bar{c}$), we have that u_i and z_i are interchangeable over $Span(\bar{c}, u_1, \ldots, u_{i-1})$.

Recall that $\langle h \rangle = \langle h' \rangle \oplus \langle h'' \rangle$, and since $|h'| = rank(h') = rank(c'u) = |c'u|$, it follows that $\langle h'' \rangle$ must be torsion. Furthermore, the projection of $\bar{c}''$ onto U must be trivial, for otherwise the rank of $\bar{c}''$ over $\bar{c}'$ would not be zero. Therefore, $\bar{c}$ is contained in $\langle \bar{c}' \rangle^* \oplus \langle h'' \rangle$.

Thus, for some f.g. $W \leqq G$,

$$\langle \bar{c}, \bar{u}, \bar{b} \rangle = \langle \bar{c}' \rangle^* \oplus \langle \bar{h}'' \rangle \oplus \langle \bar{z} \rangle \oplus W,$$

where $\bar{z}$ is interchangeable with $\bar{u}$ over $\bar{c}$ in a strong sense (see above), and $\bar{c}$ is contained in $\langle \bar{c}' \rangle^* \oplus \langle h'' \rangle$.

Recall $\bar{v}$ was the other tuple independent over $\bar{c}$ and of the same length as $\bar{w}$. As above, (but ignoring $\bar{b}$) we can split $\langle \bar{c}\bar{v} \rangle$ into a direct sum

$$\langle \bar{c}' \rangle^{**} \oplus \langle \bar{f}'' \rangle \oplus \langle \bar{y} \rangle,$$

where $\bar{c}$ is in $\langle \bar{c}' \rangle^{**} \oplus \langle \bar{f}'' \rangle$, $\langle \bar{f}'' \rangle$ is torsion, and $\bar{y}$ is interchangeable with $\bar{v}$ over $\bar{c}$ in the strong sense as we had above (for $\bar{u}$). Note that W perhaps contains infinite cyclic summands, some of which may interact with $\langle \bar{y} \rangle$. But the rank of G is infinite, and thus we can replace W by an isomorphic subgroup W' such that there is no such interaction. The new f.g. subgroup $\langle \bar{c}' \rangle^* \oplus \langle \bar{h}'' \rangle \oplus \langle \bar{z} \rangle \oplus W'$ will be isomorphic to the original one under an isomorphism that fixes $\bar{c}$ and $\bar{u}$. Thus, we assume $\langle \bar{y} \rangle \cap W = 0$. We have that $W = T(W) + X$, where $T(W)$ is torsion, and X is free abelian whose generating basis is independent over $\bar{c}\bar{y}\bar{z}$. Consider the torsion group $T = \langle \bar{f}'' \rangle + \langle \bar{h}'' \rangle + T(W)$. Also, consider $C = \langle \bar{c}' \rangle^* + \langle \bar{c}' \rangle^{**}$ whose rank is equal to the rank of $\bar{c}$. It follows that $T + C$ contains $\bar{c}$. By the choice of X, its generating basis is independent over $\bar{c}\bar{y}\bar{z}$. We arrive at two isomorphic direct decompositions,

$$(C + T) \oplus X \oplus \langle \bar{z} \rangle \text{ and } (C + T) \oplus X \oplus \langle \bar{y} \rangle,$$

both are embeddable into G. Consider the natural isomorphism τ that fixes $(C + T) \oplus X$ and maps $\bar{z}$ to $\bar{y}$. Then τ fixes $\bar{c}$. Furthermore, the tuple $\bar{u}$ is mapped to some tuple $\bar{w}$ which is interchangeable, in the strong sense as above, with the tuple $\bar{y}$ over $\bar{c}$. Thus, it also interchangeable (in this strong sense) with the tuple $\bar{v}$ over $\bar{c}$. In particular, $\bar{w}$ is independent over $\bar{c}$ and satisfies the desired property $w_i \in Span(\bar{c}, v_1, ..., v_i)$. Since τ is an isomorphism, it preserves the validity of ϕ. $\square$

Lemma 3.12. *For any $\bar{c}$, the independence diagram $\mathcal{I}(\bar{c})$ of $\bar{c}$ in G is c.e. uniformly in $\bar{c}$.*

Proof. We must effectively list all existential formulae $\exists \bar{x} \phi(\bar{c}, \bar{y}, \bar{x})$ such that $G \models \exists \bar{x} \phi(\bar{c}, \bar{u}, \bar{x})$ for some $\bar{u}$ which is independent over $\bar{c}$. We have already proven that any existential formula over $\bar{c}\bar{u}$, where $\bar{u}$ is independent over $\bar{c}$, is witnessed within a finite initial segment of a f.g. subgroup of G of the form

$$(C + T) \oplus X \oplus F,$$

where $C + T$ is a f.g. subgroup of G that contains $\bar{c}$, and X and F are free abelian of finite rank such that $\bar{u}$ is interchangeable, over $\bar{c}$, with the basis of F. Furthermore, since the rank of G is infinite, for any f.g. $(C+T)$ extending $\bar{c}$ and any given free X and F, we can isomorphically embed $(C+T) \oplus X \oplus F$ into G. We can certainly list all f.g. subgroups of G extending $\bar{c}$, all free abelian groups of finite rank, and all potential $\bar{u}$ independent over $\bar{c}$. This gives a computable enumeration of $\mathcal{I}(\bar{c})$. □

Proposition 3.10 follows from the two lemmas above and the meta-theorem from [HTMM15]. □

3.2.4. *Finalising the proof of Theorem 1.4(1).* Using Proposition 3.10 we transform G into H having a computable basis. By Lemma 3.8, H is tractable. Lemma 3.5 guarantees that $\mathrm{Hom}(H, \mathbb{T})$ is a c.e. closed subset of computable Polish group $\mathbb{A}$. We use Fact 3.1 to produce a uniformly computable (in $\mathbb{A}$) sequence of points that is dense in $\mathrm{Hom}(H, \mathbb{T})$. Since the natural group operations are computable on $\mathbb{A}$, their restrictions to $\mathrm{Hom}(H, \mathbb{T})$ are computable as well, with respect to the computable dense subset of $\mathrm{Hom}(H, \mathbb{T})$. This gives a computable Polish presentation of $\widehat{G}$.

3.3. **Proof of Theorem 1.4(2).** Let S be an infinite set of prime numbers. Observe that the (discrete, countable) group

$$G_S = \bigoplus_{p \in S} \mathbb{Z}_p$$

has a constructivisation if and only if S is computably enumerable. Therefore, it is sufficient to build a set S which is not c.e. but such that $\widehat{G}_S$ is a computable Polish group.

We will simultaneously construct S and an approximation to $\widehat{G}_S$ which will be viewed as a closed subset of $\prod_{i \in \mathbb{N}} \mathbb{T}_i$. Note that $\widehat{G}_S$ is profinite and is homeomorphic to the inverse limit of $(\bigoplus_{p \in S \restriction n} \mathbb{Z}_p)_{n \in \mathbb{N}}$. To make sure S is not c.e., it is sufficient to satisfy $W_e \neq S$ for every e, where W_e is the e'th c.e. set (as usual). To ensure $\widehat{G}_S$ is computable, at every stage we will be deciding its "current position" in $\prod_{i \in \mathbb{N}} \mathbb{T}_i$, but only with precision 2^{-s} (to be clarified). We will see that we can meet all the diagonalisation requirements without upsetting this global strategy. We now describe the diagonalisation module for W_e.

The group U will be the inverse limit of the form

$$U = \varprojlim U_n,$$

where $U_n = \bigoplus_{i < n} C_i$ under the natural surjective projections $\psi_{n+1,n} : \bigoplus_{i < n+1} C_i \rightarrow \bigoplus_{i < n} C_i$ with kernel C_{n+1}. The primary cyclic C_i will be represented as closed subgroups $\mathbb{T}_i$, and each such C_i will be defined by one of the basic modules (to be discussed below).

The basic module for e.

(1) Reserve a large fresh prime q_e and keep q_e in S and wait for q_e to enter W_e.
(2) While waiting, attempt to witness $q_e \in S$ by making a better approximation to $\mathbb{Z}_{q_e}$ within $\mathbb{T}_e$. This is done as follows. At stage s we commit ourselves to the interval $(1/q_e - 2^{-s}, 1/q_e + 2^{-s})$ and pretend that this interval contains the generator of the $\mathbb{T}_e$-projection of our group. Note that if $q_e \notin W_e$ then these intervals will shrink to $1/q_e$ making the projection isomorphic to the closed subgroup $\mathbb{Z}_{q_e}$ of $\mathbb{T}_e$.
(3) If the prime q_e enters W_e at stage s, then:
   (a) Choose a large fresh prime u_e so that for some integer k we have

$$ku_e \in (1/q_e - 2^{-s}, 1/q_e + 2^{-s}).$$

(b) Switch from approximating $\mathbb{Z}_{q_e}$ within $\mathbb{T}_e$ to approximating $\mathbb{Z}_{u_e}$, as follows. Choose n_e large enough so that

$$(ku_e - 2^{-n_e-s}, ku_e + 2^{-n_e-s}) \subset (1/q_e - 2^{-s}, 1/q_e + 2^{-s}).$$

From now on, at stage $s' > s$ we will declare that the generator of the $\mathbb{T}_e$-projection of the group is within $2^{-n_0-s'}$ of $1/u_e$. This way we build a copy of $\mathbb{Z}_{u_e}$ within $\mathbb{T}_e$. Now whenever we have to improve our approximation of the generator $1/u_e$ within $\mathbb{T}_e$, we commit ourselves to

$$(1/u_e - 2^{-n_0-s'}, 1/u_e + 2^{-n_0-s'}).$$

(c) Permanently keep u_e in S and q_e out of S. (In particular, do not allow any other basic module to use these primes.)

Construction. At stage s of the construction we let the first s basic modules to act according to their instructions. Note that there is no interaction between the basic modules. At any stage s, once we've defined our current approximation to g_i, we also define an approximation to ng_i naturally (with $|n| \leq s$), according to the definition of the computable operations in $\prod_{i \in I} \mathbb{T}_i$.

Verification. The result of the construction is a uniform approximation (by shrinking intervals) of a sequence of points in $\mathbb{T}_e$. Our group U will be represented as the subgroup of $\prod_{e \in \mathbb{N}} \mathbb{T}_e$ consisting of infinite sequences of the form

$$(n_0 g_0, n_1 g_1, n_2 g_2, \ldots, n_e g_e, \ldots),$$

where $n_e \in \mathbb{Z}$ and the g_e are the generators of $C_e \leqq \mathbb{T}_i$.

Note that we can change our mind about the generator of C_e and switch from approximating $C_e \cong \langle 1/q_e \rangle$ to $C_e \cong \langle 1/u_e \rangle$ within $\mathbb{T}_e$, but this happens at most once for a fixed e. This switch will be consistent with what we've declared so far because we choose the approximation of $1/u_e$ to be a sufficiently small interval (see substage (b) in (3) above). We simply add a new very small interval around $1/u_e$ and proceed consistently with what we've declared so far.

Note that, by construction, the closed subgroup that we've built forms a c.e. closed subspace of $\mathbb{T}$. It follows from Fact 3.1 that the group is a computable Polish group. (We note that in this copy the standard group operations are in fact effectively open.) However, we have explicitly made sure that the discrete countable dual $\widehat{U} = G_S = \bigoplus_{p \in S} \mathbb{Z}_p$ of U has no constructivisation.

4. Proof of Theorem 1.7

4.1. **Proof of Theorem 1.7(1).** Recall the definitions of a computable function between computable Polish spaces and the formal definition of a computable Polish group (Def. 2.1 and Def. 2.3). Fix an effective listing $G_0, G_1, \ldots$ of (partially) computable Polish spaces in which every G_i is additionally equipped with a pair of c.e. sets that are interpreted as names of partial operations on G_i. We need to measure the complexity of the index sets of profinite and compact connected groups. The first step is to measure the complexity of being a compact group.

Proposition 4.1. *The index set* $CPGr = \{i : G_i$ *is a compact Polish group*$\}$ *is* Π^0_3*-complete.*

Proof of Proposition 4.1. Recall that compactness is equivalent to total boundedness. Thus, we need to state that for every rational q there exits a cover of M by (closed) basic balls of size at most q, which is a Π^0_3-property. See Nies and M. [MN13] for further details. Given a triple (G, W, U), where G is a (partial) computable structure on a Polish space and W, U are c.e. sets, we need to guess whether W and U are names of computable group operations on G.

Lemma 4.2. *Let G and M be compact computable Polish spaces. Then*

$$\{e : W_e \text{ is a name of a computable } f : G \to M\}$$

is Π^0_3*, uniformly in G and M.*

Proof. For technical convenience, we will use the following uniform variation of Definition 2.1.

Definition 4.3. Let f be a continuous function between Polish metric spaces M and N. A $*$-name of f the any collection of pairs of basic open balls (B, C) such that $f(B) \subseteq cl(C)$, and for every $x \in M$ and every $\epsilon > 0$ there exists $(B, C) \in \Psi$ such that $B \ni x$ and $r(C) < \epsilon$.

We can uniformly pass from a $*$-name of f (Def. 4.3) to a name of f (Def. 4.3) and back, as follows. Suppose Ψ is a name of f. Since $f(B) \subset C$ implies $f(B) \subset cl(C)$, every $*$-name is a name. Now suppose Ψ is a $*$-name of f. Using $\epsilon/2$ instead of ϵ in Def. 4.3, fix (B, C) with $r(C) < \epsilon/2$ such that $x \in B$ and $f(B) \subset cl(C)$. Replace C with an equicentric $C' \supset C$ s.t. $r(C) < r(C') < \epsilon$. We have $f(B) \subset cl(C) \subseteq C'$ and $r(C') < \epsilon$. Thus, given a $*$-name of f we can uniformly effective produce a name of f.

The uniform procedure of passing from a name to a $*$-name described above can be applied to any c.e. set W. We denote the resulting c.e. set by W^*. Then W is a name iff W^* is a $*$-name (of the same function).

Fix a c.e. set Ψ and interpret it as a set of pairs of basic open balls with rational radii:

$$\Psi = \{(C, B) : C, B \text{ basic open in } G, M \text{ resp.}\}.$$

Uniformly transform it into a c.e. set Ψ^*. To make sure that Ψ^* is a name of a computable operation, we require that Ψ^* additionally satisfies:

(1) For every $(B_0, C_0), \dots, (B_n, C_n) \in \Psi^*$, $\bigcap_i B_i \neq \emptyset$ implies $\bigcap_i C_i \neq \emptyset$,
(2) for each rational $\epsilon > 0$ there exists a finite cover $B_0, \dots B_k$ of G and $(B_0, C_0), \dots, (B_k, C_k) \in \Psi^*$ such that $r(C_i) < \epsilon$, $i = 1, \dots, k$.

We first check that (1) and (2) are (at most) Π^0_3, and then we prove that they capture the property of being a name.

Claim 4.4. *The properties (1) and (2) are (at most) Π^0_3.*

Proof of Claim. Since intersection of open sets must be witnessed by special points from the respective computable structures, it is clear that (1) is Π^0_2. It is less obvious why (2) has to be Π^0_3. The problem is that the union of $B_0, \dots B_k$ may contain all special points but still do not a cover the whole G. To show that (2) is Π^0_2 we need to slightly modify it. Consider the condition:

(2)* For each rational $\epsilon > 0$ there exists a finite cover $cl(B_0), \dots cl(B_k)$ of G and $(B'_0, C_0), \dots, (B'_k, C_k) \in \Psi^*$ such that $B_i \subseteq_{form} B'_i$ and $r(C_i) < \epsilon$, $i = 1, \dots, k$.

Here $B_i \subseteq_{form} B'_i$ means that either B_i is identical to B'_i or $B_i \subset_{form} B'_i$, see Def. 2.4. Recall that formal inclusion $\subset_{form}$ is Σ^0_1. Also, being a closed cover is a closed property and thus can be checked only for special points. It follows that (2)* is of the form $\forall\exists(\forall\&\exists)$, i.e. is Π^0_3. It remains to check that (2) $\iff$ (2)*. Since the B'_i from (2)* cover G, it follows that (2)* implies (2). For any open cover $B_0, \dots, B_k$ from (2), $G \subset cl(B_0) \cup \dots \cup cl(B_k)$. Since $B_i \subseteq_{form} B_i$ for each i, we can use the B_i themselves (instead of B'_i) to witness (2)*. $\qquad\square$

Clearly, if Ψ^* is a $*$-name of a computable operation $f : G \to M$, then Ψ^* satisfies (1) and (2) (recall G is compact).

Claim 4.5. *If Ψ^* satisfies (1) and (2) then it is a $*$-name of a computable operation.*

Proof. We define a map ψ as follows. For every $x \in G$, choose $(B, C) \in \Psi^*$ such that $x \in B$ and declare C a Ψ^*-nbhd of $\psi(x)$. (Note that if x is a computable point then this process is effective.) Then set $\psi(x)$ to be equal to any point in the intersection

$$\bigcap \{cl(B) : B \text{ is a } \Psi^*\text{-nbhd of } \psi(x)\}.$$

Property (1) imply that any two Ψ^*-nbhds of $\psi(x)$ have a non-empty intersection. Let C_n be the closure of the intersection of the first n Ψ^*-nbhd of $\psi(x)$ in any (not necessarily effective) list of such nbhds. Then (C_n) is a nested sequence of non-empty compact sets, thus it has a non-empty intersection. Property (2) guarantees that for every ϵ there exists a Ψ^*-nbhd of $\psi(x)$ of size ϵ. Therefore, the intersection is a singleton. We conclude that ψ is a (total) function.

We claim that Ψ^* is a $*$-name for ψ. Property (2) implies that for every $\epsilon > 0$ there exists $(B, C) \in \Psi^*$ such that $B \ni x$ and $r(C) < \epsilon$. It remains to show that for each $(B, C) \in \Psi^*$ we have $\psi(B) \subseteq cl(C)$. Fix $x \in B$. Then

$$\{\psi(x)\} = \bigcap \{cl(B) : B \text{ is a } \Psi^*\text{-nbhd of } \psi(x)\}.$$

Since C is a Ψ^*-nbhd of $\psi(x)$, in particular $\psi(x) \in cl(C)$. $\qquad\square$

To finish the proof of the lemma, observe that our analysis was fully uniform in the computable structures on G and M. $\qquad\square$

We return to the proof of Proposition 4.1. The product space $G \times G$ is compact. There is a uniform procedure that, given a computable Polish space G, outputs a computable presentation of $G \times G$. It follows from Lemma 4.2 that this index set of Polish spaces equipped with two well-defined computable operations is Π_3^0. To finish the proof of Prop. 4.1, recall that the group axioms are closed properties and thus can be checked only for special points. (Set $e = x \cdot x^{-1}$ for the first found x.)

Since completeness will not be used in the proof of Theorem 1.7(1), we give only a sketch. Represent a Π_3^0-predicate in the form $\forall x \exists^{<\infty} y P(x, y, z)$, and code z into the surjective inverse limit of discrete groups (G_x). For each fixed x, G_x will be the direct sum of several copies of $\mathbb{Z}_2$. There will be finitely many $\mathbb{Z}_2$-summands in all such G_x iff $\exists^{<\infty} y$ for all x. Then the group is computable Polish and totally disconnected, but it will be compact (thus, profinite) iff Π_3^0 holds. The construction can be viewed as a finite injury one. We leave details to the reader. $\qquad\square$

Proposition 4.6.
1. *The index set of connected compact Polish groups is Π_2^0-complete within compact groups.*
2. *The index set of profinite Polish groups is Π_3^0-complete within compact groups.*

Proof. We prove 1. Let G be a compact Polish group (see Prop. 4.1). Since G is compact, G is disconnected iff there exists a finite collection of basic open $B_1, \ldots, B_k, D_1, \ldots, D_n$ such that

$$cl(\bigcup_{i \leq k} B_i) \cup cl(\bigcup_{j \leq n} D_j) = G \text{ and } cl(\bigcup_{i \leq k} B_i) \cap cl(\bigcup_{j \leq n} D_j) = \emptyset.$$

If $cl(\bigcup_{i \leq k} B_i) \cup cl(\bigcup_{j \leq n} D_j) \neq G$ then there is a special point outside this set, so the property is Π_1^0. We claim that, for any basic open balls $B(x, r)$ and $B(y, q)$, the property $cl(B(x, r)) \cap cl(B(y, q)) \neq \emptyset$ is equivalent to

$$\forall \epsilon > 0 \ B(x, r + \epsilon) \cap B(y, q + \epsilon) \neq \emptyset,$$

which is clearly Π_2^0. If $cl(B(x, r)) \cap cl(B(y, q)) \neq \emptyset$ then the property above holds for every $\epsilon > 0$. On the other hand, suppose z_ϵ witnesses $B(x, r + \epsilon) \cap B(y, q + \epsilon) \neq \emptyset$. Then $(z_{2^{-n}})_{n \in \omega}$ has a converging subsequence, let z be the limit of this subsequence (we use compactness of G). It follows that $d(x, z) \leq r$ and $d(y, z) \leq q$, as required. The property $cl(\bigcup_{i \leq k} B_i) \cap cl(\bigcup_{j \leq n} D_j) = \emptyset$ can be expressed as a finite Boolean combination of such Σ_2^0-statements. It follows that connectedness is a Π_2^0-property of a compact G.

Remark 4.7. In the notation as above, $cl(B(x, r)) \cap cl(B(y, q)) = \emptyset$ iff $\exists \epsilon > 0 \ B(x, r+\epsilon) \cap B(y, q+\epsilon) = \emptyset$. Since there are only finitely many balls involved overall, we can pick the least rational ϵ witnessing the required empty intersections. Thus, using $0'$ we can uniformly produce two finite disjoint collections of *open* balls (if they exist). In the proof of the second part and in Corollary 4.8 we will alternate between using closed and open names for such clopen sets.

To see that it is Π_2^0-complete within compact groups, start enumerating rational points in $\mathbb{T}$. Add more points only if the predicate fires. We will end up with a computable copy of $\mathbb{T}$ iff the predicate fires infinitely often, and we will have a finite (thus, totally disconnected) subgroup of $\mathbb{T}$ otherwise.

We prove 2. Recall that a compact Polish group is profinite iff its neutral element 1_G has a basis of normal clopen subgroups. Recall also that a closed subgroup of a profinite group is itself profinite. Consider the following procedure. Let $D_0 = G$. At stage $s > 0$ let D_s be the first found clopen subgroup such that the diameter of D_s is at most 2^{-s} and $D_s \subseteq D_{s-1}$ (if such D_s exists at all). To find a clopen subgroup, we use $0'$ to verify the Σ_2^0-definition of a clopen set (see above). For that, search

for finitely many closed balls witnessing that the group is disconnected, where the union of the first k of them together form a normal subgroup.

Since all involved sets are clopen, it is sufficient to check the inclusion, the diameter, normality, and the group operations only for special points. Note that the set is represented as a finite union of *open* basic balls (Remark 4.7). Note that we can (uniformly in the description) list all special points that are contained in these open balls, and they will give a computable structure on the subgroup. The next iteration can use this finite description as its input instead of G. It follows that $0'$ is capable of uniformly finding such a D_s (if it exists), and thus $\forall s(D_s$ is defined) is a Π_3^0-statement equivalent to profiniteness for a compact group G.

The Π_3^0-completeness part is similar to Π_2^0-completeness above, for connectedness groups. Given a Π_3^0-predicate and an element on which the predicate needs to be tested, produce a (topological) direct product $\prod_{x \in \omega} G_x$. Make sure that G_x is a finite subgroup of $\mathbb{T}$ iff Σ_2^0 holds on x, and build a copy of $\mathbb{T}$ otherwise. Then $\prod_x G_x$ is profinite iff $\forall x$ (Σ_2^0 holds on x). Note that in any case the group is compact. This finishes the proof of the proposition, and of the first part of Theorem 1.7. $\qquad\square$

Note that the procedure described in the proof of Prop. 4.6 leads to:

Corollary 4.8. *There exists a uniformly $0'$-computable procedure which, on input a computable Polish presentation of a profinite group G, produces its $0'$-recursive presentation.*

Proof of Corollary. Using $0'$ we can list a basis of e consisting of clopen normal subgroups. For any such fixed clopen N, G/N is finite. Note that every coset of the form $\tilde{x}N$ is open and thus contains a spacial point x, in particular every coset is of the form xN for some special x. We claim that, given such an N, $0'$ can find finitely many special points $x_0 = e, x_1, \ldots, x_n$ such that $\{x_i N\}$ is a disjoint cover of G. To see why, note that $x_i N \cap x_j N \neq \emptyset$ iff for some special y, $yx_i^{-1} \in N$ and $yx_j^{-1} \in N$, both events are c.e. in the finite description of the clopen set N by finitely many basic *open* balls (see Remark 4.7). Also, since left-translation is a self-homeomorphism of G onto itself and N is clopen, each $x_i N$ is clopen as well. Thus, $\{x_i N\}$ is a (closed) cover iff for every special y there is an i such that $yx_i^{-1} \in N$; if we view the latter as a finite union of *closed* balls (Remark 4.7) then the statement becomes Π_1^0 and thus can be decided using $0'$. Similarly, the group structure upon $\{x_i\}$ mod N can be reconstructed effectively and uniformly, in N. Simply search for an x_k such that $x_i x_j x_k^{-1} \in N$ (this is an effective search in the open name of N) and then declare $x_i x_j =_N x_k$ in G/N. Note that the procedure above is uniform in the description of N. Therefore, the $0'$-computable list of such normal clopen subgroups leads to a $0'$-computable inverse system of finite groups whose inverse limit represents G. $\qquad\square$

4.2. **Proof of Theorem 1.7(2).** It is not hard to see that the upper bound is Σ_1^1, we give details below.

Fact 4.9. *The topological group isomorphism problem for compact topological groups is Σ_1^1.*

Proof. Let X and Y be special dense subsets in G_i and G_j, respectively. We first explain why the existence of a surjective topological homomorphism is Σ_1^1. We are asking for a function $f : X \times \omega \to Y$ such that:

(1) for each $x \in X$ and $n \in \omega$, $d(f(x,n), f(x, n+1)) < 2^{-n}$;
(2) $\lim_n f(X, n)$ is dense in G_j;
(3) $\lim_n f(\cdot, n) \upharpoonright X$ is uniformly continuous;
(4) $\lim_n f(\cdot, n) \upharpoonright X$ is a group-homomorphism.

By uniform continuity, it is sufficient to define it on a dense set. Thus, if we have (1) and (3) then we have a unique continuous extension $\bar{f}$ of $\lim_n f(X, n)$. Since continuous images of compact sets are compact, and compact subsets of Polish spaces are closed, surjectivity follows from the continuity of $\bar{f}$ and (2). Finally, (4) guarantees that $\bar{f}$ is a group-homomorphism. To make sure that f is 1-1, we also ask for a $g : Y \times \omega \to X$ which determines a continuous surjective homomorphism $\bar{g}$ from G_j onto G_i, and such that $\bar{f} \circ \bar{g} = Id_{G_i}$. The latter property is closed, and therefore if it fails then it must be witnessed by special points in X. In particular, it is sufficient to check it only for special points. It remains to note that all mentioned conditions are Arithmetical in $f \oplus g$. $\qquad\square$

It follows from Fact 4.9 and (1) of Theorem 1.7 that the isomorphism problems for connected and totally disconnected compact Polish groups are both Σ_1^1.

We now prove Σ_1^1-completeness. Our proof of Theorem 1.4(1) is not uniform. Nonetheless, it is fully uniform when restricted to torsion-free constructive groups of rank ω. As we noted in the proof of Thm. 1.4(1), the only obstacle is the absence of a computable basis. If the rank is infinite then the proof of Dobrica's theorem becomes uniform, as explained in the first part of the proof of Prop. 3.10. Every torsion abelian group also has a computable basis – it is empty. Thus, Theorem 1.4(1) is uniform for torsion constructive groups as well. In each of these two special cases, we have a uniformly effective reduction $G \to \widehat{G}$ such that $G \cong H$ iff $\widehat{G} \cong \widehat{H}$. The isomorphism problems for constructive torsion and torsion-free abelian groups of rank ω are both Σ_1^1-complete ([DM08, GK02]). (In fact, the groups witnessing Σ_1^1-completeness in [DM08] have computable bases, so we don't even need Dobrica's result in this case.) It remains to note that the duals of discrete torsion abelian groups are profinite, and the duals of discrete torsion-free abelian groups are connected [Pon66].

5. Profinite abelian groups

Recall Def. 1.2 of a recursive profinite group. A recursive profinite group can be viewed as the collection of [infinite] paths through a computably branching tree with no dead ends. In such a presentation, every [infinite] path represents an element of the group, and the operations are represented by computable operators acting on this topological space (see [LR81, Smi81b]). We can define an (ultra)metric on this space, similarly to how it is done for Cantor space. With respect to this metric, the presentation becomes a computable Polish group. In this presentation, we can effectively list basic clopen sets. In particular, a map $\phi : H_1 \to H_2$ between recursive profinite groups H_1 and H_2 is computable if it the pre-image of any basic clopen set of H_2 can be effectively listed as a *disjoint* union of basic clopen sets of H_2. We will use this property below.

5.1. **Proof of Theorem 1.9.** Consider a profinite abelian P. We can view P as the inverse limit of a system of finite groups $0 \leftarrow A_0 \leftarrow A_1 \leftarrow \ldots$, where all maps are surjective. It is well-known (see [Fuc70] for more detail) that $\widehat{P}$ is the direct limit of the system $0 \to A_0 \to A_1 \to \ldots$, where the injective embeddings are can be effectively reconstructed from the respective maps in the inverse system for P. (To acquire a better understanding of this, see Example 2.5 but restrict yourself to finite cyclic groups.) In contrast to the general case of arbitrary compact Polish abelian groups, the above correspondence between the direct and the inverse systems is uniformly effective. This gives (1) of Theorem 1.9. Part (2) of Theorem 1.9 follows from the fact that non-injective maps in the direct system canonically and effectively correspond to non-surjective maps in the inverse system (see e.g. Proposition 30 on p.37 of [Mor77]). Since we need an algorithmic version of this fact, the reader might want to see a more detailed and "constructive" explanation (see below).

Suppose $\widehat{P}$ is a c.e.-presentation of a (discrete) torsion abelian group. Assume that at every stage s we either adjoin a new generator to A_s, or we declare one generator of A_s equal to zero (in A_{s+1}). We explain only the new scenario in which elements of A_s are declared equal to 0. Suppose at stage s we have a finite group A_s and a finite inverse system $0 \leftarrow_{\psi_0} A_1 \leftarrow_{\psi_1} A_2 \ldots \leftarrow_{\psi_{s-1}} A_s$ in which the maps are not necessarily onto. Suppose at $s + 1$ we discover $a \in A_s$ is equal to 0. Form a direct decomposition $A_s = B_s \oplus \langle b \rangle$ where $mb = a$ for some $m \in \mathbb{Z}$. For that, extend $\langle a \rangle$ to a pure cyclic subgroup $\langle b \rangle$ of A_s. Then $A_{s+1} = B \oplus \mathbb{Z}_m$, where elements of $\mathbb{Z}_m$ can be identified with cosets of $\langle b \rangle$ modulo $\langle a \rangle$. Define ψ_{s-1} to be identity on B, and let ψ_{s-1} match every element of $\mathbb{Z}_m$ with some representative of the respective coset in A_s. It follows that the limit of of $0 \leftarrow_{\psi_0} A_0 \leftarrow_{\psi_1} A_1, \ldots$, is equal to $\widehat{\widehat{P}} \cong P$. Indeed, with the help of $0'$ we can transform the above inverse system into an equivalent system with surjective maps and whose limit is P.

Now suppose we have $0 \leftarrow_{\psi_0} \widehat{A_1} \leftarrow_{\psi_1} \ldots$ whose limit is P, but where the maps are not necessarily onto. We can fix two full decompositions of A_s and $\psi(A_{s+1})$ such that the cyclic direct summands of A_s extend the cyclic direct summands of $\psi_{s+1}(A_{s+1})$ (see [Fuc70]). We have already used this fact in the proof of Proposition 3.10. Thus, we can effectively refine it to an equivalent system with cyclic factors, i.e. we can assume $A_s/\psi_{s+1}(A_{s+1})$ is cyclic for each i. In particular, it follows that there exists an element $b \in A_s$ such that $A_s = B \oplus \langle b \rangle$ and $\psi_{s+1}(A_{s+1}) = B \oplus \langle mb \rangle$ for some $m \in \mathbb{Z}$. In the discrete dual, declare $b = mb$ and proceed according to this new equality. The resulting finite group can be identified with A_{s+1}. We can (non-computably) replace $(A_s)_{s \in \omega}$ with an equivalent direct system which is nested under inclusion and has $\widehat{P}$ as its limit. This gives (1) and (2) of Theorem 1.9.

We prove (3). Recall that a homeomorphism witnessing $G \cong \widehat{\widehat{G}}$ can be chosen in the canonical way:

$$g \to \langle \cdot, g \rangle,$$

where $\langle \chi, g \rangle = \chi(g)$ for any character $\chi \in \widehat{G}$. It follows that the identification is fully effective if we restrict ourselves to computable profinite and constructive torsion discrete groups. *In particular, we can effectively identify G and its second dual.*

Suppose we have a computably categorical (c.c.) torsion group G, we need to show that $\widehat{G}$ is c.c. as a profinite group. Let H_1 and H_2 be two computable profinite groups homeomorphic to $\widehat{G}$. As noted above, we can uniformly effectively produce constructive torsion groups U_1 and U_2 whose duals can be effectively identified with H_1 and H_2, respectively. (More formally, we effectively identify $\widehat{\widehat{H_i}}$ with H_i, then we set $U_i = \widehat{H_i}$.)

Let $\phi : U_2 \to U_1$ be a computable isomorphism of constructive groups U_1 and U_2. It is well-known [Mor77] that the map

$$\widehat{\phi}(\chi) = \chi \circ f$$

where $\chi \in H_1 = \widehat{U_1}$ and $\chi \circ f \in H_2 = \widehat{U_2}$, is a topological group-isomorphism of H_1 onto H_2. We claim that this isomorphism and its inverse are both computable.

We can compute $\chi(a)$ (in $\mathbb{T}$) effectively and uniformly in χ [represented as an infinite path] and $a \in U_i$. It is crucial that we can compute the rational equal to $\chi(a)$, and not merely as a sequence converging to $\chi(a)$. It follows that $\chi(a)$ is determined by a and a finite initial segment of χ.

Since all procedures described above are effective, we conclude that $\widehat{\phi}(\chi) = \chi \circ f$ is a computable group-homeomorphism of H_1 onto H_2. Since $f^{-1} : U_1 \to U_2$ is computable, a symmetric argument shows that the inverse of $\widehat{\phi}$ is computable as well.

Now suppose U_1 and U_2 are constructivisations of the discrete G. We effectively and uniformly pass to computable profinite H_1 and H_2 whose duals can be effectively identified with U_1 and U_2, respectively. Let $f : H_1 \to H_2$ be a computable topological group-isomorphism. As before, we define $\widehat{f} : U_2 \to U_1$ by the rule $\widehat{f}(\chi) = \chi \circ f$. We claim that $\widehat{f}$ is a computable isomorphism. More specifically, we need to effectively and uniformly find $u \in U_1$ such that $\widehat{f}(\chi) = u$.

Every element of U_2 is a character of H_2, and each such character χ can be uniquely associated with its kernel K_χ which is a compact subgroup of H_2. Thus, the character χ is uniquely described by the $\mathbb{T}$-images of the finitely many cosets in H_2/K_χ. Note that K_χ is a finite union of basic clopen sets, and we can cover H_2 by finitely many translations of K_χ, without repetition and intersection. As the result we obtain a finite disjoint collection of basic clopen balls $B_1, \ldots, B_k$ such that any two elements of [a fixed] B_i are equal modulo K_χ, and furthermore the finite list $\chi(B_1), \ldots, \chi(B_i) \in \mathbb{T}$ uniquely describes χ among other characters in U_2. Furthermore, if H_2 is a computable profinite group, then we can compute such a description of any element in U_2. Indeed, as we build $U_2 = \widehat{H_2}$, we monitor the new elements that enter U_2 and list the descriptions accordingly. The same argument applies to U_1.

We can effectively express both K_χ and its pre-image $f^{-1}(K_\chi)$ as a disjoint union of finitely many basic clopen sets in H_2 and H_1 respectively. We can also find the disjoint finite covers of H_2 and H_1 by translations of K_χ and $f^{-1}(K_\chi)$, respectively. Using the disjoint covering of $f^{-1}(K_\chi)$ by basic clopen balls and the disjoint covering of H_1 by $f^{-1}(K_\chi)$, and we can effectively find finitely many disjoint basic clopen balls $B_1, \ldots, B_k$ such that for any $\sigma, \tau \in B_i$ we have $\chi \circ f(\sigma) = \chi \circ f(\tau)$ and such that these images completely determine the character $\chi \circ f$. It remains to find the unique character $u \in U_1$ that has the same description by basic open balls, and set $\widehat{f}(\chi) = u$. (Note it follows that f is necessarily effectively open.)

5.2. **Applications of Theorems 1.4 and 1.9 to profinite groups.** Some corollaries have already been stated in the introduction. In this section we discuss several further applications to co-r.e. profinite groups.

Using limitwise monotonic sets [Khi98, KKM13, DKT11, KNS97], we can produce a c.e.-presented p-group that has no constructivization (Khisamiev [Khi98], see also [Mel14] for a proof).

Corollary 5.1. For every prime p there exists a co-r.e. abelian pro-p group which has no recursive presentation.

As we noted in [Mel14], if a (discrete) reduced abelian p-group is not a direct sum of cyclic groups, then every c.e.-presentation of the group can be transformed into it constructivisation. Thus, we have:

Corollary 5.2. *Suppose P is a co-c.e. abelian pro-p group whose dual $\widehat{P}$ is a reduced group of Ulm type > 1. Then P has a recursive presentation.*

The abelian group W witnessing Theorem 1.4(2) is profinite. Since its dual has no constructivisation, Theorem 1.9(1) implies that W has no recursive presentation, thus proving Cor. 1.6. We note that the group witnessing Theorem 1.4(2) has a co-r.e. presentation.

We push the corollary above a bit further. Recall that Haar measure on a compact group G is the unique left-invariant probability measure on Borel subsets of G; see e.g. [Pon66] for a formal definition. Fix some effective listing of basic open balls in a computable Polish G. We say that $X \subseteq \omega$ is a name of an open set $U \subseteq G$ if $U = \bigcup_{i \in X} B_i$.

Definition 5.3. *We say that a measure μ on G is computable if $\mu(\bigcup_{i \in X} B_i)$ is a real uniformly computable in X.*

Recall that Corollary 1.5 states that there exists a computable Polish abelian group upon which Haar measure is not computable.

Proof of Corollary 1.5. We claim that the computable Polish presentation U of the dual of $G_S = \bigoplus_{p \in S} \mathbb{Z}_p$ from the proof of Theorem 1.4(1) has this property. We explicitly made sure that S is not c.e. Recall that U was built as a c.e closed subgroup of $\prod_{n \in \mathbb{N}} \mathbb{T}_n$ so that, for every n, the projection U_n of U to $\mathbb{T}_n$ was isomorphic to $\mathbb{Z}_p$ for some $p \in S$. (We identify the c.e. closed set U with the computable Polish presentation of U given by Fact 3.1.) We claim that if Haar measure μ on U was computable then S would be c.e. So assume μ is computable.

For a basic open interval $I \subset \mathbb{T}_n$, let

$$\mathbb{I} = \mathbb{T}_0 \times \mathbb{T}_1 \times \ldots \mathbb{T}_{n-1} \times I \times \mathbb{T}_{n+1} \times \ldots.$$

Then $\mathbb{I}$ is a basic open set in U. Furthermore, if I has rational end-points, then $\mathbb{I}$ has a computable name. In particular, its Haar measure must be a (uniformly) computable real. For every n and for every element u of U_n there exists a basic open interval $I_u \subset \mathbb{T}_n$ containing u such that

$$\mu(\mathbb{I}_u) = 1/p,$$

for $p \in S$ such that $U_n \cong \mathbb{Z}_p$. At a stage s we have disjoint current 2^{-s}-approximations $I_{u,s}$ to such intervals, and we also have our current best guess O_s on the order of U_n so that $1/O_s$ agrees with $\mu_s(I_{u,s})$ up to 2^{-s}. We simply wait until a late enough s such that $I_{n,s}$ is so small that enumerating more elements into U_n would have to reduce $\mu(I_{u,s})$ at least twice. (We invite the reader to reconstruct the elementary routine details.) After this stage O_s must be stable. This gives an algorithm for enumerating S, a contradiction. $\qquad\square$

One can show the effective functor from Theorem 1.9 also preserves computable dimension, relative computable categoricity, and degree spectra. See [AK00] for the definitions. These notions can be naturally extended to the category of inverse-limit presentations of profinite groups up to topological group isomorphism (we omit details). As a consequence of [MN], there exists a recursive profinite group which is computably categorical but not relatively computably categorical. It also follows from [MN] that the computable dimension of a recursive profinite abelian group is either 1 or ω. As a consequence of the main result in [KKM13], there exists a pro-p abelian group whose degree spectrum contains all non-computable Δ_2^0-degrees, but does not contain $\mathbf{0}$. We note that proving all these results directly with profinite groups seems totally infeasible.

6. A brief conclusion

The area of computable topological groups is wide open, we suggest only a few further directions that seem most relevant to the main results of the paper. As we noted in the introduction, one can ask many questions related to Theorem 1.4 and, more generally, to the effective content of Pontryagi Duality. For instance, is there any reasonable (i.e., Arithmetical) uniform upper bound on the complexity of $\widehat{W}$ in Theorem 1.4(2)? Does it remain ineffective in presence of computable Haar measure? Is (1) of Theorem 1.4 *provably* non-uniform? Khoussainov suggested looking at Pontryagin Duality from a different, "local" perspective. We could fix a computable locally compact G and ask questions about the complexity of characters in $\widehat{G}$. For example, is there a constructive G which has no computable non-trivial character?

We leave open whether there is a computable compact Polish group such that all its computable Polish presentations have no algorithm for Haar measure. We also suspect that there exists a computable Polish profinite group such that all its X-recursive presentations have the property $X \geq_T 0'$.

References

[AK00]   C. Ash and J. Knight. *Computable structures and the hyperarithmetical hierarchy*, volume 144 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, 2000.

[Bar95]   E. Barker. Back and forth relations for reduced abelian p-groups. *Ann. Pure Appl. Logic*, 75(3):223–249, 1995.

[BHW08]   V. Brattka, P. Hertling, and K. Weihrauch. A tutorial on computable analysis. In *New computational paradigms*, pages 425–491. Springer, New York, 2008.

[Boo59]   W. Boone. The word problem. *Annals of Math*, 70:207–265, 1959.

[DK86]   R. Downey and S. Kurtz. Recursion theory and ordered groups. *Ann. Pure Appl. Logic*, 32(2):137–151, 1986.

[DKT11]   R. Downey, A. Kach, and D. Turetsky. Limitwise monotonic functions and applications. In *Proceedings of STACS 2012*, pages 56–85, 2011.

[DM08]   R. Downey and A. Montalbán. The isomorphism problem for torsion-free abelian groups is analytic complete. *J. Algebra*, 320(6):2291–2300, 2008.

[Dob81]   V. Dobrica. Constructivizable abelian groups. *Sibirsk. Mat. Zh.*, 22(3):208–213, 239, 1981.

[Dob83]   V. P. Dobritsa. Some constructivizations of abelian groups. *Sibirsk. Mat. Zh.*, 24(2):18–25, 1983.

[Dow97]   R. Downey. On presentations of algebraic structures. In *Complexity, logic, and recursion theory*, volume 187 of *Lecture Notes in Pure and Appl. Math.*, pages 157–205. Dekker, New York, 1997.

[Dow98]   R. Downey. Computability theory and linear orderings. In *Handbook of recursive mathematics, Vol. 2*, volume 139 of *Stud. Logic Found. Math.*, pages 823–976. North-Holland, Amsterdam, 1998.

[EG00]   Y. Ershov and S. Goncharov. *Constructive models*. Siberian School of Algebra and Logic. Consultants Bureau, New York, 2000.

[Ers]   Yu. Ershov. Problems of solubility and constructive models [in russian]. Nauka, Moscow (1980).

[Fol16]   Gerald B. Folland. *A course in abstract harmonic analysis*. Textbooks in Mathematics. CRC Press, Boca Raton, FL, second edition, 2016.

[Fuc70]   L. Fuchs. *Infinite abelian groups. Vol. I*. Pure and Applied Mathematics, Vol. 36. Academic Press, New York, 1970.

[Fuc73]   L. Fuchs. *Infinite abelian groups. Vol. II*. Academic Press, New York, 1973. Pure and Applied Mathematics. Vol. 36-II.

[GK02]   S. Goncharov and J. Knight. Computable structure and antistructure theorems. *Algebra Logika*, 41(6):639–681, 757, 2002.

[GMKT]   Noam Greenberg, Alexander G. Melnikov, Julia F. Knight, and Daniel Turetsky. Uniform procedures in uncountable structures. *To appear*.

[Gon80]   S. S. Gončarov. Autostability of models and abelian groups. *Algebra i Logika*, 19(1):23–44, 132, 1980.

[Gon81]   S. Goncharov. Groups with a finite number of constructivizations. *Dokl. Akad. Nauk SSSR*, 256(2):269–272, 1981.

[Gon97]   Sergei S. Goncharov. *Countable Boolean algebras and decidability*. Siberian School of Algebra and Logic. Consultants Bureau, New York, 1997.

[GR93]   Xiaolin Ge and J. Ian Richards. Computability in unitary representations of compact groups. In *Logical methods (Ithaca, NY, 1992)*, volume 12 of *Progr. Comput. Sci. Appl. Logic*, pages 386–421. Birkhäuser Boston, Boston, MA, 1993.

[Hig61]   G. Higman. Subgroups of finitely presented groups. *Proc. Roy. Soc. Ser. A*, 262:455–475, 1961.

[HTMM15]   Matthew Harrison-Trainor, Alexander Melnikov, and Antonio Montalbán. Independence in computable algebra. *J. Algebra*, 443:441–468, 2015.

[Khi98]   N. Khisamiev. Constructive abelian groups. In *Handbook of recursive mathematics, Vol. 2*, volume 139 of *Stud. Logic Found. Math.*, pages 1177–1231. North-Holland, Amsterdam, 1998.

[KKM13]   I. Kalimullin, B. Khoussainov, and A. Melnikov. Limitwise monotonic sequences and degree spectra of structures. *Proc. Amer. Math. Soc.*, 141(9):3275–3289, 2013.

[KNS97]   B. Khoussainov, A. Nies, and R. Shore. Computable models of theories with few models. *Notre Dame J. Formal Logic*, 38(2):165–178, 1997.

[Lin81]   C. Lin. Recursively presented abelian groups: effective *p*-group theory. I. *J. Symbolic Logic*, 46(3):617–624, 1981.

[Lot98]   Peter Loth. *Classifications of abelian groups and Pontrjagin duality*, volume 10 of *Algebra, Logic and Applications*. Gordon and Breach Science Publishers, Amsterdam, 1998.

[LR81]   Peter La Roche. Effective Galois theory. *J. Symbolic Logic*, 46(2):385–392, 1981.

[LS01]   Roger C. Lyndon and Paul E. Schupp. *Combinatorial group theory*. Classics in Mathematics. Springer-Verlag, Berlin, 2001. Reprint of the 1977 edition.

[Mal61]   A. Mal'cev. Constructive algebras. I. *Uspehi Mat. Nauk*, 16(3 (99)):3–60, 1961.

[Mal62]   A. Mal'cev. On recursive Abelian groups. *Dokl. Akad. Nauk SSSR*, 146:1009–1012, 1962.

[Mar02]   David Marker. *Model theory*, volume 217 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2002. An introduction.

[McN15]   Timothy H. McNicholl. A note on the computable categoricity of ℓ^p spaces. In *Evolving computability*, volume 9136 of *Lecture Notes in Comput. Sci.*, pages 268–275. Springer, Cham, 2015.

[Mel13]   Alexander G. Melnikov. Computably isometric spaces. *J. Symbolic Logic*, 78(4):1055–1085, 2013.

[Mel14]   Alexander G. Melnikov. Computable abelian groups. *Bull. Symb. Log.*, 20(3):315–356, 2014.

[MM]   Alexander G. Melnikov and Antonio Montalbán. Computable polish group actions. *To appear*.

[MN]   Alexander G. Melnikov and Keng Meng Ng. Computable torsion abelian groups. *To appear*.

[MN79]   G. Metakides and A. Nerode. Effective content of field theory. *Ann. Math. Logic*, 17(3):289–320, 1979.

[MN13]   Alexander G. Melnikov and André Nies. The classification problem for compact computable metric spaces. In *The nature of computation*, volume 7921 of *Lecture Notes in Comput. Sci.*, pages 320–328. Springer, Heidelberg, 2013.

[MN16]   Alexander G. Melnikov and Keng Meng Ng. Computable structures and operations on the space of continuous functions. *Fund. Math.*, 233(2):101–141, 2016.

[Mor77]   Sidney A. Morris. *Pontryagin duality and the structure of locally compact abelian groups*. Cambridge University Press, Cambridge-New York-Melbourne, 1977. London Mathematical Society Lecture Note Series, No. 29.

[MS]   Timothy H. McNicholl and D. M. Stull. The isometry degree of a computable copy of l^p. *To appear*.

[Nov55]   P. Novikov. On the algorithmic unsolvability of the word problem in group theory. *Trudy Mat. Inst. Steklov*, 44:1–143, 1955.

[Nur74]   A. Nurtazin. *Computable classes and algebraic criteria of autostability*. Summary of Scientific Schools, Math. Inst. SB USSRAS, Novosibirsk, 1974.

[PER89]   Marian B. Pour-El and J. Ian Richards. *Computability in analysis and physics*. Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1989.

[Pon66]   L. S. Pontryagin. *Topological groups*. Translated from the second Russian edition by Arlen Brown. Gordon and Breach Science Publishers, Inc., New York-London-Paris, 1966.

[Rog87]   H. Rogers. *Theory of recursive functions and effective computability*. MIT Press, Cambridge, MA, second edition, 1987.

[Smi81a] R. Smith. Two theorems on autostability in *p*-groups. In *Logic Year 1979–80 (Proc. Seminars and Conf. Math. Logic, Univ. Connecticut, Storrs, Conn., 1979/80)*, volume 859 of *Lecture Notes in Math.*, pages 302–311. Springer, Berlin, 1981.

[Smi81b] Rick L. Smith. Effective aspects of profinite groups. *J. Symbolic Logic*, 46(4):851–863, 1981.

[Tur36]   Alan M. Turing. On computable numbers, with an application to the entscheidungsproblem. *Proceedings of the London Mathematical Society*, 42:230–265, 1936.

[Tur37]   Alan M. Turing. On computable numbers, with an application to the entscheidungsproblem. *Proceedings of the London Mathematical Society*, 43:544–546, 1937.

[Wei00]   Klaus Weihrauch. *Computable analysis*. Texts in Theoretical Computer Science. An EATCS Series. Springer-Verlag, Berlin, 2000. An introduction.

[Wei01]   Klaus Weihrauch. On computable metric spaces Tietze-Urysohn extension is computable. In *Computability and complexity in analysis (Swansea, 2000)*, volume 2064 of *Lecture Notes in Comput. Sci.*, pages 357–368. Springer, Berlin, 2001.