NEW DEGREE SPECTRA OF ABELIAN GROUPS

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Abstract. We show that for every computable ordinal of the form $\beta = \delta + 2n + 1 > 1$, where δ is zero or a limit ordinal and $n \in \omega$, there exists a torsion-free abelian group having an X-computable copy if, and only if, X is *non-low*$_\beta$.

1. Introduction

The study of the algorithmic nature of classical algebraic objects has long tradition which goes back at least to Hermann [Her26] and van der Waerden [vdW30]. We continue the tradition that goes back to Mal'cev [Mal62] and Rabin [Rab60] who initiated the systematic study of computability-theoretic aspects of countably infinite groups.

Definition 1.1 (Rabin [Rab60], Mal'cev [Mal62])**.** A countably infinite group A is *computably presentable* or simply *computable* if there exists a group B isomorphic to A so that the domain of B is ω and the operation on B is a Turing computable function on two arguments. The group B is called a computable presentation of A.

Definition 1.1 can be generalized to any algebraic structure. For instance, computable fields [MN79, EG00] and Boolean algebras [Gon97] have been studied extensively. Notice that a group is computably presentable exactly if it has a "recursive" presentation in the sense of Higman [Hig61] with decidable word problem. If the word problem is merely computably enumerable, then we say that the group is *c.e. presented*.

1.1. **Degree spectra.** Although the central objects of our studies are computable groups, noncomputable presentations appear naturally in many cases. For instance, finitely presented groups with unsolvable word problem have no computable presentation, and infinitely generated c.e. presented torsion abelian groups may have no computable copy [Khi98].

The standard approach of effective algebra to non-computable algebraic structures uses the following definition. An infinitely countable algebraic structure $\mathcal{A}$ is computable relative to a Turing degree $\mathbf{d}$ (see [Soa87]), or $\mathbf{d}$-computable, if the universe of $\mathcal{A}$ can be identified with the natural numbers ω in such a way that the atomic diagram of $\mathcal{A}$ becomes $\mathbf{d}$-computable. For instance, one may speak of structures computable in the halting problem, the natural examples being c.e. presented groups. In the context of groups, this approach is equivalent to Definition 1.1 relativized to $\mathbf{d}$.

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Jockusch suggested that the algorithmic nature of an algebraic structure $\mathcal{A}$ is best captured by the following invariant called the *degree spectrum* of $\mathcal{A}$:

$$\mathrm{DegSpec}(\mathcal{A}) := \{\mathbf{d} : \mathcal{A} \text{ is } \mathbf{d}\text{-computable}\},$$

which is the collection of all Turing degrees that can compute a copy of $\mathcal{A}$. The degree spectrum of $\mathcal{A}$ may or may not have a least element under Turing reducibility. If $\mathbf{a}$ is the least element in $\mathrm{DegSpec}(\mathcal{A})$, then $\mathbf{a}$ is called the *degree* of $\mathcal{A}$ [Ric81].

Even elementary structures such as additive subgroups of the rationals may have no degree. In this case one uses the Turing jump operator to obtain more information about such structures. Recall that $\mathbf{d}'$ stands for the halting problem for machines with oracle $\mathbf{d}$. Then a Turing degree $\mathbf{a} \geq \mathbf{0}^{(\alpha)}$ is the proper α^{th} jump degree of A if the set $\{\mathbf{d}^{(\alpha)} : \mathbf{d} \in \mathrm{DegSpec}(\mathcal{A})\}$ has $\mathbf{a}$ as its least element, and for every $\gamma < \alpha$ the set $\{\mathbf{d}^{(\gamma)} : \mathbf{d} \in \mathrm{DegSpec}(\mathcal{A})\}$ is not a cone under $\leq_T$. (Here α is of course a computable ordinal.) There has been a lot of work on degree spectra and α^{th} jump degrees of various algebraic structures; see, e.g., [Ric81, DK92, AJK90, JS91, Kni86, JS94].

1.2. **Degree spectra of torsion-free abelian groups.** Recently, there has been significant progress in understanding the degree spectra of torsion-free abelian groups. We list all results known in this direction in the next few lines.

It is not difficult to show that for every $n \leq 3$ and every degree $\mathbf{d} > \mathbf{0}^{(n)}$, there is a torsion-free group G having proper n^{th} jump degree $\mathbf{d}$ (see Downey [Dow97] for $n = 0, 1$ and Melnikov [Mel09] for $n = 2, 3$; discussed in [AKMS12]). Coles, Downey, and Slaman [CDS00] showed that, in fact, *every* torsion-free abelian group of rank 1 has a first jump degree; their result can be extended to any finite rank [Mel09]. The case of higher ordinals remained unresolved until the recent work of Andersen, Kach, Melnikov, and Solomon [AKMS12] who showed that for every computable $\alpha > 3$, each $\mathbf{d} > 0^{(\alpha)}$ can be realized as the proper α^{th} jump degree of a torsion-free abelian group.

In this paper, we continue the investigation of degree spectra of torsion-free abelian groups. We follow Soare [Soa87] in the definition of $\Sigma^0_{\omega+1}$. (Thus, $(\emptyset^{(\omega)})'$ is a $\Sigma^0_{\omega+1}$-complete set.) In [AKMS12], Andersen, Kach, Melnikov and Solomon asked whether for each computable ordinal $\alpha \geq 3$ there exists a torsion-free abelian group having proper α^{th} jump degree $0^{(\alpha)}$ (Question 1.6). We prove:

Theorem 1.2. For every computable ordinal β of the form $\delta + 2n + 1 > 1$, where δ is zero or is a limit ordinal and $n \in \omega$, there exists a torsion-free abelian group having an X-computable copy if, and only if, X is *non-low*$_\beta$.

As an immediate corollary, we obtain:

Corollary 1.3. For every β as in Theorem 1.2, there exists a torsion-free abelian group having $0^{(\beta+1)}$ as its proper $(\beta + 1)^{th}$-jump degree.

The *non-low*$_\beta$ degree spectra for such computable β are new to abelian groups in general. We note that such degree spectra cannot be realized in the class of reduced abelian p-groups of small Ulm length [KKM13]. Theorem 1.2 extends results from the earlier paper [Mel09] where the case of $\beta = 1$ was established.

Proving the theorem requires an implementation of a machinery that has recently been used to study jump degrees of torsion-free abelian groups [AKMS12]. This technique is based on ideas from classical abelian group theory [Fuc70] and also on previous work [AKMS12, DM08, FKM+11]. While the result in [AKMS12] gave a uniform and reconstructable coding of a Σ^0_α-set into the isomorphism type of a computable group, the proof of Theorem 1.2 gives an effective coding of a uniformly Σ^0_α-family of finite sets into a computable group (Proposition 2.1). Coding a family indeed requires new ideas on top of what is already contained in [AKMS12] (to be discussed). For instance, we introduce a new technical notion of a σ-shifted component that is central to the proof. We believe that this technical notion may find further applications in the future.

1.3. **A framework.** We note that, in [AKMS12] and in the present paper, the choice of primes used in the coding is somewhat irrelevant to the required properties of the constructed groups. For the sake of presentation and future applications, in this paper we use different primes in different locations of the groups we build. The modification makes the groups more rigid. Consequently, we can extract more information from their algebraic structure with less effort (see, e.g., Claim 2.16). For the sake of this modification, we need to formally and inductively define a specific class of groups. Direct sums of groups from this class include most of the torsion-free abelian groups which have been used recently in effective algebra (see, e.g., [DM08, DM13]). We hope that the general approach introduced in this paper may serve as a technical base for future systematic research in the area.

2. Proof of Theorem 1.2

Understanding the proof of Theorem 1.2 requires a background in abelian group theory [Fuc70, Fuc73]. It is expected that the reader is familiar and comfortable with the notions of linear independence, divisible and pure closure, infinite and finite divisibility, notations $p^\infty|x$ and $\frac{z}{p^\infty}$, etc. A sufficient initial segment of the theory can be also found in [DM13, AKMS12]. We use infinitary computable formulae [AK00] and assume that the reader is familiar with the foundations of computability theory [Soa87].

We claim that, in order to prove Theorem 1.2, it is sufficient to establish:

Proposition 2.1. For every infinite collection $\mathcal{R}$ of finite sets and every computable $\alpha = \delta + 2n + 2 > 2$, where $\delta = 0$ or is a limit ordinal and $n \in \omega$, there exists a torsion-free abelian group $G_{\alpha,\mathcal{R}}$ such that $G_{\alpha,\mathcal{R}}$ has an X-computable presentation if, and only if, $\mathcal{R}$ has a uniform Σ^X_α-enumeration.

To see why it is sufficient to prove Proposition 2.1, recall that for every such α there exists a family of finite sets having Σ^X_α-enumeration if, and only if, X is *non-low*$_{\alpha-1}$ (this fact follows from [Weh98][1]). The rest of the paper is devoted to the proof of Proposition 2.1.

Proof idea. We suppress α in $G_{\alpha,\mathcal{R}}$. We will effectively and uniformly in a given enumeration of $\mathcal{R}$ encode every finite set S of $\mathcal{R}$ into a separate group H_S and then put them together in a direct sum:

$$G_{\mathcal{R}} = \bigoplus_{S\in\mathcal{R}} H_S.$$

[1]For example, the main result of [Weh98] relativized to $0'$ gives a family that has a $\Sigma^{\mathbf{a}}_1$-enumeration if and only if $\mathbf{a} > 0'$. Thus, the family has a $\Sigma^{X'}_1$-enumeration, or Σ^X_2-enumeration, if and only if $X' > 0'$ if and only if X is non-low (= non-low$_1$). This way we obtain $\beta + 1 = \alpha$ in the notations of Proposition 2.1 and Theorem 1.2.

We then read S off the isomorphism type of the group using infinitary logic. The formula that we use, roughly, asks if a certain configuration of elements and relations between them occur in the group. The *main difficulty* is that the configuration does not have to be within H_S, but can be spread among several components. Thus, we have to be very careful when choosing the isomorphism types of the components, and in fact the main difficulty of the proof is that not every choice would do the job.

The first main idea is coding subsets of S instead of coding a *finite* $S \in \mathcal{R}$ itself. Let $(D_i)_{i\in\omega}$ be the standard effective enumeration of all finite subsets of ω. For every $S \in \mathcal{R}$, we shall encode

$$Age_S = \{i : D_i \subseteq S\},$$

into the isomorphism type of a direct component H_S of $G_{\mathcal{R}}$. Observe that the relation $D_i \subseteq S$ is only Σ^0_α. The first naive attempt of encoding would be defining H_S using a sequence of groups indexed by $i \in \omega$ and encoding Σ^0_α or Π^0_α-outcomes depending on whether $D_i \subseteq S$ or not. (The ordered sequence would be then put together using the "chain operation" which will be defined later.) Unfortunately, the group encoding the Σ^0_α-outcome depends on the least witness of the Σ^0_α-event (i.e. $G(\Sigma^0_\alpha(n)) \not\cong G(\Sigma^0_\alpha(n))$ when $m \neq n$), and so we get the isomorphism type $G(\Sigma^0_\alpha(n))$ for *some* n.

The second main idea allows us to circumvent the difficulty explained above. Recall that S is finite. We take all finite $\sigma \in \omega^{<\omega}$ and produce a "σ-shifted" H_S, denoted $H_{\sigma,S}$, for every such σ. The entries of the string σ will be used to specifically force the corresponding procedures to modify their computations shifting the occurrence of the potential witnesses. The construction is organized so that we necessarily list all possible combinations $G(\Sigma^0_\alpha(n))$ (for all n) among the corresponding components of $H_{\sigma,S}$ *if the true outcome is* Σ^0_α, and also so that the shift does not effect the Π^0_α-outcomes. Note that almost all outcomes are Π^0_α, since S is finite. We then set

$$H_S = \bigoplus_{\sigma\in\omega^{<\omega}} H_{\sigma,S}.$$

The σ-shifts will homogenize the group $G_{\mathcal{R}}$ and will make its isomorphism type independent of the enumeration of $\mathcal{R}$.

Some parts of the verification rely on the machinery developed in [AKMS12]. Several adjustments need to be made to the machinery, but these are not crucial. Although we will discuss and illustrate the machinery from [AKMS12], we believe that a reader aiming for a deep understanding of the verification may find a detailed study of [AKMS12] rather helpful. We will modify the coding components from [AKMS12], and this modification will make the verification more accessible. Alternatively, the reader may choose to use the Σ- and Π-coding components from [AKMS12] in place of the components defined below. In this case Claim 2.16 has to be replaced by Lemma 4.14 of [AKMS12]. Both approaches will give a proof of Proposition 2.1, but the second seems to be unsatisfactory since Lemma 4.14 of [AKMS12] is weaker than Claim 2.16 but is harder to prove.

The proof is organized as follows. To define the Σ^0_α- and Π^0_α- coding components we need to explain the general machinery from [AKMS12]; this is done in Subsection 2.1. In Subsection 2.1, we also introduce a modification to the machinery which was discussed above. For the sake of this modification, we need to formally define a certain class of groups (Definition 2.2) and a new operation on this class

(Definition 2.3). We formally define the Σ^0_α- and Π^0_α- coding components in Subsection 2.2. The effective content of Subsection 2.2 is discussed in Subsection 2.3. Then, in Subsection 2.4, we prove a proposition which demonstrates a typical application of the machinery. The proposition also gives an idea of how the coding works (locally) in the case of $\alpha = 3$. We will use the proposition in the verification. Only after we describe the machinery and gain some intuition will we finally arrive at a formal definition of $G_{\mathcal{R}}$ in Subsection 2.5. The verification is contained in Subsection 2.6.

Although we believe that a good understanding of the construction requires reading the subsections linearly, an impatient reader may look at Subsection 2.1 first, and then go to Subsection 2.5 immediately. After looking at the definition of $G_{\mathcal{R}}$, the reader may go back to Subsection 2.2 and see what the Σ- and Π-coding components are. The reader may then proceed to the verification.

2.1. **Basic operations.** We call a torsion-free abelian group with a distinguished non-zero element a *rooted* group, and we call the distinguished element the *root* of the group [AKMS12]. (We will construct an elementary rooted group (G, g) and then remove the root g from its signature.) The (rooted) groups used in [AKMS12, DM08, FKM+11] were constructed from smaller groups using certain elementary operations. To state Definition 2.3 formally, we would like to isolate these operations from concrete contexts.

(1.) *Connection by a prime*: For rooted groups (G, g) and (H, h) and a prime q, define
$$(H(\frac{h+g}{q^\infty})G, h)$$
to be the group
$$\langle H \oplus G; \frac{h+g}{q^n} : n \in \omega\rangle$$
with root h.

(2.) *Chain operation*: Given rooted groups $(G_k, g_k)_{k\in\omega}$ and primes $(q_k)_{k\in\omega}$, let
$$(G_k \left(\frac{g_k + g_{k+1}}{q_k^\infty}\right)_{k\in\omega} G_{k+1}, g_0)$$
be the group
$$\langle \bigoplus_{k\in\omega} G_k; \frac{g_k + g_{k+1}}{q_k^m} : g_k \in G_k, k, m \in \omega\rangle$$
with root g_0.

(3.) *Branching operation*: Given rooted (H, h) and $(G_k, g_k)_{k\in\omega}$ and $J \subseteq \omega$ ($J \neq \emptyset$), let $(H(\frac{h+g_j}{q^\infty})_{j\in J} G_j, h)$ be the group
$$\langle H \oplus \bigoplus_{j\in J} G_j; \frac{h+g_j}{q^m} : h \in H, g_j \in G_j, j \in J, m \in \omega\rangle$$
with root h.

(4.) *Prime closure*: Given a rooted group (G, g) and a set of primes P, we define $[(G, g)]_P$, also written $([G]_P, g)$, to be
$$\langle g/p^n : g \in G, p \in P, n \in \omega\rangle \subseteq \mathcal{D}(G),$$

with the same root g. Here $\mathcal{D}(G)$ stands for the divisible closure of G. We write $[G]_p$ instead of $[G]_{\{p\}}$ for a singleton $\{p\}$.

In the following, we call the operations above *elementary*.

Definition 2.2 (Elementary rooted groups)**.** We call a rooted group *elementary* if it can be constructed from rooted groups of the form $([\mathbb{Z}]_P, p^k)$ using a finite sequence of elementary operations. Here p is a prime, P is a finite set of primes that may or may not contain p, and $k \in \omega$. (These parameters do not have to be the same for different sub-components. If $P = \emptyset$, then $[\mathbb{Z}]_P = \mathbb{Z}$.)

Suppose (G, g) is an elementary rooted group. We associate (G, g) with a labeled rooted tree of finite height. The idea is that this tree can later be used to visualize the group or to change the primes that were used in the definition of the group. Formally, we define the labeled tree by induction, as follows.

i. If $(G, g) = ([\mathbb{Z}]_P, p^k)$, then its tree contains only one node which is labeled by (P, p^k).
ii. If $(G, g) = (H(\frac{h+g}{q^\infty})U, g)$, then put an edge between the roots of the already defined structural trees of (H, h) and (U, g), and we label this edge by $\{q\}$. Declare the root of the structural tree of (U, g) to be the root of the structural tree of (G, g).
iii. If $(G, g) = (G_k\left(\frac{g_k+g_{k+1}}{q_k^\infty}\right)_{k\in\omega} G_{k+1}, g_0)$, then connect the root of the already defined structural tree of (G_k, g_k) and the root of the structural tree of (G_{k+1}, g_{k+1}) by an edge and label this edge by $\{q_k\}$. Declare the root of the structural tree of (G_0, g_0) to be the root of the structural tree of (G, g).
iv. If $(G, g) = (H(\frac{h+g_j}{q^\infty})_{j\in J} G_j, h)$, then for each $j \in J$ connect the root of the structural tree of (H, h) to the root of the structural tree of (G_j, g_j) and label it by $\{q\}$. Declare the root of the structural tree of (H, h) to be the root of the new structural tree.
v. If $(G, g) = [(H, g)]_S$, then replace every label of the form (P, p^k) by $(P \cup S, p^k)$, and replace every label of the form W by $W \cup S$.

Clearly, groups having identical structural trees are isomorphic. Note that an elementary rooted group (G, g) may have more than one structural tree. Nonetheless, we will always associate a group with some *specifically chosen* structural tree (given by the construction, say), and we call it *the* (structural) tree of (G, g). Once the tree is fixed, it makes sense to speak about sub-components, or blocks, of G, those being groups naturally corresponding to subtrees of the tree.

Definition 2.3 (Prime substitution)**.** Let S be the set of all primes that occur in the labels of the structural tree of an elementary rooted group (G, g), and let $\phi : S \to S'$ be a bijection of S onto a set of primes S'. We define $(G, g)_\phi$ as follows. In each label of the structural tree of (G, g), replace every prime p by $\phi(p)$. Then pass to the elementary group corresponding to the new labeled tree.

It is clear that the class of elementary rooted groups is closed under the operation of prime substitution. All our groups will be direct sums of elementary rooted groups. We can extend the operation of prime substitution to direct sums in the obvious way.

2.2. **Groups encoding Σ^0_α and Π^0_α outcomes.** We define groups $G(\Pi^0_\beta)$, $G(\Sigma^0_\beta(m))$ and $G(\Sigma^0_\beta)$ by recursion. The definition below is similar to Definition 3.1 from [AKMS12], but is not the same. More specifically, we use the operation *prime substitution* instead of *prime closure* at intermediate steps.

We use letters p, q, v, u, d with subscripts to denote distinct primes. For the sake of effectiveness, we assume that the subscripts correspond to effective listings of distinct primes. Every ordinal $\beta \leq \alpha$ will be identified with its notation, and hence every limit ordinal will be associated with a specific effective sequence $(\gamma_i)_{i\in\omega}$ of smaller ordinals having β as their supremum. We will be using the operation of prime substitution for which we need to define, by transfinite recursion, bijections from sequences of primes to sets of new fresh primes (see Definition 2.3). To avoid the unnecessary formalism, we will not define these maps formally and will take for granted that it can be done:

Notation 2.4. We write $\phi_{\beta,k}$ for the map indexed by an ordinal β and a number k, and assume that $\phi_{\beta,k}$ maps, effectively and uniformly, primes to new fresh primes which were not used at previous inductive steps. We also assume that the ranges of the $\phi_{\beta,k}$ do not overlap for different subscripts. Whenever we use any of the $\phi_{\beta,k}$ for prime substitution, we assume that all the primes in the group under the operation are in the domain of $\phi_{\beta,k}$.

Definition 2.5. In the following, δ always denotes 0 or a limit ordinal.

- For $\beta = 2$, define $G(\Sigma^0_\beta(m))$ to be the group $[\mathbb{Z}]_{p_2}$ with root q_2^m, and let $G(\Pi^0_\beta)$ be the group $[\mathbb{Z}]_{p_2,q_2}$ with root 1.
- For $\beta \geq 3$ of the form $\delta + 2n + 1$, define
$$G(\Sigma^0_\beta) = [\mathbb{Z}]_{p_\beta} \left(\frac{r + r_k}{q_\beta^\infty}\right)_{k\in\omega} A_k,$$
where r_k is the root of A_k, the element $r = 1$ in $[\mathbb{Z}]_{p_\beta}$ is the root of $G(\Sigma^0_\beta)$, $A_k = G(\Pi^0_{\beta-1})$ for k even, and $A_k = G(\Sigma^0_{\beta-1}(m))$ for each $k = 2\langle i, m\rangle + 1$.
- For $\beta \geq 3$ of the form $\delta + 2n + 1$, define
$$G(\Pi^0_\beta) = [\mathbb{Z}]_{p_\beta} \left(\frac{r + r_k}{q_\beta^\infty}\right)_{k\in\omega} A_k,$$
where r_k is the root of A_k, the element $r = 1$ in $[\mathbb{Z}]_{p_\beta}$ is the root of $G(\Pi^0_\beta)$, and $A_k = G(\Sigma^0_{\beta-1}(m))$ for each $k = \langle i, m\rangle$.
- For $\beta \geq 2$ of the form $\delta + 2n + 2$ and $m \in \omega$, define
$$G(\Sigma^0_\beta(m)) = H_k \left(\frac{r_k + r_{k+1}}{v_{\beta,k}^\infty}\right)_{k\in\omega} H_{k+1},$$
where r_k is the root of H_k, r_0 is declared the root of $G(\Sigma^0_\beta(m))$, and $H_k = [G(\Sigma^0_{\beta-1})]_{\phi_{\beta,k}}$ if $0 \leq k \leq m$ and $H_k = [G(\Pi^0_{\beta-1})]_{\phi_{\beta,k}}$ otherwise. (For future notational convenience, we assume that $\phi_{\beta,0}$ makes r_0 infinitely divisible by p_β.)
- For $\beta \geq 2$ of the form $\delta + 2n + 2$, define
$$G(\Pi^0_\beta) = H_k \left(\frac{r_k + r_{k+1}}{v_{\beta,k}^\infty}\right)_{k\in\omega} H_{k+1},$$

where r_k is the root of H_k, r_0 is declared the root of $G(\Pi^0_\beta)$, and $H_k = [G(\Sigma^0_{\beta-1})]_{\phi_{\beta,k}}$ for all k. (For future notational convenience, we assume that $\phi_{\beta,0}$ makes r_0 infinitely divisible by p_β.)

- For $\beta \geq \omega$ a limit ordinal, define

$$G(\Sigma^0_\beta(m)) = H_k\left(\frac{r_k + r_{k+1}}{v^\infty_{\beta,k}}\right)_{k\in\omega} H_{k+1},$$

where r_k is the root of H_k, r_0 is declared the root of $G(\Sigma^0_\beta(m))$, and $H_k = [G(\Sigma^0_{\gamma_k})]_{\phi_{\beta,k}}$ if $0 \leq k \leq m$ and $H_k = [G(\Pi^0_{\gamma_k})]_{\phi_{\beta,k}}$ otherwise. (For future notational convenience, we assume that $\phi_{\beta,0}$ makes r_0 infinitely divisible by p_β.)

- For $\beta \geq \omega$ a limit ordinal, define

$$G(\Pi^0_\beta) = H_k\left(\frac{r_k + r_{k+1}}{v^\infty_{\beta,k}}\right)_{k\in\omega} H_{k+1},$$

where r_k is the root of H_k, r_0 is declared the root of $G(\Pi^0_\beta)$, and $H_k = [G(\Pi^0_{\gamma_k})]_{\phi_{\beta,k}}$ for every k. (For future notational convenience, we assume that $\phi_{\beta,0}$ makes r_0 infinitely divisible by p_β.)

2.3. **Effective content of Definition 2.5.** In Definition 2.5, every alternation of quantifiers corresponds to either an application of the chain operation or the branching operation. We use the branching operation for $\beta = 3$, and then the chain operation for $\beta = 4$, etc. Each group naturally reflects a Σ^0_n-fact. Hence, it is straightforward to establish:

Lemma 2.6. Suppose S is a Σ^0_β set, uniformly in β, where β is a computable ordinal $\leq \alpha$. There exists a uniform procedure which, given e, β, constructs a computable group H_e such that $H_e \cong G(\Sigma^0_\beta)$ (or $G(\Sigma^0_\beta(m))$ for *some* m) if $e \in S$, and $H_e \cong G(\Pi^0_n)$ otherwise.

Proof. The proof proceeds by effective transfinite recursion. Recall that for all computable ordinals $\leq \alpha$ we use a fixed effective notation. Note that the primes which can be used in a given location of a group are specified by a sequence of transformations $(\phi_{\rho_k,k})_k$, where the ρ_k form a descending sequence of computable ordinals. We can compute the composition of the corresponding maps at every component of the group. The rest follows from the definition of the groups and from the well-known fact that we can choose a presentation of a Σ^0_β-predicate having stable witnesses (i.e., if $\exists x P(x)$ then $\forall y \geq x P(y)$.) $\square$

2.4. **Illustration of the technique.** The proposition below illustrates, roughly, how we will extract the Σ^0_3-outcome from a direct sum of $G(\Sigma^0_3)$- and $G(\Pi^0_3)$-subcomponents. Here Σ^0_3 naturally corresponds to "there exists a sub-component which is infinitely divisible by w". Informally, the proposition says that a "encodes" a Σ^0_3-outcome if, and only if, it is a linear combination of h_i each of which "encodes" a Σ^0_3-outcome. Proposition 2.7(1) will be used in the verification in a much more general context not necessarily corresponding to Σ^0_3 (to be explained).

Proposition 2.7. Let u, w, p, q be distinct primes. Suppose

$$A = \bigoplus_{i\in I}[H_i(\frac{h_i + g_{i,k}}{p^\infty})_{k\in K_i} G_{i,k}],$$

is a computable group where $I \subseteq \omega$, $K_i \subseteq \omega$ are non-empty, $(H_i, h_i) \cong ([\mathbb{Z}]_q, 1)$, and either $G_{i,k} \cong ([\mathbb{Z}]_u, w^{k_{i,j}})$ with $k_{i,j} \in \omega$, or $G_{i,k} \cong ([\mathbb{Z}]_{\{u,w\}}, 1)$. Then there exists a $\mathcal{L}^c_{\omega_1\omega}$ formula Ψ of complexity Σ^c_3 such that $A \models \Psi(a)$ if and only if

(1) $a = \sum_i m_i h_i$, where $m_i \in \mathbb{Z}$, and
(2) for each $m_i \neq 0$ in $a = \sum_i m_i h_i$ there exists k such that $G_{i,k} \cong ([\mathbb{Z}]_{u,w}, 1)$.

The formula can be produced uniformly in the primes u, w, p, q. The formula does not depend on I and K_i[2].

Proof. We need a lemma:

Lemma 2.8. Let $I \subseteq \omega$ be a non-empty set, and let

$$A = \bigoplus_{i \in I} [H_i(\frac{h_i + g_{i,k}}{p^\infty})_{k \in K_i} G_{i,k}],$$

where each of the H_i and $G_{i,k}$ are prime closures of $\mathbb{Z}$ by non-empty sets of primes not containing p, and with roots h_i and $g_{i,k}$, respectively[3]. Assume a is a non-zero element of A such that $p^\infty | a$. Then

$$a = \sum_i (c_i h_i + \sum_{f \in F_i} d_{i,f} g_{i,f})$$

where $c_i, d_{i,f}$ are rationals and $F_i \subseteq K_i$ are finite sets such that

$$c_i = \sum_{f \in F_i} d_{i,f}$$

for every i.

Remark 2.9. Notice that for some (possibly, for all) indices i, the coefficients c_i in $a = \sum_i (c_i h_i + \sum_{f \in F_i} d_{i,f} g_{i,f})$ may be zero.

Proof of Lemma 2.8. By the definition of G,

$$a = \sum_{i,j} r_{i,j}(h_i + g_{i,j}),$$

where $r_{i,j}$ are rationals, for otherwise a would not be infinitely divisible by p. Let $B_i = H_i(\frac{h_i+g_{i,k}}{p^\infty})_{k \in K_i} G_{i,k}$. The groups B_i can be viewed as elementary rooted groups with structural trees T_i of height 1. Taking the restriction of T_i to vertices that occur in $a_i = \sum_j r_{i,j}(h_i + g_{i,j})$ with non-zero coefficients, we obtain a finitely branching tree of height 1. Suppose $g_{i,j}$ is so that $r_{i,j} \neq 0$. The element

$$a_i' = a_i - r_{i,j}(h_i + g_{i,j})$$

is either 0 or corresponds to a sub-tree of T_i having fewer successors. In either case, by the inductive hypothesis, there exists a decomposition

$$a_i' = c_i' h_i + \sum_{f \in F_i'} d_{i,f}' g_{i,f}$$

such that $c_i' = \sum_{f \in F_i'} d_{i,f}'$. Then clearly

$$a_i = (c_i' + r_{i,j})h_i + \sum_{f \in F_i'} d_{i,f}' g_{i,f} + r_{i,j} g_{i,j}$$

[2]Although it does not help to prove the proposition, we note that in all actual applications I and K_i will be computable with all possible uniformity.

[3]Note that we have no further assumptions on the cardinalities of the non-empty sets $K_i \subseteq \omega$.

and $c'_i + r_{i,j} = \sum_{f \in F'_i} d'_{i,f} + r_{i,j}$. $\square$

The formula Ψ states that $a \neq 0$, $q^\infty | a$, and there exists a non-zero x such that $p^\infty | (a+x) \wedge u^\infty | x \wedge w^\infty | x$. Note that every element of the desired form satisfies the formula trivially. Suppose a satisfies Ψ. By Lemma 2.8,

$$a + x = \sum_i (c_i h_i + \sum_{f \in F_i} d_{i,f} g_{i,f}),$$

where $F_i \subseteq K_i$ are finite sets and

$$c_i = \sum_{f \in F_i} d_{i,f}$$

for every i. The rest of the proof relies on the elementary properties of direct decompositions that can be found in [Fuc70, Fuc73].

Embed the group A into its p-closure

$$[A]_p = \bigoplus_{i \in I} \left([H_i]_p \oplus \bigoplus_{k \in K_i} [G_{i,k}]_p \right).$$

Since infinite divisibility is preserved under isomorphic embeddings, we have $[A]_p \models q^\infty | (a+x)$. Suppose $a = b_a + d_a$, where $b_a \in \bigoplus_i [H_i]_p$ and $d_a \in \bigoplus_{i, k \in K_i} [G_{i,k}]_p$, and similarly $x = b_x + d_x$. Recall that $u^\infty | x$, and thus $b_x = 0$. Since $q^\infty | a$, we have $d_a = 0$. Recall that $a + x = \sum_i (c_i h_i + \sum_{f \in F_i} d_{i,f} g_{i,f})$. We conclude that $a = \sum_i c_i h_i$ and $x = \sum_{i, f \in F_i} d_{i,f} g_{i,f}$. By our assumption, $w^\infty | x$, and therefore $w^\infty | g_{i,f}$ for every i and $f \in F_i$. The latter gives (2) of the proposition.

We now prove (1). Recall that $c_i = \sum_{f \in F_i} d_{i,f}$, where $c_i \in [\mathbb{Z}]_{p,q} \cap [\mathbb{Z}]_{p,u,w} = [\mathbb{Z}]_p$. It remains to show $c_i \in \mathbb{Z}$. Since divisibility is preserved under projections onto direct components, it is sufficient to prove that for any choice of integers $m, k > 0$ and each i the element $mh_i/p^k \in [A]_p$ does not belong to A. Suppose we have $mh_i/p^k \in A$, $p \not| m$, then

$$mh_i/p^k = \sum_i \left(r_i h_i + \sum_j s_{i,j}(h_i + g_{i,j}) + \sum_l \eta_{i,l} g_{i,l} \right),$$

where $r_i \in [\mathbb{Z}]_q$, $s_{i,j} \in [\mathbb{Z}]_p$, and $\eta_{i,l} \in [\mathbb{Z}]_{u,w}$. Since $h_i, g_{i,j}$ are linearly independent for different choices of i and j,

$$mh_i/p^k = r_i h_i + \sum_j s_{i,j}(h_i + g_{i,j}) + \sum_l \eta_{i,l} g_{i,l}.$$

We have $m/p^k = r_i + \sum_j s_{i,j}$; and for every j there is an l such that $s_{i,j} = -\eta_{i,l}$ (and indeed $l = j$). We conclude that $s_{i,j} \in [\mathbb{Z}]_{u,w} \cap [\mathbb{Z}]_p = \mathbb{Z}$ and thus $m/p^k \in [\mathbb{Z}]_q \cap [\mathbb{Z}]_p = \mathbb{Z}$, i.e. $k = 0$. $\square$

Remark 2.10. If we remove the conjunct $w^\infty | x$ from the formula Ψ from the proof of Proposition 2.7, we will obtain a syntactical necessary and sufficient condition for an element to satisfy (1) of Proposition 2.7.

In the verification, we will be applying Proposition 2.7 and Remark 2.10 to various sub-groups of elementary rooted groups. These applications will require a slight modification of the proposition which will be repeated over and over again in different contexts. We explain this modification below.

We follow the notation from Proposition 2.7. The structural tree of the larger computable group G containing A will possibly have edges connecting the elements playing the roles of g_i and $h_{i,k}$ to some other elements of G outside A. These edges will be labeled by primes, say by s_i and $v_{i,k}$, respectively. We will have $s_i \neq v_{j,k}$ for any i, j, k, and s_i and $v_{i,k}$ will be unequal to any of the primes playing the roles of u, w, p, q. A typical application of Proposition 2.7 can be described as follows:

- Consider the least (super)group $H \supseteq G$ in the divisible closure of $G \supseteq A$ such that $H \models s_i^{\infty}|g_j$ and $H \models v_{j,k}^{\infty}|h_{i,k}$.
- The least pure subgroup S of H containing A will detach as a direct summand of H.
- By the choice of Ψ, we will have $G \models \Psi(g)$ implies $H \models \Psi(g)$ and $g \in S$ (indeed, $S \models \Psi(g)$).
- Then we can repeat the proof of Proposition 2.7(1) and get Proposition 2.7(1) but perhaps with $m_i \in \bigcup_i [\mathbb{Z}]_{s_i} \leqq \mathbb{Q}$.
- By Lemma 2.8, the choice of Ψ and $[\mathbb{Z}]_{v_{j,k}} \cap [\mathbb{Z}]_{s_i} = \mathbb{Z}$ will imply that $m_i \in \mathbb{Z}$.

We conclude that Proposition 2.7 goes through in this more general situation. We will omit the repeated argument above in all actual applications of Proposition 2.7 and Remark 2.10.

2.5. **Coding family $\mathcal{R}$ into $G_{\mathcal{R}}$.** Given a finite set S, define its *age* to be

$$Age_S = \{i : D_i \subseteq S\},$$

where $(D_i)_{i\in\omega}$ is the canonical listing of all finite sets of natural numbers.

Recall that $\alpha = \delta + 2n + 2$, where δ is 0 or a limit ordinal and $n \in \omega$. We also fix injective and effective maps $\psi_{\alpha,k}$, $k \in \omega$ (to be used for the operation of substitution) which are different from the $\phi_{\beta,k}$ and also consistent with Definition 2.5 (i.e., they effectively map the primes used in the corresponding $G(\Sigma)$- or $G(\Pi)$-component H_k to new fresh primes which do not overlap for different k). In the following, we suppress α in $\psi_{\alpha,k}$. We identify finite strings in $\omega^{<\omega}$ with functions from $\omega^\omega \to \{0, 1\}$ having finite support.

Definition 2.11. Given any finite set S and any finite string $\sigma \in \omega^{<\omega}$, define

$$H_{\sigma,S} = B_{S,\sigma,k} \left(\frac{r_{S,\sigma,k} + r_{S,\sigma,k+1}}{w_{\alpha,k}^{\infty}} \right)_{k\in\omega} B_{S,\sigma,k+1},$$

where $B_{S,\sigma,k} \cong [G(\Sigma_\alpha^0(\sigma(k)))]_{\psi_k}$ if $k \in Age_S$, and $B_{S,\sigma,k} \cong [G(\Pi_\alpha^0)]_{\psi_k}$, otherwise. Define

$$H_S = \bigoplus_{\sigma\in\omega^{<\omega}} H_{\sigma,S}.$$

Finally, let

$$G_{\mathcal{R}} = \bigoplus_{S\in\mathcal{R}} H_S.$$

We can effectively choose ψ_k to be consistent with Definition 2.5. Consequently, we have:

Lemma 2.12. There exists a uniform procedure which, given any Σ^0_α-*enumeration* of $\mathcal{R}$, outputs a computable presentation of the group

$$G_{\mathcal{R}} = \bigoplus_{S \in \mathcal{R}} H_S.$$

Proof. Note that $i \in Age_S$ is a uniformly Σ^0_α-fact. Consequently, we can apply Lemma 2.6, an effective enumeration of $\omega^{<\omega}$, and the fact that the ψ_k can be effectively and consistently defined. It is crucial that $\mathcal{R}$ consists of finite sets. $\square$

2.6. **Reconstructing $\mathcal{R}$ from $G_{\mathcal{R}}$.** We use the same notations as in the previous subsection. According to its definition, the group $G_{\mathcal{R}}$ is of the form:

$$G_{\mathcal{R}} = \bigoplus_{S \in \mathcal{R}, \sigma \in \omega^{<\omega}} \left(B_{S,\sigma,k} \left(\frac{r_{S,\sigma,k} + r_{S,\sigma,k+1}}{w^\infty_{\alpha,k}} \right)_{k \in \omega} B_{S,\sigma,k+1} \right),$$

where $B_{S,\sigma,k} \cong [G(\Sigma^0_\alpha(\sigma(k)))]_{\psi_k}$ if $k \in Age_S$, and $B_{S,\sigma,k} \cong [G(\Pi^0_\alpha)]_{\psi_k}$, otherwise. We state a lemma that relies on algebraic and combinatorial techniques developed in [AKMS12].

Lemma 2.13. There exists a uniform collection $(\Phi_k)_{k \in \omega}$ of computable infinitary Σ^c_α-formulae such that $G_{\mathcal{R}} \models \Phi_k(x)$ if and only if

$$x = \sum_{(S,\sigma) \in I} m_{S,\sigma} r_{S,\sigma,0},$$

where $m_{S,\sigma} \in \mathbb{Z} \setminus \{0\}$, and for some S such that $(S, \sigma) \in I$ we have $k \in Age_S$ (equivalently, $r_{S,\sigma,k}$ is the root of $B_{S,\sigma,k} \cong [G(\Sigma^0_\alpha(\sigma(k)))]_{\psi_k}$).

We first apply Lemma 2.13 to prove Proposition 2.1, and then we prove Lemma 2.13.

Proof of Proposition 2.1. Lemma 2.6 implies that every Σ^0_α-family $\mathcal{R}$ can be uniformly transformed into a computable copy of $G_{\mathcal{R}}$. We prove that if $G_{\mathcal{R}}$ is computable then $\mathcal{R}$ has a Σ^0_α-enumeration. If we succeed, then a straightforward relativization will prove the proposition.

Given an oracle $\mathcal{Y}$ for the (α-1)'th iteration of the Turing jump, enumerate the sets

$$U_x = \{k : G_{\mathcal{R}} \models \Phi_k(x)\},$$

where $x \in G_{\mathcal{R}}$, $x \neq 0$. If $x = \sum_{(S,\sigma) \in I} m_{S,\sigma} r_{S,\sigma,0}$, then Lemma 2.13 implies that

$$U_x = \bigcup_{S:(S,\sigma) \in I} Age_S.$$

Acting effectively in $\mathcal{Y}$, we start enumerating a sequence of finite sets

$$\emptyset = D_{i_1} \subseteq D_{i_2} \subseteq \dots$$

listing indices in U_x. If D_{i_n} has already been defined, then we wait for a $j \in U_x$ such that $D_{i_n} \subset D_j$. If we find such a j, then we set $i_{n+1} = j$. We set $i_{n+1} = i_n$, otherwise. We then repeat the same procedure for i_{n+1}, etc. Notice that the sequence $(D_{i_n})_{n \in \omega}$ has to stabilize on some finite set which is equal to one of the elements in $\{S : (S, \sigma) \in I\}$. Indeed, we may have $D_{i_{n+1}} \supset D_{i_n}$ only if there exists an S in the finite family $\{S : (S, \sigma) \in I\}$ of finite sets such that $S \supseteq D_{i_{n+1}} \supset D_{i_n}$. On the other hand, if D_{i_n} has been defined and there exists at least one S in $\{S : (S, \sigma) \in I\}$ such that $S \supset D_{i_n}$, then there will be a stage at which we will define $D_{i_m} \supset D_{i_n}$. Also notice that if x is not of the form $\sum_{(S,\sigma)\in I} m_{S,\sigma} r_{S,\sigma,0}$, then the sequence consists of only the empty set (by Lemma 2.13). We conclude that for each $x \neq 0$, the union $C_x = \cup_{n\in\omega} D_{i_n}$ of the sequence defined by the procedure above is a $\mathcal{Y}$ c.e. set uniformly in x, and furthermore $C_x = S$ for some S in $\{S : (S,\sigma) \in I\}$ corresponding to x.

We define a $\mathcal{Y}$-effective enumeration

$$\{R_x : x \in G_{\mathcal{R}}\}$$

of the family $\mathcal{R}$ by the rule $R_x = \{(x,j) : j \in C_x\}$. As we have already mentioned, either $R_x = \emptyset$ (we can assume $\emptyset \in \mathcal{R}$) and then $R_x \in \mathcal{R}$ trivially, or $R_x \neq \emptyset$ but still $R_x \in \mathcal{R}$. Also, every set in $\mathcal{R}$ clearly will be listed. This establishes Proposition 2.1 and, hence, Theorem 1.2. □

Proof of Lemma 2.13. We have

$$G_{\mathcal{R}} = \bigoplus_{S\in\mathcal{R},\sigma\in\omega^{<\omega}} \left(B_{S,\sigma,k}\left(\frac{r_{S,\sigma,k} + r_{S,\sigma,k+1}}{w^{\infty}_{\alpha,k}}\right)_{k\in\omega} B_{S,\sigma,k+1} \right),$$

where $B_{S,\sigma,k} \cong [G(\Sigma^0_\alpha(\sigma(k)))]_{\psi_k}$ if $k \in Age_S$, and $B_{S,\sigma,k} \cong [G(\Pi^0_\alpha)]_{\psi_k}$, otherwise.

Suppose $x \in G_{\mathcal{R}}$ is not equal to zero.

Claim 2.14. There exists a formula of complexity Σ^c_3 such that x satisfies the formula if and only if $x = \sum_{(S,\sigma)\in I} m_{S,\sigma} r_{S,\sigma,0}$ where $m_{S,\sigma} \in \mathbb{Z}$ for each (S,σ).

Proof of Claim. The formula can be produced using Remark 2.10 with the right choice of primes and index sets in Proposition 2.7. More specifically, the formula says that $x \neq 0$ and $\psi_0(p_\alpha)^\infty | x$, and there exists y such that $\psi_1(p_\alpha)^\infty | y$ and $w^\infty_{\alpha,0} | (x+y)$. □

Given k and $x = \sum_{(S,\sigma)\in I} m_{S,\sigma} r_{S,\sigma,0}$, we would like to check, using a formula, whether $r_{S,\sigma,k}$ is the root of a $G(\Sigma^0_\alpha(\sigma(k)))$-type component. We first prove:

Claim 2.15. Let $x = \sum_{(S,\sigma)\in I} m_{S,\sigma} r_{S,\sigma,0}$ where $m_{S,\sigma} \in \mathbb{Z}$. For every k we can uniformly produce a Σ^c_3 formula Θ such that $\Theta_k(x,y)$ if and only if $y = \sum_{(S,\sigma)\in I} m_{S,\sigma} r_{S,\sigma,k}$. (That is, y is a linear combination of the roots of $B_{S,\sigma,k}$ and, furthermore, the corresponding coefficients are equal to those of $r_{S,\sigma,0}$ in x.)

Proof of Claim. We prove the claim by induction. The case $k = 0$ is Claim 2.14. Suppose we have produced $\Theta_{k-1}(x,\cdot)$. Consider the pure subgroup generated by the roots of the $B_{S,\sigma,k-1}$-subcomponents and $B_{S,\sigma,k}$-subcomponents. Define $\Theta_k(x,y)$ to be the formula

$$(\exists z) \left(\Theta_{k-1}(x,z) \,\wedge\, w^\infty_{\alpha,k-1} | (y+z) \wedge\, \psi_k(p_\alpha)^\infty | y \right).$$

By the inductive hypothesis, $z = \sum_{(S,\sigma)\in I} m_{S,\sigma} r_{S,\sigma,k-1}$. Since $\psi_k(p_\alpha)^\infty | y$, we have

$$y = \sum_{(S,\sigma)\in I} t_{S,\sigma} r_{S,\sigma,k},$$

where $t_{S,\sigma}$ are rationals. By the inductive hypothesis, we may assume that $\Theta_{k-1}(x,z)$ contains a conjunct of the form $\psi_{k-1}(p_\alpha)^\infty | z$. Consider the pure closure of the subgroup generated by $r_{S,\sigma,k}$ and $r_{S,\sigma,k-1}$ for various S and σ. Note that $w^\infty_{\alpha,k-1} | (y+z)$, and thus by Lemma 2.8 applied to this pure subgroup we have $t_{S,\sigma} = m_{S,\sigma}$. □

Recall that our final goal is to produce a uniform collection $(\Phi_k)_{k\in\omega}$ of computable infinitary Σ^c_α-formulae such that $G_{\mathcal{R}} \models \Phi_k(x)$ if and only if

$$x = \sum_{(S,\sigma)\in I} m_{S,\sigma} r_{S,\sigma,0},$$

where $m_{S,\sigma} \in \mathbb{Z}\setminus\{0\}$, and for some S the element $r_{S,\sigma,k}$ is the root of $B_{S,\sigma,k} \cong [G(\Sigma^0_\alpha(\sigma(k)))]_{\psi_k}$.

By Claims 2.14 and 2.15, to conclude the proof of the lemma it is sufficient to establish:

Claim 2.16. For every k we can uniformly produce a Σ^c_α formula Γ_k such that for every element of the form $x = \sum_{(S,\sigma)\in I} m_{S,\sigma} r_{S,\sigma,k}$, $G_{\mathcal{R}} \models \Gamma_k$ if and only if for some $m_{S,\sigma} \neq 0$ the corresponding $r_{S,\sigma,k}$ is the root of $B_{S,\sigma,k} \cong [G(\Sigma^0_\alpha(\sigma(k)))]_{\psi_k}$.

Remark: The proof of this claim is similar to the proof of Lemma 4.14 of [AKMS12], and is indeed simpler. For the sake of exposition, we give a proof.

Proof of Claim. The proof uses a transfinite induction on "even" α. Before we describe the induction, we do some preliminary syntactical analysis that will be used at all steps of the induction. The idea is that we use infinite divisibility to get access to the smaller components that were used in the definition of Σ^0_α- and Π^0_α-blocks. This is essentially done by several consecutive applications of Remark 2.10.

We fix k. For every σ and S, the $B_{S,\sigma,k}$-subcomponent is of the form

$$A_{S,\sigma,i}\left(\frac{a_{S,\sigma,i}+a_{S,\sigma,i+1}}{\psi_k(v_{\alpha,i})^\infty}\right)_{i\in\omega} A_{S,\sigma,i+1},$$

where either $A_{S,\sigma,i} \cong G(\Sigma^0_{\alpha-1})$ for all i, or there exists an m such that $A_{S,\sigma,i} \cong G(\Sigma^0_{\alpha-1})$ for $i \leq m$ and $A_{S,\sigma,i} \cong G(\Pi^0_{\alpha-1})$ for $i > m$. The former corresponds to $B_{S,\sigma,k} \cong G(\Pi^0_\alpha)$, and the latter corresponds to $B_{S,\sigma,k} \cong G(\Sigma^0_\alpha(m))$ where m depends on S, σ and k.

We claim that for every i we can uniformly produce a Σ^c_3-formula $\mathcal{U}_i$ such that $\mathcal{U}_i(x,y)$ holds if and only if

$$y = \sum_{(S,\sigma)\in I} m_{S,\sigma} a_{S,\sigma,i}.$$

Indeed, we may take the formula witnessing Claim 2.14 and replace $w_{\alpha,k-1}$ by $\psi_k(v_{\alpha,i})$ in the formula, and we also replace $\psi_k(p_\alpha)$ by the prime that labels the roots $a_{S,\sigma,i}$ of $A_{S,\sigma,i}$. The same proof will witness that the formula has the desired property. A straightforward syntactical analysis of the formula $\mathcal{U}_i$ shows that it is of the form

$$(\exists z_0)\ldots(\exists z_i)\mathcal{X}_i(x,\bar{z},y),$$

where $\mathcal{X}_i(x,\bar{z},y)$ is Π^c_2.

We are ready to consider the basic case $\alpha = 4$ of the induction. We restrict ourselves to the subgroup

$$\bigoplus_{(S,\sigma)} A_{S,\sigma,i},$$

where S ranges over $\mathcal{R}$ and σ over $\omega^{<\omega}$. By Proposition 2.7, there is a Σ^c_3-formula $\mathcal{W}_{i,3}$ such that $\mathcal{W}_{i,3}(y)$ holds if and only if $y = \sum_{(S,\sigma)} n_{S,\sigma} a_{S,\sigma,i}$ where for each $n_{S,\sigma} \neq 0$, the corresponding $a_{S,\sigma,i}$ is the root of a $G(\Sigma^0_3)$-type component. In fact, this formula can be produced with all possible uniformity. The desired Σ^0_4 formula for the basic case $\alpha = 4$ is

$$(\exists y) \bigvee_i (\exists z_0) \ldots, (\exists z_i) \left(\mathcal{X}_i(x, \bar{z}, y) \ \wedge \ \neg \mathcal{W}_{i,3}(y) \right).$$

Indeed, the first conjunct inside the parentheses guarantees $m_{\sigma,S} = n_{\sigma,S}$, and the second conjunct says that $B_{S,\sigma,k} \cong [G(\Sigma^0_4(\sigma(k)))]_{\psi_k}$ with $\sigma(k) \le i$.

Suppose $\alpha > 4$. By our assumption, α is an "even" successor ordinal. We claim that we can uniformly produce $\Sigma^c_{\alpha-1}$-formulae $\mathcal{W}_{i,\alpha-1}$ such that $\mathcal{W}_{i,\alpha-1}(y)$ holds if and only if $y = \sum_{(S,\sigma)} n_{S,\sigma} a_{S,\sigma,i}$ where for some $n_{S,\sigma} \neq 0$, the corresponding $a_{S,\sigma,i}$ is the root of a $G(\Sigma^0_{\alpha-1})$-type component. If we succeed in producing such formulae, then $(\exists y) \bigvee_i (\exists z_0) \ldots, (\exists z_i) \left(\mathcal{X}_i(x, \bar{z}, y) \ \wedge \ \neg \mathcal{W}_{i,\alpha-1}(y) \right)$ will satisfy the desired properties as in the basic case $\alpha = 4$ described above.

We explain how we produce $\mathcal{W}_{i,\alpha-1}(y)$. In the following, we fix k and i and we sometimes suppress k and i in subscripts. In the next few lines we use Remark 2.10 and Proposition 2.7(1) restricted to the relevant pure subgroups of $G_{\mathcal{R}}$.

We can produce a Σ^c_3-formula that ensures $y = \sum_{(S,\sigma)} n_{S,\sigma} a_{S,\sigma,i}$, where $n_{S,\sigma}$ are integers. (Indeed, this step is not really necessary since $\mathcal{X}_i(x, \bar{z}, y)$ implies $n_{S,\sigma} = m_{S,\sigma}$.) Using primes $\psi_k(p_{\alpha-2})$ and $\psi_k(q_{\alpha-2})$, we can produce a Σ^c_3 formula $\mathcal{Z}$ that holds on z if and only if $z = \sum_{(S,\sigma,s)} l_{S,\sigma,s} d_{S,\sigma,s}$, where $d_{S,\sigma,s}$ are the roots of various $G(\Sigma^0_{\alpha-2}(m))$-type and $G(\Pi^0_{\alpha-2})$-type subcomponents of $A_{S,\sigma,i}$. Let $D_{S,\sigma,s}$ denote the subcomponent with root $d_{S,\sigma,s}$. Lemma 2.8 implies that for every i,

$$\sum_s l_{S,\sigma,s} = n_{S,\sigma}.$$

Furthermore, using a variation of Claim 2.16 with the right choice of primes (e.g., use $\psi_k(v_{\alpha-2,j})$), we can produce a uniform sequence of Σ^c_3-formulae $\{\mathcal{F}_j\}_{j\in\omega}$ such that $\mathcal{F}_j(z, c_j) \wedge \mathcal{Z}(y, z)$ holds if and only if $c_j = \sum_{(S,\sigma,s)} l_{S,\sigma,s} k_{S,\sigma,s,j}$, where $k_{S,\sigma,s,j}$ is the root of $K_{S,\sigma,s,j}$ which is the j'th subcomponent of $D_{S,\sigma,s}$ that was used in its definition via the chain operation, counting from its root. Adjusting the argument described after the proof of Proposition 2.7, we can ensure that the coefficients $l_{S,\sigma,s}$ are indeed integers[4].

The subcomponent $K_{S,\sigma,s,j}$ is of a $G(\Sigma^0_{\gamma_j})$-type or of a $G(\Pi^0_{\gamma_j})$-type for some $\gamma_j < \alpha - 2$ (note that $\alpha - 2$ could be a limit ordinal). We can effectively compute (the $\le \alpha$-notation for) γ_j. By the inductive hypothesis, we can effectively produce the formula $\mathcal{W}_{i,\gamma_j}$. Replacing several primes in $\mathcal{W}_{i,\gamma_j}$ according to the prime substitution that we used in the definition of the group, we can effectively pass to a formula $\mathcal{W}'_{\gamma_j}$ which holds on $c_j = \sum_{(S,\sigma,s)} l_{S,\sigma,s} k_{S,\sigma,s,j}$ if and only if each of the $k_{S,\sigma,s,j}$ is the root of a $G(\Sigma^0_{\gamma_j})$-type subcomponent.

The desired formula $\mathcal{W}_{i,\alpha-1}(y)$ can be set equal to

$$(\exists z) \left(\mathcal{Z}(y, z) \wedge \bigwedge_{j\in\omega} (\exists c_j) [\mathcal{F}_j(z, c_j) \wedge \mathcal{W}'_{\gamma_j}(c_j)] \right).$$

Indeed, the formula says that there exists a z which is a linear combination, with integer coefficients, of "immediate successors" of (the summands of) y in the structural tree; furthermore, these immediate successors are the roots of $G(\Sigma_{\alpha-2})$-type

[4] This is one of the main advantages of our coding when compared to [AKMS12].

components (as witnessed by any infinite sequence of $(c_j)_{j\in\omega}$). Each such z satisfies the formula trivially. On the other hand, if z satisfies the formula, then the analysis preceding its definition illustrates that z must be indeed of the desired form. This observation concludes the proof of Claim 2.16. □

□

3. Conclusion

We expect that our machinery can be used to extend Theorem 1.2 to every computable ordinal. The "even" ordinals, however, seem to require further technical adjustments or/and new ideas. The issue comes from the algebraic side of the coding, namely from the embeddability relation between components corresponding to different outcomes. (For more intuition and a discussion of the machinery, see [AKMS12].) We also state:

Question 3.1. *For which n is non-low$_n$ the degree spectrum of a completely decomposable group?*

The case of $n = 1$ was established in [Mel09].

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