URI ANDREWS. *A new spectrum of recursive models using an amalgamation construction.* ***J. Symbolic Logic***, vol. 73 no. 3 (2011), pp. 883–896.

BAKHADYR KHOUSSAINOV AND ANTONIO MONTALBÁN. *A computable $\aleph_0$-categorical structure whose theory computes true arithmetic.* ***J. Symbolic Logic***, vol. 72 no. 2 (2010), pp. 728–740.

Many classical proofs in algebra and model theory are algorithmic in nature. For example, when we show that every field is contained in an algebraically closed field, we usually *construct* the closure. Another illustrative example is the proof of the Completeness Theorem for a first-order language via the Henkin construction. Although these proofs are typically given via "algorithms", the algorithmic nature of these and similar results became fully accessible only after Mal'cev and Rabin (early 1960's) independently came up with the following definition:

Definition A countably infinite algebraic structure $\mathcal{A}$ is *computable* or *constructive* if there exists a bijection $\nu : \mathbb{N} \to \mathcal{A}$ such that the ν-preimages of operations and relations on $\mathcal{A}$ become (uniformly) computable sets of natural numbers. The map ν is called a computable presentation (or computable copy) of $\mathcal{A}$.

The definition generalizes the notion of a recursively presented group in which the word problem is solvable, and also the notion of an explicitly given field suggested by van der Waerden and formalized by Fröhlich and Shepherdson. A computable model $\mathcal{M}$ is *decidable* or *strongly constructive* if there is a constructivization ν of $\mathcal{M}$ such that every ν pre-images of definable sets are uniformly computable.

The effective content of the Henkin construction is rather straightforward: A complete consistent first-order theory T is decidable if and only if T has a decidable model (folklore). The algorithmic nature of the closure theorem for fields is more interesting: Every computable field F can be embedded into its computable algebraic closure $\overline{F}$ via a computable isomorphism, but locating the image of F within $\overline{F}$ does not have to be an algorithmically decidable problem (Rabin, based on the work of Fröhlich and Shepherdson).

The two basic examples discussed above belong to the field known as *computable model theory* which examines the algorithmic nature of algebraic structures and their theories. One of the central topics in computable model theory is the study of relations between algorithmic complexity of a theory and its models. In this review we concentrate on model-theoretic restrictions and consider $\aleph_0$- and $\aleph_1$-categorical theories and their models.

The first paper under review studies computable models of $\aleph_1$-categorical theories. Recall that a first-order theory is κ-categorical if all its models of cardinality κ are isomorphic. In 1965, Morley showed that if a theory in a countable language is categorical in some uncountable cardinality, then it is categorical in all uncountable cardinalities. Baldwin and Lachlan (1971) proved that countable models of an uncountably categorical theory (which is not countably categorical) can be listed in a chain $\mathcal{A}_0 \preceq \mathcal{A}_1 \preceq \ldots \preceq \mathcal{A}_\omega$ of proper elementary embeddings, where $\mathcal{A}_0$ is the prime model and $\mathcal{A}_\omega$ is the saturated model of the theory. Which models from the Baldwin-Lachlan elementary chain have constructivizations? Starting from Goncharov (1978), there has been a line of study into the effective content of the Baldwin and Lachlan theorem. For a uncountably categorical theory T which is not countably categorical, its *spectrum* is the collection of i corresponding to computable $\mathcal{A}_i$ in the Baldwin-Lachlan chain. Various authors including Goncharov, Kudaibergenov, Hirschfeldt, Khoussainov, Nies, Semukhin and Shore showed that $\{0\}$, $\{0, \ldots, n\}$, $\{\omega\}$, ω, $\omega + 1 \setminus \{0\}$ and $\{1, \ldots, \alpha\}$ for every $\alpha \in (1, \ldots, \omega]$, can be realized as spectra.

Andrews showed that there is an uncountably categorical theory whose only constructive models are the prime and the saturated model. Thus, set $\{0, \omega\}$ is a spectrum.

The proof of the theorem which can be found in the reviewed paper uses an alteration of the Hrishovski construction. The Hrushovski construction is essentially the Fraisse limit construction in which *strong embeddings* are used in place of usual embeddings to control categoricity of the theory. Using his construction, Hrushovski (1991) obtained a new example of a strongly minimal theory which does not fall into three classes defined by Zilber (1986), thus refuting Zilber's conjecture. The formal definition of a strong embedding is technical and can be found in the paper under review.

The very rough idea of the proof can be described as follows. Let $(\mathcal{M}_e)_{e\in\omega}$ be an effective listing of partial computable structures inherited from the enumeration of all partial computable functions. Using an alternation of the Hrushovski construction, Andrews builds a computable saturated model $\mathcal{M}$ so that for every $k > 0$ and e, if $\mathcal{M}_e$ has a transcendental k-element basis, then $\mathcal{M}_e \not\preccurlyeq \mathcal{M}$. The strategy of satisfying $\mathcal{M}_e \not\preccurlyeq \mathcal{M}$ can be viewed as an infinite game. Andrews describes a strategy which guarantees that either $\mathcal{M}_e$ has no finite basis, or is not embeddable. To make the prime model computable, Andrews names by a constant every element algebraic over $\emptyset$, thus making the signature of $\mathcal{M}$ infinite. The specific properties of the Hrushovski construction are crucial for the verification.

The second paper under review studies $\aleph_0$-categorical theories. It is well-known that a theory is $\aleph_0$-categorical if and only if the number of complete n-types is finite for every $n \in \mathbb{N}$. Since all countable models of such a theory are isomorphic, the theory T is decidable if and only if its only countable model is decidable.

Various algorithmic aspects of countably categorical theories have been investigated by Schmerl, Lerman, Knight, and others. For instance, we could consider a computable model whose theory is countably categorical and ask for the complexity of this theory. Khoussainov and Goncharov (2004) showed that for every $n \in \mathbb{N}$ this complexity can be at least at the Δ^0_{n+1} level of the Arithmetical Hierarchy. Using Marker's extensions, Khoussainov and Goncharov came up with an elegant coding of $0^{(n)}$, the canonical Δ^0_{n+1} -complete set, into the countably categorical first-order theory of a computable structure.

It has been a long standing problem if the result of Khoussainov and Goncharov can be improved to the coding of $Th(\mathbb{N}; 0, S, +, \times)$, the highest possible upper bound. Khoussainov and Montalban have resolved this problem by constructing an $\aleph_0$-categorical theory T whose only countable model is computable such that T computes the theory of true first order arithmetic. The result shows that even under very strong model-theoretic restrictions on $\mathcal{M}$, the theory of a computable model could be as hard as possible.

The proof of the above stated theorem uses a generalization of the random graph to higher dimensions. The constructed computable structure consists of "levels", each level is *either* isomorphic to the n-dimensional analog of the random graph, *or* to a graph which extends the random n-graph. For any given Σ^0_n-sentence ϕ_n in the signature of true arithmetic, Khoussainov and Montalban produce a computable n-graph G_n which is "random" if $\neg\phi_n$ holds, and which is isomorphic to the extension mentioned above, otherwise. The n-graphs G_n can be glued together so that the resulting model is countably categorical, and so that the satisfyability of ϕ_n is encoded into its Σ^0_n-theory. They ensure that the theory computes the true arithmetic by choosing an appropriate sequence of ϕ_n. Although various codings of Σ^0_n information are common in computable model theory, the use of generalized random graphs is certainly a fresh idea.

It is worth mentioning that Andrews has recently announced another proof of the result of Khoussainov and Montalban discussed above. The new proof uses a direct amalgamation construction combined with the coding of the true arithmetic rather than a generalization of the random graph.

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Entry for the Table of Contents: