ENUMERATING COMPACT SPACES

SERIKZHAN BADAEV, NIKOLAY BAZHENOV, BIRZHAN KALMURZAYEV, ALEXANDER MELNIKOV, AND MANAT MUSTAFA

Abstract. We investigate the existence of a 1-1 effective (aka Friedberg) enumerations in the class of compact Polish spaces. We first prove a number of negative results. We show that there is no Friedberg enumeration of computably compact spaces up to: isometry, homeomorphism, computable isometry, or computable homeomorphism. The main result of the article is that there exists a Friedberg enumeration of all primitive recursively compact spaces up to isometry.

1. Introduction

What does it mean for a class of structures to have a satisfactory classification? To attempt to answer this question, let us examine the standard textbook classification-type results and see what they share in common:

(1) vector spaces over (say) $\mathbb{Q}$;
(2) algebraically closed fields;
(3) finitely generated abelian groups;
(4) compact oriented surfaces;
(5) abelian p-groups of bounded order.

Vector spaces and algebraically closed fields can be classified by their dimension. The abelian group examples and compact oriented surfaces can also be classified by certain finite invariants, such as the number of handles or the number of cyclic summands of a given type. However, we cannot always hope to describe an infinite object by a finite invariant. Modern mathematical structures are too complex to be captured by finite invariants.

There is another feature that all these examples share. In each case, we can *algorithmically list* all members of the class without repetition. In his fundamental paper [20], Friedberg proved that there is a uniformly computably enumerable list of all c.e. sets with no repetition (up to the usual equality of sets). He produced such a list in spite of the fact that the index set $\{\langle i, j\rangle : W_i = W_j\}$ is Π^0_2-complete. This is known as a *Friedberg enumeration* of all c.e. sets. Also, it is well known (and is easy to see) that there is a uniform 1-1 list of well-orderings of order-type less than a fixed computable ordinal α. Motivated by these classical theorems, Goncharov and Knight [21] suggested the following definition.

Definition 1.1. Let $\mathcal{K}$ be a class of structures. We say that $\mathcal{K}$ admits a *Friedberg enumeration* (a *Friedberg numbering* or a *Friedberg list*) if there is a uniformly computable listing of all computably presentable members of $\mathcal{K}$ without repetition, up to isomorphism.

Which classes of algebraic and topological structures admit a Friedberg list? Of course, the notion depends on the notion of computable presentability in the class. It turns out that many standard classes do *not* possess a Friedberg list with respect to the most natural notion of computable presentability for the class. For instance, if we take the notion of *computable (constructive) algebraic structure* (Mal'cev [28], Rabin [35]) as our basic notion, then the following classes are easily seen to *not* have a Friedberg enumeration:

(i) Linear orders.
(ii) Boolean algebras.
(iii) Graphs.
(iv) Torsion-free abelian groups.
(v) Abelian p-groups.
(vi) Structures with two unary operations.

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(vii) Fields of a given characteristic.

The list goes on. The reason behind the lack of a Friedberg list is (essentially) the Σ^1_1-completeness of the isomorphism problem in all these classes ([21, 17, 11]). However, some relatively tame classes may also fail to have a Friedberg list. For example, there is no Friedberg enumeration of all computable additive subgroups of $\mathbb{Q}$ up to isomorphism [27]. (For more 'negative' examples, see [27].)

If we choose some other notion of algorithmic presentability as the basis of our theory, then the situation is often not much better. For instance, even if we use some other presentations (polynomial-time, 1-decidable, or c.e.) the answer in (i)–(vii) will remain negative. Also, there is no Friedberg list of all finitely presented groups, as follows easily from the results of Adyan [1, 2] and Rabin [34].

Indeed, there are very few *positive* results in the literature that assert the existence of a Friedberg list of all computable members in a class. Apart from the already mentioned elementary and classical results and a few further observations, the essentially exhaustive list of classes that are known to have a Friedberg list is as follows:

(a) Computable algebraic fields [27].
(b) Computable equivalence structures [14].
(c) Computable abelian p-groups of Ulm type $\leq n$, for any fixed $n \in \omega$ [15].

In stark contrast with (1)–(5), none of the three results above is a triviality. For instance, (b) relies on a $0'''$-technique to produce a Friedberg list. There are also several results in pure computability theory that generalise the original Friedberg theorem, the essentially complete list of references is [22, 3, 32, 33, 38, 10].

In *computable topology*, which is the main subject of the present article, the situation appears to be even more complex. Apart from the observation (4) and the somewhat related result [10] cited earlier, the only theorem known to us is the following consequence of (c): There is a Friedberg enumeration of all recursive ([37]) pro-p groups of pro-Ulm type $\leq n$ (for each fixed $n \in \omega$). It is derived in [15] from (c) using effective Pontryagin duality [29]. However, profinite groups are not a particularly interesting topological class, as all such (infinite, separable) groups are homeomorphic to 2^ω, ignoring the group operation. We see that deep results asserting the existence of a Friedberg enumeration of a class are very rare and essentially non-existent in computable analysis or effective topology. Thus, *any result showing the existence of a Friedberg enumeration for a class $\mathcal{K}$ should be viewed as a very strong positive classification-type result about $\mathcal{K}$.*

The main purpose of this note is to initiate the investigation of Friedberg enumerations in computable metric space theory [26, 13]. We shall use the fairly well-understood class of compact Polish spaces to derive a number of negative results and one unexpected *positive* result.

1.1. **Results.** Recall that a *computable Polish space* is given by a dense sequence $(x_i)_{i\in\omega}$ such that $d(x_i, x_j)$ are uniformly computable reals (Ceitin [12], Moschovakis [31]). This notion has been central to computable topology and effective descriptive set theory for many decades. In the important class of compact spaces, a slightly stronger notion of a computable compact space has proven to be much more useful. A space is *computably compact* ([30]) if it is computable Polish and, additionally, given $n \in \omega$, we can uniformly produce a 2^{-n}-cover of the space by basic open balls. The classical notion of a computably compact space has over a dozen equivalent formulations; see [13] for a detailed technical exposition and the proofs. Computable compact spaces and various techniques associated with computable compactness are central to modern effective topology and computable analysis.

Compactness is often viewed as a generalisation of being finite, and it tends to be dual to being discrete. Clearly, all finite sets admit a uniformly computable enumeration without repetitions, and recall that Friedberg [20] showed that all c.e. sets have this property as well. Thus, it is natural to anticipate that compact spaces would possess a Friedberg enumeration. This intuition is partially supported by the above-mentioned technical result in [15] about 'recursive' profinite groups; this is because the notion of a 'recursive' profinite group due to Smith [37] is equivalent to computable compactness, as explained in [13].

Unfortunately, the most straightforward potential positive results fail.

Theorem 1.2. Let $\mathcal{K}$ be the class of all computably compact Polish spaces. There is no Friedberg enumeration of $\mathcal{K}$ up to:

(1) isometric isomorphism;
(2) computable isometric isomorphism;
(3) homeomorphism;
(4) computable homeomorphism.

The rather straightforward proofs of (1), (2) and (3) of the theorem will be given in Section 3; it takes more effort to establish (4). Indeed, Theorem 1.2 represents just a subset of the numerous possible negative results. For example, even if we remove the requirement of *computable* compactness in (3) and only insist that the spaces are compact, we still arrive at the same conclusion, and using essentially the same argument. In other words, Theorem 1.2 and its proof seem to leave no hope for any meaningful positive result in this direction.

Nonetheless, as we noted earlier, compactness is a generalisation of being finite, and it tends to be dual to being discrete. The standard 1-1 list of all finite sets given by their strong indices is uniformly primitive recursive, and each non-empty c.e. set can be viewed as the range of some primitive recursive function. These observations suggest that our intuition can perhaps be rescued if we restrict ourselves to *primitive recursive* Polish spaces. To obtain the notion of a primitive recursive Polish space, simply replace 'computable' with 'primitive recursive' throughout the definition of a computable Polish space. (This notion was suggested very recently in [36] and subsequently used in [4]. However, the idea behind this definition is, of course, much older. It can be traced back to, for example, Goodstein [23], which focuses on *primitive recursive* algorithms in elementary real analysis. For some more recent applications of primitive recursive analysis, see [16, 6].) Similarly, to obtain the notion of a *primitive recursively (PR-) compact space*, we additionally require that there is a primitive recursive procedure that on input n outputs a finite 2^{-n}-cover of the space [16, 13]. The main result of this article is as follows.

Theorem 1.3. *There is a Friedberg enumeration of all primitive recursively compact Polish spaces up to isometric isomorphism.*

By that we mean that there is a total computable function γ such that for each fixed n, $\gamma(n,*)$ describes a primitive recursive procedure, not merely a computable procedure, representing the respective space. Each PR-compact Polish space is mentioned in this sequence exactly once, up to isometric isomorphism. (Clearly, we cannot possibly hope to obtain a uniformly primitive recursive Friedberg enumeration unless we allow access to an oracle for, e.g., the Ackermann function.)

The statement of the theorem can be further strengthened without significantly affecting its proof. For instance, the exact same proof seems to work for PR Polish spaces that are merely computably compact, and not necessarily primitive recursively compact. Also, the result can be sub-recursively relativised to any total computable function $g\colon \omega \to \omega$.

However, it is not difficult to see that there is no Friedberg enumeration of PR-compact spaces up to homeomorphism; this is because the Σ^1_1-completeness results from [13] central to the proof of (3) of Theorem 1.2 can be witnessed by primitive recursively compact spaces. We also suspect that we would get a negative result if we restricted ourselves to computable or even primitive recursive homeomorphisms. In other words, Theorem 1.3 (along with its sub-recursive relativisation) appears to be the most general possible positive result about Friedberg enumerations for the class of compact Polish spaces.

2. Preliminaries

Definition 2.1. A *computable* (presentation of a) *Polish space* is given by:

(1) a dense sequence $(x_i)_{i\in\omega}$, perhaps with repetitions, and
(2) a computable function f which, given $i, j, s \in \omega$, outputs $r = \dfrac{n}{m} \in \mathbb{Q}$ such that

$$|d(x_i, x_j) - r| < 2^{-s},$$

where d is the metric on the space.

Condition (2) is equivalent to saying that the distances $d(x_i, x_j)$ are uniformly computable reals. If we view Polish spaces up to isometry, then we fix the metric d in the definition above. If we view spaces up to homeomorphism, then we require that the metric is compatible with the topology. In both cases, we also require that the metric is complete, and so $M = \overline{(x_i)_{i\in\omega}}$. This is not really a restriction in the compact case, since every compact metric space is necessarily complete.

Let M be a computable Polish space, and $(x_i)_{i\in\omega}$ be the computable dense sequence witnessing this. Points x_j from this sequence are called *special, ideal,* or (less frequently) *rational*. A *basic open ball* is a ball of the form $B(x_j, r) = \{y \in M : d(x_j, y) < r\}$. A *basic closed ball* is a ball of the form $D(x_j, r) = \{y \in M : d(x_j, y) \leq r\}$. In both cases, x_j is a special point and $r \in \mathbb{Q}$ is positive. We also always represent rational numbers as fractions when possible. In particular, a basic open ball is assumed to have its radius represented as a fraction. We say that an open set V is c.e. in a computable Polish space M if V is a c.e. union of basic open balls represented in this way. Say that a sequence of special points $(y_j)_{j\in\omega}$ is *fast Cauchy* if $d(y_j, y_{j+1}) < 2^{-j}$, for all j. The *name* of a point $x \in M$ of a computable Polish space M is the set $N^x = \{B \ni x : B \text{ is basic open}\}$.

Fact 2.2 (Folklore)**.** *For a point $x \in M$ in a computable Polish space, the following are equivalent:*

(1) N^x is a computably enumerable set of basic open balls.
(2) x is the limit of a computable fast Cauchy sequence.
(3) For every special x_i, $d(x, x_i)$ is a computable real uniformly in i.

Definition 2.3. A point x in a computable Polish M is *computable* if it satisfies the equivalent properties listed in Fact 2.2.

To state the next proposition, we need the notion of a (Type II) computable function.

Definition 2.4. Let $f\colon M \to N$ be a function between two computable Polish spaces. We say that f is (Type II) *computable* if it uniformly effectively turns fast Cauchy sequences into fast Cauchy sequences. More formally, if there is a uniform sequence of operators $(\Phi_n)_{n\in\omega}$ such that on input a sequence $(x_{i_n})_{n\in\omega}$ with $d_M(x_{i_n}, x_{i_{n+1}}) < 2^{-n}$, we have that

$$f(\lim_n x_{i_n}) = \lim_m \Phi_m^{(x_{i_n})_{n\in\omega}}$$

and

$$d_N(\Phi_m^{(x_{i_n})_{n\in\omega}}, \Phi_{m+1}^{(x_{i_n})_{n\in\omega}}) < 2^{-m},$$

for all m.

The following fact is well-known:

Fact 2.5 (Folklore)**.** *For a function $f\colon M \to N$ between two computable Polish spaces, the following are equivalent:*

(1) f is computable.
(2) f is effectively continuous, i.e., $f^{-1}(B)$ is uniformly c.e. open for each basic open B.
(3) $N^{f(x)}$ is uniformly computably enumerable relative to N^x (in the sense of enumeration reducibility).

2.1. **Computably compact spaces.**

Definition 2.6 ([30])**.** A computable Polish space M upon a dense set $(x_i)_{i\in\omega}$ is *computably compact* if there is a computable function which, given n, outputs a finite tuple $i_0, \dots, i_k$ of natural numbers such that

$$M = B(x_{i_0}, 2^{-n}) \cup \dots \cup B(x_{i_k}, 2^{-n}),$$

i.e., it is a finite open 2^{-n}-cover of the space.

Proposition 2.7. For a computable Polish space $M = \overline{(x_i)_{i\in\omega}}$, the following are equivalent:

(1) M is computably compact (Definition 2.6).
(2) For every n, one can effectively produce a finite cover of M by basic closed 2^{-n}-balls.
(3) There is a computably enumerable list of *all* finite open covers of the space by basic open balls.
(4) There is an effective procedure which, given an enumeration of a countable cover of the space by basic open balls, outputs its finite sub-cover.
(5) There is a computable function $h\colon \mathbb{N} \to \mathbb{N}$ such that

$$M = \bigcup_{i \le h(n)} B(x_i, 2^{-n}).$$

See [13] for a proof of $(1) \leftrightarrow (3)$. The equivalence of (3) and (4) is essentially obvious; see [13] for an explanation. Finally, to see why $(1) \leftrightarrow (2)$, note that every 2^{-n} (closed or open) cover also gives a 2^{-n+1} (closed or open) cover; simply fix the (closed or open) 2^{-n+1}-balls with the same 'centres' as the given 2^{-n}-balls. This is also clearly uniform. For many more equivalent formulations of computable compactness, we cite [13].

We will also need the following well-known properties of computable compact spaces that will be used throughout, and often without an explicit reference.

Proposition 2.8. Let M be a computable compact space and N a computable Polish space.

(1) If $f\colon M \to N$ is computable, then $f(M)$ is computably compact.
(2) If $f\colon M \to N$ is a computable homeomorphism, then f^{-1} is computable too.
(3) $\mathrm{diam}(M) = \sup_{x,y\in M} d(x,y)$ is a computable real.
(4) If $f\colon M \to \mathbb{R}$ is computable, then $\sup_{x\in M} f(x)$ is a computable real.
(5) If $M \subseteq N$, then $x \mapsto d(x, M)$ is computable.

In (5), we assume that M is represented by a dense sequence of points which are uniformly computable with respect to the given computable presentation of N. (So the finite covers of M witnessing its computable compactness are centred in these points that are special in M but are merely computable in N.) Note that (4) evidently follows from (1) and (3). For a detailed exposition of the theory, and in particular a detailed verification of Proposition 2.8 and many other properties of computably compact spaces, we cite [13].

2.2. **Primitive recursive spaces.**

Definition 2.9. A *primitive recursive* (PR-) Polish space is given by:

(1) a dense sequence $(x_i)_{i\in\omega}$, perhaps with repetitions, and
(2) a primitive recursive function f which, given $i, j, s \in \omega$, outputs $r = \dfrac{n}{m} \in \mathbb{Q}$ such that

$$|d(x_i, x_j) - r| < 2^{-s},$$

where d is the metric on the space.

We identify a PR-Polish space with the respective tuple $((x_i)_{i\in\omega}, d, f)$. Indeed, $(x_i)_{i\in\omega}$ can be identified with ω, in which case d can be also omitted (because it is determined by f). Thus, a PR-Polish space is just a primitive recursive function f (satisfying the properties induced by the triangle inequality).

Recall that $B(x, r) = \{y : d(x, y) < r\}$.

Definition 2.10. A PR-Polish space $((x_i)_{i\in\omega}, d, f)$ is *primitive recursively compact* (PR-compact) if there is a primitive recursive function $h(s)$ such that, for every $s \in \omega$, the basic open balls $B(x_0, 2^{-s}), \ldots, B(x_{h(s)}, 2^{-s})$ cover the space.

Such an h is called a *modulus of compactness* of the space. If both f and h are merely computable, we get the standard notion of a computably compact space; see Proposition 2.7. The primitive recursive analogy of computable compactness has not yet been investigated systematically to this extent; but see [4]. A more systematic investigation of PR-compact spaces was left as an open problem (a challenge) in [13]. We may have f computable and h primitive recursive or vice versa, giving intermediate notions. It is not clear how natural these intermediate notions are, but in fact the latter notion has been used in [4] to construct a certain counter-example unrelated to the subject of the present article.

It is essentially only important to use in the present paper that primitive recursive Polish spaces are total computable, and that every PR-compact space is evidently computably compact, and thus we can apply Proposition 2.8. Proposition 2.8 should have a natural primitive recursive counterpart; however, it won't be necessary for our purposes.

3. The negative results. Proof of Theorem 1.2

Before we prove the theorem, we clarify our notation and terminology. We fix an effective listing $(M_i)_{i\in\omega}$ of all (partial) computable Polish spaces. Each such M_i is given by a dense sequence that can be identified with ω and a (partial) computable metric on it. (We slightly abuse our notation and identify M_i with its completion $\overline{M_i}$.)

Let $\mathcal{S}$ be a family of computably compact Polish spaces, and let $\sim$ be an equivalence relation on the class of all computable Polish spaces (e.g., $\sim$ could be the isometric isomorphism relation). An *enumeration of the family $\mathcal{S}$ up to the equivalence relation $\sim$* is a sequence $(K_i)_{i\in\omega}$ of computable Polish spaces such that:

- for every $Y \in \mathcal{S}$, there exists $i \in \omega$ such that $K_i \sim Y$;
- for every K_i, there is $Y \in \mathcal{S}$ such that $Y \sim X_i$;
- the sequence $(K_i)_{i\in\omega}$ is uniformly computably compact, i.e., for a space K_i, its distances $d^i(x^i_k, x^i_\ell)$, $k, \ell \in \omega$, and its modulus of compactness h^i (see item (5) of Proposition 2.7) are computable uniformly in i.

In other words, the sequence $(K_i)_{i\in\omega}$ uniformly effectively lists all elements of $\mathcal{S}$, up to $\sim$. An enumeration $(K_i)_{i\in\omega}$ is *Friedberg* if $K_i \not\sim K_j$ for all $i \neq j$.

The proof of Theorem 1.2 consists of three parts: those are Fact 3.1, Propositions 3.2 and 3.3 given below.

Fact 3.1. *There is no Friedberg enumeration of all computably compact Polish spaces up to isometric isomorphism. (In fact, this class does not admit an (effective) enumeration.)*

Proof. Suppose $(K_i)_{i\in\omega}$ is such an enumeration. It is well-known that the diameter

$$\mathrm{diam}(C) = \sup_{x,y\in C} d(x,y)$$

of a computably compact space C is uniformly computable in the presentation of the space. If $\delta_i = \mathrm{diam}(K_i)$, then it is easy to construct a computable real δ such that

$$\forall i \ \delta \neq \delta_i.$$

This is done using an effective Cantor-style diagonalization; alternatively, one could appeal to the Effective Baire Category Theorem (e.g., [9]) applied to the dense c.e. open sets $U_i = \mathbb{R} \setminus \{\delta_i\}$ in $\mathbb{R}$. Consider $K = [0, \delta]$ and observe $K \ncong_{iso} K_i$, for all i. However, K is clearly computably compact. □

Note that the same proof implies that there is no (Friedberg) enumeration of computably compact spaces up to computable isometry, and indeed up to X-computable isometry for any fixed X.

Proposition 3.2. *There is no Friedberg enumeration of all computably compact Polish spaces up to homeomorphism.*

Proof. Recall that we have fixed an effective listing $(M_i)_{i\in\omega}$ of all (partial) computable Polish spaces. Among all computable Polish spaces $(M_i)_{i\in\omega}$, the homeomorphism problem for computably compact Polish spaces

$$\{(i,j) : M_i \cong_{hom} M_j \text{ and } M_i, M_j \text{ are compact}\}$$

is Σ^1_1-complete ([13, Corollary 4.30]), as witnessed by uniformly computably compact Stone spaces. Let $(M_{f(i)}, M_{g(j)})$ be the pairs of uniformly computably compact Stone spaces witnessing the Σ^1_1 completeness. We have that $M_{f(i)} \cong_{hom} M_{g(j)}$ iff the Σ^1_1-outcome holds on input (i,j).

Suppose there existed a Friedberg enumeration $(K_i)_{i\in\omega}$ of all computably compact Polish spaces up to homeomorphism. Given (i,j), calculate $M_{f(i)}$ and $M_{g(j)}$. To see whether $M_{f(i)} \ncong_{hom} M_{g(j)}$, search for $k \neq m$ such that

$$M_{f(i)} \cong_{hom} K_k \text{ and } M_{g(j)} \cong_{hom} K_m.$$

As explained in the proof of [13, Corollary 4.30], for two compact Polish spaces 'being homeomorphic' is a uniformly Σ^1_1-property; we do not even need the spaces to be *computably* compact to make this conclusion. It follows that both $M_{f(i)} \ncong_{hom} M_{g(j)}$ and $M_{f(i)} \cong_{hom} M_{g(j)}$ are Σ^1_1, contradicting the Σ^1_1-completeness of the homeomorphism problem for compact spaces. Thus, no such Friedberg enumeration $(K_i)_{i\in\omega}$ can possibly exist. □

Note that the same argument shows that, up to homeomorphism, there is no Friedberg enumeration of all compact (but not necessarily computably compact) Polish spaces.

Proposition 3.3. *There is no Friedberg enumeration of all computably compact Polish spaces up to computable homeomorphism.*

Proof. We fix an effective listing $(K_i)_{i\in\omega}$ of (potential) computably compact Polish spaces. Here each computably compact space K_i is represented by a computable (pseudo)metric and a uniform sequence $(C_n)_{n\in\omega}$ of finite tuples C_n of basic closed 2^{-n}-balls that (supposedly) cover the space. The key step in the proof is the following technical lemma.

Lemma 3.4. *The computable homeomorphism problem for computably compact Polish spaces*

$$CCCH = \{\langle i,j\rangle : K_i \cong^{comp}_{hom} K_j \text{ and } K_i, K_j \text{ are computably compact}\}$$

is Σ^0_3-complete.

Proof. Firstly, we need to argue that the index set $CCCH$ is Σ^0_3. Recall that for a space K_j, we have a computable (pseudo)metric and a sequence $(C_n)_{n\in\omega}$ of finite tuples C_n of basic closed 2^{-n}-balls. It is Π^0_1 to tell whether we indeed have a (pseudo)metric, and we also state that, for all n,

$$\forall i\, x_i \in \bigcup C_n,$$

where $(x_i)_{i\in\omega}$ are special points of K_j. Since the sets C_n are simply finite collections of basic closed balls, the overall complexity of 'K_j being computably compact' is at most Π^0_1.

Claim 3.5. *Assuming K_i, K_j are computably compact, $K_i \cong^{comp}_{hom} K_j$ is Σ^0_3.*

Proof. To say that there is a computable homeomorphism between K_i and K_j, it is sufficient to state that there exists a computable, surjective and injective $f\colon K_i \to K_j$. (This is because f^{-1} is automatically computable in this case as well by Proposition 2.8(3).) If f is already total computable, then it is Π^0_1 to say that it is injective. This is because if it fails to be injective, then it has to be witnessed by special points. Since $f(K_i)$ is a computable closed set in K_j, f is surjective if for all special points y of K_j, y is at distance zero to $f(K_i)$, which is Π^0_1 as well.

It remains to state that there is a computable map f. More formally, we should have that, for some functional Φ, $f(\xi) = \lim_n \Phi^\xi(n)$, where necessarily

(1) for every n, $\Phi^\xi(n)$ is total on K_i, and
(2) for all $\xi \in K_i$,
$$d(\Phi^\xi(n), \Phi^\xi(n+1)) \leq 2^{-n},$$
where d is the metric in K_j.

Since K_i is computably compact, the totality of $\Phi^\xi(n)$ is Σ^0_1 uniformly in n. We simply wait for $\Phi(n)$ to converge on more and more inputs. Each such individual computation gives two open sets $U \subseteq K_i$ and $V \subseteq K_j$ for which $\Phi^U(n) \subseteq V$. We just wait for finitely many such U to cover K_i; this is c.e., since we can list all open covers of K_i. Since this has to hold for every n, this gives the estimate Π^0_2 for Condition (1). We now turn to Condition (2). Suppose that there is $\xi \in K_i$ such that

$$d(\Phi^\xi(n), \Phi^\xi(n+1)) > 2^{-n}.$$

Consider the uniformly computable

$$\Gamma\colon \theta \mapsto d(\Phi^\theta(n), \Phi^\theta(n+1)),$$

which is (in particular) continuous. Since $\xi \in \Gamma^{-1}(2^{-n}, \infty)$ and the latter is open, there must be a special point x_i in this open set. It follows that, in (2), we can restrict our quantification to special points only, making the condition Π^0_1 overall. We conclude that the existence of a computable $f\colon K_i \to K_j$ is Σ^0_3. ☐

We now establish Σ^0_3-completeness of the set $CCCH$. For this, we shall use computable Boolean algebras and the corresponding Stone spaces.

It is known that Stone duality is effective [24, 25], in particular, there is a uniform procedure turning a computable Boolean algebra B into the dual computably compact Stone space $\widehat{B}$. It has been shown in [7] that, furthermore, a computable Boolean algebra B is computably categorical if, and only if, the computably compact copy of $\widehat{B}$ is effectively unique up to computable homeomorphism (Theorem 1.3 of [7]). Indeed, in [7] it is established that there is a uniform procedure turning an isomorphism $f\colon B_1 \to B_2$ into the dual homeomorphism $\hat{f}\colon \hat{B}_2 \to \hat{B}_1$, and vice versa, making the diagram below (effectively and uniformly) commutative.

$$\begin{array}{ccc} B_1 & \xrightarrow{f} & B_2 \\ \downarrow & & \downarrow \\ \hat{B}_1 & \xleftarrow[\hat{f}]{} & \hat{B}_2 \end{array}$$

More formally, given B_1, B_2 and $f\colon B_1 \to B_2$, we can uniformly effectively define $\hat{B}_1, \hat{B}_2$ and the dual homeomorphism $\hat{f}\colon \hat{B}_2 \to \hat{B}_1$. Conversely, given $\hat{f}\colon \hat{B}_2 \to \hat{B}_1$, we can uniformly reconstruct (isomorphic copies of) B_1, B_2 and an isomorphism between B_1, B_2. Thus, to establish Σ^0_3-completeness of $CCCH$, it is sufficient to obtain that the index set of computably isomorphic Boolean algebras is Σ^0_3-complete. Indeed, if we have $B_1 \cong_{\Delta^0_1} B_2$, then $\hat{B}_1 \cong^{comp}_{hom} \hat{B}_2$, and vice versa, where $\cong_{\Delta^0_1}$ stands for 'being computably isomorphic'.

Claim 3.6 (Theorem 4.7(c) in [21])**.** *There is a uniform procedure which, given a Σ^0_3 predicate P, outputs a pair of (indices for) computable Boolean algebras A_x, B_x with the property:*

$$A_x \cong_{\Delta^0_1} B_x \iff P(x).$$

To establish Σ^0_3-completeness of the set $CCCH$, consider the dual spaces $\widehat{A}_x$ and $\widehat{B}_x$ of the Boolean algebras from the claim above. Lemma 3.4 is proved. □

We return to the proof of Proposition 3.3. Suppose we had a Friedberg enumeration $(C_i)_{i\in\omega}$ of all computably compact spaces up to *computable* homeomorphism. Given any computably compact K, there must be exactly one index i such that K is computably homeomorphic to C_i. Given K_i, K_j, it is Π^0_1 to say that i, j indeed define computably compact spaces; see the first few lines of the proof of Lemma 3.4. Then $K_i \not\cong^{comp}_{hom} K_j$ if, and only if, $\exists m \neq n$ such that

$$C_n \cong^{comp}_{hom} K_i \text{ and } C_m \cong^{comp}_{hom} K_j,$$

which is Σ^0_3 (as readily follows from Claim 3.5). But this contradicts the Σ^0_3-completeness established in Lemma 3.4.

Proposition 3.3 and Theorem 1.2 are proved. □

Remark 3.7. Lemma 3.4 could be also considered in the setting of *computable reducibility* on equivalence relations (in the sense of Ershov [18], Bernardi and Sorbi [8]). Let E and F be equivalence relations on ω. One says that E is *computably reducible* to F if there exists a total computable function $f(x)$ such that for all $x, y \in \omega$,

$$(x\,E\,y) \iff (f(x)\,F\,f(y)).$$

Fokina, Friedman, and Nies (Theorem 5 in [19]) proved that the computable isomorphism relation for (indices of) computable Boolean algebras is Σ^0_3-complete w.r.t. computable reducibility. This result and the proof of Lemma 3.4 together imply that computable homeomorphism for computably compact Polish spaces is also Σ^0_3-complete w.r.t. computable reducibility.

4. The positive result. Proof of Theorem 1.3

First, we give a useful preliminary result.

Lemma 4.1. *For X, Y computably compact metric spaces, $X \not\cong_{iso} Y$ is Σ^0_1.*

Proof. We modify the proof of an unpublished result of Nies and Melnikov (see Section 4.2 in [13] for a proof) and argue that the space of isometric isomorphisms $Iso(X, Y)$ between X and Y can be realised as a Π^0_1 class $\subseteq 2^\omega$, in the sense that the paths through this class are in 1-1 effective correspondence with the members of $Iso(X, Y)$. Let h be a computable compactness modulus of Y, which means that the first $h(n)$ balls (in some fixed uniform list of all open 2^{-n} balls) cover the space. Suppose the special points of Y are given by the sequence $(r_i)_{i\in\mathbb{N}}$, and let $(p_i)_{i\in\mathbb{N}}$ be the dense computable sequence in X. Instead of defining our class inside 2^ω, we will define a computably bounded tree B; this, of course, is effectively equivalent to using 2^ω.

Definition 4.2. The n-th level of B is given by Gödel numbers of (some) tuples $\overline{r} = \langle r_{j_0}, r_{j_1}, \dots, r_{j_{n-1}}\rangle$ from

$$\{r_0, \dots, r_{h(n)}\}^n$$

that satisfy the Π^0_1 condition

$$|d_Y(r_{j_i}, r_{j_k}) - d_X(p_i, p_k)| \leq 2^{-n+1}$$

for each $i < k < n$. (Recall that h is a computable compactness modulus for Y.)

We view these tuples $\overline{r}$ as possible isometric images of $\langle p_0, \dots, p_{n-1}\rangle$, up to an error of 2^{-n+1}. Thus, we require the Π^0_1 condition that $|d_Y(r_{j_i}, r_{j_k}) - d_X(p_i, p_k)| \leq 2^{-n+1}$ for each $i < k < n$. For a tuple $\overline{u}$ at level n and a tuple $\overline{v}$ at level $n+1$, we posit as a further Π^0_1 condition that $\overline{v}$ is a child of $\overline{u}$ if $d(u_i, v_i) \leq 2^{-n}$ for each $i < n$. We let B consist of all strings σ such that for each $n < |\sigma|$, $\sigma(n)$ is on level n, and if $n > 0$ then $\sigma(n)$ is a child of $\sigma(n-1)$. Then B is Π^0_1; furthermore, clearly there is a function $\hat{h} \leq_T h$ that bounds any $f \in [B]$.

We claim that $[B]$ codes the space of (not necessarily surjective) isometries $I(X,Y)$, in the sense that there is a map from $[B]$ onto $I(X,Y)$. Furthermore, we claim that the map is computable in the sense that there is a computable functional that turns any infinite path through B into an isometry from X to Y.

This is verified below.

Suppose there is an isometric embedding $\Theta : X \to Y$. Then let $\pi(n)$ be a tuple of special points on level n that is at distance less than 2^{-n} from $\langle \Theta(p_0), \dots, \Theta(p_{n-1}) \rangle$. Then $\pi \in [B]$, and using π we can effectively reconstruct Θ.

Now suppose $f \in [B]$. We claim that f uniformly computes an isometric embedding $\Theta_f \colon X \to Y$. For each i, we have a Cauchy sequence $s^i_n = f(n)_i$ (where $n > i$). Thus f uniformly computes the function Θ_f given by $\Theta_f(i) = \lim_{n>i} f(n)_i$. For each $i < k < n$ we have

$$|d_Y(s^i_n, s^k_n) - d_X(p_i, p_k)| \leq 2^{-n+1}.$$

Thus, Θ_f is an isometric embedding. Note that this is all uniformly effective. This finishes the verification of the properties of B which, as we conclude, indeed 'effectively codes' $I(X,Y)$.

We further refine B to code the collection of all *surjective* isometries $X \to Y$.

Claim 4.3. *For an isometry $f \colon X \to Y$, 'being onto' is uniformly* $\Pi^0_1(f)$.

Proof. We have that $f(X)$ is compact and thus closed, therefore f is not onto iff

$$\exists i\, d_H(r_i, f(X)) = \sup_{y \in f(X)} d(r_i, y) > 0,$$

where d_H is the Hausdorff distance induced by the metric in Y. The space $f(X)$ is computably compact relative to f, and this is uniform. In particular, the Hausdorff distance to $f(X)$ is f-computable; and this is also uniform in f; see [13]. This makes 'f being not onto' $\Sigma^0_1(f)$. □

Recall that the correspondence between paths through B and isometries from X to Y was uniform. Thus, there is a Turing functional $\Phi \colon [B] \to I(X,Y)$ witnessing this uniformity, which may or may not be defined outside of $[B]$. By the claim above, '$\Phi^\Theta \colon X \to Y$ being not onto' in $\Sigma^0_1(\Phi^\Theta)$ which gives rise to a c.e. open set of strings. It follows that B can be further pruned to a tree B' representing a Π^0_1 class coding exactly the surjective isometries $f \colon X \to Y$, and this is uniform.

We have that $X \cong_{iso} Y$ iff $[B'] \neq \emptyset$. By the usual effective compactness argument applied to $[B']$, if we have $[B'] = \emptyset$, then it will be effectively recognised at a finite stage. It follows that $X \cong_{iso} Y$ is uniformly c.e. in the indices of X and Y.

Lemma 4.1 is proved. □

We give further prerequisites for our construction (of the desired Friedberg enumeration). Let $(P_i)_{i\in\omega}$ be the uniformly computable listing of all (potential) primitive recursively (PR-) compact metric spaces. Each such P_i is represented by:

(1) a primitive recursive pseudo-metric on ω,
(2) a primitive recursive function that, given n, gives a finite tuple of basic *closed* balls having radii at most 2^{-n} that cover the space[1].

By the same argument as in the appendix to [5], we may assume that the sequence $(P_i)_{i\in\omega}$ has the property that each P_i is primitive recursive. However, the uniformly computable sequence is of course not uniformly primitive recursive. A good way to think about it is to imagine that the running time of P_i is getting increasingly slower with larger i. However, for each fixed i, the running time of P_i is primitive recursive. The index of the primitive recursive running time function can be obtained primitively recursively in i. (As we have already mentioned in the introduction, our Friedberg list will also have these features.)

Of course, some such P_i will actually be 'wrong' in the sense that either they are not (pseudo)metric spaces, or the covers are not actually covering all points. If P_i is not 'wrong', then we identify P_i with the completion of ω w.r.t. the induced metric, and we say that P_i represents a PR-compact space.

[1]This is equivalent to Definition 2.10. If $B_0, \dots, B_k$ is a closed cover, then double the radii to obtain an open cover. Conversely, if we have an open cover, consider the respective closed balls.

Lemma 4.4. *It is Π^0_1 to tell whether P_i represents a PR-compact space.*

Proof. The axioms of (pseudo)metric spaces are Π^0_1. Since we use closed covers, it is Σ^0_1 to tell whether a given finite tuple of basic closed balls $D_0, \dots, D_m$ covers the entire space. Indeed, the union of $D_0, \dots, D_m$ is closed, so if there is a point outside this union, then there is a special such point. □

Fix a primitive recursive presentation $\mathbb{U}$ of the Urysohn space induced by the rational Urysohn space; its primitive recursiveness has been verified in [4].

Proposition 4.5 ([4])**.** *There is a uniformly primitive recursive procedure which, given a primitive recursive Polish space M, produces a primitive recursive isometric embedding $f\colon M \to \mathbb{U}$.*

We introduce several conventions:

(1) Given a finite approximation $P_i[s]$ that does not violate the Π^0_1 condition given by Lemma 4.4, up to 2^{-s}, we may perform a few steps in the calculation of $f\colon P_i[s] \to \mathbb{U}$ and calculate the images of the first s special points in P_i up to 2^{-s}. The properties of the Urysohn space guarantee that these partial 2^{-s}-embeddings can be constructed. We may perform this computation at the background until we need it (if ever).

    If at some stage $s+1$ we discover that P_i is 'wrong', we will still have $P_i[s]$ and $f[s](P_i[s])$ defined up to 2^{-s}. The image $f[s](P_i[s])$ can be viewed as a finite collection of 2^{-s}-balls, each being the 2^{-s}-image of one of the special points in $P_i[s]$. We can always pick an arbitrarily large finite number of pairwise non-equal points in these finitely many 2^{-s}-balls in the Urysohn space.

(2) Also, at stage s, we will modify each (potential) closed 2^{-s}-cover of P_i and turn it into an open 2^{-s+1}-cover of the space. We guarantee that at every stage every point of $P_i[s]$ has to be contained inside at least one ball in the cover. (If we used closed covers, this would have been problematic.) This can be done with the speed of P_i.

In the construction, we define a uniformly computable sequence of PR-compact spaces $(F_i)_{i\in\omega}$ and meet the following requirements:

R_e : P_e represents a compact space $\Rightarrow$ $\exists k\ P_e \cong_{iso} F_k$,

$N_{i,j}$: $i \neq j \to F_i \ncong_{iso} F_j$,

for all $i, j, e \in \omega$.

Recall that $I(X, Y)$ denotes the set of all (not necessarily surjective) isometries from X to Y; see Definition 4.2 and the paragraph after it.

4.1. **The case of only two R-strategies.** We begin with enumerating $P_0 = F_0$. Without loss of generality, we may assume R_0 is met. (For example, we may assume P_0 is the singleton space.)

To meet R_1, monitor P_1. Meanwhile, initiate the uniform enumeration of the discrete spaces $F_s = \{0, \dots, s\} \subseteq \mathbb{R}$ under the standard metric on the real line $\mathbb{R}$. We call these spaces *junk spaces of the first kind*, or simply *1-junk*.

Wait either for P_1 to be discovered being 'wrong' (this is Σ^0_1) or for a Σ^0_1-confirmation of $P_0 \ncong_{iso} P_1$. If one discovers that P_1 is 'wrong', then never introduce it into the sequence $(F_i)_{i\in\omega}$ and declare R_1 met. Also, if we never see that $P_0 \ncong_{iso} P_1$, then we never attempt to incorporate P_1 into the constructed sequence $(F_i)_{i\in\omega}$.

Remark 4.6. In the second case, we actually wait for *both* $I(P_0, P_1)$ and $I(P_1, P_0)$ to be empty. (This will be important when we have more than two strategies.) The former uses only covers of P_1, and the latter uses only covers of P_0 (recall the construction from Lemma 4.1). For instance, any extension $R \supseteq P_1$ within the cover of P_1 witnessing $I(P_0, P_1) = \emptyset$ will also satisfy that $I(P_0, R) = \emptyset$. (And the same can be said about P_0.)

So suppose $P_0 \ncong_{iso} P_1$ is discovered at a stage s, and so far it looks like P_1 is not 'wrong'. There will be only finitely many 1-junk spaces whose diameter is less than (or equal to) the diameter of P_1 — we can have an upper bound for the latter based on the covers of P_1 we have considered so far. We will not introduce P_1 unless it is not isometric to these finitely many 1-junk spaces.

Suppose we finally introduce P_1 into the construction: i.e, some F_{i_0} starts copying P_1.

If at some later stage we discover that P_1 is actually wrong, then we use the Urysohn space (as explained above) to consistently extend the part of P_1 built so far to a finite (thus, PR-compact) space that has more points than each finite 'junk space' seen in the construction so far. Spaces of this sort (i.e., the 'abandoned' or 'initialised' ones) will be called *junk spaces of the second kind*, or simply *2-junk*. Note that the modulus of PR-compactness of the space can be computed with the speed of P_1.

4.2. **The case of three R-strategies.** To introduce P_1, we wait for $I(P_0, P_1) = I(P_1, P_0) = \emptyset$ unless P_1 is declared 'wrong' (see Remark 4.6). Before we introduce P_2 we wait for $P_2 \not\cong_{iso} P_0$ and $P_2 \not\cong_{iso} P_1$. This also includes the case when P_1 has not yet been introduced, which must be because it is currently looking isomorphic to some other space in the list. (But we still check $P_2 \not\cong_{iso} P_1$ directly, though it is perhaps not necessary.)

If P_1 was introduced but then discovered wrong, we do not introduce P_2 unless it is not isometric to the finitely many 1-junk spaces of diameter close to the diameter of P_2 *and* also not isometric to the 2-junk space left in place of P_1 (after P_1 was eliminated from $(F_i)_{i\in\omega}$).

4.3. **The general case.** Each junk space will be given a priority. When we attempt to introduce P_e into the sequence $(F_i)_{i\in\omega}$, we will check whether it is not isometric to the finitely many spaces in the F-sequence that are of higher priority. (Those will include all spaces currently copying P_i with $i < e$.) We will also not introduce P_e unless it is not isometric to any of the 1-junk spaces having their diameter smaller than the (potential) diameter of P_e; there are only finitely many such spaces. (If F_j copied P_k and P_k is now 'dead' and turned into 2-junk, F_j still has priority of P_k.)

For the general case, we naturally extend the case of three R-strategies, as follows:

(1) If we need to introduce another F_s at a stage s, but there is no P_j available to copy into F_s, then we pick n very large and set F_s equal to the discrete 1-junk space $\{0, \dots, n\} \subseteq \mathbb{R}$ of diameter $n+1$. In particular, it can be chosen so that its diameter is guaranteed to be larger than the diameters of all P_j, $j \leq s$.
(2) To introduce (an F-space copying) P_j, we first verify that it is not isometric to:
   (a) P_i, $i < j$,
   (b) the finitely many 1-junk spaces having their diameters less than or equal to (our best guess for) the diameter of P_i, $i < j$,
   (c) the finitely many 2-junk spaces that used to copy some P_i with $i < j$.

In particular, if F_k is copying some P_j, then (by Remark 4.6) F_k will never be isomorphic to any of the P_i, $i < j$, even if P_j is eventually discovered to be 'wrong'. Also, and for the same reason, F_k will never be isomorphic to any of the finitely many F_t in the construction with $t < k$. This is because either their diameter is too big, or we will explicitly require F_k to be non-isometric to them before allowing F_k to copy P_j.

Finally, when $m > k$, then either F_m is a 1-junk whose diameter is way too large, or it is a 2-junk, or it is copying some P_ℓ, $\ell > j$. If F_m is a 2-junk structure, then it must be that at some earlier stage F_m attempted to copy P_ℓ, for $\ell > j$. But this was possible because we discovered that P_ℓ (if it is not wrong) cannot possibly be isometrically isomorphic to P_j. When P_ℓ died, we replaced it with a finite 2-junk space while keeping the new points inside the cover witnessing that P_j cannot possibly be isometrically isomorphic to P_ℓ, in the sense of Remark 4.6. This argument also shows that in the case when F_m is copying some P_ℓ, we have that P_ℓ is not isometrically isomorphic to F_k.

In the construction, we simply let the strategies act according to their instructions, as described above. By a straightforward induction, each requirement is met. Indeed, the argument above shows that the constructed sequence $(F_i)_{i\in\omega}$ is a Friedberg enumeration for the family of all PR-compact Polish spaces. The reader should also note that the index of the primitive recursive running time of F_i is primitive recursive in i; we give more details. Indeed, each F_i is either a 1-junk space, or eventually copies some P_j, or is a 2-junk space. If it is a 1-junk space, and it was initially declared to be such a space, then its running time can be assumed to be linear. If F_i was introduced to copy some P_j, where j is determined by the construction, then its computation will be that of P_j, up to a primitive recursive delay. This will also remain true if P_j is later discovered to be wrong, in which case we switch gears in the construction and turn F_i into a 2-junk space. In this case, the speed with which the metric in F_i will be calculated will primitively recursively depend on the primitive recursive presentation of $\mathbb{U}$.

In summary, in all cases the running time of F_i primitively recursively depends on the running time of P_j, $\mathbb{U}$, and the running time of the overall construction. Note that the construction is also primitive recursive in the sense that we are not waiting for any unbounded search to finish its work before we finish a stage. (All searches that are implemented in the construction are spread over many steps, thus keeping the running time of each individual step primitively recursively bounded.) It follows that the running time of F_i is primitive recursive, and furthermore its index can be obtained primitively recursively from i.

Theorem 1.3 is proved.

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Kazakh-British Technical University, Almaty, Kazakhstan
Email address: s.badaev@kbtu.kz

Kazakh-British Technical University, Almaty, Kazakhstan
Email address: n.bazhenov.1@gmail.com

Kazakh-British Technical University, Almaty, Kazakhstan
Email address: birzhan.kalmurzayev@gmail.com

Victoria University of Wellington, New Zealand
Email address: alexander.g.melnikov@gmail.com

Nazarbayev University, Astana, Kazakhstan
Email address: manat.mustafa@nu.edu.kz