Enumerations and Completely Decomposable Torsion-Free Abelian Groups ⋆

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Abstract. We study possible degree spectra of completely decomposable torsion-free Abelian groups.

1 Introduction

Computable torsion-free Abelian groups were studied by several authors , including Mal'tsev, Smith, Khisamiev, Goncharov, Dzgoev, Dobrica, Downey, and many others.

We study the possible degree spectra of torsion-free Abelian groups restricting our attention to the completely decomposable ones. This class is quite natural and can be nicely described in algebraic terms.

For a torsion-free Abelian group the key notion is its rank. First, we give a short observation of possible degree spectra in the case of finite rank. Then we study some special classes of completely decomposable groups of infinite rank trying to give necessary and sufficient conditions for these groups to be computable. We suggest a partial solution of the problem due to Khisamiev about S-divisible groups.

For an arbitrary structure, the degree spectrum can be enough complicated and reach, but does not contain the degree $\mathbf{0}$, or low degrees. Wehner [12] built a graph that has presentations exactly in the noncomputable degrees (see also [9] for an alternative proof). Miller [7] built a linear order, that has all possible non-computable Δ^0_2 copies, but does not have a computable one. These results give us examples of a somewhat algorithmic "anomaly", and show the variety and richness of pure and applied computable algebra.

The main result of this paper is the following theorem:

Theorem. *For any family R of finite sets there exists a completely decomposable torsion-free Abelian group G_R of infinite rank, such that G_R has an X-computable copy if and only if R has a Σ^X_2-computable enumeration.*

As a corollary, we get a result that adds to the list, mentioned above, of structures with "anomalous" degree spectrum:

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Theorem. *There exists a completely decomposable torsion-free Abelian group G of infinite rank, such that G has an X-computable copy if and only if $X' >_T 0'$, i.e. has exactly non-low copies.*

2 Definitions

2.1 Computability

We need some basic facts from classical computability theory, group theory, computable model theory. For a more comprehensive background see also [10], [11] and [4].

Definition 1. A set A is *recursive with respect to a set* B (in symbols, A $\leq_T$ B), if its characteristic function is B-recursive. That means that it can be computed by a Turing machine with "oracle" B. If $A \leq_T B$ and $B \leq_T A$ then we write $A \equiv_T B$. It is obvious that $\equiv_T$ is an equivalence relation. The equivalence classes of $\equiv_T$ are called *degrees.*

Definition 2. Let $K = \{x : \Phi_x^A(x) \downarrow\} = \{x : x \in W_x^A\}$. This set is denoted by A' and called the *jump* of the set A.

The superscript A in $\Phi_x^A(x)$ means that Φ is (partial) recursive with respect to the set A. It is clear how to define the n-th jump of A (denoted by $A^{(n)}$), i.e. $A^{(n)} = (A^{(n-1)})'$, where we let $A^{(0)} = A$. The jump operation is well defined on degrees, and the iteration of jumps induces a hierarchy, called arithmetical hierarchy:

$$X \in \Sigma_n^Y \leftrightarrow \text{X is r.e. in } Y^{(n-1)}.$$

Definition 3. We say that a set $A \leq_T \emptyset'$ is *low* if $A' \equiv_T \emptyset'$, and it is n*-low* if $A^{(n)} \equiv_T \emptyset^{(n)}$.

Definition 4. Let $A, B \subseteq \omega$. Then A is e-reducible to B (in symbols: $A \leq_e B$) if, for some r.e. relation R,

$$x \in A \Leftrightarrow (\exists u)(R(x,y) \wedge D_u \subseteq B),$$

where D_u denotes the finite set with canonical index u.

This relation originates an equivalence relation, denoted by the symbol $\equiv_e$, and partitions the collection of subsets of ω in equivalence classes called *e-degrees.*

Definition 5. Let $R = \{R_i | i \in \omega\}$ and $\nu : \omega \xrightarrow{\text{onto}} R$ (this map is called a numbering of R). The set $S_\nu \leftrightharpoons \{\langle m, i\rangle | m \in \nu(i)\}$ is called an *enumeration* of R. We also refer to ν as an enumeration of R, using the simple fact, that from S_ν we can recover the map ν, and vice versa.

We will follow the tradition of the theory of numberings in defining what is meant by a computable enumeration: An enumeration ν is called Σ_n^X*-computable* if $S_\nu \in \Sigma_n^X$ (and X-computable if $S_\nu \in \Sigma_1^X$).

Note that if there is a map $\nu : \omega \xrightarrow{\text{onto}} R$ with $S_\nu \in \Sigma_n^X$, but $dom(\nu) \subsetneq \omega$, then we can define a Σ_n^X-computable numbering $\pi : \omega \to^{\text{onto}} R$ with $dom(\pi) = \omega$ by the following procedure:

Let f be a $0^{(n-1)}$-computable function that enumerates $\{m : \langle m, i\rangle \in S_\nu\}$ without repetitions with $dom(f) = \omega$ (such a function of course exists) and let $\pi(k) = \nu f(k)$, for all k.

Let $\mathcal{A}$ be a countable model.

Definition 6. $\mathcal{A}$ is X-computable (constructive) is its open diagram $D_0(\mathcal{A})$ (in an appropriate enumeration of its domain and signature) is X-recursive.

Definition 7. The set $DSp(\mathcal{A}) \leftrightharpoons \{deg(D_0(\hat{\mathcal{A}})) : \hat{\mathcal{A}} \simeq \mathcal{A}\}$ is called the *degree spectrum* of $\mathcal{A}$. If the degree spectrum of $\mathcal{A}$ is an upper cone over $\mathbf{a}$, then $\mathbf{a}$ is called the degree of $\mathcal{A}$.

The set $\{\mathbf{x}' : \mathbf{x} \in DSp(\mathcal{A})\}$ is called the *jump (degree) spectrum* of $\mathcal{A}$. If the jump spectrum of $\mathcal{A}$ is an upper cone above $\mathbf{b}$, then $\mathbf{b}$ is called the *jump degree* of $\mathcal{A}$.

The following fact is well-known and can be found in [4].

Proposition 1. *If A is a computable model and B is an r.e. submodel of A, then B has a computable copy.*

2.2 Algebraic Background

We suppose that the reader knows the definition of a group and some basic definitions and facts from group theory. If the reader does not remember what is, say, a homomorphism and a subgroup, he should better first find out about it in any introductory book in group theory.

Now let G be a countable group. It will be convenient for us to use the following definition which is equivalent to the definition a computable model of a group.

Definition 8. A group $\langle G, \cdot\rangle$ is called *computable* if $\mid G \mid \subseteq \omega$ is a recursive set and the operation $\cdot$ is presented by some recursive function.

We define *A-computable* groups by replacing the word "recursive" with "A-recursive" in the above definition.

We restrict our attention to groups with commutative operation. They also are known as Abelian groups. For Abelian groups there is a tradition to write "+" instead of "·", "0" instead of "1", and "−" for inverse elements. We will also write na, $n \in \omega$, to denote the element $\underbrace{a + a + \ldots + a}_{n \text{ times}}$.

The simplest examples of Abelian groups are $\langle Z, +\rangle$ and $\langle Z_k, +\rangle$. (The universe of Z_k is $\{0, 1, \ldots, k-1\}$, and the operation is $+ \mod k$).

The striking difference we can see is that in Z, if $0 = na$ then $a = 0$ or $n = 0$. In Z_k every element a satisfies $ka = 0$.

Definition 9. We say that an Abelian group $\langle A, +\rangle$ is *torsion-free* if for all elements $a \neq 0$ and any natural $n \neq 0$, $na \neq 0$.

So, Z is torsion-free, while Z_k is not (Z_k is a *torsion* group, and any element of this group has a *torsion*, i.e. an n such that $na = 0$).

Now we will restrict our attention to the class of *countable torsion-free Abelian groups.*

Now look at V and V^k, where V is the rational number Q (as a vector space) and V^k is the k-dimensional vector space over Q. Note that V and V^k are both Abelian groups if we forget about scalar multiplication (in this case we will refer to V^k as an Abelian group). We can easily classify all vector spaces over Q by their dimensions.

We want to define the "dimension" of a (torsion-free) Abelian group. There are two equivalent approaches.

Definition 10. Elements $g_0, \ldots, g_n$ of a torsion-free Abelian group G are *linearly independent* if, for all $c_0, \ldots, c_n \in Z$, the equality $c_0g_0+c_1g_1+\ldots+c_ng_n = 0$ implies that $c_i = 0$ for all i. An infinite set is *linearly independent* if every finite subset of this set is linearly independent. A maximal linearly independent set is called a basis, and the cardinality of any basis is called the *rank* of G (often denoted as $rk(G)$).

This is the classic approach. But often it is not easy to catch the idea of this definition. The next definition is more understandable.

Definition 11. A countable torsion-free Abelian group G has *finite rank* k if G can be embedded into V^k and cannot be embedded into V^{k-1}. It has *infinite rank* if it cannot be embedded into V^k, for any k. It is embeddable only into the countable vector space V^∞ over Q, having infinite dimension.

As for vector spaces, it can be proved that the notion of rank is proper, i.e. all maximal linearly independent sets have the same cardinality. We will not prove the equivalence of these two definitions. But it will be useful to illustrate how we can embed our group into the indicated vector space.

Let $B = \{g_i : i \in I\}$ be a basis of G. Fix a vector space $V^{card(I)}$ and the standard (orthogonal) basis $W = \{v_i : i \in I\}$ of $V^{card(I)}$. We define φ on the elements of the basis:

$$\varphi g_i = v_i.$$

$$\varphi 0 = 0$$

Now look at the definition of a basis: for any nonzero $g \in G$ there is the unique equivalence $ng = \sum c_ig_i$, where only finitely many $c_i \neq 0$, and this equivalence is reduced, i.e. the (double) greatest common divisor $(n, (c_0, c_1, \ldots))$ is 1. This condition can be easily proven.

Now define φ on this element g:

$$\varphi g = (\frac{1}{n}) \sum c_iv_i.$$

It is easy to check that $\varphi(a+b) = \varphi(a) + \varphi(b)$ and it is an injection, i.e. an isomorphic embedding.

Now fix a canonical listing of the prime numbers:

$$p_1, p_2, \ldots, p_n, \ldots$$

Definition 12. Let $g \in G$, $g \neq 0$. We write $p^k|g$ if $(\exists h \in G)(p^k h = g)$ and define

$$h_p(g) = \begin{cases} \max\{k : G \models p^k|g\}, \text{if this maximum exists,} \\ \infty, \text{else.} \end{cases}$$

Then $\chi(g) = (h_{p_1}(g), \ldots, h_{p_n}(g), \ldots)$ is called the characteristic of an element g. A set

$$\{\langle i, j\rangle : j \leq h_{p_i}, h_{p_i} > 0\}$$

we will call the *characteristic sequence of g*. There is a natural one-to-one correspondence between these two objects, but we will distinguish them.

Definition 13. Given two characteristics, $(k_1, \ldots, k_n, \ldots)$ and $(l_1, \ldots, l_n, \ldots)$, we say that they are *equivalent*, $(k_1, \ldots, k_n, \ldots) \simeq (l_1, \ldots, l_n, \ldots)$, if $k_n \neq l_n$ only for a finite number of indices n, and only if these k_n and l_n are finite. This relation is obviously an equivalence relation, and the corresponding equivalence classes are called *types*.

We can correctly define the type of a group G in the case when $rk(G) = 1$. Indeed, every two elements must be linearly dependent, therefore there is only a finite difference in their characteristics. We denote the type of G as $\mathbf{t}(G)$.

We need also additional explanations of some notions. We say that h is a k-root of g, if $kh = g$. We say that g has a k-root if such an h exists. For a torsion-free Abelian group, if there exists a k-root then it is unique; if not, then the difference of two such roots will have a torsion. The characteristic of a given element of G describes all possible prime p_i-roots that this element has in G.

According to the previous definitions, G has rank 1 if and only if it can be presented as a subgroup of Q.

Theorem 1 (Baer [1], see also [11]). *Let G and H be torsion-free Abelian groups of rank 1. Then G and H are isomorphic if and only if they have the same type.*

Proof (Sketch). We can choose any nonzero $g \in G$ and $h \in H$, and it will be necessarily $\chi(g) \simeq \chi(h)$. Then we extract the finite number of roots obtaining $g' \in G$ and $h' \in H$ with identical characteristics. We define the isomorphism $\varphi : G \to H$ starting with $g' \to h'$.

The next definition is very important:

Definition 14. An Abelian group G is the *direct sum* ($G = A \bigoplus B$) of groups A and B (*direct summands* of G) if G can be presented as a domain $\{(a, b) : a \in A, b \in B\}$ with an operation $+$ naturally defined as addition on each component:

$$(a, b) + (c, d) = (a + c, b + d).$$

The groups A and B are isomorphic to the subgroups $\langle\{(a, 0) : a \in A\}, +\rangle$ and $\langle\{(0, b) : b \in B\}, +\rangle$ respectively.

Given finitely many groups it is easy to define the direct sum of these groups by iteration of the previous definition. For an infinite number of groups (say, $(A_i)_{i\in\omega}$) we define the direct sum as follows:

1. The domain consists of infinite sequences $(a_0, a_1, a_2, \ldots, a_i, \ldots)$, each $a_i \in A_i$, such that the set $\{i : a_i \neq 0\}$ is finite.

2. The operation is defined as in the previous definition:

$$\begin{aligned}(a_0, a_1, a_2, \ldots, a_i, \ldots) &+ (b_0, b_1, b_2, \ldots, b_i, \ldots)\\ &= (a_0 + b_0, a_1 + b_1, a_2 + b_2, \ldots, a_i + b_i, \ldots)\end{aligned}$$

It is not difficult to show that:
1. $rk(\bigoplus_{i\in\{1,2,\ldots,n\}} H_i) = n$ if $rk(H_i) = 1$, for all i.
2. If $H = \bigoplus_{i\in I} H_i$, $G = \bigoplus_{i\in I} G_i$ and for all $i \in I$, $H_i \cong G_i$, then $H \cong G$.

Definition 15. A torsion-free Abelian group is *completely decomposable* if it can be presented as the direct sum of groups having rank 1.

This class of torsion-free Abelian groups has a very good property - any completely decomposable group has a unique decomposition (as a direct sum) up to isomorphism (see [11] for more information). This means that in an algebraic sense the isomorphism type of a completely decomposable group is the collection of types of its summands (possibly with repetitions).

3 On Some Classes of Torsion-Free Abelian Groups

Mal'tsev was perhaps the first logician who systematically studied computable torsion-free Abelian groups. In mid sixties he pointed out a necessary and sufficient condition for a torsion-free Abelian group of rank 1 to be computably presented:

Proposition 2 (Mal'tsev,[6]).

1. *Let H be a subgroup of (any computable copy of) Q. Then H has a computable copy if and only if H has an r.e. domain.*
2. *Let $rk(G) = 1$. Then G has a computable copy if and only if the characteristic sequence of any nonzero element of G is r.e..*

Proof (Sketch).

1. If H has r.e. domain, then it has a computable copy (it is an r.e. submodel of a computable model). Now suppose H has a computable copy G, and $\varphi : G \longrightarrow H$ is an isomorphism. Fix some nonzero element g in G, and let $\varphi g = h$. Now return to the comments after the definition of rank and define a
   textbfrecursive isomorphism $\psi : G \rightarrow H$ starting with $\psi g = h$. To proof that the isomorphism can be presented by some recursive function uses the following:
   A. The one-element sets $\{g\}$ and $\{h\}$ are bases of G and Q respectively.
   B. G is computable, so there exists an effective procedure for checking the equivalence $nv = mg$ in G.
   C. To define a map, enumerate all $v \in G$. As there exists the unique reduced equivalence $nv = mg$, you can enumerate all possible coefficients until $G \models nv = mg$. Now map v to $(m/n)h \in Q$.

   H will be r.e. as it is the range of a recursive function.
2. An algorithm that enumerates the characteristic sequence of some element g in fact gives us a computable monotonic approximation of the characteristic of g (think about it). Fix a computable presentation of Q. We will enumerate a subgroup of Q isomorphic to G (and φ is an isomorphism), such that $\varphi g = 1 \in Q$.
   We will approximate the characteristic of g.
   If the i-th component h_i of characteristic increases (say, by k) at some step, we add the corresponding $p_i^{k+h_i}$-root to 1 in the subgroup (i.e. add $1/p_i^{k+h_i}$ to the generators of the subgroup), and so on. We need also close this partial map under addition - it can be done effectively.
   We will obtain an r.e. subgroup of Q. By Baer's theorem (Theorem 1) it will be isomorphic to G. It must have a computable copy as an r.e. subgroup of a computable group.

   Now let G be a computable group of rank 1. Fix g in G. Enumerate G and approximate the greatest k, such that $(\exists h)(p^k h = g)$, for all primes p. This procedure is an effective monotonic approximation of $\chi(g)$, and therefore it gives us an enumeration of the characteristic sequence of g.

This proposition can be generalized to the case of any finite rank. In spite of the fact that we study the class of completely decomposable groups, the following proposition is true in a more general case.

Proposition 3. *Let G be a torsion-free Abelian group of finite rank n. Then G has a computable copy if and only if it is isomorphic to some r.e. subgroup of (any computable presentation of) V^n.*

Proof. Any r.e. subgroup of computable V^n has a computable copy.

Now let G be a computable group. Fix a basis $\{g_1, \ldots, g_n\}$ of G and a basis $\{v_1, \ldots, v_n\}$ of V^n. Define an embedding into V^n as we did in the previous section. Again it is easy to see that this embedding is recursive, therefore it enumerates some r.e. subgroup of V^n.

Suppose G^X is a computable in X presentation of a torsion-free Abelian group G of finite rank. By the previous proposition, you can define an X-recursive embedding of G^X into $V^{rk(G)}$ with range H. Now fix a copy G^Y of G that is computable in Y. Choosing the same basis up to isomorphism, you can define a recursive in Y embedding **with the same range** from G^Y into $V^{rk(G)}$. This gives us the following:

Lemma 1. *Let G be a torsion-free Abelian group of finite rank n and let V^n be a computable n-dimensional vector space over Q. Then*

$$\{X : G \textit{ has } X\textit{-computable copy}\} = \{X : H \textit{ is } X\textit{-r.e.}\},$$

where H is a subgroup of V^n isomorphic to G, as defined above.

This property can be useful and it provides the connection between the degree spectrum of such groups and e-degrees.

Theorem 2 (Selman,[8]). *$A \leq_e B$ if and only if $\{X : B$ r.e. in $X\} \subseteq \{X : A$ r.e. in $X\}$.*

An example of the application is the following simple fact:

Proposition 4. *If a torsion-free Abelian group of finite rank n has X-computable copy for any $X >_T 0$, then it has a computable copy.*

Proof. By Lemma 1, $\{X : G$ has X-computable copy$\} = \{X : H$ is X-r.e.$\}$, for $H \subseteq V^n$. But if a set H is r.e. in X, for all $X >_T 0$, then it must be r.e. If it is not r.e., then by Selman's theorem

$$(\forall Y)(Y >_e 0_e \rightarrow Y \geq_e H).$$

It is well-known that this is impossible (for instance, there exists a set $R >_e 0$ which is incomparable with H).

It was also claimed by the author (unpublished) that this method can be applied to prove that if a torsion-free Abelian group of finite rank has X-computable copy for any non-low X, then it has a low copy. But this is a corollary of a very deep and difficult result from [2]. Using the technique demonstrated above we can extract from [2] a more general information: **any torsion-free Abelian group of finite rank has a jump degree.**

We need the following definition:

Definition 16. Let G be an Abelian group, and H be a subgroup of G. We say that H is a *serving* subgroup of G if the following property holds:

For all $k \in \omega$ and $g \in H$, if $G \models (\exists h)(kh = g)$ then $H \models (\exists h)(kh = g)$.

Any element of a serving subgroup H of a torsion-free Abelian group G has the same characteristic in G and in H. Note that not every subgroup is serving. Say, Z is not a serving subgroup of Q. But if H is a direct summand of G, then it is not difficult to check that H is a serving subgroup. The range of the embedding of G into $V^{rk(G)}$ is not necessarily a serving subgroup - it is serving if and only if G is $V^{rk(G)}$ itself.

Proposition 5. *Let G be a computable torsion-free Abelian group, and let H be a serving subgroup of G. If $rk(H)$ is finite, then H has a computable copy.*

Proof. Fix a basis $\{g_1, \ldots, g_n\}$ of H. Every element h of H satisfies the unique reduced equivalence:

$$mh = m_1 g_1 + \ldots + m_n g_n.$$

But G is a computable, therefore we can enumerate $h \in G$ and all tuples $(m_1, \ldots, m_n)$ for this fixed h. Then we effectively check the equivalence for each h and each tuple. That means that we can enumerate all the solutions h of this family of equations over the basis.

But H is a serving subgroup, so all these solutions h will be inside H. So H is an r.e. subgroup of a computable group G, and thus it must have a computable copy.

It was mentioned before that any completely decomposable torsion-free Abelian group G can be (uniquely up to permutations of indices) presented as

$$G = \bigoplus_{i \in I} G_i,$$

where each G_i has rank 1. The following problem arises:

Question 1. Given a collection of characteristics (types) of all G_i and their complexities, what can we say about the degree spectrum of G?

It can be easily proved that if the sequence $(G_i)_{i \in I}$ is uniformly computable, then G is computable, [5]. Also we will use these simple corollaries:

Proposition 6. *Let G be a completely decomposable group, and let $G = \bigoplus_{i \in I} G_i$ ($rk(G_i) = 1$ for all i) be the unique decomposition of G. If a family of characteristic sequences $\{S(g_i) : 0 \neq g_i \in G_i\}$ has a computable numbering, then G has a computable copy.*

Proof. We can define an r.e. subgroup of a computable group Q^ω. We embed each G_i into the i-th copy of Q. Given a numbering of a family $\{S(g_i) : 0 \neq g_i \in G_i\}$ we can produce enumerations of $S(g_i)$ uniformly in i and, therefore, enumerate G_i (as we did before) as a subgroup of the i-th copy of Q uniformly in i. Then we use the fact that any r.e. torsion-free Abelian group has a computable copy (this is a well-known result of Khisamiev, it can be found in [5] and it was reproved in [3]). Or we can say that G has a computable copy as an r.e. subgroup of a computable group.

Proposition 7. *Let G be a completely decomposable group, and let $G = \bigoplus_{i \in I} G_i$ ($rk(G_i) = 1$ for all i) be the unique decomposition of G. If $\{\mathbf{t}(G_i) : i \in I\}$ is a finite set, then G has a computable copy if and only if each direct summand G_i has.*

Proof. It is easy to see that the sequence $(G_i)_{i\in I}$ is uniformly computable if all G_i are computable.

Now let G be computable. Any G_i is a serving subgroup of G, therefore it must be computable.

If we study degree spectra of models, the first question that arises is doomed to look like the following:

Question 2. For any Turing degree **a**, is there a completely decomposable torsion-free Abelian group with degree **a**?

Proposition 8 (Downey, after Knight, see [3]). *For any Turing degree* $\boldsymbol{a}$ *there is a torsion-free Abelian group of rank* 1 *with degree* $\boldsymbol{a}$*.*

Proof. Given $A \in \mathbf{a}$, encode $A \oplus \overline{A}$ in the characteristic of the group $G_A \subseteq Q$ (we fix an element 1 in G_A):

$$h_{p_i}(1) = \begin{cases} 1, & \text{if } i \in A \oplus \overline{A} \\ 0, & \text{else.} \end{cases}$$

Use now Mal'tsev's observation: exactly the r.e. rational subgroups have computable copies, and a rational subgroup is r.e. if and only if the characteristic sequence of any nonzero element of this group is r.e. But $S(1)$ is r.e. in B exactly if $A \leq_T B$.

The following proposition answers our question:

Proposition 9. *For any nonzero* $\alpha \in \omega \cup \{\omega\}$ *and for any Turing degree* $\boldsymbol{a}$ *there exists a torsion-free completely decomposable Abelian group of rank* α *with degree* $\boldsymbol{a}$*.*

Proof. Fix $A \in \mathbf{a}$. Define $H = (\bigoplus_{i\in(\alpha-1)} H_i) \bigoplus G_A$, if α is finite, and $H = (\bigoplus_{i\in\omega} H_i) \bigoplus G_A$ otherwise. Here each $H_i \cong Z$ and G_A is the group from the previous proposition. Relativizing Proposition 7 we obtain that H has B-computable copy if and only if G_A has (indeed, all H_i are computable and the collection of types of the direct summands in the decomposition of H is finite). This finishes the proof.

To continue we need the following definition:

Definition 17 (Khisamiev). Let S be the collection of prime numbers. A completely decomposable torsion-free Abelian group G_S is S-divisible if it can be presented as

$$G_S = \bigoplus_{p\in S} Q_{p^\infty},$$

where Q_{p^∞} denotes the subgroup of $\langle Q, +\rangle$ generated by fractions $\{\frac{1}{p^k} : k \in \omega\}$.

Khisamiev asked the author the following question:

Question 3 (Khisamiev). Is it true that an S-divisible group has a computable copy if and only if S is Σ_3?

The next proposition is a partial answer:

Theorem 3. *Let G_S be an S-divisible group and let $G = (\bigoplus_{i\in\omega} H_i) \bigoplus G_S$ where each H_i is isomorphic to Z. Then G has a computable copy if and only if S is Σ_3.*

Proof. It is obvious that if G is computable then S is Σ_3:

$$p \in S \Leftrightarrow (\exists g \in G)(\forall n)(\exists h \in G)(p^n h = g).$$

Fix a canonical computable listing of all primes:

$$(p_0, p_1, \dots, p_s, \dots)$$

Now let S be Σ_3. Then there exist a computable relation T, such that

$$p_s \in S \Leftrightarrow (\exists x)(\exists^\infty y)T(x, y, s)$$

and a procedure that **enumerates T without repetitions** (it is well-known fact that such presentations of Σ_3 sets, and such procedures exist). We will enumerate $\{S(g_i) : 0 \neq g_i \in G_i\} \bigcup \{S_j : j \in \omega, (\forall j)(S_j = \emptyset)\}$, where each G_i will be a direct summand of G_S in its complete decomposition and each S_j corresponds to some copy of Z in the decomposition of G. By Proposition 6, the existence of such a uniform enumeration implies that G must be computably presentable.

It will be useful to keep in mind the form of an isomorphic decomposition of G that we want to enumerate by the end of our proof:

$$\bigoplus_{x,s,k\in\omega} G_{x,p_s,k}.$$

In the following, the algorithm $S_{x,s,k}$ will correspond to $S(g), 0 \neq g \in G_{x,p_s,k}$.

In describing the algorithm, we omit specifying the stage at which the various parameters are evaluated, as this will be clear from the context. If on the other hand, this is necessary, and p, say, is a parameter, then we write $p[t]$ to denote the values of p at stage t.

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Step 0.

Set $S_{x,s,k} = \emptyset$, for all x, s, k.

Set $k(s, x) = 0$, $M(s) = 0$ and $N(s) = \emptyset$, for all s.

Step t.

For each $s \le t$ enumerate $T_s = \{\langle x, y\rangle : \langle x, y, s\rangle \in T\}$. For each s two cases are possible:

1. We were waiting until, at a step m with $M(s) < m \leq t$, a pair $\langle x, y\rangle$ has appeared in the enumeration of T_s. Stop enumerating T_s and set $S_{x,s,k(s,x)} = S_{x,s,k(s,x)} \cup \{\langle s, r+1\rangle\}$, where $r = \max\{j : \langle s, j\rangle \in S_{x,s,k(s,x)}\}$, if this maximum exists, and $r = 0$ otherwise.
   Set $N(s) = N(s) \cup \{x\}$ and $M(s) = m$.
   For all $x' > x$, such that $x' \in N(s)$, set $k(s, x') = k(s, x') + 1$.
2. We have done t steps in the procedure that enumerates T_s and have not found any pair at stages m, with $M(s) < m \leq t$. Then stop enumerating and do nothing.

Now go to **step** $t + 1$.

Informal description.

For any fixed s, we must check whether $p_s \in S$. We enumerate $T(x, y, s)$ and for each s and x we formulate the hypothesis $H_{s,x}$ that there exist infinitely many y such that $T(x, y, s)$. We put these hypotheses in the order:

$$H_{s,0} \succ H_{s,1} \succ \ldots \succ H_{s,x} \succ H_{s,x+1}, \ldots$$

and define corresponding characteristic sequences:

$$\{S_{0,s,k}\}_{k\in\omega} \cup \{S_{1,s,k}\}_{k\in\omega} \cup \ldots \cup \{S_{x,s,k}\}_{k\in\omega} \cup \{S_{x+1,s,k}\}_{k\in\omega} \ldots$$

For any s and x, we start with enumerating $S_{x,s,0}$ ($k = 0$). If at some step we find some new y for $x' < x$, then we will forget about $S_{x,s,k}$ and start enumerating $S_{x,s,k+1}$. To do it for an arbitrary k, we need $k(x, s)$. This corresponds to the order of hypotheses, because $H_{s,x'}$ is more preferable, so we temporarily reject $H_{s,x}$.

On the other hand, we need to reject all $\{H_{s,x} : x > x'\}$. To do it effectively we need a set $N(s)$ of hypotheses that have ever been temporarily accepted. We can restrict our rejection only to finitely many $\{H_{s,x} : x \in N(s)\}$.

As was mentioned before, we wait for a moment when a new y in the enumeration appears, for fixed x, s. To ensure that we have found a y that has never been enumerated before for these fixed x, s, we need a counter $M(s)$ and a procedure that enumerates T without repetitions.

The idea can be formulated in a short way as follows: $p_s \in S$ if and only if there exists a hypothesis that is accepted starting from some step and will be never rejected.

$\star\ \star\ \star$

Lemma 2. *Let $p_s \in S$ and $x_s = \min\{x : (\exists^\infty y)T(x, y, s)\}$.*

1. *For any x such that $0 \leq x \leq x_s$ there exists a natural number K_x such that, starting from some step t, $k(s, x) = K_x$.*
2. *If $x > x_s$ then two cases are possible:*
   (a) *If $x \in N(s)$ since some step t, then for any natural number K there exists a step t, such that $k(s, x) > K$ since this step (i.e. $k(s.x)$ is unbounded);*
   (b) *If $x \notin N(s)$ for any step, then $k(s, x) = 0$.*

Proof. 1. If $x = 0$ then $k(s, 0) = 0$ and it will never be changed ($K_0 = 0$). Let $x > 0$. By our construction, $k(s, x)$ can be changed (increased) at step t if and only if there exists $x' < x$ such that case 1 of algorithm runs at step t for this x' (i.e. some new pair $\langle x', y\rangle$ has appeared). But for any x', $0 \leq x' < x$, there exist only finitely many y, such that $T(x', y, s)$. This implies that there exists some step t such that after this step $k(s, x)$ will be never increased. This final value we denote by K_x.

2. (a) Suppose $x \in N(s)$ starting from step t. There exist infinitely many y such that a pair $\langle x_s, y\rangle$ appears in the enumeration of T_s. That means that $k(s, x)$ will be increased infinitely often after step t ($x > x_s$, see the algorithm). Therefore it is unbounded.
   (b) By construction $k(s, x)$ can be increased at step t only if $x \in N(s)$.

Lemma 3. *If $p_s \notin S$ then for any x there exists a natural number K_x such that starting from some step t, $k(s, x) = K_x$.*

Proof. As in the previous lemma, case 1.

Lemma 4. *Fix some s. There exist x_s and k such that $\bigcup_{t\in\omega} I_t(S_{x_s,s,k})$ is an infinite set if and only if $p_s \in S$. If such a pair $\langle x_s, k\rangle$ exists, then there exists only one such pair (for this fixed s).*

Proof. First, let $p_s \in S$. Suppose $x_s = \min\{x : (\exists^\infty y)T(x, y, s)\}$. By Lemma 2 (1) there exists a natural number K_{x_s} such that, starting from some step t, $k(s, x_s) = K_{x_s}$. But there exist infinitely many y, such that $T(x, y, s)$. Therefore Case 1 of the algorithm and the operation

"...set $S_{x_s,s,K_{x_s}} = S_{x_s,s,K_{x_s}} \cup \{\langle s, r + 1\rangle\}$, where $r = \max\{j : \langle s, j\rangle \in S_{x_s,s,K_{x_s}}\}$, if this maximum exists, and $r = 0$, otherwise ..."

will be performed infinitely often starting from this step t. So, the cardinality of $S_{x_s,s,K_{x_s}}$ will increase unboundedly.

We need to show that only one such set will increase unboundedly. But $x_s = \min\{x : (\exists^\infty y)T(x, y, s)\}$, therefore for any $x < x_s$ there exist only finitely many y such that $T(x, y, s)$. That means that $\bigcup_{t\in\omega} S_{x,s,k}[t]$ will be finite (moreover, most of the sets in the union will be empty). Indeed, a new element can be added to $S_{x,s,k}$ if and only if some new pair $\langle x, y\rangle$ appears in the enumeration of T_s.

If $x > x_0$, then by Lemma 2 (2) two cases are possible:

(a) If $x \in N(s)$ since some step t, then $k(s, x)$ is unbounded. But if we set $k(s, x) = k(s, x) + 1$, we will never return to enumerating $S_{x,s,k(s,x)-1}$ again. So it will be never increased, and will stay finite.
(b) If $x \notin N(s)$ for any step t, then for all x and k, $S_{x,s,k}[t] = \emptyset$. To convince oneself of this, one needs only check that $S_{x,s,k}$ can be increased only if $x \in N(s)$.

Finally, we need to show that for all $k \neq K_{x_s}$, the cardinality of $S_{x_s,s,k}[t]$ is bounded by some finite h. If $k < K_{x_s}$ then we again observe that if we set $k(s, x_s) = k(s, x_s) + 1$, we will never return to the enumeration of $S_{x_s,s,k(s,x_s)-1}$

again. If $k > K_{x_s}$, then we will never start the enumeration of $S_{x_s,s,k}$, because we need $k = k(s, x_s)$ for this.

Now suppose that there exists x_s and k such that $\bigcup_{t\in\omega} S_{x_s,s,k}[t]$ is infinite. As was mentioned before, $S_{x_s,s,k}$ can be increased infinitely many times if and only if a new pair $\langle x_s, y\rangle$ appears in T_s infinitely many times.

The algorithm effectively enumerates a family of characteristic sequences that corresponds to some completely decomposable group

$$H = \bigoplus_{x,s,k\in\omega} G_{x,p_s,k}.$$

By Proposition 6, H has a computable presentation.

We need to show that $G \cong H$, i.e.

$$H \cong (\bigoplus_{i\in\omega} H_i) \bigoplus G_S,$$

where each $H_i \cong Z$ and $G_S = \bigoplus_{p\in S} Q_{p^\infty}$.

Lemma 5. *Let $H_S = \bigoplus_{s\in S} G_{x_s,p_s,K_{x_s}}$ (where notations are as defined above). Then*

$$H_S \cong G_S.$$

Proof. By Lemma 4, for any fixed s, $p_s \in S$ if and only if $\bigcup_{t\in\omega} S_{x_s,s,K_{x_s}}[t]$ is infinite. But it follows that if $p_s \in S$, then $\bigcup_{t\in\omega} S_{x_s,s,K_{x_s}}[t] = S(g)$ for some nonzero $g \in Q_{p_s^\infty}$. Indeed, we can add to $S_{x_s,s,K_{x_s}}$ only pairs of the form $\langle s, j\rangle$. So $\bigcup_{t\in\omega} S_{x_s,s,K_{x_s}} = \bigcup_{j\in\omega, j\neq 0}\langle s, j\rangle$, and this is exactly the characteristic sequence of $1 \in Q_{p_s^\infty}$. Therefore these groups have the same type. By Baer's Theorem (Theorem 1),

$$Q_{p_s^\infty} \cong S_{x_s,s,K_{x_s}}.$$

Finally, $H_S \cong G_S$ as direct sums of isomorphic groups.

Lemma 6. *1. For any fixed $s \in S$, if $x \neq x_s$ or $k \neq K_{x_s}$ then $G_{x,p_s,k} \cong Z$.*
2. If $s \notin S$, then $G_{x,p_s,k} \cong Z$ for any x, k.

Proof. 1. If $x \neq x_s$ or $k \neq K_{x_s}$, then by Lemma 4 $\bigcup_{t\in\omega} S(x, s, k)[t]$ is a finite set. But any finite characteristic sequence corresponds to a characteristic with *finitely many finite* h_p. All such characteristics belong to one so-called zero type, the type of the group Z. Now use Baer's Theorem (Theorem 1) to conclude part 1.
2. If $s \notin S$, then by Lemma 4 $\bigcup_{t\in\omega} S(x, s, k)[t]$ is again a finite set.

By Lemma 5 and Lemma 6, $H \cong G$.

The next simple class of completely decomposable groups can be also nicely described:

Definition 18. Let S be a collection of prime numbers. A completely decomposable torsion-free Abelian group G_S is S-none-divisible if it can be presented as

$$G_S = \bigoplus_{p \in S} Q^{p^\infty},$$

where Q^{p^∞} denotes the subgroup of $\langle Q, + \rangle$ generated by the fractions $\{\frac{1}{m} : (p, m) = 1, p \in S\}$.

The following proposition is dual to the one of the S-divisible case.

Proposition 10. *Let G_S be an S-none-divisible group and let $G = (\bigoplus_{i \in \omega} H_i) \bigoplus G_S$, where each H_i is isomorphic to Q. Then G has a computable copy if and only if S is Σ_2.*

Proof. First, it is easy to verify that given a computable copy of such a group, the relation $p \in S$ can be expressed by a Σ_2 formula.

Note that for any Σ_2 set S there exists a recursive relation T, such that $s \in S$ if and only if $(\exists^{<\infty} x)T(x, s)$. We again fix this relation and an effective procedure that enumerates it without repetitions. We also suppose that S is not co-finite (the co-finite case is simple).

⋆ ⋆ ⋆

Step 0. Set $R_s = \emptyset$ and $M(s) = 0$, for all s.
Step t. For each $s \leq t$ enumerate $T_s = \{x : \langle x, s \rangle \in T\}$. For each s two cases are possible:

1. We were waiting until, at some step m, $M(s) < m \leq t$, x has appeared in the enumeration of T_s. Stop enumerating T_s and set $R_s = R_s \cup \{\langle s, r+1 \rangle\}$, where $r = \max\{j : \langle s, j \rangle \in R_s\}$, if this maximum exists, and $r = 0$, otherwise. Set $M(s) = m$.
2. We have done t steps in the procedure that enumerates T_s and have not found any element at stages m, with $M(s) < m \leq t$. Then stop enumerating and do nothing.

⋆ ⋆ ⋆

As we see, the construction is simpler than in the previous theorem. Let $R^s = \{\langle k, j \rangle : (s, k) = 1\}$ and $R(s) = R_s \cup R^s$. The collection of characteristic sequences $\{R(s) : s \in \omega\}$ must have a computable numbering. By Proposition 6, this collection corresponds to some computable completely decomposable torsion-free Abelian group H. To prove that $G \cong H$, we need only mention that:

1. $s \in S$ if and only if $(\exists^{<\infty} x)T(x, p_s)$ if and only if $\bigcup_{t \in \omega} R_s[t]$ is finite.
2. Let G_s be a subgroup of Q with characteristic sequence $R(s)$. Then

$$G_s \cong \begin{cases} Q, \text{if } R_s \text{ is infinite,} \\ Q^{p_s^\infty}, \text{else.} \end{cases}$$

Indeed, in the first case any nonzero element of G_s will have the same characteristic as (any nonzero element of) Q has. In the second case the characteristics of the fixed element of G_s and any nonzero element of $Q^{p_s^\infty}$ will have only a finite difference in the s-th place. Now we use Baer's Theorem (Theorem 1) in each case to prove the isomorphism.

3. G and H are isomorphic as direct sums of respectively isomorphic groups.

4 Enumeration as a Computable Invariant

Now we are ready to formulate and proof the main result.

Theorem 4. *For any family R of finite sets there exists a completely decomposable torsion-free Abelian group G_R of infinite rank, such that G_R has an X-computable copy if and only if R has a Σ_2^X-computable enumeration.*

Proof. Given a Σ_2^X-computable enumeration ν^X of $R = \{R_k : k \in \omega\}$ (the only interesting case is when R is infinite) we build a computable in X enumeration of a family of characteristic sequences. Each characteristic sequence in this family will correspond to some $G_{k,m}$ in the unique (up to permutations of indices) complete decomposition of the group $G_R = \bigoplus_{k\in\omega} \bigoplus_{m\in\omega} G_{k,m}$, where $rk(G_{k,m}) = 1$ and for any m,

$$\chi_{k,m}(n) = \begin{cases} 0, & D_n \subseteq R_k, \\ \infty, & \text{else.} \end{cases}$$

Here $\chi_{k,m}(n)$ is the characteristic of some nonzero element of $G_{k,m}$.

Then, for the converse, we need to construct some Σ_2^X-computable enumeration if we have some X-computable presentation of G_R.

We fix a family of finite sets denoted by R and its Σ_2^X-computable enumeration ν^X with corresponding $S_\nu^X = \{\langle i, k\rangle | i \in \nu^X(k)\} \in \Sigma_2^X$. Without loss of generality, we can assume that $\emptyset \in R$.

We will use the fact that every Σ_2^X-relation can be presented as $\{\langle i, k\rangle : (\exists^{<\infty} x) P^X(x, \langle i, k\rangle)\}$, where P^X is some recursive in X relation. We fix T^X and an effective procedure that enumerates it without repetitions such that

$$S_\nu^X = \{\langle i, k\rangle : (\exists^{<\infty} x) T^X(x, \langle i, k\rangle)\}.$$

Again, we stick to the convention that we denote by $p[t]$ the evaluation at step t of any parameter p defined by the algorithm.

⋆ ⋆ ⋆

Step 0**.**

For all x, k, i set $S_{k,m} = \emptyset$, $m_{x,i,k} = n_{x,i,k} = 0$, $Y_{i,k} = \emptyset$, $d_{i,k} = 0$.

Step t**.**

Substep $t, 1$**.**

For all $i, k \le t$ do:

for each $x \in Y_{i,k}$ (if such an x x exists) and m, n such that

$$m_{x,i,k} < m \leq m_{x,i,k} + t$$

$$n_{x,i,k} < n \leq n_{x,i,k} + t$$

(say, in the increasing lexicographic order), set

$$S_{k,m} = \begin{cases} S_{k,m} \cup \{\langle n, r+1 \rangle\}, & \text{where } r = \max\{j : \langle n, j \rangle \in S_{k,m}\}, \\ & \text{if } i \in D_n, \\ S_{k,m}, & \text{if } i \notin D_n. \end{cases}$$

Set $m_{x,i,k} = m_{x,i,k} + t$ and $n_{x,i,k} = n_{x,i,k} + t$.

Substep $t, 2$.

For each $i, k \leq t$ enumerate T.

For each pair $\langle i, k \rangle$ two cases are possible:

1. We were waiting until, at some step d of the procedure that enumerates T, for which $d_{i,k} < d \leq t$, $\langle x, \langle i, k \rangle \rangle$ has appeared in the enumeration of T. Stop enumerating T and set $Y_{i,k} = Y_{i,k} \cup \{x\}$.
2. We have done t steps in the procedure that enumerates T and have not found any element at stages d such that $d_{i,k} < d \leq t$. Then stop enumerating and do nothing.

In either case set $d_{i,k} = t$.

⋆ ⋆ ⋆

The algorithm builds an X-computable enumeration of $\{S_{k,m} : k, m \in \omega\}$. By (relativized) Proposition 6 there exists a computable completely decomposable torsion-free Abelian group G^X with the same family of characteristic sequences as its direct summand in its complete decomposition.

Let $\theta_{k,m}$ be a characteristic that corresponds to a characteristic sequence $S_{k,m}$, for each k, m.

Lemma 7. *1. Let $D_n \subseteq \nu^X(k)$. Then for all m there exists $h \in \omega$ such that*

$$\theta_{k,m}(n) = h.$$

2. Let $D_n \nsubseteq \nu^X(k)$. Then for any m,

$$\theta_{k,m}(n) = \infty.$$

Proof. 1. Suppose $D_n = \{i_1, \ldots, i_s\}$. By our assumption, each $i_j \in \nu^X(k)$. Therefore $(\exists^{<\infty} x) T(x, \langle i_j, k \rangle)$. Look at **Substep** $t, 2$. A set $Y_{i_j,k}$ will not grow larger from some step on, i.e. $Y_{i_j,k}$ is a finite set. But for all m, at **Substep** $t, 2$ the n-th component $\{v : \langle n, v \rangle \in S_{k,m}\}$ of $S_{k,m}$ will increase only once for each $x \in \bigcup_t Y_{i_j,k}[t]$: we have

$$S_{k,m} = S_{k,m} \cup \{\langle n, r+1 \rangle\} \text{ where } r = \max\{j : \langle n, j \rangle \in S_{k,m}\}, \text{ if } i_j \in D_n$$

only if $x \in Y_{i_j,k}$ and $n_{x,i_j,k} < n$.
This is true for any $i_j \in D_n$. Therefore after some step, the n-th component of $S_{k,m}$ will stop increasing. This is equivalent to

$$\theta_{k,m}(n) = h, \text{ for some } h \in \omega.$$

2. There exists i such that $i \in D_n$ but $i \notin \nu^X(k)$. Therefore $(\exists^\infty x)T(x, \langle i, k\rangle)$. By our construction, $Y_{i_j,k}$ will increase unboundedly at **Substeps** $t, 2$, i.e. $\bigcup_t Y_{i_j,k}[t]$ is an infinite set.
The n-th component $\{v : \langle n, v\rangle \in S_{k,m}\}$ of $S_{k,m}$ will increase once for each $x \in \bigcup_t I_t(Y_{i,k})$, i.e.

$$S_{k,m} = S_{k,m} \cup \{\langle n, r+1\rangle\} \text{ where } r = \max\{j : \langle n, j\rangle \in S_{k,m}\}, \text{ if } i \in D_n$$

only if $x \in Y_{i,k}$ and $n_{x,i,k} < n$.
This is equivalent to

$$\theta_{k,m}(n) = \infty.$$

By Lemma 7, $G^X \cong G_R$. Indeed, look at the definition of G_R. As R consists of finite sets, then there exist only finitely many numbers n, such that $D_n \subseteq \nu^X(k)$ and $\theta_{k,m}(n) = h_n$, $h_n \in \omega$. The isomorphism follows from the facts:

1. $\theta_{k,m} = \theta_{k,m'}$, for all m, m'.
2. For any k there exists k' such that $\theta_{k,m} \simeq \chi_{k',m}$.
3. For any k there exists k' such that $\chi_{k,m} \simeq \theta_{k',m}$.

Finally, we need only Baer's Theorem (Theorem 1) to show that $G^X \cong G_R$, as they are direct sums of respectively isomorphic groups.

Given a Σ_2^X enumeration of R we can build a computable group G^X. It is a presentation of

$$G_R = \bigoplus_{k\in\omega}\bigoplus_{m\in\omega} G_{k,m}.$$

Now we need the converse, i.e. construct some Σ_2^X-computable enumeration if we have some X-computable presentation of G_R.

Proposition 11. *There exists an algorithm, that for any computable in X presentation G^X of G_R (defined above for R) gives a Σ_2^X-enumeration of R.*

Proof. We define a partial X-recursive function r as follows:

$$r(g,n,k) = \begin{cases} 1, \text{ if } G^X \models p_n^k | g, \\ r(g,n,k)\uparrow, \text{ else.} \end{cases}$$

We define also an X'-recursive function $\hat{r}$:

$$\hat{r}(g,n,k) = \begin{cases} 1, \text{ if } r(g,n,k)\downarrow = 1, \\ 0, \text{ if } r(g,n,k)\uparrow. \end{cases}$$

Using $\hat{r}$ we can check (with oracle X') the existence of prime roots for any $g \in G^X$. If there is a pair $\langle n, k\rangle$, such that $\hat{r}(g, n, k) = 0$, then g has only finitely many p_n-roots.

We identify elements of G^X with there codes.

⋆ ⋆ ⋆

Step 0 Let all m_g^0 be undefined.
Step t
Substep $t, 1$ For each $g \in G^X$ such that $g \leq t$ and m_g is undefined, compute $\hat{r}(g, m, k)$ for $m, k \leq s$. If there exist $m, k \leq t$ such that $\hat{r}(g, m, k) = 0$, then set $m_g = m$ [1]. For every such g add to the enumeration all pairs

$$\{\langle j, g\rangle : j \in D_{m_g}\}.$$

Substep $t, 2$ For each $g \in G^X$ such that $g \leq t$ and m_g is defined, compute $\hat{r}(g, m, k)$ for $m, k \leq t$. If there exist $m, k \leq t$, such that $\hat{r}(g, m, k) = 0 \wedge D_{m_g} \subset D_m$, then for every such g, add to the enumeration the pairs

$$\{\langle j, g\rangle : j \in D_m \setminus D_{m_g}\},$$

and then set $m_g = m$.

⋆ ⋆ ⋆

Lemma 8. *The described algorithm builds an enumeration of R.* [2].

Proof. Remember that

$$G_R = \bigoplus_{k \in \omega} \bigoplus_{m \in \omega} G_{k,m},$$

where $rk(G_{k,m}) = 1$ and $G_{k,m} \cong G_{k,m'}$ (for any m, m'). For some $g_{k,m} \in G_{k,m}$ we have the following:

$$\chi_{g_{k,m}}(n) = \begin{cases} 0, \text{ if } D_n \subseteq R_k, \\ \infty, \text{ else.} \end{cases}$$

Let $g \in G$. Then $g = r_{k_1,m_1} g_{k_1,m_1} + \ldots + r_{k_t,m_t} g_{k_t,m_t}$ for some $g_{k_j,m_j} \in G_{k_j,m_j}, j = 1, \ldots, t$. But G is the direct sum, and g is a linear combination of linear independent elements. It is easy to see that

$$(\forall k)[(\chi_g(k)) < \infty \Longleftrightarrow \bigvee_{j=1,\ldots,t} (D_k \subseteq R_{k_j})].$$

That means that g codes the *union* of all subsets, coded by its components.

The algorithm lets us move higher and higher along the subsets of *some* R_{k_j}, until we reach this R_{k_j}. After we reach it, there will be no pairs of the form $\langle l, g\rangle$ added to enumeration. That means that we will enumerate R_{k_j}. We will enumerate all the elements of R, and only these elements can be enumerated.

[1] We can let m be the minimal one with this property.
[2] As was mentioned earlier, given an enumeration defined as a partial mapping (i.e. $dom\nu \subsetneq \omega$) we can transform it into π: $dom\pi = \omega$.

Lemma 9. *The algorithm builds a Σ_2^X-computable enumeration.*

Proof. The function $\hat{r}(g, n, k)$ is recursive in X'. This means that the procedure is effective in X', and the enumeration is Σ_2^X.

Lemma 1 and Lemma 2 together provide the proof of our proposition.

We have found a correspondence between Σ_2^X-enumerations of R and computable in X presentations of G_R. This completes the proof of the theorem.

The following result is one of the possible applications of the previous theorem:

Theorem 5. *There exists a completely decomposable torsion-free Abelian group G of infinite rank, such that G has X-computable copy if and only if $X' >_T 0'$, i.e. has exactly non-low copies. Moreover, each direct summand in the complete decomposition of this group can be computably presented.*

Proof. The proof is based on the following well-known result due to Wehner:

Proposition 12 ([12]). *There exists a family F that consists of finite sets such that F has an X-computable enumeration if and only if $X >_T 0$.*

Firstly, we relativize this proposition. It gives us a family of finite sets that has Σ_2^X $(X' >_T 0')$ enumerations, but has no Σ_2 enumeration. Then we apply the construction from the previous theorem, to this family of sets.

As was mentioned before, any torsion-free Abelian group of finite rank has a jump degree. We can see from the previous theorem that this condition fails in the case of infinite rank.

Corollary 1. *There exists a completely decomposable torsion-free Abelian group G of infinite rank with no jump degree.*

References

1. Baer, R., *Abelian Groups Without Elements of Finite Order,* Duke Math. J., 3 (1937), 68-122.
2. Coles, R., Downey, R., Slaman, T., *Every Set Has a Least Jump Enumeration.* J. London Math. Soc. (2) 62 No.3, 641-649.
3. Downey, R., *On Presentations of Algebraic Structures.* Complexity, Logic and Recursion Theory, ed. Sorbi, Marcel Dekker, 157–205 (1997).
4. Ershov Y., Goncharov S., *Constuctive Models.* Novosibirsk, Nauchnaya kniga, 1999.
5. Khisamiev, N., *Constructive Abelian Groups.* Handbook of Recursive Mathematics, vol. 139, Elsevier, 1998.
6. Mal'tsev, A. I., *On Recursive Abelian Groups.* transl. in: Soviet Math. Dokl.,3.
7. Miller, R., *The Δ_2^0-Spectrum of a Linear Order*, preprint.
8. Selman, A., Arithmetical Reducibilities I, Zeit. Math. Log. Grund. Math. 17 (1971), 335-370.

9. Slaman, T., *Relative to Ani Non-Recursive Set.* Proc. Amer. Math. Soc., 126(1998), pp.2117-2122.
10. Soare, R.I., *Recursively Enumerable Sets and Degrees.* Springer-Verlag, Berlin Heidelberg, 1987.
11. Fuchs, L., *Infinite Abelian Groups. Vol. II.* Academic Press, 1973.
12. Wehner, S., *Enumerations, Countable Structures and Turing Degrees.* Proc. Amer. Math. Soc., 126 (1998), pp.2131-2139.