COMPUTABLE GELFAND DUALITY

PETER BURTON, CHRISTOPHER J. EAGLE, ALEC FOX, ISAAC GOLDBRING, MATTHEW HARRISON-TRAINOR, TIMOTHY H. MCNICHOLL, ALEXANDER MELNIKOV, AND TEERAWAT THEWMORAKOT

Abstract. We establish a computable version of Gelfand Duality. Under this computable duality, computably compact presentations of metrizable spaces uniformly effectively correspond to computable presentations of unital commutative C^* algebras.

1. Introduction

One of the most foundational results in the study of operator algebras is the Gelfand Duality Theorem. By means of this theorem a commutative unital C^* algebra A can be represented as the C^* algebra of continuous functions from a compact metrizable space X into the field of complex numbers $\mathbb{C}$. Specifically, X may be chosen to be the *spectrum* $\Delta(A)$ of A (the set of nonzero homomorphisms of A into $\mathbb{C}$) with the Gelfand topology (the weakest topology in which all evaluation maps are continuous). By the Banach-Stone Theorem, the space X is unique up to homeomorphism. Here, we present a computable Gelfand duality within the framework of effective metric structure theory. Specifically, we prove the following.

Theorem 1.1. *Suppose X is a compact metrizable space. If $C^*(X)$ is computably presentable, then X has a computably compact presentation.*

Moreover, our proof of Theorem 1.1 is nearly fully uniform. Namely, it provides an effective operator that given a name of a (not necessarily computable) presentation of a C^* algebra A and a corresponding name of the unit of A, produces a name of a presentation of X and of a total boundedness function for X. For the reader who is unfamiliar with computable presentations of metric structures (such as metric spaces and Banach spaces), we give precise definitions for the setting of C^* algebras in Section 2. These definitions are drawn from the recent work of A. Fox [13]. For now, let us conceive of a computable presentation of a metric structure as a dense sequence by means of which the operations and the metric can be computably approximated. A computably compact presentation of a metric space provides enough information to compute arbitrarily tight covers of the space. These presentations have a number of important features not enjoyed by an arbitrary presentation. For example, they provide for the computation of maxima of computable functions. We discuss names of presentations in Section 6. Roughly speaking, a

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name of a presentation is a point in Baire space that completely describes the presentation.

We note that the converse of Theorem 1.1 has been proven by A. Fox [13]. *We thus have a classification of the commutative unital C^* algebras that are computably presentable.*

Part of the significance of our work stems from our proof of Theorem 1.1 which gives a concise demonstration of the effectiveness of the Gelfand duality with only the rudiments of operator algebras and computable analysis. The classical proofs of Gelfand duality typically rely on the Banach-Alaoglu Theorem. The most effective version of this theorem that we are aware of is due to V. Brattka [7]. This version effectively embeds the closed unit ball of a dual into a computably compact presentation of a certain metric space. However, the embedded image does not satisfy the computable compactness criterion considered here. Our proof of Theorem 1.1 is structured so as to avoid the need for an effective version of the Banach-Alaoglu Theorem. In particular, we show that the information provided by a presentation of $C^*(X)$ can be used to identify the points of X by means of a certain family of *vector names* which are judiciously constructed sequences of vectors of A. In a sequence of papers, Banaschewski et. al. set forth a constructive proof of Gelfand duality in the context of toposes [2], [4], [3], [5], [11]. By contrast, our proof of Theorem 1.1 takes place in a fairly concrete setting and requires only a minimal knowledge of computability and functional analysis.

Theorem 1.1 also advances the recently emerged program of effective metric structure theory which seeks to understand metric structures (such as Banach spaces) from the perspective of computable structure theory, that is, studying which structures have computable presentations and identifying presentations that are essentially the same, that is, computably isomorphic. We refer the reader to the texts by Ash and Knight and Montalban for a much more expansive treatment of classical computable structure theory [1], [28]. The origins of effective metric structure theory go back at least as far as the seminal work of Pour-El and Richards [32]. However, the spark for the fairly recent development of the theory is the 2013 paper of A. Melnikov [26]. The insights in the latter led to a significant amount of work on the computable structure theory of Banach spaces, in particular Lebesgue spaces [20, 9, 8] and Stone spaces [6]. Recently, A. Fox has extended this activity to the realm of C^* algebras [13], and our efforts build on his.

Perhaps most consequentially, Theorem 1.1 adds to the list of recently discovered *computable dualities* such as the computable Stone duality and the computable Pontryagin duality [6], [25]. We discuss these and other computable dualities in Section 2. Computable dualities have already led to the solution of several open problems [18, 15, 24]. The computability of the Gelfand duality connects the computability of C^* algebras with the well-developed area of computably compact Polish spaces [12]. One would therefore expect Theorem 1.1 to lead to new discoveries in computable operator algebras. Indeed, in Section 6, we combine Theorem 1.1 with known results in computable algebra and topology to produce interesting examples of C^* algebras that do not have computable presentations.

The paper is organized as follows. Section 2 summarizes background from functional analysis and computable analysis. In Section 3, we attend to a few preliminary matters of a purely classical nature. In particular, we introduce the concept of a *vector name* of a point and prove some classical properties of such names.

In Section 4, we present preliminary results on the computability of the unit and related findings on the computability of certain post-composition operators. These properties will then be used in the proof of Theorem 1.1 which is given in Section 5. In Section 6, we discuss some consequences of Theorem 1.1. We also discuss its uniformity. Section 7 summarizes our results and presents some directions for future work.

2. Background

We assume knowledge of the fundamentals of computability theory as expounded in [10]. Fix an effective enumeration $(\phi_e)_{e\in\mathbb{N}}$ of the computable partial functions from $\mathbb{N}$ into $\mathbb{N}$.

$\mathbb{C}$ is the field of scalars for each C^* algebra considered herein. $\mathbb{N}$ denotes the set of nonnegative integers.

We follow the computability theory of operator algebras developed by A. Fox [13]. This framework is an extension of the computability theory for Banach spaces put forth by Pour-El and Richards [32]. Fix a C^* algebra A. We say $(A, (v_n)_{n\in\mathbb{N}})$ is a *presentation* of A if $(v_n)_{n\in\mathbb{N}}$ generates a dense subalgebra of A. If $A^\# = (A, (v_n)_{n\in\mathbb{N}})$ is a presentation of A, then each vector in the subalgebra of A generated by $(v_n)_{n\in\mathbb{N}}$ over the field of rational scalars is a *rational vector* of $A^\#$.

By means of standard techniques, we can generate an effective indexing of the rational vectors of a presentation. Usually, it is not necessary to provide the details of such an indexing, but for the sake of later developments we will be more precise. Specifically, we index the rational vectors of a presentation by means of rational $*$-polynomials as follows. Let $x_0, x_1, \ldots$ be pairwise distinct indeterminants, and let $\mathcal{U}$ denote the free $*$-algebra generated by $X = \{x_0, x_1, \ldots\}$ over $\mathbb{Q}(i)$. Fix an effective enumeration $(\mathfrak{p}_j)_{j\in\mathbb{N}}$ of $\mathcal{U}$. By effective, we mean that from m, n we can compute j, k, s so that $\mathfrak{p}_j = \mathfrak{p}_m + \mathfrak{p}_n$, $\mathfrak{p}_k = \mathfrak{p}_m\mathfrak{p}_n$, and $\mathfrak{p}_s = \mathfrak{p}_m^*$. When $A^\# = (A, (u_n)_{n\in\mathbb{N}})$ is a presentation of A, and when $\mathfrak{q} \in \mathcal{U}$, let $\mathfrak{q}[A^\#]$ denote the vector of A obtained from $\mathfrak{p}$ by substituting u_j for x_j for each $j \in \mathbb{N}$. We call $\mathfrak{p}_j[A^\#]$ the *j-th rational vector of $A^\#$*.

We note that given indices of rational vectors u and v, it is possible to compute indices of uv, $u + v$, and u^*. We also note that this indexing is not necessarily injective, nor can we necessarily effectively determine if two numbers index the same rational vector.

A presentation $A^\#$ is *computable* if the norm is computable on the rational vectors, that is, there is an algorithm that given $k \in \mathbb{N}$ and an index of a rational vector v of $A^\#$, computes a rational number q so that $|q - \|v\|\,| < 2^{-k}$. A is *computably presentable* if it has a computable presentation.

The *standard presentation* of the C^* algebra $\mathbb{C}$ is defined by setting $v_n = 1$ for all $n \in \mathbb{N}$. The rational vectors of this presentation are precisely the rational points of the plane. We identify $\mathbb{C}$ with its standard presentation, and no other presentation of $\mathbb{C}$ is considered.

Fix a presentation $A^\#$ of A, and let v_0 be a vector of A. v_0 is a *computable vector* of $A^\#$ if there is an algorithm that given $k \in \mathbb{N}$ computes a rational vector v of $A^\#$ so that $\|v_0 - v\| < 2^{-k}$. An index of such an algorithm is an *$A^\#$-index* of v_0. A sequence $(u_n)_{n\in\mathbb{N}}$ of vectors of A is a *computable sequence* of $A^\#$ if u_n is a computable vector of $A^\#$ uniformly in n, that is, if there is an algorithm that given $n, k \in \mathbb{N}$ computes a rational vector v of $A^\#$ so that $\|v - u_n\| < 2^{-k}$.

We define computability of operators and functionals via names as follows. An $A^{\#}$*-name* of v_0 is a sequence $(u_k)_{k\in\mathbb{N}}$ of rational vectors of A so that $\|u_k - v_0\| < 2^{-k}$ for all $k \in \mathbb{N}$. It follows that v_0 is a computable vector of $A^{\#}$ if and only if v_0 has a computable name. Since we identify $\mathbb{C}$ with its standard presentation, and as no other presentations of $\mathbb{C}$ are considered, we simply refer to a $\mathbb{C}$-name as a name.

Suppose T is an n-ary operator on A. We say T is a *computable operator of* $A^{\#}$ if there is an oracle Turing machine that given $A^{\#}$-names of vectors $v_1, \ldots, v_n$ computes an $A^{\#}$-name of $T(v_1, \ldots, v_n)$. An $A^{\#}$*-index* of T is an index of such a machine. T is *intrinsically computable* if T is a computable operator of every computable presentation of A. T is *uniformly intrinsically computable* if there is an algorithm that given an index of a computable presentation A^+ produces an A^+-index of T. It is easily shown that if T is a computable operator of $A^{\#}$, and if $v_1, \ldots v_n$ are computable vectors of $A^{\#}$, then $T(v_1, \ldots, v_n)$ is a computable vector of $A^{\#}$. Furthermore, an $A^{\#}$-index of $T(v_1, \ldots, v_n)$ can be computed from $A^{\#}$ indices of T, v_1, …, v_n.

Computability of functionals is defined similarly. We also define intrinsically computable functionals and uniformly intrinsically computable functionals in the same way that we defined intrinsically computable operators and uniformly intrinsically computable operators. Again, it is easily shown that if v_0 is a computable vector of $A^{\#}$, and if F is a computable functional of $A^{\#}$, then $F(v_0)$ is a computable point of the plane. Furthermore, from an $A^{\#}$-index of v_0 and an $A^{\#}$-index of F, it is possible to compute an index of $F(v_0)$.

It follows from these definitions that the addition, multiplication, and involution of A are uniformly intrinsically computable operators of A. In addition, for each $s \in \mathbb{Q}(i)$, the multiplication-by-s operator is a uniformly intrinsically computable operator of A uniformly in s. The norm is an intrinsically computable functional of A. An additional useful principle is the following. If $A^{\#} = (A, (v_n)_{n\in\mathbb{N}})$ is a computable presentation, then a bounded linear functional F on A is computable if $(F(v_n))_{n\in\mathbb{N}}$ is computable.

We will use moduli of convergence to demonstrate the computability of certain limits. These are defined as follows. If $(v_n)_{n\in\mathbb{N}}$ is a convergent sequence of vectors of A, then a *modulus of convergence for* $(v_n)_{n\in\mathbb{N}}$ is a function $g : \mathbb{N} \to \mathbb{N}$ so that $\|v_m - \lim_n v_n\| < 2^{-k}$ whenever $m \geq g(k)$. It is easily shown that if $(v_n)_{n\in\mathbb{N}}$ is a computable sequence of $A^{\#}$ that has a computable modulus of convergence, then $\lim_n v_n$ is a computable vector of $A^{\#}$.

It is well-known that a computable function $g : \mathbb{R}^n \to \mathbb{R}$ can be effectively approximated on compacta by rational polynomials. This principle does not hold for computable functions of one or more complex variables. However, as the involution on $\mathbb{C}$ gives access to the real and imaginary parts of a complex number, we can nevertheless effectively approximate a computable $g : \mathbb{C}^n \to \mathbb{C}$ on compacta by rational $*$-polynomials.

Suppose (X, d) is a complete metric space. A *computable presentation* of (X, d) consists of a dense sequence $(p_n)_{n\in\mathbb{N}}$ of points of X so that the array $(d(p_m, p_n))_{m,n\in\mathbb{N}}$ is computable; that is, there is an algorithm that given $m, n, k \in \mathbb{N}$ computes $q \in \mathbb{Q}$ so that $|q - d(p_m, p_n)| < 2^{-k}$. If X is a Polish space, then a computable presentation of X consists of specifying a compatible complete metric d and a computable presentation of (X, d). Suppose X is a Polish space and $X^{\#} = (X, d, (p_n)_{n\in\mathbb{N}})$ is a computable presentation of X. We say that $X^{\#}$ is *computably compact* if from $k \in \mathbb{N}$ it is possible to compute $n_0, \ldots, n_t \in \mathbb{N}$ so that $X \subseteq \bigcup_j B(p_{n_j}; 2^{-k})$. Since (X, d) is complete, if X has a computably compact presentation, then X is compact. The terminology 'computably compact' was coined by Mori, Tsujii, and Yasugi [29]. As its name suggests, this notion is restricted to compact spaces, but it can be generalized to locally compact spaces; e.g., [31, 34, 23]. One remarkable feature of this notion is that it is exceptionally robust; at least *nine* equivalent formulations of computable compactness can be found in [12, 16, 30].

As mentioned in the introduction, Theorem 1.1 contributes to the program of computable dualities. These dualities include the following.

(D1) A Stone space B has a computably compact presentation if and only if $C(B; \mathbb{R})$ is computably presentable [6].

(D2) A (discrete, countable) torsion-free abelian group G is computably presentable if and only if its connected compact Pontryagin dual $\widehat{G}$ has a computably compact presentation [25, 18].

(D3) If T is a discrete, countable, and torsion Abelian group, then the following are equivalent [25, 12].
  (a) T is computably presentable.
  (b) The profinite Pontryagin dual $\widehat{T}$ has a computably compact presentation.
  (c) $\widehat{T}$ has a recursively profinite presentation.

(D4) If B is a countable discrete Boolean algebra, then the following are equivalent [14, 15].
  (a) B has a computable presentation.
  (b) The Stone space $\widehat{B}$ of B has a computably compact presentation.
  (c) $\widehat{B}$ has a computable presentation.

(D5) The computably locally compact totally disconnected groups are exactly the duals of the computable (discrete, countable) meet groupoids of their compact cosets [24, 23].

(D6) A profinite group is recursively presented if and only if it is topologically isomorphic to the Galois group of a computable field extension [27, 17, 33].

A few further dualities can be found in [22, 12, 18]. We refer the reader to [12] for a rather detailed exposition of some aspects of this unified theory.

3. Preliminaries from classical analysis

Fix a compact metrizable space X, and let $A = C^*(X)$. A *vector name* of $p \in X$ is a sequence $(f_n)_{n\in\mathbb{N}}$ of vectors of A so that $\{p\} = \bigcap_{n\in\mathbb{N}} f_n^{-1}(\frac{1}{2}, \infty)$. It follows from Urysohn's Lemma that every point of X has a vector name. Our approach to proving Theorem 1.1 is to use vector names to identify points. However, as we shall see later, not every vector name lends itself to computability; we need to use names that have a certain amount of structure. Accordingly, we define a vector name $(f_n)_{n\in\mathbb{N}}$ to be *well structured* if $f_{n+1}^{-1}(\frac{1}{4}, \infty) \subseteq f_n^{-1}(\frac{2}{3}, \infty)$ and $\|f_n\| \leq 2/3 + 2^{-n}$. The following lemma captures the feature of well structured names that we will exploit in Section 5; namely, it will be used to show that if $a \in X$ has a computable well structured name, then the evaluation-at-a functional is computable.

Lemma 3.1. *Suppose $(f_s)_{s\in\mathbb{N}}$ is a well structured name of $a \in X$, and let $g \in C(X; [0, 1])$. Then, the following are equivalent.*

(1) *There exists $s \in \mathbb{N}$ so that $\|f_s(1 - g)\| < \frac{1}{3}$.*

(2) $g(a) > \frac{1}{2}$.
(3) $\|f_s(1-g)\| < \frac{1}{3}$ *for all sufficiently large* $s \in \mathbb{N}$.

Proof. First suppose $\|f_s(1-g)\| < \frac{1}{3}$ for some $s \in \mathbb{N}$. Since $f_s(a) > \frac{2}{3}$, we must have $g(a) > \frac{1}{2}$.

Next suppose $g(a) > \frac{1}{2}$, and set $\epsilon = \frac{1}{2}(g(a) - \frac{1}{2})$. Let $V = g^{-1}(\frac{1}{2} + \epsilon, \infty)$. We first show that for all sufficiently large s, $f_s(t)(1-g(t)) < \frac{1}{3}$ when $t \in V$. To this end, choose $N_0 \in \mathbb{N}$ so that $(\frac{1}{2} - \epsilon)(\frac{2}{3} + 2^{-N_0}) < \frac{1}{3}$. Suppose $s \geq N_0$ and $t \in V$. Then, by the definition of V, $1 - g(t) < \frac{1}{2} - \epsilon$. Since $(f_s)_{s\in\mathbb{N}}$ is well structured, $f_s(t) \leq \frac{2}{3} + 2^{-s}$. Hence, $(1-g(t))f_s(t) < \frac{1}{3}$.

Now, we show that for all sufficiently large s, $(1-g(t))f_s(t) \leq \frac{1}{4}$ for all $t \in X \setminus V$. First, set $K_s = \overline{f_s^{-1}(1/4, \infty)}$. Since $(f_s)_{s\in\mathbb{N}}$ is well structured, $K_{s+1} \subseteq K_s$. Since $(f_s)_{s\in\mathbb{N}}$ names a, $a \in \bigcap_s K_s$. However, since $(f_s)_{s\in\mathbb{N}}$ is well structured, $K_{s+1} \subseteq f_s^{-1}(1/2, \infty)$, and so $\bigcap_s K_s = \{a\}$. Thus, $\bigcap_s K_s \setminus V = \emptyset$. By Cantor's Theorem, $K_s \subseteq V$ for all sufficiently large s. If $t \in X \setminus V$, and if $K_s \subseteq V$, then $f_s(t) \leq 1/4$ and so $f_s(t)(1-g(t)) \leq \frac{1}{4}$. ☐

We say that a vector name $(f_n)_{n\in\mathbb{N}}$ is *adequately structured* if $f_{n+1}^{-1}(\frac{1}{2}, \infty) \subseteq f_n^{-1}(\frac{2}{3}, \infty)$. Adequately structured names will serve as an intermediate step towards constructing well structured names. The process of building a well structured name from one that is adequately structured is as follows. For all $t \in \mathbb{R}$, let

$$\psi(t) = \begin{cases} \frac{1}{2}t & t < \frac{1}{2} \\ \frac{5}{2}(t - \frac{1}{2}) + \frac{1}{4} & \frac{1}{2} \leq t < \frac{2}{3} \\ t & t \geq \frac{2}{3} \end{cases}$$

We now have the following lemma. [1]

Lemma 3.2. *If* $(f_s)_{s\in\mathbb{N}}$ *is an adequately structured name of* $a \in X$*, then* $(\min\{\psi \circ |f_s|, 2/3 + 2^{-s}\})_{s\in\mathbb{N}}$ *is a well structured name of* a.

Proof. Let $\widehat{f_s} = \min\{\psi \circ |f_s|, 2/3 + 2^{-s}\}$. By definition of ψ, we have $\widehat{f}_{s+1}^{-1}(1/4, \infty) \subseteq \widehat{f}_s^{-1}(2/3, \infty)$.

We now claim that $(\widehat{f_s})_{s\in\mathbb{N}}$ is a vector name of a. By the choice of ψ, $\widehat{f}_s^{-1}(1/4, \infty) = f_s^{-1}(1/2, \infty)$. Thus, $\{a\} = \bigcap_s \widehat{f}_s^{-1}(1/4, \infty)$, and so $\bigcap_s \widehat{f}_s^{-1}(1/2, \infty) \subseteq \{a\}$. At the same time,

$$a \in \widehat{f}_{s+1}^{-1}(1/4, \infty) \subseteq \widehat{f}_s^{-1}(2/3, \infty) \subseteq \widehat{f}_s^{-1}(1/2, \infty).$$

Thus, $a \in \bigcap_s \widehat{f}_s^{-1}(1/2, \infty)$. By definition, $\left\|\widehat{f_s}\right\| \leq 2/3 + 2^{-s}$. Thus, $(\widehat{f_s})_{s\in\mathbb{N}}$ is well structured. ☐

We note that Lemma 3.1 fails if $(f_s)_{s\in\mathbb{N}}$ is merely an adequately structured name; in particular the implication of (1) by (2) fails.

Finally, we will use the following lemma to demonstrate that a sequence is dense in X by relating it to the density of a sequence in $C(X; [0,1])$. The proof is a standard argument via Urysohn's Lemma.

Lemma 3.3. *Suppose* $(g_n)_{n\in\mathbb{N}}$ *is dense in* $C(X; [0,1])$*, and fix* $r \in (0,1)$*. Furthermore, suppose* $(p_n)_{n\in\mathbb{N}}$ *is a sequence of points of* X *so that for each* $n \in \mathbb{N}$*, if* $\|g_n\| > r$*, then there exists* $k \in \mathbb{N}$ *so that* $g_n(p_k) > r$*. Then* $(p_n)_{n\in\mathbb{N}}$ *is dense in* X*.*

[1] We thank Konstantyn Slutsky for suggesting these names and for the proof of Lemma 3.2.

Proof. Let $t_0 \in X$, and let $\epsilon > 0$. By Urysohn's Lemma, there is a continuous $\lambda : X \to [0,1]$ so that $\lambda(t) = 1$ when $t \in \overline{B}(t_0; \epsilon/2)$ and $\lambda(t) = 0$ when $t \in X \setminus B(t_0; \epsilon)$. Hence, there exists n so that $g_n(t) > r$ when $t \in \overline{B}(t_0; \epsilon/2)$ and $g_n(t) < r$ when $t \in X \setminus B(t_0; \epsilon)$. Take $k \in \mathbb{N}$ so that $g_n(p_k) > r$. Then $p_k \in B(t_0; \epsilon)$, establishing the desired conclucion. □

4. Computability-theoretic preliminaries

We begin by addressing the computability of the unit. These considerations will have some effect on the uniform computability of certain post-composition operators and in turn will influence the uniformity of Theorem 1.1.

Suppose A is a unital C^* algebra. We say that A is *computably unital* if $\mathbf{1}_A$ is a computable vector of every computable presentation of A. We say that A is *uniformly computably unital* if an $A^\#$-index of $\mathbf{1}_A$ can be computed from an index of $A^\#$.

The following has been proven by A. Fox. We include a proof for the sake of completeness.

Theorem 4.1. *Every commutative unital C^* algebra is computably unital.*

Proof. Let A be a commutative unital C^* algebra. Suppose $A^\#$ is a computable presentation of A. Let $\delta_0 = \frac{1}{2}(1 - 2^{-1/2})$, and fix a rational vector v_0 of $A^\#$ so that $\|v_0 - \mathbf{1}_A\| < \delta_0$.

Let $k \in \mathbb{N}$. It is required to compute a rational vector v so that $\|v - \mathbf{1}_A\| < 2^{-k}$. Set $\epsilon_0 = 2^{-(2k+1)}$. Search for a rational vector v so that $\|v^2 - v\| < \epsilon_0$ and so that $\|v - v_0\| < \delta_0$. Since the rational vectors are dense in A and the norm is continuous, it follows that this search terminates. It remains show that $\|v - \mathbf{1}_A\| < 2^{-k}$. By way of contradiction, suppose $\|v - \mathbf{1}_A\| \geq 2^{-k}$. Without loss of generality, suppose $A = C^*(X)$ for some compact metrizable space X. Hence, there exists $t_0 \in X$ so that $|v(t_0) - 1| \geq 2^{-k}$. Let $\alpha = v(t_0)$, and let $c = v(t_0)^2 - v(t_0)$. Let $\beta \in \mathbb{C}$ be the other root of $z^2 - z - c$. Thus, $\alpha\beta = -c$ and $\beta = 1 - \alpha$. Since $|c| < \epsilon_0$, $\min\{|\alpha|, |\beta|\} < \sqrt{\epsilon_0}$. However, as $\sqrt{\epsilon_0} < 2^{-k} \leq |\alpha - 1|$, $|\alpha| < \sqrt{\epsilon_0}$. Hence,

$$\begin{aligned}|v(t_0) - v_0(t_0)| &\geq ||v(t_0)| - |v_0(t_0)|| \\ &\geq |v_0(t_0)| - |v(t_0)| \\ &\geq 1 - \delta_0 - \sqrt{\epsilon_0} \\ &\geq \delta_0.\end{aligned}$$

This is a contradiction. □

The proof of Theorem 4.1 is nonuniform. As we shall see, our only obstacle to a fully uniform proof of Theorem 1.1 is the uniform computability of the unit. Hence, we now explore how uniform can Theorem 4.1 be made. Our best result in this direction is the following.

Proposition 4.2. *Let X be a compact metrizable space, and suppose X has a finite number of connected components. Then, $C^*(X)$ is uniformly computably unital.*

Proof. Let $A = C^*(X)$. Let n_0 denote the number of connected components of X. Let $C_1, \ldots, C_{n_0}$ denote the connected components of X. Fix a computable presentation $A^\#$ of A. Let $k \in \mathbb{N}$, and suppose it is required to compute a rational vector v_0 of $A^\#$ so that $\|v_0 - \mathbf{1}_A\| < 2^{-k}$. Let:

$$\begin{aligned} k_0 &= k+3 \\ \epsilon_1 &= \frac{1}{2}\min\{(1-2^{-k_0}n_0)^2, 2^{-(k_0+1)}/n_0\} \\ \epsilon_2 &= \frac{1}{2}\min\{\frac{1}{4}, (2^{-k_0}/n_0)^2\}. \end{aligned}$$

Search for rational vectors $v_1, \ldots, v_{n_0}$ that satisfy the following conditions:

(1) $|\,\|v_n\| - 1| < \frac{1}{2}$.
(2) $\|v_m v_n\| < \epsilon_1$ when $m \neq n$.
(3) $\left\|v_n^2 - v_n\right\| < \epsilon_2$.

Thus, $\|v_n\| > \frac{1}{2}$. Set $v = \sum_{n=1}^{n_0} v_n$.

Let I_n denote the indicator function of C_n. By letting v_n approach I_n for each n, it is seen that this search terminates. It only remains to show $\|v - \mathbf{1}_A\| < 2^{-k}$.

Let us say that v_m is *approximately supported on* C_n if $|v_m(t) - 1| < 2^{-k_0}/n_0$ for all $t \in C_n$. We first claim that for each $m \in \{1, \ldots, n\}$, there exists $n \in \{1, \ldots, n_0\}$ so that v_m is approximately supported on C_n. Towards this end, take $t_0 \in X$ so that $|v_m(t_0)| > \frac{1}{2}$, and take the unique $n \in \{1, \ldots, n_0\}$ so that $t_0 \in C_n$.

For each $t \in C_n$, let $c(t) = v_m(t)^2 - v_m(t)$. For each $t \in C_m$, let $\alpha(t) = v_m(t)$, and let $\beta(t)$ be the other root of $z^2 - z - c(t)$. Thus, $\alpha(t) + \beta(t) = 1$, and $\alpha(t)\beta(t) = -c(t)$. By (3), $|c(t_0)| < \epsilon_2$. Since $|v_m(t_0)| > \frac{1}{2}$, $|v_m(t_0)| > \sqrt{\epsilon_2}$. Thus, $|1 - v_m(t_0)| = |\beta(t_0)| < \sqrt{\epsilon_2} < 2^{-k}/n_0$.

By way of contradiction, suppose $t_1 \in C_n$ and $|1 - v_m(t_1)| \geq 2^{-k}/n_0$. Thus, $|\beta(t_1)| = |1 - v_m(t_1)| > \sqrt{\epsilon_2}$. Hence, $|\alpha(t_1)| = |v_m(t_1)| < \sqrt{\epsilon_2}$. Since $|v_m(t_0)| > \frac{1}{2} > \sqrt{\epsilon_2}$, by connectedness there exists $t_2 \in C_n$ so that $1/2 > |v_m(t_2)| > \sqrt{\epsilon_2}$. Thus, $|1 - v_m(t_2)| > \frac{1}{2}$. Putting all this together, we obtain $|v_m(t_2)(v_m(t_2) - 1)| > \sqrt{\epsilon_2} \cdot \frac{1}{2} > \epsilon_2$, yielding a contradiction.

Now, we claim that for each $m \in \{1, \ldots, n_0\}$, there is exactly one $n \in \{1, \ldots, n_0\}$ so that v_m is approximately supported on C_n. By way of contradiction suppose otherwise. By our first claim and the pigeonhole principle, there exist $m, m', n \in \{1, \ldots, n_0\}$ so that $m \neq m'$ and v_m, $v_{m'}$ are approximately supported on C_n. Then, $\min\{|v_m(t_0)|, |v_{m'}(t_0)|\} > 1 - 2^{-k}/n_0$. So, $|v_m(t_0)v_{m'}(t_0)| > \epsilon_1$. But, $\|v_m v_{m'}\| < \epsilon_1$- a contradiction.

We now conclude there is an injective map $j : \{1, \ldots, n_0\} \to \{1, \ldots, n_0\}$ so that v_m is approximately supported on $C_{j(m)}$ for each $m \in \{1, \ldots, n_0\}$. Hence, j is surjective and $\mathbf{1}_A = \sum_{m=1}^{n_0} I_{j(m)}$. Therefore,

$$\|v - \mathbf{1}_A\| \leq \sum_{m=1}^{n_0} \left\|v_m - I_{j(m)}\right\|$$

We estimate $\left\|v_m - I_{j(m)}\right\|$ as follows. Let $m \in \{1, \ldots, n_0\}$. If $t \in C_{j(m)}$, then $|v_m(t) - I_{j(m)}(t)| < 2^{-k_0}/n_0$. Suppose $t \notin C_{j(m)}$. Let $t \in C_n$, and let $m' = j^{-1}(n)$. Thus $m' \neq m$. By definition of j, $|v_{m'}(t) - 1| < 2^{-k_0}/n_0$. Thus, by (1)

$$\begin{aligned} |v_m(t)v_{m'}(t) - v_m(t)| &\leq (1 + \frac{1}{2})(2^{-k_0}/n_0) \\ &< 2^{-k_0+1}/n_0. \end{aligned}$$

Hence,

$$\begin{aligned} |v_m(t)| &\leq |v_m(t) - v_m(t)v_{m'}(t)| + |v_m(t)v_{m'}(t)| \\ &\leq 2^{-k_0+1}/n_0 + \epsilon_1. \end{aligned}$$

Thus, $\left\|v_m - I_{j(m)}\right\| < 2^{-k_0+1}/n_0 + \epsilon_1$. Hence,

$$\|v - \mathbf{1}_A\| \leq 2^{-k_0+1} + n_0\epsilon_1 < 2^{-k_0+2} < 2^{-k}.$$

□

We now turn to the computability of post-composition operators. We start with post-composition operators induced by rational $*$-polynomials. We summarize our results in the following proposition. We believe these findings are simple enough so as not to require a formal proof. At the same time, we believe they are useful enough to warrant a formal statement.

Proposition 4.3. *Suppose $A = C^*(X)$, and let $A^\#$ be a computable presentation of A.*

(1) *If $p : \mathbb{C}^n \to \mathbb{C}$ is a rational $*$-polynomial, then the post-composition operator $(f_1, \ldots, f_n) \mapsto p(f_1, \ldots, f_n)$ is a computable operator of $A^\#$.*
(2) *In addition, if $p(\vec{0}) = 0$, then an index of this operator can be computed from p and an index of $A^\#$.*
(3) *If $p(\vec{0}) \neq 0$, then an index of this operator can be computed from p, an index of $A^\#$, and an $A^\#$-index of $\mathbf{1}_A$.*

We now use Proposition 4.3 to establish the conditions under which a post-composition operator is computable.

Proposition 4.4. *Suppose $A = C^*(X)$, and let $A^\#$ be a computable presentation of A.*

(1) *If $g : \mathbb{C}^n \to \mathbb{C}$ is computable, then the post-composition operator $(f_1, \ldots, f_n) \mapsto g(f_1, \ldots, f_n)$ is a computable operator of $A^\#$.*
(2) *In addition, if $g(\vec{0}) = 0$, then an index of this operator can be computed from an index of g and an index of $A^\#$.*
(3) *If $g(\vec{0}) \neq 0$, then an index of this operator can be computed from an index of g, an index of $A^\#$, and an $A^\#$-index of $\mathbf{1}_A$.*

Proof. Given $A^\#$-names of $f_1, \ldots, f_n \in A$, we compute an $A^\#$-name of $g(f_1, \ldots, f_n)$ as follows. First, compute a positive integer M so that $\max\{\|f_1\|, \ldots, \|f_n\|\} < M$. Let $R_M = \{(z_1, \ldots, z_n) \in \mathbb{C}^n \ : \ |z_j| \leq M\}$.

We then compute a sequence $(p_k)_{k\in\mathbb{N}}$ of rational $*$-polynomials so that $|p_k(q) - g(q)| < 2^{-k}$ whenever $q \in R_M$. Thus, for each $k \in \mathbb{N}$, $\|p_k(f_1, \ldots, f_n) - g(f_1, \ldots, f_n)\| < 2^{-k}$. By Proposition 4.3, it follows that we may compute for each k a rational vector u_k so that $\|u_k - p_{k+1}(f_1, \ldots, f_n)\| < 2^{-(k+1)}$. Thus, $(u_k)_{k\in\mathbb{N}}$ is an $A^\#$-name of $g(f_1, \ldots, f_n)$.

By inspection, all of the steps in the above procedure are uniform in n, an index of g, an index of $A^\#$, and an $A^\#$-index of $\mathbf{1}_A$. If $g(\vec{0}) = 0$, then we can choose p_k so that $p_k(\vec{0}) = 0$. Hence, in this case, only an index of g and an index of $A^\#$ are required to compute an index of the post-composition operator induced by g. □

Corollary 4.5. *If X is a compact metrizable space, then* $|\ |$, $\max$, *and* $\min$ *are uniformly intrinsically computable operators of $C^*(X)$.*

5. Proof of Theorem 1.1

Suppose X is a compact metrizable space, and let $A = C^*(X)$. For each $p \in X$ and $f \in A$, let $\widehat{p}(f) = f(p)$, so $\widehat{p}$ is the evaluation-at-p functional.

Fix a computable presentation $A^\#$ of A. In order to simplify exposition, throughout this section, all computability is referent to this particular presentation. Consequently, when we say that a vector, sequence, or operator is computable, we mean that it is a computable vector, sequence, or operator of $A^\#$.

By standard techniques, we may compute an effective and injective enumeration $(\kappa_n)_{n\in\mathbb{N}}$ of a dense sequence of rational vectors of $A^\#$. For all $a, b \in X$, let

$$d(a,b) = \sum_{n\in\mathbb{N}} 2^{-n} \frac{|\kappa_n(a) - \kappa_n(b)|}{1 + |\kappa_n(a) - \kappa_n(b)|}.$$

It is well-known that d is a metric that is compatible with the topology of X. For all $p, q \in X$, let $D_p(q) = d(p,q)$. Thus, $D_p \in A$ for each $p \in X$.

Our goal now is to build a computably compact presentation of (X, d). The following theorem reduces the complexity of this task while also motivating the method of our proof.

Theorem 5.1. *Suppose $(a_n)_{n\in\mathbb{N}}$ is a dense sequence of points of X so that $\widehat{a_n}$ is a computable functional uniformly in n. Then, $(D_{a_n})_{n\in\mathbb{N}}$ is a computable sequence of vectors and $(X, d, (a_n)_{n\in\mathbb{N}})$ is a computably compact presentation of X.*

Proof. We first note that

$$D_{a_n} = \sum_{t\in\mathbb{N}} 2^{-t} \frac{|\kappa_t - \kappa_t(a_n)\cdot \mathbf{1}_A|}{\mathbf{1}_A + |\kappa_t - \kappa_t(a_n)\cdot \mathbf{1}_A|}. \tag{5.1}$$

It is easy to verify that the series in Equation (5.1) (viewed as a sequence of partial sums) has a computable modulus of convergence. Thus, we only need to demonstrate that each term of this series is computable uniformly in n, t. To facilitate this, for all $n, t \in \mathbb{N}$, set:

$$\begin{aligned} u_{n,t} &= |\kappa_t - \kappa_t(a_n)\cdot \mathbf{1}_A| \\ v_{n,t} &= \mathbf{1}_A + u_{n,t}. \end{aligned}$$

We note that since $v_{n,t}(p) \geq 1$ for all $p \in X$, $v_{n,t}$ is invertible and $\left\|v_{n,t}^{-1}\right\| \leq 1$. By Theorem 4.1 and Corollary 4.5, $u_{n,t}$ and $v_{n,t}$ are computable vectors uniformly in n, t. It only remains to show that $v_{n,t}^{-1}$ is computable uniformly in n, t. This can be accomplished by a simple search procedure as follows. Given $k \in \mathbb{N}$, search for a rational vector u so that $\|uv_{n,t} - \mathbf{1}_A\| < 2^{-k}$. Since $\left\|v_{n,t}^{-1}\right\| \leq 1$, it follows that $\left\|u - v_{n,t}^{-1}\right\| < 2^{-k}$.

Since $\widehat{a_n}$ is computable uniformly in n, it now follows that $(d(a_m, a_n))_{m,n\in\mathbb{N}}$ is a computable array of real numbers. whence it follows that $X^\# = (X, d, (a_n)_{n\in\mathbb{N}})$ is a computable presentation of X. All that remains is to demonstrate the computable compactness of $X^\#$. We accomplish this as follows. For every $k \in \mathbb{N}$ and finite nonempty $F \subseteq \mathbb{N}$, $X = \bigcup_{n\in F} B(a_n; 2^{-k})$ if and only if $\|\min_{n\in F} D_{a_n}\| < 2^{-k}$. Since $(a_n)_{n\in\mathbb{N}}$ is dense in X, and since X is compact, for each $k \in \mathbb{N}$, there *is* a finite $F \subseteq \mathbb{N}$ so that $X = \bigcup_{n\in F} B(a_n; 2^{-k})$. By Corollary 4.5, from F it is possible to compute $\|\min_{n\in F} D_{a_n}\|$. It follows that $X^\#$ is a computably compact presentation. □

Thus, we now seek to produce a dense sequence $(a_n)_{n\in\mathbb{N}}$ of points of X so that $\widehat{a_n}$ is computable uniformly in n. To this end, for each $n \in \mathbb{N}$, set $u_n = \min\{|\kappa_n|, \mathbf{1}_A\}$. It follows from Corollary 4.5 that $(u_n)_{n\in\mathbb{N}}$ is computable. Furthermore, when $f \in C(X; [0, 1])$, $\|u_n - f\| \leq \||\kappa_n| - f\| \leq \|\kappa_n - f\|$. Thus, $(u_n)_{n\in\mathbb{N}}$ is dense in $C(X; [0, 1])$. It then follows that $(u_n)_{n\in\mathbb{N}}$ generates a dense subalgebra of $C^*(X)$.

Ensuring the computability of $\widehat{a_n}$ is simplified somewhat by the following principle which is perhaps of independent interest.

Proposition 5.2. *Suppose $p \in X$ has a computable well structured name. Then, $\widehat{p}$ is computable. Furthermore, an index of $\widehat{p}$ can be computed from an index of a well structured name of p.*

Proof. Fix a well structured name $(f_s)_{s\in\mathbb{N}}$ of p. Since $\|\widehat{p}\| \leq 1$, and since $(u_m)_{m\in\mathbb{N}}$ generates a dense subalgebra of A, it suffices to show $(\widehat{p}(u_m))_{m\in\mathbb{N}}$ is computable.

Let $(q_r)_{r\in\mathbb{Q}\cap(0,1)}$ be a computable family of polynomials with the following properties:

- q_r maps $[0, 1]$ into $[0, 1]$.
- $q_r(x) > \frac{1}{2}$ if and only if $x > r$.

Let $k, m \in \mathbb{N}$. It is required to compute a rational number q so that $|\widehat{p}(u_m)-q| < 2^{-k}$. We search for $r_0, r_1 \in \mathbb{Q} \cap [0, 1]$ and $s \in \mathbb{N}$ so that the following hold.

(1) $0 < r_1 - r_0 < 2^{-k}$.
(2) $r_0 = 0$ or $\|f_s(1 - q_{r_0} \circ u_m)\| < \frac{1}{3}$.
(3) $r_1 = 1$ or $\|f_s(1 - q_{1-r_1} \circ (1 - u_m))\| < \frac{1}{3}$.

Set $q = r_0$.

We first show that this search terminates. To this end, we first consider the case $u_m(p) = 0$. If we take $r_0 = 0$ and $r_1 = 2^{-(k+1)}$, then $q_{1-r_1}(1 - u_m(p)) > \frac{1}{2}$ and so by Lemma 3.1 there exists s so that $\|f_s(1 - q_{1-r_1} \circ (1 - u_m))\| < \frac{1}{3}$. Now, suppose $u_m(p) = 1$. In this case, we take $r_1 = 1$ and $r_0 = 1 - 2^{-(k+1)}$. We then have $q_{r_0}(u_m(p)) > 1/2$ and so there exists s so that $\|f_s(1 - q_{r_0} \circ (u_m))\| < \frac{1}{3}$. Finally, suppose $0 < u_m(p) < 1$. Choose rational numbers $r_0, r_1 \in (0, 1)$ so that $0 < r_1 - r_0 < 2^{-k}$ and so that $r_0 < u_m(p) < r_1$. Thus, $q_{r_0}(u_m(p)) > \frac{1}{2}$ and $q_{1-r_1}(1 - u_m(p)) > \frac{1}{2}$. It then follows from Lemma 3.1 that there exists s so that $\|f_s(1 - q_{r_0} \circ u_m)\| < \frac{1}{3}$ and so that $\|f_s(1 - q_{1-r_1} \circ (1 - u_m))\| < \frac{1}{3}$. Hence, in all cases, the above search terminates.

It only remains to show $|q - u_m(p)| < 2^{-k}$. It follows from Lemma 3.1 that $r_0 \leq u_m(p) \leq r_1$. Since $r_1 - r_0 < 2^{-k}$, we have $|q - u_m(p)| < 2^{-k}$. □

In light of Lemma 3.3 and Proposition 5.2, much of the task at hand now reduces to the following.

Lemma 5.3. *If $f \in C(X; [0, 1])$ is a computable vector, and if $\|f\| > \frac{2}{3}$, then there is a computable adequately structured name that begins with f. Furthermore, an index of such a name can be computed from an index of f.*

Proof. Fix a polynomial p over $\mathbb{Q}$, e.g., $p(x) = 16x^5 - 40x^4 + 32x^3 - 8x^2 + x$, with the following properties.

(1) p maps $[0, 1]$ onto $[0, 1]$.
(2) For all $x \in \mathbb{R}$, $p(x) = x$ if and only if $x \in \{0, \frac{1}{2}, 1\}$.
(3) $p(x) > x$ when $x \in (-\infty, 0) \cup (\frac{1}{2}, 1)$.
(4) $p(x) < x$ when $x \in (0, \frac{1}{2}) \cup (1, \infty)$.

Let p^k denote the k-th iterate of p. It follows that for each $x \in [0,1]$,

$$\lim_k p^k(x) = \begin{cases} 1 & \frac{1}{2} < x \leq 1 \\ \frac{1}{2} & x = \frac{1}{2} \\ 0 & 0 \leq x < \frac{1}{2}. \end{cases}$$

We construct $(f_s)_{s\in\mathbb{N}}$ by stages. At stage s, we define f_s, and we may declare a $j \in \mathbb{N}$ to be *incorporated.*

Stage 0**:** Set $f_0 = f$. No j is incorporated at stage 0.

Stage $s+1$**:** We first define a function h_s by cases as follows. If there is no $j \leq s$ so that $\|f_s + u_j\| > \frac{5}{3}$ and so that j has not been incorporated by the end of stage s, then set $h_s = \frac{3}{4} f_s$. Otherwise, let j_s be the least such j, and set $h_s = \frac{3}{10}(f_s + u_{j_s})$; we also incorporate j_s at stage $s+1$.

By way of induction, $h_s \in C(X;[0,1])$ and $2/3 < \|f_s\| \leq 1$. Thus, $1 > \|h_s\| > \frac{1}{2}$. By the properties of p, there is a natural number k so that $\|p^k \circ h_s\| > \frac{3}{4}$; let k_s be the least such number. Set $f_{s+1} = p^{k_s} \circ h_s$.

We now show $f_{s+1}^{-1}(1/2,\infty) \subseteq f_s^{-1}(2/3,\infty)$. Suppose $f_{s+1}(t) > \frac{1}{2}$. By the properties of p, $h_s(t) > \frac{1}{2}$. If no j is incorporated at $s+1$, then $h_s = \frac{3}{4} f_s$ and so $f_s(t) > \frac{2}{3}$. Suppose j_s is incorporated at $s+1$. Then, $f_s(t) + u_{j_s}(t) > \frac{5}{3}$, and so $f_s(t) > \frac{2}{3}$.

We now demonstrate that $(f_s)_{s\in\mathbb{N}}$ names a point. By construction, for each $s \in \mathbb{N}$, there exists $x_s \in X$ so that $f_{s+1}(x_s) > \frac{3}{4}$. Since X is compact and metrizable, there is an $a \in X$ and an increasing sequence $(s_j)_{j\in\mathbb{N}}$ so that $\lim_j x_{s_j} = a$. We show that $(f_s)_{s\in\mathbb{N}}$ names a. To this end, let $C = \bigcap_{s\in\mathbb{N}} f_s^{-1}(1/2,\infty)$. We note that $C = \bigcap_{s\in\mathbb{N}} \overline{f_s^{-1}(1/2,\infty)}$. By a standard argument, $a \in C$. By way of contradiction suppose $b \in C - \{a\}$. By Urysohn's Lemma, there is a continuous $\lambda : X \to [0,1]$ so that $\lambda(a) = 1$ and $\lambda(b) = 0$. Hence, there exists r so that $u_r(a) > \frac{11}{12}$ and so that $u_r(b) < \frac{2}{3}$. Let t_0 be the least stage so that every $r' < r$ that is incorporated has been incorporated by the end of stage t_0. Let m be the least integer so that $s_m + 1 > t_0$. Then, $u_r(a) + f_{s_m+1}(a) > \frac{3}{4} + \frac{11}{12} = \frac{5}{3}$. Thus, if r has not been incorporated by stage $s_m + 1$, it is at stage $s_m + 2$. So, let $s+1$ be the stage at which r is incorporated. Then, $h_s(b) < \frac{1}{2}$, and so $f_{s+1}(b) < \frac{1}{2}$- a contradiction since $b \in C$. ☐

There is a computable $e : \mathbb{N} \to \mathbb{N}$ so that $\text{ran}(e) = \{n \in \mathbb{N} \ : \ \|u_n\| > \frac{2}{3}\}$. It follows from Lemma 5.3 that there is a uniformly computable sequence $(\Lambda_n')_{n\in\mathbb{N}}$ of adequately structured names so that Λ_n' originates with $u_{e(n)}$. Let $a_n \in X$ be the point named by Λ_n'. By Lemma 3.3, $(a_n)_{n\in\mathbb{N}}$ is dense in X (take $r = \frac{2}{3}$). Now, set $\Lambda_n = \min\{\psi \circ \Lambda_n', 2^{-n}\}$ where ψ is the function defined in Section 3. Thus, by Lemma 3.2, Λ_n also names a_n. In addition, by Proposition 4.4, $(\Lambda_n)_{n\in\mathbb{N}}$ is computable. Hence, by Proposition 5.2, $(\widehat{a_n})_{n\in\mathbb{N}}$ is computable. Therefore, by Theorem 5.1, X has a computably compact presentation.

6. Applications and extensions

Since isometric isomorphisms map computable presentations onto computable presentations, Theorem 1.1 easily implies the following.

Corollary 6.1. *Suppose A is a unital commutative C^* algebra. If A is computably presentable, then the spectrum of A has a computably compact presentation.*

In addition, we can now give nontrivial examples of operator algebras that do not have computable presentations.

Corollary 6.2.

(C1) *There exists a compact space X so that X is computably presentable but $C^*(X)$ has no computable presentation.*

(C2) *There exists a commutative unital C^*-algebra that has a Y-computable presentation if and only if Y is not low (i.e., $Y' > \emptyset'$).*

(C1) follows from the existence of a compact computably presentable metric space X that is not homeomorphic to any metric space with a computably compact presentation [18, 12]. Notably, perhaps the most elegant way to produce such a space uses (D2) stated in Section 2, and one more effective duality established in [22] which we will not state here. Using this sequence of effective dualities, the existence of such a space X can be reduced to the old result of Mal'cev [19] characterising computable subgroups of $(\mathbb{Q}, +)$. Similarly, in (C2) the existence of such a pathological space can be reduced to the existence of a torsion-free abelian group that has exactly the non-low presentations established in [21]; see also [22] for a detailed explanation.

By inspection, the steps of the proof of Theorem 1.1 are uniform. That is, from an index of a presentation $C^*(X)^\#$ and a $C^*(X)^\#$-index of the unit, we may compute an index of a computably compact presentation of X. However, uniformity holds in a more general sense which we describe now.

To begin, we set forth a method of naming presentations of C^* algebras. Let A be a C^* algebra, and fix a presentation $A^\# = (A, (u_n)_{n\in\mathbb{N}})$. The set $D(A^\#) = \{(r, j, r') \ : \ r, r' \in \mathbb{Q} \cap (0, \infty) \ \wedge \ r < \left\|\mathfrak{p}_j[A^\#]\right\| < r'\}$ is the *diagram* of $A^\#$. We say that $f \in \mathbb{N}^\mathbb{N}$ is a *name* of $A^\#$ if $D(A^\#) = \{(r, j, r') \ : \ \exists k \in \mathbb{N} f(k) = \langle r, j, r'\rangle\}$. It follows that $A^\#$ is computable if and only if it has a computable name.

Suppose $\mathcal{M} = (X, d)$ is a metric space and $\mathcal{M}^\# = (\mathcal{M}, (s_j)_{j\in\mathbb{N}})$ is a presentation of $\mathcal{M}$. The *diagram* of $\mathcal{M}^\#$ is $D(\mathcal{M}^\#) = \{(r, j, k, r') \ : \ r < d(s_j, s_k) < r'\}$. A *name* of $\mathcal{M}^\#$ is an $f \in \mathbb{N}^\mathbb{N}$ so that $D(\mathcal{M}^\#) = \{(r, j, k, r') \ : \ \exists t \in \mathbb{N} f(t) = \langle r, j, k, r'\rangle\}$.

A *total boundedness function* of $\mathcal{M}^\#$ is an $f \in \mathbb{N}^\mathbb{N}$ so that for each $j \in \mathbb{N}$, $f(j)$ is a code of a finite $F \subseteq \mathbb{N}$ so that $X = \bigcup_{k\in F} B(s_k; 2^{-j})$. A computably presented metric space is compact if and only if it has a total boundedness function.

Now, we note that the steps in the proof of Theorem 1.1 are fully uniform in that they require only the name of a presentation $C^*(X)^\#$ and an $C^*(X)^\#$-name of the unit. We are thus led to the following.

Theorem 6.3. *There is an oracle Turing machine that, given a name of a presentation $A^\#$ of a commutative unital C^* algebra A and an $A^\#$-name of $\mathbf{1}_A$, produces a name of a presentation of $\Delta(A)^\#$ and a total boundedness function of $\Delta(A)^\#$.*

We note that A. Fox [13] gives a computable operator which, given the name of a presentation of a compact X and a total boundedness function, produces a name of a presentation of $C(X)$ and a name of the unit. Thus a total boundedness function and a unit are, in some sense, the right data to make the Gelfand duality effective.

In addition, by Theorem 4.1, we obtain the following.

Corollary 6.4. *There is a Turing machine that, given the name of a presentation $A^{\#}$ of a commutative unital C^* algebra A whose spectrum has finitely many connected components, and the number of connected components of $\Delta(A)$, produces a name of a presentation of $\Delta(A)^{\#}$ and a total boundedness function of $\Delta(A)^{\#}$.*

7. Conclusion

We have shown that the Gelfand duality for C^* algebras holds effectively, and that this effective duality is highly uniform provided enough information about the unit is given. In doing so, we have contributed to the program of studying the effective content of classical mathematics by providing another effective duality principle and by advancing the computable theory of operator algebras. The essential content of our main result is that from a (presentation of) a C^* algebra $C^*(X)$, one can compute a complete description of the points of X. Our theorem connects the computability theory of C^* algebras with the well-developed computability theory of Polish spaces [12]. Along the way, we have produced a number of less important though useful results on the computability of the unit.

Just as the dualities of classical mathematics build bridges between seemingly unrelated disciplines, effective dualities transfer methods and theorems between disparate branches of computable mathematics. One would then expect Theorem 1.1 to lead to new discoveries in the computability theory of operator algebras such as Corollary 6.2.

At this point, we are naturally led to the following question which was raised by T. McNicholl a few years ago (see also [6]).

Question 7.1 (McNicholl)**.** *Is Banach-Stone Duality effective? That is, if the Banach space $C(X)$ has a computable presentation, does it follow that X has a computably compact presentation?*

The effective duality (D1) stated above gives a positive answer in the case when the domain is totally disconnected. It has recently been announced by Melnikov and Ng that indeed Banach-Stone Duality *fails* computably, in the sense that there is a compact metrizable space X so that the Banach space $C(X)$ has a computable presentation but X does not have a computably compact presentation.

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Email address: pburton@iastate.edu

Email address: eaglec@uvic.ca

Email address: foxag@uci.edu

Email address: isaac@math.uci.edu

Email address: matthar@umich.edu

Email address: mcnichol@iastate.edu

Email address: alexander.g.melnikov@gmail.com

Email address: teerawat.thew@hotmail.com