COMPUTABLE COMPLETELY DECOMPOSABLE GROUPS

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ABSTRACT. A completely decomposable group is an abelian group of the form $\bigoplus_i H_i$, where $H_i \leq (Q, +)$. We show that every computable completely decomposable group is Δ_5^0-categorical. We construct a computable completely decomposable group which is not Δ_4^0-categorical, and give an example of a computable completely decomposable group G which is Δ_4^0-categorical but not Δ_3^0-categorical. We also prove that the index set of computable completely decomposable groups is arithmetical.

1. INTRODUCTION

1.1. **Effective classification of algebraic structures.** The concern of this paper is *effective* or *computable* mathematics, where we will be dealing with the algorithmic nature (or lack thereof) of basic objects of mathematics. Effective structure theory and, more particularly, effective algebra are concerned with studying familiar mathematical objects such as groups, rings and fields, but where the objects are given with computable domains and such that the operations are computable functions.

To understand the effective content of mathematics we need to study effectively presented structures, because it is upon such structures we are able to run algorithms or meaningfully show that certain algorithms are not present. This kind of study has roots in the early 20th century, such as the work of Grete Hermann [23], van der Waerden [45]. This effective philosophy (called then "explicit procedures") can be found in early editions of, say, van der Waerden [44]. An interesting analysis of how non-computable procedures were introduced to 20th century mathematics can be found in Metakides and Nerode [38].

The use of computability theory as a tool in the effective mathematics can be traced to Frölich and Shepherdson [15], Mal'cev [32] and Rabin [40]. For example, it is possible to show that every computable field has a computable algebraic closure (Rabin [40]) but that the closure is not necessarily computably unique (Frölich and Shepherdson [15]). Metakides and Nerode [37] took this field analysis further to classify exactly when the algebraic closure was computably unique : precisely when the field *also* had an algorithm to effectivize the usual method of adjoining roots. Metakides and Nerode also studied other aspects of effective field theory. Such studies can be generalized to other algebraic structures such as linear orderings [9],

Boolean algebras [20], and other structures [2, 12, 14]. In passing we remark that some of the arguments of [15] simply used the newly available model of computable functions to formalize the earlier arguments of van der Waerden, which apparently go back to Emmy Noether.

Our specific concern for the present paper is the effective aspects of the isomorphism type of a structure. In classical algebra, the standard classification tool is to represent a structure up to isomorphism, in effective algebra, the standard classification is to think of an *effective* structure up to *effective* isomorphisms. In the field example above, we would be thinking of a computably unique computable algebraic closure as be ones where any other computable algebraic closure was computably isomorphic to it.

Quite aside from the basic natural interest in effective procedures in algebraic structures, we remark that computable structure theory often reveals deeper algebraic facts about familiar structures. For example, the hidden meaning in the field example above is that *there must be some other method of constructing algebraic closures not using the the standard root adjoining method.* The whole enterprise of combinatorial group theory, looking at effective procedures in finitely presented groups, is another clear vindication of a program looking at effective aspects of algebraic structures. In this tradition, the results in this paper require significant new algebraic understanding which, we believe, is of an independent interest; as well as a further development of the notions (such as excellent S-bases) from our earlier paper [8].

1.2. Computable abelian groups.
Mal'cev had a deep interest in the algebraic theory of infinite abelian groups (see, e.g., his early paper [33]). Mal'cev immediately tested the new effective approach to algebra on the class of abelian groups (Mal'cev [32]). In his paper [32], Mal'cev defines an abelian group to be *recursive* (*computable*) if there is an effective listing of its elements under which the operation of the group becomes a recursive (computable) function. This numbering of the universe of the group is called a computable presentation or constructivisation of the group. Computable groups are often called constructive [14, 2].

Computable abelian groups have been intensively studied. For (an incomplete) survey of results in the field see, e.g., Khisamiev [26] and Downey [12]. See also our paper [8] for a discussion of some recent results in the area.

Modern computable abelian group theory combines methods of computable model theory (see Ershov and Goncharov [14], Ash and Knight [2]) and pure abelian group theory (Fuchs [16, 17]). Advanced topics of computable abelian group theory require both new algebraic and recursion-theoretic methods, some of these methods have recently found applications outside the theory [7, 25].

1.3. Computably categorical groups.
In this paper our underlying program is to study effective isomorphisms between computable abelian groups. Mal'cev was probably the first to ask for a necessary and sufficient condition on an abelian group to have a unique computable presentation. Mal'cev [31] also discovered that the group $\bigoplus_{i\in\mathbb{N}}(Q,+)$ has two computable presentations which are not computably isomorphic.

If a group has a unique computable presentation, up to a computable isomorphism, then it is usually called *computably categorical* or *autostable*. Nurtazin [39] showed that a computably presentable torsion-free abelian group is computably

categorical if, and only if, its rank is finite. Goncharov [19, 14] and, independently, Smith [42] characterized computably categorical abelian p-groups. It is also known that an abelian group has either one or infinitely many computably non-isomorphic computable presentations [19].

There is no systematic algebraic theory of mixed abelian groups, so it is not surprising that the general case of mixed abelian groups is still open (problem posed by Goncharov; see [14] and [11]). Nonetheless, computably categorical abelian groups are reasonably well understood.

1.4. **Higher categoricity.** If a computable abelian group is not computably categorical, it is natural to ask the following question: "How close to being computably categorical is the group?". For instance, we could ask for a characterization of abelian groups computably categorical relative to the halting problem: every two computable copies of such a group has an isomorphism decidable by a Turing machine with an oracle for the halting problem. Such groups are called Δ_2^0-categorical. We can iterate the halting problem and define Δ_n^0-categorical groups in a similar manner.

Very little is known about Δ_n^0-categorical algebraic structures (not necessarily groups) for $n \geq 2$. McCoy [34] characterized Δ_2^0-categorical linear orders and Boolean algebras under some extra effectiveness conditions. Recently, using a rather difficult machinery Harris [21] characterized Δ_n^0-categorical Boolean algebras for every finite n. It is also known that in general Δ_{n+1}^0-categoricity does not imply Δ_n^0-categoricity in the classes of linear orders [3], abelian p-groups [5], and ordered abelian groups [36].

There is a little hope to describe Δ_2^0-categorical computable abelian groups even when restricted to the class of torsion-free groups. The main difficulty is the absence of satisfactory algebraic invariants which would describe these groups up to a (not necessarily computable) isomorphism, as it was shown in [10]. Nonetheless, there is one class of torsion-free abelian groups which is classically rather well-understood. These groups are called *completely decomposable*.

1.5. **Completely decomposable groups.** In 1937, Baer [4] introduced the class of completely decomposable groups. A torsion-free abelian group is completely decomposable if it is isomorphic to

$$\bigoplus_{i \in I} H_i,$$

where H_i is a subgroup of the rationals Q under addition, for every $i \in I$. Algebraic properties of completely decomposable groups and their pure subgroups have been intensively studied, especially in the case of finite rank (see Arnold [1] and Butler [6]). These studies can be generalized to broader classes such as almost completely decomposable groups [30]. The subject has become too broad to be sufficiently covered in a short introduction, see [22, 18] for an extended survey. For a detailed exposition of the theory of completely decomposable and almost completely decomposable groups see [17, 1, 30].

Khisamiev and Krykpaeva [28] were the first to look at completely decomposable groups from the computability-theoretic point of view (there are also some early observations related to the subject mentioned in [26]). Remarkably, even a basic question of the theory of completely decomposable groups, when considered from

the effective point of view, may lead to a difficult problem with an unexpected solution; see, e.g., the surprising result of Khisamiev [27]. More results on computable completely decomposable groups can be found in [27, 13, 35].

There is a lot more hope to obtain a satisfactory description of higher computable categoricity in the special class of completely decomposable groups. Recently Downey and Melnikov [8] studied computable *homogeneous* completely decomposable groups. These are completely decomposable groups in which all elementary summands are isomorphic. The authors showed that every homogeneous completely decomposable group is Δ_3^0-categorical, and also obtained a full description of Δ_2^0-categorical groups of this form in terms of *semi-low sets* (see Soare [43]). It is rather unexpected that the specific recursion-theoretic notion of semi-low sets appears in a description of Δ_2^0-categorical groups. The purpose of this paper is to extend the results from [8] to the general case of arbitrary completely decomposable groups.

1.6. **Results.** The isomorphism type of a completely decomposable group is fully determined by the types of its elementary summands [4, 17], and each elementary summand can be described by its *type* [4, 17]. Classically, the collection of types of elementary components may encode a complicated countable partial ordering [17], and one may expect that there is no arithmetical upper bound on the complexity of isomorphisms of such groups. A careful analysis of the situation in the case of *computable* groups leads to a somewhat unexpected result:

Theorem 3.1. Every computable completely decomposable group is Δ_5^0-categorical.

The proof of Theorem 3.1 exploits methods from [8] as well as a new algebraic notion of a *labeled regular set* which is central to the proof. A labeled regular set is a collection of elements which possess some nice properties (Definition 3.6) together with a function which indicates characteristics of elements of the set. Using the algebraic machinery developed in the proof of Theorem 3.1, we show:

Theorem 3.17. The index set of computable completely decomposable groups is arithmetical. More specifically, the index set is Σ_7^0.

This theorem is not a straightforward consequence of Theorem 3.1, and its proof needs some extra work: we need a listing of isomorphism types of all completely decomposable groups. We don't know if the index set of computable completely decomposable groups is Σ_7^0-complete, but we know Theorem 3.1 is sharp:

Theorem 4.3. There is a computable completely decomposable group which is not Δ_4^0-categorical.

Thus, Theorem 3.1 can not be improved to Δ_4^0. The proof is of some technical interest as it uses a new algebraic strategy. Theorems 3.1 and 4.3 also provide us with a natural class of algebraic structures which are arithmetically categorical and, at the same time, categorical at the high level of Δ_5^0. All other sharply Δ_5^0-categorical structures that we know are rather specific from the purely algebraic point of view.

It could be the case that every Δ_4^0-categorical completely decomposable group was already Δ_3^0-categorical, similarly to well-orderings [3]. We show that it does not happen in the class of completely decomposable groups:

Theorem 4.1. There is a computable completely decomposable group which is Δ_4^0-categorical but not Δ_3^0-categorical.

The groups constructed in Theorem 4.1 and Theorem 4.3 are the first known examples of torsion-free abelian groups which are Δ_n^0-categorical but not Δ_{n-1}^0 categorical for $n > 3$; examples for $n \leq 3$ can be found in [8].

1.7. **The structure of the paper.** Section 2 contains the necessary background. In Section 3 we prove Theorem 3.1 and Theorem 3.17. The proof of Theorem 3.17 relies on the techniques developed in the proof of Theorem 3.1. We illustrate Theorem 4.1 and Theorem 4.3 in Section 4. We first prove Theorem 4.1 which is simpler, and then prove Theorem 4.3 using the algebraic machinery developed for Theorem 4.1.

2. Background

We use known definitions and facts from computability theory and the theory of abelian groups. Standard references are [43] for computability and [16] for the theory of torsion-free abelian groups.

2.1. **Abelian groups.** Every abelian group is a module over Z, and the notions of linear independence and rank of an abelian group can be taken directly from the module theory. It is not hard to see that a torsion-free abelian group A has rank 1 if and only if $A \leqq \langle Q, + \rangle$. (We write $A \leqq B$ if A is a subgroup of B.) Not every abelian group is a Q-module; consequently, the operation of division by an integer is typically a partial operation.

Notation 2.1. Given $R \subseteq Q$ and $X \subseteq G$, denote by $[X]_R$ the set of sums

$$\sum_{x \in X} r_x x$$

where $r_x \in R$ for every x, and $r_x = 0$ for almost all x. We also assume G contains $r_x x$, for every $x \in X$.

We fix the canonical listing of the prime numbers:

$$p_0, p_1, \ldots, p_n, \ldots.$$

If an integer n divides an element g, we write $n|g$.

Definition 2.2 (Characteristic and h_i). Suppose G is a torsion-free abelian group. For $g \in G$, $g \neq 0$, and a prime number p_i, set

$$h_i(g) = \begin{cases} \max\{k : p_i^k | g \text{ in } G\}, \text{if this maximum exists,} \\ \infty, \text{otherwise.} \end{cases}$$

The sequence $\chi_G(g) = (h_0(g), h_1(g), \ldots)$ is called the *characteristic* of the element g in G.

For a torsion-free group G, a subgroup H of G is a pure subgroup of G if and only if $\chi_H(h) = \chi_G(h)$ for every $h \in H$.

Definition 2.3. Let $\alpha = (k_0, k_1, \ldots)$ and $\beta = (l_0, l_1, \ldots)$ be two characteristics. Then we write $\alpha \leq \beta$ if $k_i \leq l_i$ for all i, where ∞ is greater than any natural number.

Definition 2.4 (Type). Two characteristics, $\alpha = (k_0, k_1, \ldots)$ and $\beta = (l_0, l_1, \ldots)$, are *equivalent*, written $\alpha \sim \beta$, if $k_n \neq l_n$ only for finitely many n, and k_n and l_n are finite for these n. The equivalence classes of this relation are called *types*.

We write $\mathbf{t}(g)$ for the type of an element g. If $G \leq \langle Q, + \rangle$ (equivalently, if G has rank 1) then all non-zero elements of G have equivalent types, by the definition of rank. Hence, we can correctly define the type of G to be $\mathbf{t}(g)$ for a non-zero $g \in G$, and denote it by $\mathbf{t}(G)$. The following theorem was known to Levi in 1917 [29]:

Theorem 2.5 (Baer [4]; Levi [29]). *Let G and H be torsion-free abelian groups of rank 1. Then G and H are isomorphic if and only if $\mathbf{t}(G) = \mathbf{t}(H)$.*

The next simplest class of torsion-free abelian groups is the class of *completely decomposable* groups.

Definition 2.6 (Completely decomposable groups). A torsion-free abelian group is called *completely decomposable* if G is a direct sum of groups each having rank 1. A completely decomposable group is *homogeneous* if all its elementary summands are isomorphic.

It is known that any two decompositions of a completely decomposable group into direct summands of rank 1 are isomorphic [4]. We will refer to this fact implicitly. Fix any complete decomposition of G into elementary summands. For a type $\mathbf{f}$, denote by $G_{(\mathbf{f})}$ the sum of all elementary summands of G having type $\mathbf{f}$. If the group G has no elementary summands of type $\mathbf{f}$, then we set $G_{(\mathbf{f})} = 0$. We have:

$$G = \bigoplus_{\mathbf{f}} G_{(\mathbf{f})},$$

where $\mathbf{f}$ ranges over all types. Whenever we are given a completely decomposable group, we usually have in mind a complete decomposition of it.

Definition 2.7. For a torsion-free abelian group A and a type $\mathbf{f}$, denote by $A_{\mathbf{f}}$ the subgroup generated by elements of having types $\geq \mathbf{f}$, and denote by $A_{\mathbf{f}}^{\star}$ the subgroup of A generated by elements of having types $> \mathbf{f}$.

Remark 2.8. Note that, in general, $A_{\mathbf{f}}^{\star}$ may contain elements of type $\mathbf{f}$. For example, consider a group having elementary components of only two types:

$$A = A_{(\mathbf{s})} \oplus A_{(\mathbf{t})},$$

where $inf\{\mathbf{s}, \mathbf{t}\} = \mathbf{f}$ and both $\mathbf{s}$ and $\mathbf{t}$ are strictly greater than $\mathbf{f}$. We have $A_{\mathbf{f}}^{\star} = A$. As can be easily seen, the group A contains elements of type $\mathbf{f}$. More specifically, every element having non-zero projections onto both summands has this property.

Fact 2.9. *Let G be a completely decomposable group, and let $G = \bigoplus_{t} G_{(t)}$ be its decomposition in homogeneous completely decomposable summands. For every type $\mathbf{f}$,*

$$G_{\mathbf{f}} = \bigoplus_{t \geq f} G_{(t)}$$

and

$$G_{\mathbf{f}}^{\star} = \bigoplus_{t > f} G_{(t)}.$$

Proof. Clearly, $\bigoplus_{\mathbf{t} \geq \mathbf{f}} G_{(\mathbf{t})}$ is contained in $G_{\mathbf{f}}$. For every element g of G, let $g = \sum_{\mathbf{t}} g_{\mathbf{t}}$ be its decomposition into projections onto the homogeneous summands $G_{(\mathbf{t})}$. Here $\mathbf{t}$ ranges over all types, and $g_{\mathbf{t}} = 0$ for almost every $\mathbf{t}$. Note that the type of

g is the infimum of the types of the projections. Therefore, only projections onto the components of types $\geq \mathbf{f}$ may occur if $\mathbf{t}(g) \geq \mathbf{f}$. This shows

$$G_\mathbf{f} = \bigoplus_{\mathbf{t} \geq \mathbf{f}} G_{(\mathbf{t})}.$$

The proof for $G_\mathbf{f}^\star = \bigoplus_{\mathbf{t} > \mathbf{f}} G_{(\mathbf{t})}$ is similar. $\qquad\square$

2.2. **Computability.** We assume that the reader is familiar with the elementary facts of computability theory and computable model theory which are contained in the first few chapters of [43] and [2, 14]. Our notations are standard. For instance, $0'$ denotes the halting problem, and $0''$ stands for the halting problem for Turing machines with oracles for $0'$, etc.

A set A is Turing reducible to B if the characteristic function of A can be computed by a Turing machine with oracle B. Turing reducibility is classically used to compare the degrees of unsolvability of two countable sets.

Another common notion is the arithmetical hierarchy: a set is Δ_n^0 if it is Turing reducible to $0^{(n)}$ (we also say computable in $0^{(n)}$); a set is Σ_n^0 if it can be listed by a Turing machine with oracle $0^{(n)}$. It is well-known that the hierarchy is proper. We will also use the following:

Lemma 2.10 (Folklore). *For every n and every Σ_n^0 set S, there exists a computable predicate R such that*

$$e \in S \Leftrightarrow (\exists x)(Q_1 y_1) \ldots (Q_{n-1} y_{n-1}) R(e, x, y_1, \ldots, y_{n-1}),$$

and for every $e \in S$ there exists exactly one x witnessing $e \in S$. (Here Q_i is either $\forall$ or $\exists$.)

Proof. The cases $n = 1, 2, 3$ can be easily illustrated directly. For $n > 3$, one uses relativization and the effective uniformity of cancellation of repeated quantifiers. $\qquad\square$

It is also well-known (see Rogers [41]) that the prefix $\forall x \exists y$ can be replaced by $\exists^\infty z$ equivalently, possibly under a uniform change of the underlying computable predicate R.

In computable model theory, one often uses first-order formulas which admit infinite computable conjunctions and disjunctions [2]. For example, we can use an infinite disjunction to express that an element of $(Z, +)$ is a power of some other element, as follows:

$$(\exists x \in Z) \bigvee_{n \in \omega, n > 1} nx = z.$$

Also, we could write

$$\bigvee_{x \in Z, n \in \omega, n > 1} nx = z.$$

Recall that nx is an abbreviation, and we can not write $(\exists n)$ in the language of abelian groups. Nonetheless, this syntactical nonsense is not an obstacle if we wish to write a program which searches for such an x in a computable representation of Z. The collection of all z satisfying the above property is, in general, Σ_1^0 as a set.

The $\exists$ quantifier and infinite computable conjunctions play similar roles in computable model theory. Using a simple analogy illustrated above, we could define

the hierarchy of computable infinitary formulas $\Sigma_1^c \subset \Sigma_2^c \subset \ldots$ based on the number of alternations of quantifiers and infinite disjunctions and conjunctions. The hierarchy has a natural correspondence with Σ_n^0 subsets of computable models [2]. We will be implicitly using the natural correspondence between Σ_n^0 sets and Σ_n^c, and even write Σ_n^0 in place of Σ_n^c.

3. Proofs of Theorem 3.1 and Theorem 3.17

Theorem 3.1. *Every completely decomposable group is Δ_5^0-categorical.*

Proof idea. The first obvious idea would be to provide a uniform version of the first main result from [8]. Nonetheless, there are certain purely algebraic difficulties.

The first difficulty is that a homogeneous component of a completely decompsable group is not necessarily invariant under automorphisms of G. Thus, we have to deal with quotient groups merely *isomorphic* to homogeneous components, not with the components themselves.

The second difficulty is that we can not build isomorphisms on these quotients separately and then take their union, because the maps may overlap unpredictably. As a consequence, the proof has to run on all homogeneous components simultaneously. To implement this idea, we introduce a new algebraic notion of a labeled regular set which enables us to build a "regular" decomposition of a given computable completely decomposable group.

The proof of the theorem is divided into several parts. The first part contains the necessary definitions which are too specific to be included into the preliminaries. In the second part we give the construction which builds an isomorphism between any two copies of the group, and in the third part we verify the construction and show it is computable in $0^{(4)}$.

Proof. Before we describe the main procedure, we need several definitions.

Definitions and notations. The following definition is from [8]:

Definition 3.2. Let S be a set of primes, and let G be a torsion-free abelian group. If $S \neq \emptyset$, then we say that elements $b_1, \ldots, b_k$ of G are S-independent in G if $p| \sum_{i \in \{1, \ldots, k\}} m_i b_i$ in G implies $\bigwedge_{i \in \{1, \ldots, k\}} p|m_i$, for all integers $m_1, \ldots, m_k$ and $p \in S$.

Observe that if S is non-empty, then S-independence implies linear independence. For suppose a collection of elements was S-independent but not linearly independent. Then a non-trivial linear combination of these elements would be equal to zero, impying that each non-zero coefficient of the combination was divisible by any power of $p \in S$, a contradiction. Based on the above observation, we say that elements are $\emptyset$-independent if they are simply linearly independent. For $S \neq \emptyset$, a maximal S-independent subset does not have to be maximal linearly independent (see [8]).

Definition 3.3. A maximal S-independent subset of G is said to be an S-*basis* of G. An S-basis is *excellent* if it is also a maximal linearly independent subset of G.

Notation 3.4. From now on, G stands for a completely decomposable group. In the following, $(\mathbf{f}_i)_{i \in I}$ stands for the listing of types of homogeneous components

of G without repetition, and for every $i \in I$,

$$\alpha_i = (k_{i,j})_{j \in \omega}$$

is a characteristic of type $\mathbf{f}_i$. Here I is either ω or finite. Define also

$$P_i = \{p_j : k_{i,j} = \infty\},$$

where $p_0, p_1 \ldots$ is the standard listing of primes.

In the notation above, we do not assume that the homogeneous component corresponding to $\mathbf{f}_i$ is necessarily non-zero; however, we assume that G contains at least one element of type $\mathbf{f}_i$, for each $i \in I$ (the latter does not imply the former).

Notation 3.5. For β a characteristic and A an abelian group, define $A[\beta] = \{g \in G : \beta \leq \chi(g)\}$. For S a subset of the set of all primes, let

$$\widehat{S} = \{p : p \text{ is prime and } p \notin S\}.$$

Also, define $Q(\beta)$ to be the canonical rational subgroup corresponding to characteristic $\beta = \{l_i : i \in \omega\}$:

$$Q(\beta) = \langle \{\frac{1}{p_i^n} : n \leq l_i\} \rangle.$$

In the subgroup $Q(\beta) \leqq Q$, the multiplicative identity of Q has characteristic β.

Before we proceed, we should give some intuition of what we are going to do. Imagine our completely decomposable group is homogeneous. In the notations introduced above, it means I is a singleton $\{0\}$, and $\mathbf{f}_0$ is the type of G. This situation is described in detail in [8]; we briefly overview the results from [8] here. Let $\alpha = \alpha_0$ be the distinguished characteristic of type $\mathbf{f}_0$, and let $P = P_0 = \{p_j : k_{0,j} = \infty\}$ be the primes which correspond to the positions of ∞ in $\alpha = \alpha_0$. In [8] we show that in the homogeneous case we have

$$G[\alpha] \cong \bigoplus_{j < rk(G)} Q^{(P)},$$

where $Q^{(P)}$ stands for (the additive group of) the localization of integers by the primes P. Furthermore, every excellent $\widehat{P}$-independent basis ($\widehat{P}$ stands for the collection of primes not in P) of $G[\alpha]$ generates G over the group $Q(\alpha)$ which is the subgroup of $(Q, +)$ containing 1 with $\chi(1) = \alpha$.

If G is not homogeneous, we need to have a listing of elements which are good enough to (in a certain sense) generate the whole group. The characteristics of these elements may be unequal in this case. Thus, we need to have a labeling which maps every element to its characteristic. Before we proceed, note that $(G[\alpha_i] + G_{\mathbf{f}_i}^\star)/G_{\mathbf{f}_i}^\star$ is homogeneous completely decomposable of the from $\bigoplus_{j < r(i)} Q^{(P_i)}$, where $r(i)$ is the rank of the homogeneous component of G having type $\mathbf{f}_i$ (to be shown in Fact 3.9). Taking into account our comments, the definition below is a way to "stretch" S-independence over various homogeneous components of G and various S. Fact 2.9 explains why we need to consider certain factor-groups and not simply subgroups.

Definition 3.6 (Labeled regular set). Let $(\mathbf{f}_i)_{i \in I}$ be the the listing of types of homogeneous components of G without repetition (Notation 3.4). A *labeled regular set* is a pair (σ, v), where:

    (1) σ is a set of elements of G;

(2) $v : \sigma \to I$ is a function (a labeling) with the property $\mathbf{t}(a) = \mathbf{f}_{v(a)}$, for every $a \in \sigma$ (recall that $\mathbf{t}(a)$ stands for the type of a);

(3) for every i, if $v^{-1}(i) \neq \emptyset$ then $v^{-1}(i)$ is a set of representatives of $\widehat{P}_i$-independent classes in $(G[\alpha_i] + G^\star_{\mathbf{f}_i})/G^\star_{\mathbf{f}_i}$, where α_i is a distinguished characteristic of type $\mathbf{f}_i$ (here P_i is the collection of primes corresponding to ∞ in $\mathbf{f}_i$, and $\widehat{P}_i$ are the primes which are not in P_i).

Recall Notations 2.1 and 3.5. Given a labeled regular set (σ, v), let

$$Span\,(\sigma, v) = \sum_{i \in \omega} [v^{-1}(i)]_{Q(\alpha_i)},$$

where $[\emptyset]_{Q(\alpha_i)} = 0$. We will show that the sum above is in fact direct:

$$Span\,(\sigma, v) = \bigoplus_{i \in \omega} [v^{-1}(i)]_{Q(\alpha_i)},$$

and, furthermore, every homogeneous summand of this direct decomposition splits into elementary components, each elementary component being the span of an element of σ over the corresponding $Q(\alpha_i)$.

Thus, for every labeled regular set (σ, v), the subgroup $Span\,(\sigma, v)$ is a completely decomposable group of rank $card(\sigma)$ with homogeneous components $[v^{-1}(i)]_{Q(\alpha_i)}$, where $v^{-1}(i)$ is an excellent $\widehat{P}_i$-basis of $[v^{-1}(i)]_{Q(\alpha_i)}$. We are ready to describe the procedure which builds an isomorphism.

Building an isomorphism. Without loss of generality, we may assume that a computable completely decomposable $G = \{g_0 = 0, g_1, \ldots\}$ has infinite rank (for otherwise it is computably categorical). The construction below builds a sequence of labeled regular sets which together span the whole group G.

Construction. Define stage-by-stage a sequence of labeled regular sets

$$(\sigma_0, v_0), (\sigma_1, v_1), \ldots$$

starting with $(\emptyset, \emptyset)$. At stage j we search for a labeled regular set (τ, w) such that $\sigma \subset \tau$ and $Span\,(\tau, w)$ contains g_j. We will show in Lemma 3.7 that the procedure produces an infinite sequence $(\sigma_j, v_j)_{j \in \omega}$.

End of construction.

Consider $B = \bigcup_{j \in \omega} \sigma_j$ and $U = \bigcup_{j \in \omega} v_j$. As we have observed, S-independence implies linear independence, therefore B is a basis of G. Furthermore, as we will show in Fact 3.12,

$$G = \bigoplus_{i \in I} [U^{-1}(i)]_{Q(\alpha_i)},$$

where $(\mathbf{f}_i)_{i \in I}$ is the listing of types of homogeneous components, and for every $i \in I$,

$$\alpha_i = (k_{i,j})_{j \in \omega}$$

is a characteristic of type $\mathbf{f}_i$. We will also see in Fact 3.12 that we have more:

$$G = \bigoplus_{b \in B} R_b,$$

where R_b is the span of b over $Q[\alpha_j]$ with $j = U(b)$. Thus, if we are given (B, U), we can uniformly construct a "regular" decomposition of G into elementary components with U pointing the characteristic of a given "regular" element b in this decomposition.

It remains to observe that we may run this process on any other computable copy D of G and obtain a pair (T, V), where T is a basis and V is a function mapping elements of T into their characteristics. Given (B, U) and (T, V), we stage-by-stage map $b \in B$ to a rational multiple of $c \in T$ having the same characteristic as b, and then extend this map to an isomorphism of G onto D in the obvious way.

In the verification below we prove the main algebraic Lemma 3.7 and the facts mentioned above, and we also check which oracle is sufficient to run the procedure on G.

Verification. The main algebraic part of the verification is contained in the lemma below.

Lemma 3.7. *For every labeled regular set (σ, v) and every element $g \in G$ there is a labeled regular set (τ, u) such that $\sigma \subseteq \tau$ and $g \in Span\,(\tau, u)$.*

Proof. We need another notation. Let P be a set of primes which is not necessarily the set of all primes, and let $Q^{(P)}$ be the additive subgroup of the rationals $(Q, +)$ generated by fractions of the form $\frac{1}{p^m}$, where $p \in P$ and $m \in \omega$. Let r be a cardinal number. Define

$$V_{P,r} = \bigoplus_{i < r} Q^{(P)}.$$

Let β be a characteristic. We need:

Fact 3.8. *Let β be a characteristic of type $\mathbf{f}$. In the notations introduced above, $G[\beta] = H[\beta] \oplus C$, where $C \leq G_{\mathbf{f}}^{\star}$ and $H = G_{(\mathbf{f})}$ which is the sum of elementary components of G having type $\mathbf{f}$.*

Proof. By Fact 2.9,

$$G_{\mathbf{f}} = G_{(\mathbf{f})} \oplus G_{\mathbf{f}}^{\star}.$$

By its definition, $G[\beta] \subseteq G_{\mathbf{f}}$. For every $g \in G_{\mathbf{f}}$, $\chi(g) \geq \beta$ implies the projection of g onto $G_{(\mathbf{f})}$ has characteristic $\geq \beta$. Also, every element $H[\beta]$ can be realized as a projection of a $g \in G_{\mathbf{f}}$ with $\chi(g) \geq \beta$. The fact now follows. $\square$

By the definition of a labeled regular set, if $v^{-1}(i) \neq \emptyset$ then $v^{-1}(i)$ is a set of representatives of $\widehat{P}_i$-independent classes in $(G[\alpha_i] + G_{\mathbf{f}_i}^{\star})/G_{\mathbf{f}_i}^{\star}$. We need more information about the local structure of these factor groups.

For β a characteristic, consider the group $(G[\beta] + G_{\mathbf{f}}^{\star})/G_{\mathbf{f}}^{\star}$. By Fact 3.8,

$$(G[\beta] + G_{\mathbf{f}}^{\star})/G_{\mathbf{f}}^{\star} \cong H[\beta],$$

where $H = G_{(\mathbf{f})}$. The group H is homogeneous completely decomposable of type $\mathbf{f}$. An easy modification of the first part of Theorem 4.10 from [8] implies:

Fact 3.9. *For any characteristic β,*

$$H[\beta] \cong V_{P,r},$$

where $P = \{p_i : \beta_i = \infty\}$, and r is the rank of H.

By Fact 3.9,

$$H[\alpha_i] \cong V_{P_i, k},$$

where $H = G_{(\mathbf{f}_i)}$ and k is the rank of H. The definition of S-independence implies:

Fact 3.10. *For a characteristic $\beta \in \mathbf{f}$, let $B \subset (G[\beta] + G_{\mathbf{f}}^\star)/G_{\mathbf{f}}^\star$ be $\widehat{P}$-independent, and let C be any set of representatives of B. Then the projection of C onto $G_{(\mathbf{f})}$ is $\widehat{P}$-independent in $H[\beta]$, where $H = G_{(\mathbf{f})}$.*

By Fact 3.10, the projection of $v^{-1}(i)$ onto $H = G_{(\mathbf{f}_i)}$ is $\widehat{P}_i$-independent within $H[\alpha_i]$. The second part of Theorem 4.10 from [8] gives:

Fact 3.11. *Let P and $\beta \in \mathbf{f}$ be as in the previous fact. If a set B is an excellent $\widehat{P}$-basis of $(G[\beta] + G_{\mathbf{f}}^\star)/G_{\mathbf{f}}^\star$, then $G_{\mathbf{f}}/G_{\mathbf{f}}^\star$ is generated by B over $Q(\beta)$, where $Q(\beta)$ is the subgroup of $(Q, +)$ containing 1 in which $\chi(1) = \beta$.*

In Fact 3.11 "generated" means "generated mod $G_{\mathbf{f}}^\star$". Since S-independence implies linear independence, we have:

Fact 3.12. *In the notations introduced above, $Span\,(\sigma, v) \cong \bigoplus_{i \in \omega}[v^{-1}(i)]_{Q(\alpha_i)}$.*

Note that $Span\,(\sigma, v)$ is contained in $A \leqq G$ which is a direct sum of finitely many elementary components of G. Thus, the projection of $v^{-1}(i)$ onto $H = G_{(\mathbf{f}_i)}$ is contained in $A_{(\mathbf{f}_i)}[\alpha_i]$ which is isomorphic to $V_{P_i,k}$ with $k \in \omega$. By Proposition 4.5 of [8], the projection of $v^{-1}(i)$ can be extended to an excellent $\widehat{P}_i$-basis of $A_{(\mathbf{f}_i)}[\alpha_i]$. Note that, considering the pre-image of this extension under the projection onto H, we may choose representatives C_i of an excellent $\widehat{P}_i$-basis of $(A[\alpha_i] + A_{\mathbf{f}_i}^\star)/A_{\mathbf{f}_i}^\star$ so that these representatives are contained in A.

Let τ be the union of the C_i, where i ranges over the set $J = \{i : A_{(\mathbf{f}_i)} \neq 0\}$, and let u be a function which maps every element of τ into its characteristic. We prove by induction that

$$Span\,(\tau, u) = A.$$

The group A is of finite rank, and the partial ordering $\{\mathbf{f}_i : i \in J\}$ of the types of its elementary components is finite. We argue by induction on the number of types in this partial ordering, as follows. By Fact 3.11, for every $i \in J$ the factor-group $A_{\mathbf{f}_i}/A_{\mathbf{f}_i}^\star$ is generated over $Q(\alpha_i)$ by the classes corresponding to C_i. Let $j \in J$ be such that $\mathbf{f}_j$ is maximal in $\{\mathbf{f}_i : i \in J\}$. By Fact 2.9,

$$A_{\mathbf{f}_j}^\star = 0.$$

Consequently, C_j generates $A_{\mathbf{j}}$ over $Q(\alpha_i)$.

Let $\rho = \tau - C_j = \bigcup_{i \in J - \{j\}} C_i$, and let w be the restriction of u onto ρ. By the induction hypothesis,

$$A/A_{(\mathbf{f}_j)} = Span\,(\rho, w)/A_{(\mathbf{f}_j)}.$$

Therefore, every element of $\bigoplus_{i \neq j} A_{(\mathbf{f}_i)}$ is generated by elements of ρ and elements of $A_{(\mathbf{f}_j)}$. This shows $Span\,(\tau, u) = A$ and, by the choice of A, we have $g \in Span\,(\tau, u)$

$\square$

The recursion-theoretic content of the proof is contained in:

Lemma 3.13. *The sequence of labeled regular sets from the main construction is Σ_5^0.*

Proof. Recall that $G = (g_0, g_1, \ldots)$ is a computable completely decomposable group, and let $G = \bigoplus_{i \in I} G_{(\mathbf{f}_i)}$ be its decomposition into homogeneous completely decomposable components. For every j, let $\beta_j = \chi(g_j)$. We need an enumeration of characteristics which correspond to different types.

Fact 3.14. *There exists a Σ_4^0 set $\mathcal{J} \subseteq \omega$ such that*
- (a.) *for every $i \in I$ there exists $j \in \mathcal{J}$ such that $\beta_j \in \boldsymbol{f}_i$;*
- (b.) *$\beta_i \not\sim \beta_j$, for every $i, j \in \mathcal{J}$.*

Proof. It is sufficient to show that the relation $\{(i,j) : \beta_i \sim \beta_j\}$ is Σ_3^0. Note that there is a 1-1 correspondence between β_i and the set of pairs

$$X_i = \{(j,k) : p_k^j | g_i\}.$$

The family of sets $(X_i)_{i \in \omega}$ has a uniform enumeration. Also, every X_i is associated to the corresponding element g_i in an effectively uniform way. It remains to observe that $\beta_i \sim \beta_j$ if, and only if, $X_i =^\star X_j$, for every i, j.

$\square$

Note that $(\mathbf{f}_j)_{j \in \mathcal{J}}$ are not necessarily exactly the types which correspond to non-zero $G_{(\mathbf{f}_i)}$ in the decomposition of G. Using $\mathcal{J}$ we establish a $0'''$-computable uniform enumeration of pairwise non-equivalent c.e. characteristics $(\alpha_i)_{i \in I}$ only *covering* all types of non-zero components present in the complete decomposition of G.

Fact 3.15. *For every j:*
- (1.) *$G[\alpha_j]$ is Π_2^0 uniformly in j;*
- (2.) *$G_{\boldsymbol{f}_j}^\star$ is Σ_4^0 uniformly in j.*

Proof. We have

$$g \in G[\alpha_j] \Leftrightarrow \chi(g) \geq \alpha_j \Leftrightarrow (\forall k)(\forall n)(p_k^n | g_j \to (n,k) \in \alpha_j)$$

which is Π_2^0 and

$$g \in G_{\mathbf{f}_j}^\star \leftrightarrow \bigvee_{k \in \omega} (\exists g_1, \ldots, g_k \in G)(\exists n \in \mathbb{N})\left(\chi(g_i) \not\sim \alpha_j \wedge \chi(g_i) \geq \alpha_j \wedge ng = \sum_{1 \leq i \leq k} g_i\right)$$

which is Σ_4^0, because $\chi(g_i) \not\sim \alpha_j$ is Π_3^0 as we have observed in the proof of Fact 3.14, and $\chi(g_i) \geq \alpha_j$ is Π_2^0. $\square$

As a consequence of this fact, $(G[\alpha] + G_{\mathbf{f}_j}^\star)/G_{\mathbf{f}_j}^\star$ has a Σ_5^0 set of representatives. We need more:

Fact 3.16. *Given i and elements $g_1, \ldots, g_k \in G[\alpha_j]$, the statement "the classes of $g_1, \ldots, g_k$ are $\widehat{P}_i$-independent in $(G[\alpha_j] + G_{\boldsymbol{f}_j}^\star)/G_{\boldsymbol{f}_j}^\star$" is Π_4^0 uniformly in the indices of elements and in α_j.*

Proof. It is sufficient to require that, for every coefficients $m_1, \ldots, m_k$ and for every prime p,

$$[(\exists y)(\exists x)(x \in G[\alpha] \wedge y \in G_{\mathbf{f}_j}^\star \wedge p \notin P_j \wedge px + y = \sum_{s \leq k} m_s g_s)] \Rightarrow \bigwedge_{s \leq k} p|m_s.$$

which is Π_4^0, by the preceding facts. $\square$

Thus, $0^{(4)}$ can build a sequence of labeled regular sets generating the whole group G.

$\square$

The verification is finished, and the theorem is proved. $\square$

Our machinery enables us to prove:

Theorem 3.17. *The index set of computable completely decomposable groups is arithmetical. More specifically, the index set is Σ_7^0.*

Proof idea. Using the machinery from the proof of Theorem 3.1 and effective theory of types [8, 35], we show that there is a $0^{(4)}$ listing of $0^{(4)}$ computable completely decomposable groups such that every computable one is isomorphic to a group from the listing. We could relativize Theorem 3.1 and conclude that the index set is at most Σ_{11}^0. A more careful analysis allows us to drop the upper bound down to Σ_7^0. The idea is that a certain partial relativization is sufficient.

Proof. A characteristic $\alpha = (n_i)_{i \in \omega}$ can be viewed as a set of pairs $\{(k, i) : k \leq n_i\}$ we call the corresponding *characteristic sequence* [35]. The reader should keep in mind that all elements of a computable torsion-free abelian group have c.e. characteristic sequence. The following fact is easy but helpful.

Fact 3.18. *There exists a uniform enumeration of all c.e. characteristic sequences.*

Proof. Given an enumeration of a c.e. set, effectively and uniformly transform it into an enumeration of a characteristic sequence closing every column $\{s : (s, i) \in W_e\}$ downwards. □

Identifying characteristics and corresponding characteristic sequences, let $(\beta_j)_{j \in \omega}$ be the uniform enumeration from Fact 3.18. The isomorphism type of a completely decomposable group $G = \bigoplus_{\mathbf{f}} G_{(\mathbf{f})}$ is uniquely determined by the set

$$\{(\mathbf{f}, k) : rank\,(G_{(\mathbf{f})}) = k\}.$$

We may replace every type in the set above by a characteristic of that type, and still get a full invariant describing G up to an isomorphism, but modulo the characteristic equivalence. The proof of Theorem 3.1 actually illustrates:

Fact 3.19. *For every computable completely decomposable group G, there is a $0^{(4)}$ enumeration of a set of the form*

$$\{(j, s) : rank\,(G_{(\mathbf{t}(\beta_j))}) \geq s > 0\},$$

where $\beta_i \not\sim \beta_j$ and $G = \bigoplus_j G_{(\mathbf{t}(\beta_j))}$.

On the other hand, every uniformly computable family of characteristic sequences can be realized as one coresponding to a direct decomposition of a computable completely decomposable group:

Fact 3.20. *For every c.e. set $S = \{(j, s) : s \geq 1\}$ such that $\{\beta_j : (j, 1) \in S\}$ is a set of pairwise non-equivalent characteristics in the uniform enumeration of all characteristics $(\beta_j)_{j \in \omega}$, there exists a computable completely decomposable group of the form $\bigoplus_{j:(j,1) \in S} \left(\bigoplus_{k:(j,k) \in S} Q(\beta_j) \right)$.*

Proof. The proof is not difficult and can be left to the reader. See [35] for similar constructions □

Note that in the fact above, we can effectively and uniformly determine the membership of an element to an elementary summand of the resulting group. Groups with this property are called strongly decomposable [28] or effectively completely decomposable [24]. A partial relativization of Fact 3.20 shows that, for any given Σ_n^0 listing of *c.e.* characteristics (with repetitions), we can uniformly construct the

corresponding $0^{(n-1)}$ strongly decomposable group; furthermore, the characteristic sequence of every elementary component of this group is c.e.

Summarizing the above, every computable completely decomposable group has a corresponding Σ^0_5 family of *c.e.* characteristic sequences, and every such sequence can be associated to a $0^{(4)}$-strongly decomposable group in a uniform way. Fix a uniform $0^{(4)}$ enumeration of all $0^{(4)}$-c.e. sets of the form $S = \{(j,s) : s \geq 1\}$ (see Fact 3.20). We obtain:

Fact 3.21. *There is a Σ^0_5 listing $(A_i)_{i\in\omega}$ of $0^{(4)}$-strongly decomposable groups containing all isomorphism types of computable completely decomposable groups (possibly with repetitions).*

Note that every group A_i from the enumeration provided by Fact 3.21 and Fact 3.20 have a $0^{(4)}$-computable complete decomposition algorithm: they are provided by a basis, each element of the basis belongs to a separate component. It is crucial that characteristics of elements of the complete decomposition basis of A_i are c.e. (whereas the indices of these characteristics are merely c.e. in $0^{(4)}$). By the third part of the proof of Theorem 3.1, if a computable completely decomposable group G is isomorphic to A_i, then this isomorphism is in fact Δ^0_5: it suffices to build a sequence of labeled regular sets in G, take their union, and then map each element from the union to an element of the base of A_i having an equal (c.e.) characteristic ($0^{(4)}$ certainly can determine if two c.e. characteristics are equivalent or equal). Notice that we are implicitly using the uniqueness of the complete decomposition of G.

Thus, a computable structure is a completely decomposable group if, and only if, it is isomorphic to one of the groups $(A_i)_{i\in\omega}$ from Fact 3.21 via a $0^{(4)}$-computable isomorphism. Given a computable structure $\mathcal{M}_j$, we ask "is there i and a $0^{(4)}$-isomorphism from A_i onto $\mathcal{M}_j$?" which is uniformly Σ^0_7. The theorem now follows. $\square$

4. Proofs of Theorem 4.1 and Theorem 4.3

Theorem 4.1. *There is a computable completely decomposable group which is Δ^0_4-categorical but not Δ^0_3-categorical.*

Proof idea. We have to build two computable isomorphic c.d. groups with no $0''$ isomorphism between them, and also make sure the isomorphism type is Δ^0_4-categorical.

Recall the proof of Theorem 3.1. The only reason the isomorphism could be not simpler than Δ^0_4 is that, in general, $0''$ can not determine if an element a belongs to an elementary component (a is a "true" element of a given type), or is a linear combination of elements of greater types (a is a "fake" element). We write $\Theta(a)$ if $a = x + y$ for some x, y with the property $\mathbf{t}(x) > \mathbf{t}(a)$ and $\mathbf{t}(y) > \mathbf{t}(a)$.

We build a computable completely decomposable group A in which Θ is Σ^0_3-hard. Furthermore, we show that A has a computable copy B with Θ decidable, which implies A is not Δ^0_3-categorical.

The groups A and B are computable copies of $\bigoplus_{i\in\omega} Z \oplus \bigoplus_{i\in\omega} Q^{(p)} \oplus \bigoplus_{i\in\omega} Q^{(q)}$, where $p \neq q$ are primes, and $Q^{(u)}$ denotes the localization of Z by u (to be precise, the additive group of the localization). The group B is a "nice" copy in which the complete decomposition is computable, and the listing of types of the summands is effective.

The group A encodes a Σ_3^0 set into its specifically chosen elements. For every e and x, we express an element $a_e \in A$ with $\chi(a_e) = (0, 0, \ldots, 0, \ldots)$ as:

$$a_e = b_{x,e} + d_{x,e},$$

making $p^\infty | b_{x,e}$ and $q^\infty | d_{x,e}$ if, and only if, x is least for which the predicate fires infinitely often on input e. Without loss of generality, we may assume that every e has at most one x witnessing $e \in S$ (see Lemma 2.10).

Notice that, for every a_e there exists infinitely many x such that $a_e = b_{x,e} + d_{x,e}$. Interestingly, the resulting group can be made completely decomposable and isomorphic to $\bigoplus_{i \in \omega} Z \oplus \bigoplus_{i \in \omega} Q^{(p)} \oplus \bigoplus_{i \in \omega} Q^{(q)}$. It remains to observe the group is Δ_4^0-categorical.

Proof. We are going to construct a computable copy A of the group $G = \bigoplus_{i \in \omega} Z \oplus \bigoplus_{i \in \omega} Q^{(p)} \oplus \bigoplus_{i \in \omega} Q^{(q)}$. The group A will encode a Σ_3^0-complete set S into a set of elements $\{a_e : e \in \omega\}$ so that

(1) $$e \in S \leftrightarrow \Theta(a_e).$$

We also fix a computable copy B of G having Θ computable. The group $\bigoplus_{i \in \omega} Z \oplus \bigoplus_{i \in \omega} Q^{(p)} \oplus \bigoplus_{i \in \omega} Q^{(q)}$ is Δ_4^0-categorical; it follows from the proof of Theorem 3.1 and the fact that $G_{\mathbf{0}}^\star$ is Σ_3^0, where $\mathbf{0}$ is the type of Z.

The requirements. We need to satisfy, for every e, the requirements

$$\mathbf{R}_e : A \models \Theta(a_e) \leftrightarrow e \in S,$$

and the global requirement

$$\mathbf{I} : A \text{ is isomorphic to } G.$$

Every R-requirement will have its witnesses:

The witnesses for R_e. We fix a computable predicate U such that:

(2) $$e \in S \Leftrightarrow (\exists x)(\exists^\infty y) U(x, y, e).$$

Recall that a_e is a witness corresponding to e. We also pick computable linearly independent sets of elements $\{b_{x,e} : x \in \omega\}$ and $\{d_{x,e} : x \in \omega\}$. We immediately declare

(3) $$a_e = b_{x,e} + d_{x,e},$$

for every x. We also keep $\chi(a_e) = (0, 0, \ldots, 0, \ldots)$ at all stages. From now on, we omit the subscript e in $b_{x,e}$ and $d_{x,e}$ if it is clear from the context which R_e these elements correspond to.

The strategy for R_e. As soon as a new y appears such that $U(x, y, e)$ holds, make b_x and d_x divisible by one extra power of p and q, respectively, by introducing two new elements to the group.
End of strategy.

Note that, if $e \in S$ then $\mathbf{t}(b_x) > \mathbf{t}(a_e)$ and $\mathbf{t}(d_x) > \mathbf{t}(a_e)$ for x least so that $(\exists^\infty y) U(x, y, e)$ holds. The characteristics of b_x and d_x will be equivalent to a_e, otherwise. Thus, we will have:

$$A \models \Theta(a_e) \Leftrightarrow e \in S,$$

and R_e will be met.

Construction. Let all R_e strategies act according to their instructions, starting with the computable abelian group

$$\langle a_e, b_{x,e}, d_{x,e} \; : \; e, x \in \omega \,|\, a_e = b_{x,e} + d_{x,e} \rangle \oplus \bigoplus_{i \in \omega} Z \oplus \bigoplus_{i \in \omega} Q^{(p)} \oplus \bigoplus_{i \in \omega} Q^{(q)}$$

sitting within its computable divisible hull. Whenever an element is declared divisible by a new power of a prime, enumerate a suitable new element from the hull into the group, as well as all elements witnessing consequences of this new divisibility condition.
End of construction.

It is clear that $\mathbf{R}_e$ is met, for every e. We prove:

Claim 4.2. *The group A built by the construction is isomorphic to G. (Thus, $\mathbf{I}$ is met.)*

Proof. It is sufficient to show that, for every e, the pure subgroup A_e of A generated by $\{a_e, b_{x,e}, d_{x,e} \; : \; x \in \omega\}$ is completely decomposable and its elementary summands are among $Z, Q^{(p)}$ and $Q^{(q)}$. The proof splits into two cases, depending on the outcome of the strategy.

The first case corresponds to $e \in S$. We may assume, up to a change of notations, that the least x witnessing $\exists^\infty y$ is 0. We omit the subscript e in the definition of A_e:

$$\langle a, \frac{b_0}{p^\infty}, \frac{d_0}{q^\infty}, \frac{b_1}{p^{n_1}}, \frac{d_1}{q^{n_1}}, \frac{b_2}{p^{n_2}}, \frac{d_2}{q^{n_2}}, \ldots \,|\, a = b_x + d_x \rangle,$$

where $\frac{g}{n^\infty}$ denotes the set $\{\frac{g}{n^c} : c \in \omega\}$, and $n_x \in \omega$, for every x. We claim that

$$A_e \cong D = Q^{(p)}t_0 \oplus Q^{(q)}t_1 \oplus \bigoplus_{x \in \mathbb{N}} Zu_x,$$

where $\{t_0, t_1, u_i : i \in \mathbb{N}\}$ is the complete decomposition base of D.

Let v_x, w_x be integers such that $p^{n_x}v_x + q^{n_x}w_x = 1$. Using these integers, we define an embedding ψ from A_e to D. We set:

(0) $\psi(b_0) = t_0$ and $\psi(d_0) = t_1$;
(1) $\psi(b_1) = p^{n_1}q^{n_1}u_1 + p^{n_1}v_1(t_0 + t_1)$ and $\psi(d_1) = -p^{n_1}q^{n_1}u_1 + q^{n_1}w_1(t_0 + t_1)$;
$\cdots$
(x) $\psi(b_x) = p^{n_x}q^{n_x}u_x + p^{n_x}v_x(t_0 + t_1)$ and $\psi(d_x) = -p^{n_x}q^{n_x}u_x + q^{n_x}w_x(t_0 + t_1)$;
$\cdots$.

By the choice of v_x, w_x, the divisibility conditions on the generators in the definition of A_e are preserved under the map ψ. Furthermore, $\psi(b_x) + \psi(d_x) = t_0 + t_1 = \psi(b_0) + \psi(b_1)$, for every x. Therefore, the map ψ can be extended to a homomorphism of the whole A_e to D. We denote this extension by ψ as well.

Using the first and the second row of the definition of ψ and the divisibility conditions, one can easily show that $q^{n_1}u_1$ and $p^{n_1}u_1$ belong to the image of A_e under ψ. Consequently, $u_1 = (p^{n_1}v_1 + q^{n_1}w_1)u_1 \in \psi(A_e)$.

We show that ψ is an injection. Let

$$\eta : \langle a, \frac{b_0}{p^\infty}, \frac{d_0}{q^\infty}, \frac{b_1}{p^{n_1}}, \frac{d_1}{q^{n_1}}, \frac{b_2}{p^{n_2}}, \frac{d_2}{q^{n_2}}, \ldots \rangle \to A_e$$

be the canonical epimorphism. We view every element of A_e as a word from $H = \langle a, \frac{b_0}{p^\infty}, \frac{d_0}{q^\infty}, \frac{b_1}{p^{n_1}}, \frac{d_1}{q^{n_1}}, \frac{b_2}{p^{n_2}}, \frac{d_2}{q^{n_2}}, \ldots \rangle$, modulo $Ker\,\eta$. It is sufficient to show that the image of any such word is zero if, and only if, it belongs to $Ker\,\eta$.

Every element y of H has the form $r_0 b_0 + r_1 d_0 + \sum_x f_x b_x + \sum_x s_x d_x$, where all the sums are finite, and all coefficients are rational. Assume $\phi\eta y = 0$. We can safely assume that all coefficients in $r_0 b_0 + r_1 d_0 + \sum_x f_x b_x + \sum_x s_x d_x$ are integers. The kernel of η contains integer multiples of $a - b_x - d_x$, for every x. Therefore, we can rewrite the sum into an equivalent modulo $Ker\,\eta$ one:

$$y \sim y' = r_0' b_0 + r_1' d_0 + \sum_x f_x' b_x.$$

We have $0 = \psi\eta(y') = r_0' t_0 + r_1' t_1 + \sum_x f_x'(p^{n_x} q^{n_x} u_x + p^{n_x} v_x(t_0 + t_1))$. The set $\{t_0, t_1, u_x : x \in \mathbb{N}\}$ is a basis of G. Consequently, the set $\{t_0, t_1, (p^{n_x} q^{n_x} u_x + p^{n_x} v_x(t_0 + t_1)) : x \in \mathbb{N}\}$ is a basis of G. Thus, $r_0' = r_1' = f_x' = 0$, for every x, as desired.

The second case corresponds to $e \notin S$. We omit the subscript e in the definition of A_e:

$$A_e = \langle a, \frac{b_x}{p^{n_x}}, \frac{d_x}{q^{n_x}} : x \in \omega \,|\, a = b_x + d_x \rangle,$$

where $n_x \in \omega$, for every x. In this case

$$A_e \cong Z t_0 \oplus Z t_1 \oplus \bigoplus_{x \in \mathbb{N}} Z u_x,$$

the isomorphism is induced by

(0) $\psi(b_0) = p^{n_0} t_0$ and $\psi(d_0) = q^{n_0} t_1$;

    . . .

(x) $\psi(b_x) = p^{n_x} q^{n_x} u_x + p^{n_x} v_x(p^{n_0} t_0 + q^{n_0} t_1)$, and
     $\psi(d_x) = -p^{n_x} q^{n_x} u_x + q^{n_x} w_x(p^{n_0} t_0 + q^{n_0} t_1)$;

    . . . .

The proof is similar to the previous case and can be left to the reader.

We conclude that $A = \bigoplus_{e \in \omega} A_e \oplus B \cong B$. $\square$

If there were a $0''$ isomorphism, we would be able to decide $\Theta(a_e)$ uniformly, using $0''$. A contradiction. $\square$

Theorem 4.3. *There exists a computable completely decomposable group which is not Δ_4^0-categorical.*

Proof idea. We have to build two computable isomorphic c.d. groups with no $0'''$ isomorphism between them. Note that the property Θ from the previous theorem is, in general, Σ_4^0:

(4) $\Theta(a) \leftrightharpoons (\exists x, y \in G)[\mathbf{t}(x) > \mathbf{t}(a) \text{ and } \mathbf{t}(y) > \mathbf{t}(a) \text{ and } a = x + y].$

We show that the Σ_4^0 upper bound is sharp. We construct two copies of a c.d. group, one copy having Θ decidable in $0'''$, and another copy with Θ being Σ_4^0-complete.

The first main idea is to use infinitely many primes to encode a Σ_3^0 or a Π_3^0 outcome. The second main idea is in using finite divisibility, not infinite divisibility, by each of these primes. Then, only the presence of infinitely different divisibilities will effect the characteristic of an element. Also, this strategy allows us to use an

analog of Claim 4.2 for relatively prime numbers which are not necessarily primes themselves.

Proof. We are constructing two isomorphic computable completely decomposable groups, A and B. The group A will encode a Σ_4^0-complete set S into a set of elements $\{a_e : e \in \omega\}$ so that

(5) $$e \in S \leftrightarrow \Theta(a_e).$$

The group B will have Θ decidable in $0'''$. The group B will be of the form $\bigoplus_{i \in \omega} R_i c_i$ with $\{c_i : i \in \omega\}$ a decidable set and $R_i \leq Q$, for each i. The group A will be of the form $A = D \oplus V$ where the computable subgroup V is isomorphic to B, and the computable subgroup D will contain $\{a_e : e \in \omega\}$ and will be merely isomorphic to a summand of B (of infinite rank).

The requirements. We need to satisfy, for every e, the requirements

$$\mathbf{R}_e : A \models \Theta(a_e) \leftrightarrow e \in S,$$

and the global requirements

$$\mathbf{I} : A \text{ is isomorphic to } B.$$

Every R-requirement will have its witnesses:

The witnesses for R_e. We fix a computable predicate U such that:

(6) $$e \in S \Leftrightarrow (\exists x)(\exists^\infty y)(\forall z)U(x,y,z,e).$$

Without loss of generality, we may assume that the exists at most one x witnessing $e \in S$, for each e (see Lemma 2.10). The element a_e will be a witness corresponding to e. We also pick computable linearly independent sets of elements $\{b_{x,e} : x \in \omega\}$ and $\{d_{x,e} : x \in \omega\}$. We immediately declare

(7) $$a_e = b_{x,e} + d_{x,e},$$

for every x. From now on, we omit the subscript e in $b_{x,e}$ and $d_{x,e}$ if it is clear from the context which R_e these elements correspond to.

The strategy for R_e. We are going to make sure that, if $e \in S$, then $\mathbf{t}(b_x) > \mathbf{t}(a_e)$ and $\mathbf{t}(d_x) > \mathbf{t}(a_e)$ for x least so that $(\exists^\infty y)(\forall z)U(x,y,z,e)$ holds, and we make the characteristics of b_x and d_x equivalent to a_e, otherwise. If we succeed, we will have:

$$A \models \Theta(a_e) \Leftrightarrow e \in S,$$

and R_e will be met.

Let $i : \omega^2 \times \{0,1\} \to \omega$ be an injective computable function. Every $y \in \omega$ corresponding to x is associated to a pair of positions, $i(x,y,0)$ and $i(x,y,1)$, in the characteristics of a_e, b_x and d_x.

We start by making b_x divisible by $p_{i(x,y,0)}$, and d_x divisible by $p_{i(x,y,1)}$. (Note that "divisible" does not mean "infinitely divisible" here.) We keep a_e non-divisible by $p_{i(x,y,0)}$ and $p_{i(x,y,1)}$ until (if ever) we see $z \leq s$ such that $U(x,y,z,e)$ does not hold. As soon as we find such a z, we make b_x and d_x both divisible by $p_{i(x,y,0)}$ and $p_{i(x,y,1)}$. Note that a_e is immediately declared divisible by these primes as well. For each $x_0 \neq x$, we make b_x and d_x both divisible by $p_{i(x_0,y,0)}$ and $p_{i(x_0,y,1)}$ whenever

a_e becomes divisible by these primes, for every i. We keep b_x and d_x not divisible by these primes, otherwise.

End of strategy.

The main local strategy above makes the types of a_e and b_x different at infinitely many positions if $e \in S$, and equivalent (not necessarily equal) otherwise. The same could be said about d_x. Notice that different R requirements have no interactions. We will verify that **C** and **I** are met shortly after the construction:

Construction. Let all R_e strategies act according to their instructions, starting with the abelian group

$$\langle a_e, b_{x,e}, d_{x,e} \, : \, e, x \in \omega \,|\, a_e = b_{x,e} + d_{x,e} \rangle$$

sitting within its computable divisible hull. Whenever an element is declared divisible by a new power of a prime, enumerate a suitable new element from the hull into the group, as well as all consequences of this new divisibility condition.

End of construction.

We have not defined the group A yet. Let D be the computable group built by the construction. Let L_e denote the subgroup of Q having its characteristic equal to $\chi(a_e)$, and also let $K_{e,x}$ and $U_{e,x}$ be rational subgroups having characteristics equal to $\chi(b_{x,e})$ and $\chi(b_{x,e})$ respectively, for every x and e. The construction is effective, therefore the mentioned characteristics are uniformly computable. Consequently, the direct sum of all corresponding rational subgroups is computable:

$$V_0 = \bigoplus_{e \in \omega} \left[L_e \oplus \bigoplus_{x \in \omega} (K_{e,x} \oplus U_{e,x}) \right],$$

and so is its countably infinite direct power $V = V_0^\infty$. Finally, define

$$A = D \oplus V,$$

and also set

$$B = V.$$

The claim below is straightforward:

Claim 4.4. *The requirement $\mathbf{R}_e$ is met, for every e.*

Proof. By the construction and the strategy for $\mathbf{R}_e$, there is a (necessarily unique) x such that $\mathbf{t}(b_{x,e}) > \mathbf{t}(a_e)$ and $\mathbf{t}(d_{x,e}) > \mathbf{t}(a_e)$ if, and only if, $e \in S$.   □

Claim 4.5. *The requirement $\mathbf{I}$ is met.*

Proof. The proof is almost literally the same as the proof of Calim 4.2; the only difference is that we have to deal with relatively prime numbers which are not necessarily primes themselves.

The Σ_3^0 outcome corresponds to the situation in which, for all x, both $b_{x,e}$ and $d_{x,e}$ will be declared divisible by finitely many primes $\{p_{i(x,y,0)} \, : \, y \in Y_0\}$ and $\{p_{i(x,y,1)} : y \in Y_1\}$, respectively. The relatively prime finite products $\prod_{y \in Y_0} p_{i(x,y,0)}$ and $\prod_{y \in Y_1} p_{i(x,y,1)}$ can be used in place of p and q from the proof of Claim 4.2 to show that the element a_e belongs to a homogeneous completely decomposable summand of A having type $\mathbf{t}(a_e)$.

The Π_3^0 outcome corresponds to the situation when there exists a unique x_0 for which $b_{x_0,e}$ and $d_{x_0,e}$ will be declared divisible by infinitely many primes which

do not divide a_e. In this case we use the idea from the proof of Claim 4.2 (with $b_{x_0,e}$ and $d_{x_0,e}$ playing the role of b_0 and d_0), together with the modified approach described above. In this case a_e is in the direct summand of A having two elementary components of types $\mathbf{t}(b_{x_0,e})$ and $\mathbf{t}(d_{x_0,e})$, and infinitely many components of types $\mathbf{t}(a_e)$. The uniqueness of element x_0 witnessing the Π_3^0 outcome is crucial for this proof.

In both cases, the elementary components have types which appear in the elementary decomposition of $B \cong V$. $\qquad\square$

It remains to prove:

Claim 4.6. *The property Θ in B is decidable relative to $0'''$.*

Proof. We prove that the existence of computable elementary decomposition of B allows us to replace the infinite disjunction "there exists a pair of elements" by a finite disjunction.

Given an element $c \in B = \bigoplus_i R_i c_i$, we can effectively find its *finite* decomposition

$$c = \sum_{i \in J_0} m_i c_i,$$

where $m_i \in \omega$, for each i. Then $\Theta(c)$ holds if, and only if, there exists a partition of the finite set J_0 into two non-empty sets I_0 and I_1 such that $x_0 = \sum_{i \in I_0} m_i c_i$ and $x_1 = \sum_{i \in I_1} m_i c_i$ have the property $\mathbf{t}(x_i) > \mathbf{t}(c)$, for $i = 0, 1$. The latter is a Π_3^0 condition, because it is equivalent to the existence of infinitely many integers which divide x_i and do not divide c. The claim follows. $\qquad\square$

We have shown that the groups A and B have the desired properties. The theorem is proved. $\qquad\square$

5. CONCLUSION

We leave open:

Problem 5.1. Describe Δ_n^0 and categorical computable completely decomposable groups for $1 < n \leq 5$.

We also don't know if Theorem 3.17 is sharp. There are several questions which may result a better understanding of Problem 5.1, we state two of them. Observe the following:

Fact 5.2. *Let $G = A \oplus B$, where A and B are completely decomposable homogeneous groups of rank ω, and $\mathbf{t}(A) < \mathbf{t}(B)$. Then G is not Δ_2^0-categorical.*

Proof. Encode a Σ_2^0-complete set into a presentation of G, and also consider a "nice" presentation of G. $\qquad\square$

The fact above suggests that, in many cases, Δ_n^0-categoricity is determined by the partial order of types of elementary summands, not by the types themselves. Virtually nothing is known about the complexity of partial orders of types in computable completely decomposable groups. Any progress towards the problem below could help in understanding effective categoricity of completely decomposable groups:

Problem 5.3. Describe partial orders which can be realized as orders of types of elementary summands in computable completely decomposable groups.

The partial orders of groups from Theorem 4.1 and 4.3 are computable, but there are groups encoding undecidable ones.

The proof of Theorem 4.1 suggests that the level of categoricity of a group is related to the ability to compute a complete decomposition of the group, and the proof of Theorem 3.17 uses effective decompositions to reduce the complexity from Σ_{11}^0 to Σ_7^0. The problem of the existence of a computable complete decomposition has been studied in literature. For instance, Khisamiev and Krykpaeva [28] introduced the notion of a *strongly decomposable group*: this is a completely decomposable group which has a computable copy with an effective listing of elementary summands. For further information on strongly decomposable groups see Khisamiev [27], and see [24] for an application of strongly decomposable groups in the study of degrees of orderings on abelian groups.

We say that a completely decomposable group G is *totally Δ_n^0-decomposable* if every computable copy of a completely decomposable group G is effectively decomposable relative to $0^{(n-1)}$. The case of $n = 1$ is equivalent to the rank of a group being finite: if a computable group is effectively decomposable it has a computable basis, and a group of infinite rank always has a copy with no linear dependence algorithm [39]. Thus, computable categoricity and total Δ_1^0 decomposability coincide. Also, the proof of Theorem 3.1 essentially shows that every computable completely decomposable group is totally Δ_5^0-decomposable. Even if Δ_n^0 categoricity is not equivalent to total Δ_n^0-decomposability in general, the problem below is of an independent interest:

Problem 5.4. Describe totally Δ_n^0-decomposable groups, for $n < 5$.

The cases where $n > 1$ seem to be less straightforward. We conclude by several examples of totally Δ_2^0-decomposable groups. The free abelian group of infinite rank is totally Δ_2^0-decomposable (follows from [8]); the direct sum of infinitely many copies of $Q^{(p)}$ and the free abelian group is totally Δ_3^0-decomposable (follows from the proof of Theorem 3.1), and it can not be improved to Δ_2^0; Theorems 4.3 and 4.1 provide sharp examples for $n = 4$ and 5.

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