COMPUTABLE DISTRIBUTIVE LATTICES

NIKOLAY BAZHENOV, ANDREY FROLOV, ISKANDER KALIMULLIN, AND ALEXANDER MELNIKOV

Abstract. It is known that the class of (not necessarily distributive) countable lattices is HKSS-universal, and it is also known that the class of countable linear orders is not universal with respect to degree spectra and computable categoricity. We investigate the intermediate class of *distributive* lattices.

We build the distributive lattice with the degree spectrum $\{\mathbf{d} : \mathbf{d} \neq \mathbf{0}\}$. It is not known whether a linear order with this property exists. We also show that there is a computably categorical distributive lattice which is not relatively Δ^0_2-categorical. It is well-known that no linear order can have this property. We leave open whether countable distributive lattices are HKSS-universal.

1. Introduction

Our investigations contribute to the general program that aims to understand computability-theoretic properties of countably infinite algebraic structures from common classes (such as groups, fields, linear orders etc). In this paper we study computability-theoretic properties of the class of distributive lattices. One way of comparing different classes is to study certain *effective invariants* that can be realized by a structure in the class. For example, the *degree spectrum* of a structure is the collection of all Turing degrees that can compute a copy of the structure. It is well-known that any low_4 Boolean algebra is isomorphic to a recursive one [1]. In contrast, there exists an abelian group that has *non-low*$_3$-degrees serving as its degree spectrum [2]. We can definitely conclude that the classes of countable Boolean algebras and countable abelian groups are substantially different from the computability-theoretic point of view. But which class is computationally "harder"? There are several ways to compare computability-theoretic complexity of two classes of structures, see e.g. [3, 4, 5].

Suppose we want to compare classes using some specific computability-theoretic invariant (or property) P such as degree spectra. Hirschfeldt, Khoussainov, Shore, and Slinko [3] introduced the notion of a class of structures which is *complete with respect to property* P. We follow [6] and call such classes *HKSS-complete*, and the property will usually be clear from the context. For the formal definition of a HKSS-complete class, the reader is referred to [3, Definition 1.21] (see also [6, Definition 4.6]). The idea is the following: If a countable structure has some interesting computability-theoretic property, then for any HKSS-complete class K, there is a structure $\mathcal{S} \in K$ possessing the same property. In other words, the notion of HKSS-completeness tries to capture the intuitive notion of computability-theoretic universality of the class. As has been discovered in [5], HKSS-completeness is typically witnessed by a $\mathcal{L}^c_{\omega_1\omega}$-definable functor from any countable class to the universal class, we omit details. The most important effective properties include

Date: August 31, 2016.

degree spectra, effective dimension, expansion by constants, and degree spectra of relations, see [3] for definitions and discussion. We typically aim to show that a class is complete with respect to all these properties, and indeed we are not aware of any natural example of a class that would be complete with respect to some of these properties but not the other ones.

Hirschfeldt, Khoussainov, Shore, and Slinko ([3, Theorem 3.3]) proved that the class of *non-distributive* lattices is HKSS-complete. Hence, there is the following natural question:

Problem. Is the class of distributive lattices universal from the computability-theoretic point of view? More formally, is the class of distributive lattices HKSS-complete?

One special nice class of distributive lattices, namely the class of linear orders, has been studied intensively (see surveys [7, 8]). Although it is known that linear orders are not universal with respect to degree spectra (e.g., Richter [9]), many complex degree spectra can be realised in the class of linear orders [8, 10]. It is also well-known that linear orders are not universal with respect to computable categoricity and computable dimension. Recall that the Δ^0_α-dimension of a structure $\mathcal{M}$ is the number of computable copies of the structure up to Δ^0_α-isomorphism. A structure is Δ^0_α-categorical if its Δ^0_α-dimension is 1. Much work has been done on Δ^0_α-categoricity and Δ^0_α-dimension of linear orders. For instance, Goncharov and Dzgoev [11] and independently, Remmel [12] proved that for a computable linear order $\mathcal{L}$, the following conditions are equivalent:

(1) $\mathcal{L}$ is computably categorical,
(2) $\mathcal{L}$ is relatively computably categorical,
(3) the successor relation of $\mathcal{L}$ is a finite set.

Furthermore, Goncharov and Dzgoev [11] also proved that the computable dimension (i.e., the Δ^0_1-dimension) of a computable linear order is either 1 or ω. Ash [13] obtained a complete and satisfactory description of Δ^0_α-categorical well-orders, for any computable α. McCoy [14, 15] studied Δ^0_2- and Δ^0_3-categorical linear orders. Frolov [16] has proved that there exists a linear order that is Δ^0_3-categorical but not relatively Δ^0_3-categorical.

Nonetheless, not much is known about distributive lattices which are not necessarily linearly ordered. Selivanov [17] investigated computably enumerable distributive lattices. Turlington [18] proved that for any Turing degree $\mathbf{d}$, there is a countable distributive lattice with the degree spectrum $\{\mathbf{c} : \mathbf{c} \geq \mathbf{d}\}$. Bazhenov [19] showed the following: For any computable successor ordinal $\alpha \geq 4$ and any non-zero natural number n, there is a computable distributive lattice with Δ^0_α-dimension n.

Our results provide a good evidence that the class of countable distributive lattices *could be universal*. In particular, our results show that computable distributive lattices can possess some effective properties that computable linear orders cannot have. We leave open whether countable distributive lattices are HKSS-complete.

As the first main result, we prove:

Theorem 1.1. *There exists a countable distributive lattice whose degree spectrum is exactly the non-computable Turing degrees.*

In Section 3 we prove a more general result that allows to code any family of finite sets into a countable distributive lattice. To prove Theorem 1.1 it remains to use the well-known result [20]. We note that it is still open whether there exists a linear order with such degree spectrum.

The second main result is:

Theorem 1.2. *There is a computably categorical distributive lattice which is not relatively Δ^0_2-categorical.*

As far as we know, this is the first known example of this sort from a natural enough algebraic class that would not be HKSS-complete. (Here "natural" means "not specifically made up to satisfy the property"). In Section 4, we construct a computable distributive lattice which is computably categorical but not relatively computably categorical. This weaker result has a simpler proof. Note that this weaker result contrasts with the theorem on linear orders ([11, 12]) mentioned above. In Section 5, we extend the technique of Section 4 to prove Theorem 1.2.

2. Preliminaries

We treat lattices as structures in the language $L_0 = \{\vee^2, \wedge^2\}$. Recall that a lattice is *bounded* if it has the least element 0 and the greatest element 1. For a lattice $\mathcal{D}$, $PO(\mathcal{D})$ denotes the partial order corresponding to $\mathcal{D}$.

Assume that $\mathcal{L}$ is a linear order, and $\{\mathcal{A}_n\}_{n\in\mathcal{L}}$ is a sequence of partial orders. The $\mathcal{L}$-*sum* of the sequence $\{\mathcal{A}_n\}_{n\in\mathcal{L}}$ is the structure on the universe $\{(x,n) : n \in \mathcal{L},\ x \in \mathcal{A}_n\}$. The ordering on the $\mathcal{L}$-sum is defined as follows: $(x,n) \leq (y,m)$ iff $n <_{\mathcal{L}} m$ or $(n = m)\&(x \leq_{\mathcal{A}_n} y)$. Let $\mathbf{Sum}(\mathcal{A}_n;\mathcal{L})$ denote the $\mathcal{L}$-sum of the sequence $\{\mathcal{A}_n\}_{n\in\mathcal{L}}$.

It is not difficult to prove the following claim (e.g., use the ideas from [18, pp. 8–10]).

Lemma 2.1. *Suppose that $\mathcal{L}$ is a linear order and $\{\mathcal{M}_n\}_{n\in\mathcal{L}}$ is a sequence of distributive lattices. Then the order $\mathbf{Sum}(PO(\mathcal{M}_n);\mathcal{L})$ is a distributive lattice. Moreover, if the order $\mathcal{L}$ and the sequence $\{\mathcal{M}_n\}_{n\in\omega}$ are both computable, then there is a computable copy of the distributive lattice $\mathbf{Sum}(PO(\mathcal{M}_n);\mathcal{L})$.*

Lemma 2.1 allows us to use the following notation.

Notation 2.1. If $\mathcal{L}$ is an X-computable linear order and $\{\mathcal{M}_n\}_{n\in\mathcal{L}}$ is an X-computable sequence of distributive lattices, then by $\mathbf{Sum}(\mathcal{M}_n;\mathcal{L})$ we denote the natural X-computable copy of the distributive lattice $\mathbf{Sum}(PO(\mathcal{M}_n);\mathcal{L})$.

If the order $\mathcal{L}$ has exactly k elements, then we will sometimes write $\mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \ldots \oplus \mathcal{M}_k$ in place of $\mathbf{Sum}(\mathcal{M}_i;\mathcal{L})$.

Let η denote the standard computable copy of the order of rationals. Fix a computable sequence of computable infinite sets $\{Y_k\}_{k\in\omega}$ with the following properties: $\bigcup_k Y_k = \mathbb{N}$, $Y_i \cap Y_j = \emptyset$ for $i \neq j$, and for any i and j, if $a, b \in Y_i$ and $a <_\eta b$, then there is an element c such that $c \in Y_j$ and $a <_\eta c <_\eta b$.

Assume that $\{\mathcal{M}_n\}_{n\in\omega}$ is an X-computable sequence of distributive lattices. If $a \in \eta$ and $a \in Y_k$, then set $\mathcal{N}_a = \mathcal{M}_k$. The *shuffle sum* of the sequence $\{\mathcal{M}_n\}_{n\in\omega}$ (denoted by $\mathbf{Shuf}(\mathcal{M}_n)$) is the structure $\mathbf{Sum}(\mathcal{N}_a;\eta)$.

Proposition 2.1 (Selivanov [17, Lemma 1])**.** *There exists a computable sequence of finite distributive lattices $\{\mathcal{D}^0_n\}_{n\in\omega}$ such that for any $i \neq j$, $\mathcal{D}^0_i$ does not embed into $\mathcal{D}^0_j$.*

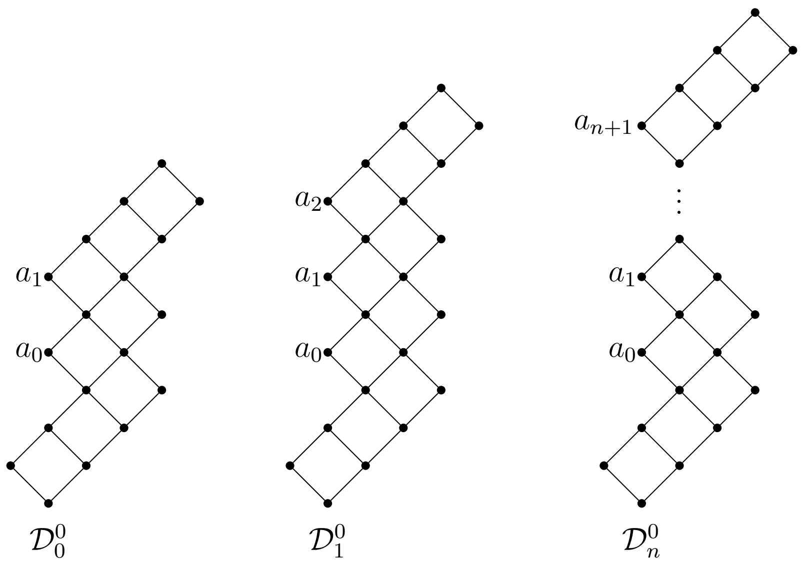

FIGURE 1. The lattices $\mathcal{D}^0_i$.

Assume that $\mathcal{A}$ and $\mathcal{B}$ are bounded lattices. Then $(\mathcal{A} \oplus_0 \mathcal{B})$ denotes the quotient lattice of $(\mathcal{A} \oplus \mathcal{B})$ modulo the congruence

$$E_0 = \left\{(1^{\mathcal{A}}, 0^{\mathcal{B}}), (0^{\mathcal{B}}, 1^{\mathcal{A}})\right\} \cup \{(x, x) : x \in \mathcal{A} \cup \mathcal{B}\}.$$

For a tuple of natural numbers $\bar{a} = (a_1, a_2, \ldots, a_n)$, let $\mathcal{D}^0(\bar{a})$ denote the lattice $(\mathcal{D}^0_{a_1} \oplus_0 \mathcal{D}^0_{a_2} \oplus_0 \ldots \oplus_0 \mathcal{D}^0_{a_n})$.

3. Degree spectra

The first main result of the paper (Theorem 1.1) follows from the more general fact below.

Theorem 3.1. *Let $\mathcal{S}$ be a countable family of finite subsets of ω. There is a countable distributive lattice $\mathcal{D}_{\mathcal{S}}$ such that the degree spectrum of $\mathcal{D}_{\mathcal{S}}$ is equal to the enumeration spectrum of the family $\mathcal{S}$.*

Proof. For a finite set $F = \{a_0 <_\omega a_1 <_\omega \ldots <_\omega a_{n-1}\}$, let

$$Perm(F) = \left\{(0, a_{\sigma(0)} + 1, a_{\sigma(1)} + 1, \ldots, a_{\sigma(n-1)} + 1) : \sigma \text{ is a permutation of } n\right\}.$$

Set $Perm(\emptyset) = \{(0)\}$. Fix a function $h\colon \omega \to \omega^{<\omega}$ such that the range of h is equal to the set

$$U = \bigcup_{F \in \mathcal{S}} Perm(F).$$

For $k \in \omega$, let $\mathcal{M}_k = \mathcal{D}^0(h(k))$. We claim that the lattice $\mathcal{D}_{\mathcal{S}} = \mathbf{Shuf}(\mathcal{M}_k)$ satisfies the desired conditions.

First, assume that the Turing degree of a set X belongs to the enumeration spectrum of $\mathcal{S}$. In other words, there is a computable function $f(x)$ such that

$$\mathcal{S} = \left\{W^X_{f(n)} : n \in \omega\right\}.$$

For $n, s \in \omega$, we define the finite set

$$P_{n,s} = Perm\left(W^X_{f(n),s}\right).$$

We build the X-computable sequence of lattices. At step s, we work with a finite collection of finite lattices $\mathcal{A}^{t,s}_j$, $t \leq s$, $j \leq p(t,s)$. We ensure the following conditions:

(1) for any $n \leq s$ and $\bar{a} \in P_{n,s}$, there is an index $j \leq p(n,s)$ such that the lattices $\mathcal{D}^0(\bar{a})$ and $\mathcal{A}^{n,s}_j$ are isomorphic,
(2) for any $n \leq s$ and $j \leq p(n,s)$, there is a tuple $\bar{a} \in P_{n,s}$ such that $\mathcal{A}^{n,s}_j \cong \mathcal{D}^0(\bar{a})$, and
(3) $\mathcal{A}^{n,s}_j \subseteq \mathcal{A}^{n,s+1}_j$ for all n and j.

For example, assume that for some s, we have $W^X_{f(0),s} = \{0,1\}$ and $W^X_{f(0),s+1} = \{0,1,2\}$. Then at step s, we had $p(0,s) = 1$ and there were finite lattices $\mathcal{A}^{0,s}_0 \cong \mathcal{D}^0(0,1,2)$ and $\mathcal{A}^{0,s}_1 \cong \mathcal{D}^0(0,2,1)$. At step $s+1$, we set $p(0,s+1) = 5$ and we "glue" new copies of $\mathcal{D}^0(3)$ on top of the lattices $\mathcal{A}^{0,s}_0$ and $\mathcal{A}^{0,s}_1$. We define $\mathcal{A}^{0,s+1}_j$, $2 \leq j \leq 5$, as fresh copies of the lattices $\mathcal{D}^0(0,3,1,2)$, $\mathcal{D}^0(0,3,2,1)$, $\mathcal{D}^0(0,1,3,2)$, and $\mathcal{D}^0(0,2,3,1)$, respectively.

For $n \in \omega$ and $j \leq \lim_s p(n,s)$, set $\mathcal{N}_{n,j} = \bigcup_{s\in\omega} \mathcal{A}^{n,s}_j$. Then it is not difficult to prove that the structure $\mathbf{Shuf}(\mathcal{N}_{n,j})$ is an X-computable isomorphic copy of the lattice $\mathcal{D}_{\mathcal{S}}$. Since the structure $\mathcal{D}_{\mathcal{S}}$ is not automorphically trivial, the result of Knight [21] implies that $\deg_T(X)$ lies in the degree spectrum of $\mathcal{D}_{\mathcal{S}}$.

Now assume that $\mathcal{A}$ is an isomorphic copy of the lattice $\mathcal{D}_{\mathcal{S}}$, and $\mathbf{a}$ is the Turing degree of the atomic diagram of $\mathcal{A}$. First, we prove the following algebraic lemma.

Lemma 3.1. *Assume that $k \in \omega$ and $g\colon \mathcal{D}^0_k \hookrightarrow \mathcal{D}_{\mathcal{S}}$ is an isomorphic embedding. Then there are an index $n \in \omega$ and a lattice $\mathcal{N} \subseteq \mathcal{D}_{\mathcal{S}}$ with the following properties:*

(1) *$\mathcal{N}$ is a copy of $\mathcal{M}_n$ in $\mathcal{D}_{\mathcal{S}}$,*
(2) *$h(n) = (a_0, a_1, \dots, a_{t-1}, k, a_{t+1}, a_{t+2}, \dots, a_p)$ for some $p \in \omega$ and $t \leq p$, and*
(3) *$\mathcal{N} = \mathcal{N}_0 \oplus_0 \mathcal{N}_1 \oplus_0 \dots \oplus_0 \mathcal{N}_{t-1} \oplus_0 g(\mathcal{D}^0_k) \oplus_0 \mathcal{N}_{t+1} \oplus_0 \mathcal{N}_{t+2} \oplus_0 \dots \oplus_0 \mathcal{N}_p$, where $\mathcal{N}_i \cong \mathcal{D}^0_{a_i}$ for all i.*

Proof. Here we give the sketch of the proof. The omitted details can be easily reconstructed or found in [17] or [18, pp. 68–71]. Note that if x and y are incomparable elements in $\mathcal{D}_{\mathcal{S}}$, then they must belong to the same copy of some $\mathcal{M}_j$ in $\mathcal{D}_{\mathcal{S}}$. Hence, all the elements of $g(\mathcal{D}^0_k)$ belong to the lattice $\mathcal{N}$ which is the copy of $\mathcal{M}_n$ (in $\mathcal{D}_{\mathcal{S}}$) for some n.

Assume that $h(n) = (a_0, a_1, \dots, a_p)$ and $\mathcal{N} = \mathcal{N}_0 \oplus_0 \mathcal{N}_1 \oplus_0 \dots \oplus_0 \mathcal{N}_p$, where $\mathcal{N}_i \cong \mathcal{D}^0_{a_i}$ for all $i \leq p$. Then one can use essentially the same argument as in [18] to show that there is a number $t \leq p$ such that $g(\mathcal{D}^0_k) \subseteq \mathcal{N}_t$. Therefore, we may assume that g is an isomorphic embedding from $\mathcal{D}^0_k$ into $\mathcal{D}^0_{a_t}$. By Proposition 2.1, g is an isomorphism and $a_t = k$. $\square$

We describe the construction of an $\mathbf{a}$-c.e. enumeration $\nu^{\mathcal{A}}$ of the family $\mathcal{S}$. At step s, we build a finite sequence of tuples $\bar{a}^s_0, \bar{a}^s_1, \dots, \bar{a}^s_{q(s)}$ and a sequence of functions $g^s_0, g^s_1, \dots, g^s_{q(s)}$ such that for every $i \leq q(s)$, g^s_i is an embedding of the lattice $\mathcal{D}^0(\bar{a}^s_i)$ into $\mathcal{A}$, and $g^s_i \subseteq g^{s+1}_i$. Set $q(0) = -1$.

At step $s+1$, search for the least $b \le s$ and $j \le q(s)$ such that there is an embedding $g^* \colon \mathcal{D}^0(\bar{a}^s_j, b) \hookrightarrow \mathcal{A}$ with the following properties: $g^s_j \subseteq g^*$ and $y \le s$ for all y from the range of g^*. If such numbers b and j exist, then define $\bar{a}^{s+1}_j = (\bar{a}^s_j, b)$ and $g^{s+1}_j = g^*$. After that, find the (least under Gödel numbering) function g' such that g' embeds $\mathcal{D}^0_0$ into $\mathcal{A}$ and $g' \cap g^s_i = \emptyset$ for all $i \le q(s)$. Set $q(s+1) = q(s)+1$, $\bar{a}^{s+1}_{q(s+1)} = (0)$, and $g^{s+1}_{q(s+1)} = g'$. If $i \le q(s)$ and $i \ne j$, then define $\bar{a}^{s+1}_i = \bar{a}^s_i$ and $g^{s+1}_i = g^s_i$.

For $n \in \omega$, let

$$\nu^{\mathcal{A}}(n) = \{m \in \omega : \exists s \big(\text{the number } (m+1) \text{ occurs in the tuple } \bar{a}^s_n\big)\}.$$

It is easy to see that the enumeration $\nu^{\mathcal{A}}$ is $\mathbf{a}$-c.e. Moreover, Lemma 3.1 implies that for every n, we have $\nu^{\mathcal{A}}(n) \in \mathcal{S}$. The choice of the function g' in the construction above guarantees that for every $F \in \mathcal{S}$, there is a number m such that $\nu^{\mathcal{A}}(m) = F$. Thus, $\nu^{\mathcal{A}}$ is an enumeration of $\mathcal{S}$ and $\mathbf{a}$ lies in the enumeration spectrum of $\mathcal{S}$. This concludes the proof of Theorem 3.1. □

Wehner [20] proved that there is a family $\mathcal{S}_0$ of finite sets such that the enumeration spectrum of $\mathcal{S}_0$ contains precisely the non-zero degrees. This and Theorem 3.1 imply that distributive lattices can realize the degree spectrum obtained by Slaman [22] and Wehner [20], thus we have Theorem 1.1.

4. Relative computable categoricity

Before we prove Theorem 1.2 we give a detailed proof of the simpler result below. Theorem 1.2 will have to deal with a more complicated coding components, but many features of the below construction will also appear in the proof of Theorem 1.2.

Theorem 4.1. *There is a computable distributive lattice $\mathcal{A}$ which is computably categorical but not relatively computably categorical.*

Proof. Fix an effective enumeration $\{(\Theta_e, \bar{c}_e)\}_{e\in\omega}$ of all c.e. families of existential L_0-formulas, where Θ_e is a family of formulas with parameters from $\bar{c}_e$. Without loss of generality, we may assume that every Θ_e satisfies the following conditions:

(1) every formula $\psi(\bar{x}, \bar{c}_e) \in \Theta_e$ is of the form

$$\psi = \exists y_1 \exists y_2 \dots \exists y_n \big(\phi_1(\bar{x}, \bar{y}, \bar{c}_e) \ \& \ \dots \ \& \ \phi_k(\bar{x}, \bar{y}, \bar{c}_e)\big),$$

where every ϕ_i is either atomic, or negation of an atomic formula, and

(2) if ϕ is an atomic subformula of some $\psi \in \Theta_e$, then ϕ satisfies one of the following:
   - (2a) $\phi = (t_0 = t_1)$, where each t_i is either a constant from $\bar{c}_e$ or a variable,
   - (2b) $\phi = (t_0 * t_1 = t_2)$, where $* \in \{\vee, \wedge\}$ and each t_i is either a constant from $\bar{c}_e$ or a variable.

We also fix an effective enumeration $\{\mathcal{M}_e\}_{e\in\omega}$ of all computable (partial) L_0-structures.

The computable distributive lattice $\mathcal{A}$ will be isomorphic to the shuffle sum $\mathbf{Shuf}(\mathcal{B}_k)$, where each of the structures $\mathcal{B}_k$ has the following property: there is a number $r \in \omega$ such that $\mathcal{B}_k$ is isomorphic to one of the lattices $\mathcal{D}^0(2r)$, $\mathcal{D}^0(2r, 2r+1)$, or $\mathcal{D}^0(2r+1, 2r)$.

We build the structure $\mathcal{A}$ and satisfy the following requirements:

$\mathcal{R}$: $\mathcal{A}$ is a shuffle sum.

$\mathcal{I}_e$: If $\mathcal{M}_e \cong \mathcal{A}$, then $\mathcal{M}_e \cong_{\Delta^0_1} \mathcal{A}$.
$\mathcal{S}_j$: $(\Theta_j, \bar{c}_j)$ is not a Scott family for the structure $\mathcal{A}$.

Fix a computable sequence of computable infinite, pairwise disjoint sets $\{Y_k\}_{k\in\omega}$ such that $\bigcup_k Y_k = \mathbb{N}$. In addition, we assume that for any i and j, if $a, b \in Y_i$ and $a <_\eta b$, then there is an element c such that $c \in Y_j$ and $a <_\eta c <_\eta b$.

We say that a number a is a *k-colored box* if $a \in Y_k$. A lattice $\mathcal{B}$ *lies in the box a at stage s* if $\mathcal{B}$ is the sublattice of $\mathcal{A}_s$ on the universe $\{(x, a) : x \in \omega\} \cap |\mathcal{A}_s|$. A box a is *empty at stage s* if $\mathcal{A}_s$ has no elements of the form (x, a), $x \in \omega$. We use the following convention: if $a <_\eta b$ and $(x, a), (y, b) \in \mathcal{A}_s$, then $(x, a) <_{\mathcal{A}_s} (y, b)$.

We ensure the following property: if two different k-colored boxes are non-empty at the end of stage s, then they must contain isomorphic finite lattices.

We have two basic actions for building $\mathcal{A}$:

(1) *Putting a finite lattice $\mathcal{N}$ into an empty box a*: The strategy takes a fresh copy of the structure $\mathcal{N}$ on the universe $\{(x, a) : x \in F\}$, where F is a finite set, and adds it to the lattice $\mathcal{A}_s$.

(2) *Glueing a finite lattice $\mathcal{N}$ to all k-colored boxes*: For every non-empty k-colored box a, the strategy proceeds as follows. If the box a contains a lattice $\mathcal{B}$, then the strategy "glues" a fresh copy of $\mathcal{N}$ on top of $\mathcal{B}$. After the procedure, the box a contains an isomorphic copy of $(\mathcal{B} \oplus_0 \mathcal{N})$.

At stage s, we say that a lattice $\mathcal{N}_0$ is *younger than* a lattice $\mathcal{N}_1$ if there are a color k and different k-colored boxes a_0 and a_1 such that:

(1) $\mathcal{N}_i$ lies in the box a_i at stage s, and
(2) there is a stage $t < s$ at which the box a_0 was non-empty and a_1 was empty.

Strategy for meeting $\mathcal{R}$. The only node working for the $\mathcal{R}$-requirement is $\emptyset$ (i.e., the root of the tree of strategies). This node has only one outcome `act`. At each stage s, it proceeds as follows. Assume that $k \in \omega$ and there is (the least) k-colored box a such that it contains a lattice $\mathcal{N}$ at the beginning of the stage s. For every such k, the $\mathcal{R}$-strategy finds the least empty k-colored box b and puts $\mathcal{N}$ into the box b.

Strategy for meeting $\mathcal{S}_j$. Assume that σ is an $\mathcal{S}_j$-strategy and

$$\Theta_j = \{\exists \bar{y}_i \psi_i(\bar{x}_i, \bar{y}_i, \bar{c}_j) : i \in \omega\},$$

where every ψ_i is a quantifier-free formula.

(1) Choose a large number r and find the least k such that every k-colored box is empty at the current stage. Denote such k by $c(\sigma)$. Assume that a_0 is the k-colored box with the least number. Put the lattice $\mathcal{D}^0(2r)$ into the box a_0 and fix the tuple $\bar{b} = (b_0, b_1, \dots, b_N)$ such that the newly added copy of $\mathcal{D}^0(2r)$ has the universe $\{b_0 <_\omega b_1 <_\omega \dots <_\omega b_N\}$. We choose $\bar{b}$ such that $b_i \notin \bar{c}_j$ for all $i \le N$.
(2) For every formula ψ_i, $i \le s$, search for a tuple $\bar{d}$ such that $\mathcal{A}_s \models \psi_i(\bar{b}, \bar{d}, \bar{c}_j)$.
(3) If such a formula and a tuple are found, glue the lattice $\mathcal{D}^0(2r + 1)$ to all k-colored boxes.
(4) Wait until the next stage at which σ is accessible.
(5) Find the least k_1 such that every k_1-colored box is empty. Denote such k_1 by $c_1(\sigma)$. Put a copy of $\mathcal{D}^0(2r + 1, 2r)$ into the least empty k_1-colored box.

While the strategy is searching at Step (2), it has outcome `wait`. Once it found a formula ψ_i and a tuple $\bar{d}$, it has outcome `act`.

Strategy for meeting $\mathcal{I}_e$. Assume that σ is an $\mathcal{I}_e$-node on the tree of strategies. The node σ has outcomes ∞, `0`, `1`, `2`, ... Let s be a current stage, and k be the number of stages less than s at which σ had outcome ∞.

The strategy σ builds the mapping f_σ using the back-and-forth method. We set $f_\sigma[0] = \emptyset$.

Let $B(\sigma, s)$ be the substructure of $\mathcal{A}_s$ such that it contains precisely the following elements:

(i) for each $\tau \subset \sigma$ such that $\tau\widehat{\ }\mathtt{wait} \subseteq \sigma$, the youngest $(k+1)$ lattices lying in the $c(\tau)$-colored boxes;

(ii) for each $\tau \subset \sigma$ such that $\tau\widehat{\ }\mathtt{act} \subseteq \sigma$ and τ has reached Step (5), the youngest $(k+1)$ lattices in the $c(\tau)$-colored boxes and the youngest $(k+1)$ lattices in the $c_1(\tau)$-colored boxes;

(iii) for each $\tau \not\subset \sigma$ such that τ is incomparable with $\sigma\widehat{\ }\mathtt{k}$ and τ is an $\mathcal{S}_j$-strategy, the youngest l lattices in the $c(\tau)$-colored boxes and the youngest l_1 lattices in the $c_1(\tau)$-colored boxes. Here l (l_1) is the minimum of $k+1$ and the number of non-empty $c(\tau)$-colored ($c_1(\tau)$-colored) boxes in $\mathcal{A}_s$;

(iv) for each $a \in \omega$ such that $\{(x,a) : x \in \omega\} \cap dom(f_\sigma[s-1]) \neq \emptyset$, the lattice lying in the box a.

At stage s, we search for an isomorphic embedding $g \colon B(\sigma, s) \hookrightarrow \mathcal{M}_{e,s}$ such that $g \supseteq f_\sigma[s-1]$. If such an embedding g is found, then search for a sublattice $\mathcal{C} \subseteq \mathcal{M}_{e,s}$ such that $\mathcal{C}$ is isomorphic to one of the lattices described in the conditions (i)–(iii) above, and the universe of $\mathcal{C}$ is disjoint with the range of g. If such a lattice found, then again search for an isomorphic embedding $g_1 \colon (g(B(\sigma,s)) \cup \mathcal{C}) \hookrightarrow \mathcal{A}_s$ such that $g_1 \supseteq g^{-1}$. If such g_1 found, then σ has outcome ∞ at stage s and $f_\sigma[s] = g_1^{-1}$. Otherwise, σ has outcome `k` and $f_\sigma[s] = f_\sigma[s-1]$.

Construction. For an $\mathcal{S}_j$-strategy, we order the outcomes as `act`<`wait`. For an $\mathcal{I}_e$-strategy, we order the outcomes as $\infty < \ldots <$`2`<`1`<`0`. The 0th level of the priority tree is devoted to the $\mathcal{R}$-strategy. For the other levels, each level is devoted (in some effective fashion) to one of requirements $\mathcal{S}_j$ or $\mathcal{I}_e$. As usual, at stage s we visit strategies of length at most s, and they act in order of priority.

Verification. First, notice that actions of the $\mathcal{R}$-strategy ensure that $\mathcal{A}$ is a shuffle sum of finite distributive lattices. Hence, by construction, $\mathcal{A}$ is a computable distributive lattice. The next lemma is similar to Lemma 3.1.

Lemma 4.1. *Assume that $t \in \omega$ and $g \colon \mathcal{D}^0_t \hookrightarrow \mathcal{A}$ is an isomorphic embedding. Let $r = [t/2]$. There are $b \in \omega$ and a finite lattice $\mathcal{N} \subseteq \mathcal{A}$ with the following properties:*

(1) *$g(\mathcal{D}^0_t)$ is a sublattice of $\mathcal{N}$,*

(2) *$\mathcal{N}$ is a structure on the universe $\{(x,b) : x \in \omega\} \cap |\mathcal{A}|$,*

(3) *if t is odd, then there is a lattice $\mathcal{N}_0 \cong \mathcal{D}^0(2r)$ such that $\mathcal{N} = \mathcal{N}_0 \oplus_0 g(\mathcal{D}^0_t)$ or $\mathcal{N} = g(\mathcal{D}^0_t) \oplus_0 \mathcal{N}_0$,*

(4) *if t is even, then $\mathcal{N}$ satisfies one of the following:*

  (4a) *$\mathcal{N} = g(\mathcal{D}^0_t)$, or*

  (4b) *there is a lattice $\mathcal{N}_1 \cong \mathcal{D}^0(2r+1)$ such that $\mathcal{N} = g(\mathcal{D}^0_t) \oplus_0 \mathcal{N}_1$ or $\mathcal{N} = \mathcal{N}_1 \oplus_0 g(\mathcal{D}^0_t)$.*

Lemma 4.2. *The structure $\mathcal{A}$ is computably categorical.*

Proof. Assume that a structure $\mathcal{M}_e$ is isomorphic to $\mathcal{A}$ and σ is an $\mathcal{I}_e$-strategy along the true path.

For an $\mathcal{S}_j$-strategy τ, suppose that a lattice $\mathcal{N}$ lies either in a $c(\tau)$-colored box, or in a $c_1(\tau)$-colored box. Consider the following four cases for τ:

(1) Suppose that $\tau \subset \sigma$. Since σ is on the true path, $\mathcal{N}$ will never grow once σ begins considering it. Thus, if $\mathcal{N}$ lies in the domain of f_σ, then by Lemma 4.1, $f_\sigma(\mathcal{N})$ is one of the summands in the shuffle sum $\mathcal{M}_e$. In other words, the mapping f_σ is *correct* on $\mathcal{N}$.

(2) If τ is incomparable with σ, then τ can never be visited after σ begins considering $\mathcal{N}$. Therefore, again, f_σ is correct on $\mathcal{N}$.

(3) Suppose that $\tau \supseteq \sigma\widehat{\ }\infty$. Then there are two subcases:

(3a) If the final outcome of τ is `wait`, then, as in Case (1), f_σ is correct on $\mathcal{N}$.

(3b) Assume that the final outcome of τ is `act`. If τ never reaches Step (5), then there is r such that $\mathcal{N} \cong \mathcal{D}^0(2r, 2r+1)$ and by Lemma 4.1, f_σ eventually will be correct on $\mathcal{N}$. Suppose that τ reaches Step (5). Then the lattices added by τ eventually will become isomorphic either to $\mathcal{D}^0(2r, 2r+1)$ or to $\mathcal{D}^0(2r+1, 2r)$ for some fixed r. Assume that s_0 is the first stage at which τ reaches Step (5). If at stage s_0, $\mathcal{N}[s_0]$ lies in the domain of $f_\sigma[s_0]$, then the choice of the mapping $f_\sigma[s_0]$ guarantees that $\mathcal{N}[s_0]$ is isomorphic to $\mathcal{D}^0(2r, 2r+1)$. Thus, after stage s_0, $\mathcal{N}$ never grows and f_σ is correct on $\mathcal{N}$.

(4) Suppose that $\tau \supseteq \sigma\widehat{\ }\mathtt{k}$ for some $k \in \omega$. If σ considers $\mathcal{N}$ at some stage s, then σ has had outcome ∞ more than k times by stage s. Hence, $\mathcal{N}$ will never grow and f_σ is correct on $\mathcal{N}$.

The argument above proves the following claim. If $\mathcal{N}$ is a summand from the shuffle sum $\mathcal{A}$ and there is a stage s such that $\mathcal{N}[s] \cap dom(f_\sigma[s]) \neq \emptyset$, then $f_\sigma(\mathcal{N})$ is a summand in the shuffle sum $\mathcal{M}_e$.

We now show that the node $\sigma\widehat{\ }\infty$ also belongs to the true path. In order to prove this, assume that $\mathtt{k}$ is the true outcome of σ. Thus, k is the number of all stages at which σ had outcome ∞. Let s_0 be the last stage at which σ had outcome ∞. Since $\mathcal{M}_e \cong \mathcal{A}$ and f_σ is correct on every summand from $\mathcal{A}$, one can find a stage $s > s_0$ such that σ has outcome ∞ at stage s. This is a contradiction. Hence, ∞ is the true outcome of σ.

Since f_σ uses the back-and-forth construction, f_σ is a computable isomorphism from $\mathcal{A}$ onto $\mathcal{M}_e$. □

Lemma 4.3. *The lattice $\mathcal{A}$ is not relatively computably categorical.*

Proof. Assume that σ is an $\mathcal{S}_j$-strategy along the true path. Recall that $\bar{b} = b_0, b_1, \dots, b_N$ is the tuple from Step (1) of the $\mathcal{S}_j$-strategy σ. If the true outcome of σ is `wait`, then there is no formula $\psi \in \Theta_j$ such that $\mathcal{A} \models \psi(\bar{b})$. Suppose that the true outcome of σ is `act`. Fix the formula ψ_i and the tuple $\bar{d}$ from Step (2) for σ. Let $\mathcal{N}$ be the summand of $\mathcal{A}$ such that $\bar{b} \in \mathcal{N}$. Note that $\mathcal{N} \cong \mathcal{D}^0(2r, 2r+1)$.

Since $\mathcal{A}$ is a shuffle sum, there is a summand $\mathcal{N}'$ in $\mathcal{A}$ with the following properties:

- For every $y \in \mathcal{N}'$, $y \notin (\bar{c}_j \cup \bar{d})$.
- $\mathcal{N}' = \mathcal{N}'_0 \oplus_0 \mathcal{N}'_1$, where $\mathcal{N}'_0 \cong \mathcal{D}^0(2r+1)$ and $\mathcal{N}'_1 \cong \mathcal{D}^0(2r)$.
- Fix a mapping $g\colon \bar{b} \to \mathcal{N}'_1$ such that g is an isomorphism of lattices, and set $b'_i = g(b_i)$ for all $i \leq N$. Then for every $x \in (\bar{c}_j \cup \bar{d}) \setminus \bar{b}$ and every $i \leq N$,

we have

$$(x \leq_{\mathcal{A}} b_i \text{ iff } x \leq_{\mathcal{A}} b'_i), \quad \text{and} \quad (b_i \leq_{\mathcal{A}} x \text{ iff } b'_i \leq_{\mathcal{A}} x).$$

Suppose that $d_k \in \bar{d}$. Then set $d'_k = b'_i$ if there is a number $i \leq N$ such that $d_k = b_i$. Otherwise, define $d'_k = d_k$.

Since ψ_i is a quantifier-free formula and $\exists \bar{y}_i \psi_i \in \Theta_j$, it is not difficult to show that $\mathcal{A} \models \psi_i(\bar{b}', \bar{d}', \bar{c}_j)$. Therefore, the tuples $\bar{b}$ and $\bar{b}'$ are not automorphic and they satisfy the same formula from Θ_j. Thus, Θ_j is not a Scott family for $\mathcal{A}$. □

This concludes the proof of Theorem 4.1. □

5. Relative Δ^0_2-categoricity

In this section we prove Theorem 1.2: There exists a computable distributive lattice $\mathcal{A}$ which is computably categorical but not relatively Δ^0_2-categorical.

Proof of Theorem 1.2. The proof is an extension of the proof of Theorem 4.1. The notation is the same as in Theorem 4.1.

For a computable structure $\mathcal{M}$ and a Σ^c_2-formula ψ, the statement "$\mathcal{M} \models \psi(\bar{x}, \bar{c})$" is effectively equivalent to the formula $(\forall^\infty y)\phi(\bar{x}, \bar{c}, y)$ for some computable relation ϕ. Thus, we fix an effective enumeration $\{(\Theta_j, \bar{c}_j)\}_{j\in\omega}$ of all c.e. families of formulas of this form, and we will diagonalize against $(\Theta_j, \bar{c}_j)$, $j \in \omega$.

Again, we satisfy the following requirements:

$\mathcal{R}$: $\mathcal{A}$ is a shuffle sum.
$\mathcal{I}_e$: If $\mathcal{M}_e \cong \mathcal{A}$, then $\mathcal{M}_e \cong_{\Delta^0_1} \mathcal{A}$.
$\mathcal{S}_j$: $(\Theta_j, \bar{c}_j)$ is not a Scott family for $\mathcal{A}$.

The lattice $\mathcal{A}$ will be isomorphic to the shuffle sum $\mathbf{Shuf}(\mathcal{B}_k)$, where each of the structures $\mathcal{B}_k$ is isomorphic to one of the following (for some $r, q \in \omega$):

(A) $\mathcal{D}^0\big(2^r(2q+3), 2^r(2q+1), \ldots, 2^r \cdot 3\big) \oplus_0 \mathcal{D}^0\big(2^r, 2^r \cdot 3, \ldots, 2^r(2q+1)\big)$,
(B) $\mathcal{D}^0\big(2^r(2q+1), 2^r(2q-1), \ldots, 2^r \cdot 3\big) \oplus_0 \mathcal{D}^0\big(2^r, 2^r \cdot 3, \ldots, 2^r(2q+3)\big)$,
(C) $\mathcal{D}^0\big(2^r(2q+5), 2^r(2q+3), \ldots, 2^r \cdot 3\big) \oplus_0 \mathcal{D}^0\big(2^r, 2^r \cdot 3, \ldots, 2^r(2q+1)\big)$,
(D) $\mathcal{D}^0\big(2^r(2q+1), 2^r(2q-1), \ldots, 2^r \cdot 3\big) \oplus_0 \mathcal{D}^0\big(2^r, 2^r \cdot 3, \ldots, 2^r(2q+5)\big)$, or
(E) the infinite sum $\mathcal{D}^0(\ldots, 2^r \cdot 5, 2^r \cdot 3, 2^r, 2^r \cdot 3, 2^r \cdot 5, \ldots)$.

Assume that $\mathcal{N}_0$ and $\mathcal{N}_1$ are finite lattices. We define the following additional action for building $\mathcal{A}$:

$[\mathcal{N}_0, \mathcal{N}_1]$-*Gluing for all k-colored boxes*: For every non-empty k-colored box a, one proceeds as follows. If the box a contains a lattice $\mathcal{B}$, then we glue a fresh copy of $\mathcal{N}_0$ to bottom of $\mathcal{B}$, and glue a copy of $\mathcal{N}_1$ on top of $\mathcal{B}$. In other words, the procedure constructs an isomorphic copy of $(\mathcal{N}_0 \oplus_0 \mathcal{B} \oplus_0 \mathcal{N}_1)$ in the box a.

The strategy for meeting $\mathcal{R}$ is the same as in Theorem 4.1.

Strategy for meeting $\mathcal{S}_j$. Suppose that σ is an $\mathcal{S}_j$-strategy. The node σ has outcomes ∞, `0`, `1`, `2`, ... Let s be a current stage, and k be the number of stages less than s at which σ had outcome ∞. Suppose that

$$\Theta_j = \{(\forall^\infty y)\psi_i(\bar{x}_i, \bar{c}_j, y) : i \in \omega\},$$

where every ψ_i is a computable relation. In the description below, we assume that all newly added elements do not belong to $\bar{c}_j$. Set $v_0 = w_0 = -1$ and $q_0 = 3$.

(1) Let $t < s$ be the last stage at which σ was accessible. If there is no such stage, then set $t = 0$.
(2) If $t = 0$ (i.e., s is the first stage at which σ is visited), then choose a large number r. For each $i \in \{1, 2\}$, proceed as follows. Find the least number m_i such that every m_i-colored box is empty. Denote m_i by $c_i(\sigma)$. Let a_i be the m_i-colored box with the least number. Put the lattice $\mathcal{D}^0(2^r)$ into the box a_i. Fix the tuple $\bar{b}_i$ which constitutes the universe of the newly added copy of $\mathcal{D}^0(2^r)$. If $i = 1$, then do $[0, \mathcal{D}^0(2^r \cdot 3)]$-gluing for the m_1-colored box. If $i = 2$, then do $[\mathcal{D}^0(2^r \cdot 3), 0]$-gluing for the m_2-colored box.
(3) If $t \neq 0$ and σ had outcome ∞ at stage t, then we do $[\mathcal{D}^0(2^r(q_t-2)), 0]$-gluing for all $c_1(\sigma)$-colored boxes, and $[0, \mathcal{D}^0(2^r(q_t-2))]$-gluing for all $c_2(\sigma)$-colored boxes.
(4) Find the least (under Gödel numbering) pair (v_s, w_s) such that $v_s \leq s$, $w_s \leq s$, and both $\psi_{w_s}(\bar{b}_1, \bar{c}_j, y)$ and $\psi_{w_s}(\bar{b}_2, \bar{c}_j, y)$ are true for all y with $v_s \leq y \leq s$. If there is no such pair, then set $v_s = w_s = -1$.
(5) If $t = 0$ or $(v_s, w_s) = (v_t, w_t)$, then σ has outcome $\mathtt{k}$ and $q_s = q_t$.
(6) Otherwise, do $[0, \mathcal{D}^0(2^r(q_t + 2))]$-gluing for all $c_1(\sigma)$-colored boxes and do $[\mathcal{D}^0(2^r(q_t + 2)), 0]$-gluing for all $c_2(\sigma)$-colored boxes. The strategy σ has outcome ∞ and $q_s = q_t + 2$.

Strategy for meeting $\mathcal{I}_e$. Let σ be an $\mathcal{I}_e$-node in the tree of strategies. The actions of σ are the same as in Theorem 4.1, modulo the following modifications in the definition of the structure $B(\sigma, s)$:

- In (i), we need to consider strategies $\tau \subset \sigma$ such that $\tau\widehat{\ }\mathtt{m} \subseteq \sigma$ for some $m \in \omega$.
- In (ii), we deal with strategies $\tau \subset \sigma$ such that $\tau\widehat{\ }\infty \subseteq \sigma$. Obviously, here we need not talk about Step (5).

Construction. Again, we order the outcomes as $\infty < \ldots < \mathtt{2} < \mathtt{1} < \mathtt{0}$, and the 0th level of our priority tree is devoted to the $\mathcal{R}$-strategy. As usual, the tree of strategies is organized in some effective fashion, and the strategies act in order of priority.

Verification. Note that the constructed structure $\mathcal{A}$ is a shuffle sum of distributive lattices.

Lemma 5.1. *The lattice $\mathcal{A}$ is computably categorical.*

Proof. Suppose that $\mathcal{M}_e \cong \mathcal{A}$ and σ is an $\mathcal{I}_e$-strategy along the true path. Consider an $\mathcal{S}_j$-strategy τ and a lattice $\mathcal{N} \subseteq \mathcal{A}$ such that $\mathcal{N}$ lies in a $c_i(\tau)$-colored box for some $i \in \{1, 2\}$. First, we need to prove the following:

Claim. For every $x \in \mathcal{N}$, $f^\sigma(x)$ is eventually defined. Moreover, $f_\sigma(\mathcal{N})$ is a summand in the shuffle sum $\mathcal{M}_e$.

We consider the only non-trivial case for τ: we assume that $\tau \supseteq \sigma\widehat{\ }\infty$. Let s_0 be the first stage at which τ was accessible. For simplicity, suppose that our τ chooses $r = 0$, and $\mathcal{N}[s_0] \cong \mathcal{D}^0(1, 3)$.

Since $\mathcal{M}_e$ is isomorphic to $\mathcal{A}$, there is the least stage $s_0' > s_0$ such that $\mathcal{N}[s_0] \subseteq dom(f_\sigma[s_0'])$. Similarly to Lemma 4.2, one can show that all the elements of $f_\sigma(\mathcal{N})[s_0']$ lie in the same summand of $\mathcal{M}_e$. Hence, if τ never has outcome ∞, then f_σ is correct on $\mathcal{N}$.

Let s_1 be the first stage at which τ has outcome ∞. Evidently, $s_1 \geq s'_0$. We have $\mathcal{N}[s_1] \cong \mathcal{D}^0(1,3,5)$, and after stage s_1, the strategy σ tries to extend f_σ to $\mathcal{N}[s_1]$. If σ fails to do it, then we consider the summand $\mathcal{B}$ in $\mathcal{M}_e$ which contains $f_\sigma(\mathcal{N})[s'_0]$. The failure of σ implies that the lattice $\mathcal{B}$ is isomorphic neither to $\mathcal{D}^0(1,3,5)$, nor to $\mathcal{D}^0(5,3,1)$. Thus $\mathcal{M}_e \not\cong \mathcal{A}$; a contradiction. Therefore, there is the first stage $s'_1 > s_1$ such that $\mathcal{N}[s_1] \subseteq dom(f_\sigma[s'_1])$.

If τ is never visited after stage s_1, then again, f_σ is correct on $\mathcal{N}$. Therefore, consider the first stage $s_2 > s_1$ at which τ is visited. Note that $s_2 \geq s'_1$ and $\mathcal{N}[s_2] \cong \mathcal{D}^0(3,1,3,5)$. If σ fails to extend f_σ to $\mathcal{N}[s_2]$, then the summand $\mathcal{B}$ is isomorphic neither to $\mathcal{D}^0(3,1,3,5)$, nor to $\mathcal{D}^0(5,3,1,3)$. This implies that $\mathcal{M}_e \not\cong \mathcal{A}$. Hence, we can find the least $s'_2 > s_2$ such that $\mathcal{N}[s_2] \subseteq dom(f_\sigma[s'_2])$. Using the same reasoning, one proceeds to costruct the sequence of stages $s_0 < s'_0 \leq s_1 < s'_1 \leq \dots$ such that for every $m \in \omega$,

$$\mathcal{N}[s_{2m+1}] \cong \mathcal{N}[s_{2m}] \oplus_0 \mathcal{D}^0(2m+5), \quad \mathcal{N}[s_{2m+2}] \cong \mathcal{D}^0(2m+3) \oplus_0 \mathcal{N}[s_{2m+1}],$$

and $\mathcal{N}[s_m] \subseteq dom(f_\sigma[s'_m])$.

The argument above shows that the structure $\mathcal{N}$ satisfies one of the two cases:

(a) $\mathcal{N}$ is isomorphic to one of the lattices from (A)–(D), and there is a stage s^* such that $\mathcal{N}[s^*] \subseteq dom(f_\sigma[s^*])$ and $\mathcal{N}[s] = \mathcal{N}[s^*]$ for all $s \geq s^*$; or

(b) $\mathcal{N}$ is isomorphic to the lattice from (E), and for every $x \in \mathcal{N}$, there is a stage s such that $x \in dom(f_\sigma[s])$.

In both cases, the mapping f_σ is correct on $\mathcal{N}$. This concludes our informal explanation.

Using the claim, it is not difficult to prove that the node $\sigma\widehat{\ }\infty$ belongs to the true path and f_σ is a computable isomorphism from $\mathcal{A}$ onto $\mathcal{M}_e$. □

Lemma 5.2. *The structure $\mathcal{A}$ is not relatively Δ^0_2-categorical.*

Proof. Suppose that Θ_j is a Scott family for the lattice $\mathcal{A}$. Let σ be the $\mathcal{S}_j$-strategy along the true path. Fix the tuples $\bar{b}_1$ and $\bar{b}_2$ from Step (2) for the node σ.

First, assume that there is $\psi \in \Theta_j$ such that

$$\mathcal{A} \models \psi(\bar{b}_1) \ \& \ \psi(\bar{b}_2). \tag{1}$$

Fix the least i such that the formula $\psi = (\forall^\infty y)\psi_i(\bar{x}_i, \bar{c}_j, y)$ satisfies (1). We find a stage s^* and a number v^* such that $(v_s, w_s) = (v^*, i)$ for all $s \geq s^*$. Let m be the number of all stages $t \leq s^*$ such that σ has outcome ∞ at stage t. After stage s^*, the outcome of σ is always $\mathtt{m}$. This implies that the tuple $\bar{b}_1$ lies in one of the lattices from (B), and $\bar{b}_2$ belongs to a lattice from (A). Hence, the tuples $\bar{b}_1$ and $\bar{b}_2$ are not automorphic, and Θ_j cannot be a Scott family for $\mathcal{A}$.

Now assume that there is no formula $\psi \in \Theta_j$ with the property (1). Then the value (v_s, w_s) changes infinitely often, and the true outcome of σ is ∞. Thus, each $\bar{b}_i$ lies in the infinite lattice from (E), and the tuples $\bar{b}_1$ and $\bar{b}_2$ are automorphic. This contradicts the assumption that Θ_j is a Scott family for $\mathcal{A}$. □

This concludes the proof of Theorem 1.2. □

References

[1] *J.F. Knight, M. Stob*, Computable Boolean algebras, J. Symb. Log., 65:4 (2000), 1605–1623.
[2] *A.G. Melnikov*, New degree spectra of abelian groups, Notre Dame J. Form. Logic, to appear.

[3] *D.R. Hirschfeldt, B. Khoussainov, R. A. Shore, A. M. Slinko*, Degree spectra and computable dimensions in algebraic structures, Ann. Pure Appl. Logic, 115:1–3 (2002), 71–113.

[4] *Fokina E.B., Friedman S.-D.*, Equivalence relations on classes of computable structures. In: K. Ambos-Spies, B. Löwe, and W. Merkle (editors), Mathematical Theory and Computational Practice, Proc. CiE 2009, (Lect. Notes Comp. Sci., 5635), 198–207. Springer, Berlin, 2009.

[5] *M. Harrison-Trainor, A. Melnikov, R. Miller, A. Montálban*, Computable functors and effective interpretability, J. Symb. Log., to appear.

[6] *A. Montálban*, Computability theoretic classifications for classes of structures, Proceedings of ICM 2014, to appear. URL: `https://math.berkeley.edu/~antonio/papers/ClassesOfStructures.pdf`

[7] *R.G. Downey*, Computability theory and linear orderings. In: Yu.L. Ershov, S.S. Goncharov, A. Nerode, and J.B. Remmel (editors), Handbook of Recursive Mathematics, vol. 2, (Stud. Logic Found. Math., 139), 823–976. Elsevier Sci. B.V., Amsterdam, 1998.

[8] *A. Frolov, V. Harizanov, I. Kalimullin, O. Kudinov, R. Miller*, Spectra of high_n and non-low_n degrees, J. Log. Comput., 22:4 (2012), 755–777.

[9] *L.J. Richter*, Degrees of structures, J. Symb. Log., 46:4 (1981), 723–731.

[10] *R. Miller*, The Δ^0_2-spectrum of a linear order, J. Symb. Log., 66:2 (2001), 470–486.

[11] *S.S. Goncharov, V.D. Dzgoev*, Autostability of models, Algebra Logic, 19:1 (1980), 28–37.

[12] *J.B. Remmel*, Recursively categorical linear orderings, Proc. Am. Math. Soc., 83:2 (1981), 387–391.

[13] *C.J. Ash*, Recursive labelling systems and stability of recursive structures in hyperarithmetical degrees, Trans. Am. Math. Soc., 298:2 (1986), 497–514.

[14] *C.F.D. McCoy*, Δ^0_2-categoricity in Boolean algebras and linear orderings, Ann. Pure Appl. Logic, 119:1–3 (2003), 85–120.

[15] *Ch.F.D. McCoy*, Partial results in Δ^0_3-categoricity in linear orderings and Boolean algebras, Algebra Logic, 41:5 (2002), 295–305.

[16] *A.N. Frolov*, Effective categoricity of computable linear orderings, Algebra Logic, 54:5 (2015), 415–417.

[17] *V.L. Selivanov*, Algorithmic complexity of algebraic systems, Math. Notes, 44:6 (1988), 944–950.

[18] *A. Turlington*, Computability of Heyting algebras and distributive lattices. PhD thesis, University of Connecticut, 2010.

[19] *N.A. Bazhenov*, Effective categoricity for distributive lattices and Heyting algebras, submitted.

[20] *S. Wehner*, Enumerations, countable structures and Turing degrees, Proc. Am. Math. Soc., 126:7 (1998), 2131–2139.

[21] *J.F. Knight*, Degrees coded in jumps of orderings, J. Symb. Log., 51:4 (1986), 1034–1042.

[22] *T.A. Slaman*, Relative to any nonrecursive set, Proc. Am. Math. Soc., 126:7 (1998), 2117–2122.