DECOMPOSITIONS OF DECIDABLE ABELIAN GROUPS

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ABSTRACT. We use computability-theoretic tools to measure the complexity of recognising a direct decomposition of an abelian group from its symbolic presentation. More specifically, we compare degrees of decidable categoricity of abelian groups and their natural direct summands.

We prove the following results. If G is a decidable homogeneous completely decomposable group, then the degree of decidable categoricity of G is either 0 or $0'$. As a non-trivial and unexpected application of our methods, we show that every decidable copy of a non-divisible homogeneous completely decomposable group has an algorithm for linear independence. We then look at torsion abelian groups and compare them to abelian p-groups. We prove that every decidable reduced abelian p-group of a finite Ulm type has degree of decidable categoricity $\mathbf{c} \in \{\mathbf{0}^{(n)} : n \in \omega\}$. In contrast, for every Turing degree $\mathbf{d}$ d.c.e. in and above $\mathbf{0}^{(2k+1)}$ there is a decidable reduced torsion abelian group G of a finite Ulm type such that G has degree of decidable categoricity $\mathbf{d}$.

1. Introduction

It is well-known that direct decompositions play a central role in the classification of broad natural subclasses of abelian groups. For instance, each torsion abelian group splits into the direct sum of its maximal p-subgroups. It may seem obvious that the classification of countable torsion abelian groups is thus completely reduced to the Ulm classification of their p-components. Also, consider a direct power of some fixed $H \leqq (\mathbb{Q}, +)$, say $G = \bigoplus_{i \in \omega} H$. Such groups are called homogeneous completely decomposable. It is natural to assume that classifying homogeneous completely decomposable groups is essentially the same as classifying subgroups of the rationals by their Baer types [3]. Interestingly, the classifications of torsion and homogeneous completely decomposable abelian groups are not necessarily as well-behaved as the classifications of their natural summands; details below. We have only started to understand these subtle properties of direct decompositions which are related to definability and computability. In this article we use methods of *computable structure theory* to systematically investigate into these effects.

Computable structure theory [1, 16] is a modern subject in mathematical logic invented by Mal'tsev [31, 32] and Rabin [37] in the 1960s. The main objects of such studies are *computable algebraic structures*. An algebraic structure is *computable* if its domain can be coded in such a way that the algebraic operations become Turing computable upon the codes of elements; such a coding is called a *computable copy* or a *constructivisation* of the structure. For example, a group-presentation with solvable word problem [30] can be viewed as a computable copy of the group, see [33] for details. For almost 60 years computable structure theory was a self-motivated subject that was focused on algorithmic aspects of algebra. Recently the

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theory has found applications beyond computable mathematics, most notably in relation to Vaught Conjecture in model theory [35] and to classification problems in commutative algebra [23, 14, 15, 38].

How is Turing computability related to classification? We give several examples coming from abelian group theory. In the 1970s, Fuchs and others discovered pathological decompositions and proved the existence of "large" indecomposable groups [21]. These discoveries suggested that the approach to classification of abelian groups up to isomorphism via direct decomposition has rather significant limitations, see the chapter on indecomposable groups in [21] for a discussion. Recently, this intuition has been formally clarified by Riggs [38] who used computability-theoretic tools to show that there is no reasonable characterisation of countable directly indecomposable abelian groups. The decomposability problem is naturally Σ^1_1, meaning that it involves a *second-order* existential quantification over all subgroups of the given group. Riggs showed that the problem is Σ^1_1-*complete*; this means that any other Σ^1_1-problem can be reduced to the decomposability problem for countable abelian groups. Although the result of Riggs is formally stated in terms of computable groups, it can be re-formulated in purely syntactical or topological terms to cover all countable groups. Thus, no "local" algebraic condition can possibly capture direct decomposability of a countable abelian group.

In contrast to indecomposable groups, *completely decomposable* groups have a rather developed algebraic structural theory, see [3, 21, 28]. A group is completely decomposable if it splits into a direct sum of additive subgroups of the rationals. Thier classification theory works well if we have access to at least one full decomposition of a given group. However, suppose all we have is a group presentation X in some general sense, e.g., by generators and relations. Is X completely decomposable? It seems that asking whether X admits a complete decomposition is a proper second-order question, just as in the case of indecomposable groups. Using methods of computable structure theory, Downey and Melnikov [15] showed that it takes six alternations of merely *first-order* quantifiers to say that a group is completely decomposable [15]. This upper bound is a lot better than Σ^1_1 for indecomposable groups; no finite number of first-order quantifiers can possibly capture a Σ^1_1-complete property. Completely decomposable groups do admit a reasonable classification, and the result of Downey and Melnikov formally measures the complexity of this classification.

If we can *measure* the complexity of a classification problem, then we can *compare* the complexities of two such problems. For instance, is classifying countable completely decomposable groups harder than describing their elementary summands? Here the situation is somewhat unexpected. Consider the *homogeneous* case, that is, assume that all the elementary summands of a completely decomposable group are isomorphic to some fixed $H \leqq \mathbb{Q}$. Downey and Melnikov [11] proved that reconstructing a full decomposition of a homogeneous completely decomposable $G = \oplus_i H$ based solely on its presentation requires an analysis of three first-order quantifiers, and this upper bound is optimal. In stark contrast, enumerating the Baer type of $H \leqq \mathbb{Q}$ is a quantifier-free process. Therefore, any complete decomposition of $\oplus_i H$ encodes a lot more information about the group than the Baer invariant of its elementary building block H.

In the results discussed above, the mentioned first-order quantification is often *external* in the following sense. For example, saying that two elements a and b of an abelian group are linearly (Prüfer) independent is the same as asking whether there exist integers $m, n \neq 0$ such that $ma+nb = 0$. This statement involves an existential quantification over m and n, but integers are not in the language of additive groups. However, from the perspective of Turing computability, searching for a witness in the group is the same as searching for a Gödel index of a formula of the form $ma + nb = 0$ that holds in the group. This *external* formula can be transformed into a first-order statement within the hereditarily finite expansion of the group, as clarified in [18]. Another way of viewing such formulae involves computable infinite conjunctions and disjunctions; the resulting language $\mathcal{L}^c_{\omega_1\omega}$ enjoys several nice syntactical properties such as a variant of compactness [1].

It is natural to ask if the subtle effects of direct decompositions discussed above can be described using only *internal* first-order properties, in the usual model-theoretic sense. To make the question formal we use the notion of a *decidable group* suggested by Ershov [17]. In a decidable group there is an algorithm that decides (internal) first-order statements about tuples of its elements. A computable copy with such an algorithm is called a *strong constructivisation* of the group, or its *decidable copy.* Decidable algebraic structures – and decidable abelian groups in particular – are much better behaved than the more general computable structures [17, 36, 27, 5]. Hopefully, having access to the first-order diagram of a group will allow for a much simpler description of decomposability.

Our first main result partially confirms our intuition. It is stated in terms of (Turing) degrees of categoricity [20]. The degree of categoricity of a computable algebraic structure $\mathcal{A}$ is the least Turing degree – if it exists – that can compute an isomorphism between any two computable copies of $\mathcal{A}$. Similarly, the degree of decidable categoricity [24] of a decidable $\mathcal{A}$ is the least Turing degree that can compute an isomorphism between any two decidable copies of $\mathcal{A}$. There is a tight connection between Scott families – these are $\mathcal{L}^c_{\omega_1\omega}$ back-and-forth invariants of a group – and the number of iterations of the Halting problem necessary to compute its degree of (decidable) categoricity. For example, modulo a mild assumption on uniformity, $DegCat(G) = \mathbf{0^{(n)}}$ says that the automorphism orbits of tuples $\bar{g} \in G$ can be uniformly described by $\mathcal{L}^c_{\omega_1\omega}$-formulae of complexity Σ^c_{n+1} where $n + 1$ is optimal. (As usual, $\mathbf{0^{(n)}}$ stands for the nth iterate of the Halting problem.)

Obviously, the degree of categoricity of any $H \leqq \mathbb{Q}$ is $\mathbf{0}$, which is the Turing degree of computable sets. In contrast, Downey and Melnikov [11] constructed examples of homogeneous completely decomposable groups having $\mathbf{0}''$ as their degree of categoricity. It is not hard to see the their result is equivalent to saying that calculating a complete full decomposition of such group may require $\mathbf{0}''$. Their result can be fully relativised and re-stated in terms of $\mathcal{L}^c_{\omega_1\omega}$; we omit it. We prove:

Theorem 1.1. *The degree of categoricity of any decidable homogeneous completely decomposable group is either* $\mathbf{0}$ *or* $\mathbf{0}'$*, where both possibilities can be realised.*

The result is relativisable, so it is not really restricted to (Turing) decidable copies. It follows that allowing access to the first-order diagram of a group removes some of the pathology related to decomposition, but not all of it. However, our methods imply that decidable homogeneous completely decomposable groups are rather well-behaved. Our second main result illustrates one such pleasant application of our techniques to independent sets.

Recall that linear (Prüfer) independence is not an "internal" first-order property. Also, a Prüfer independent set does not have to agree with any complete decomposition of a completely decomposable group unless the group is divisible. In our search for a uniform proof of Theorem 4.3 we discovered the following highly unexpected result which connects the seemingly unrelated properties of direct decomposability, decidability, and Prüfer independence.

Theorem 1.2. *Suppose that G is a homogeneous completely decomposable group which is not divisible. Then every decidable copy of G has an algorithm for linear independence.*

It is well-known that every computable torsion-free abelian group has a computable copy with a computable basis [10, 25], but this is typically *some other* copy of the group. We emphasise that the theorem above holds for *every* decidable copy of the group. Both the result itself and its proof are somewhat counter-intuitive. We conjecture that omitting at least one of the three premises in the statement of the theorem will allow for a counter-example. Furthermore, it seems that we cannot significantly relax the algebraic assumption on the group G in the theorem.

As a consequence of Theorem 4.4, the proof of Theorem 4.3 must be non-uniform for the following reason. The only diagonalisation tool in the case of divisible abelian groups is linear independence. However, it follows that in the non-divisible case we provably cannot use independence to diagonalise. Non-uniformity of proofs seems to be somewhat typical in the study of decidable abelian groups. The proof of our third main result on torsion groups will be "even more" non-uniform.

We turn to a discussion of torsion abelian groups. Recall that a torsion abelian group splits into the direct sum of its maximal p-subgroups. This elementary fact typically justifies the almost total elimination of torsion abelian groups from algebraic texts. Clearly, the decomposition into p-components is uniformly computable, so it may seem that the computable theory of torsion abelian groups can also be completely reduced to that of p-groups. This is unfortunately not the case. Computably categorical abelian p-groups – these are the groups whose degree of categoricity is $\mathbf{0}$ – admit an explicit algebraic description [40, 22, 33]. In contrast, Melnikov and Ng [34] have recently showed that no such explicit description is possible in the general torsion case. Intuitively, each separate p-component of a torsion abelian group can be used as an individual coding location, while abelian p-groups are "smooth".

Computable categoricity is not relativisable for torsion abelian groups [34], but the induced notion of decidable categoricity is relativisable for arbitrary structures [36]. With this in mind, we sought for a positive result saying that decidable torsion and decidable p-groups are indistinguishable from the point of view of categoricity. Our analysis of Ulm type 1 groups supported this intuition, see Section 3.1. Nonetheless, we discovered that degrees of decidable categoricity are *not* preserved under taking direct sums of p-groups of Ulm type > 1, and therefore the discussed above pathologies are intrinsically *not* first order in the model-theoretic sense.

Theorem 1.3. Let $\mathcal{K}$ be the class of all decidable torsion abelian groups of finite Ulm type[1].

[1]This means that the types of p-components of each such group are bounded by some $n \in \omega$ specific to the group.

(1) If G in $\mathcal{K}$ is a p-group, then its degree of decidable categoricity is of the form $\mathbf{0}^{(n)}$ for some $n \in \omega$.
(2) In contrast, each Turing degree d.c.e. in and above $\mathbf{0}^{(2k+1)}$ (for some $k \in \omega$) is the degree of decidable categoricity of some G in $\mathcal{K}$.

The notion of a d.c.e. in and above degree will be defined in due course; such degrees are much more general than just the finite iterates of the Turing jump in (1) of the theorem.

Recall that we mentioned that our result on torsion groups will be highly non-uniform. The proof of the first clause uses three substantially different strategies in three different cases. The proof also exploits various techniques such as computable p-basic trees [2, 12] and the Ash-Knigh-Oates jump inversion [2]. In order to build decidable presentations of p-groups in (2), we introduce a new jump inversion which allows us to pass from a Δ^0_2 computable group to a decidable one (see Proposition 3.3). All these techniques are specific to groups of finite Ulm type, and usually work only for reduced p-groups. We leave open whether (1) can be extended to the non-reduced case, or beyond groups of finite Ulm type. However, it is not hard to show that the natural upper bounds in (1) will still remain optimal (compare this to Barker [4]), but the coding part in (1) becomes an issue at transfinite levels.

The outline of the paper is as follows. Section 2 contains the preliminaries. In Section 3, prove the results on decidable torsion groups. Section 4 deals with homogeneous completely decomposable groups.

2. Preliminaries

All considered groups are abelian. By $\mathbb{P}$ we denote the set of all prime numbers. For $p \in \mathbb{P}$ and $n \in \omega$, $\mathbb{Z}(p^n)$ denotes the cyclic group of order p^n, and $\mathbb{Z}(p^\infty)$ is the quasicyclic p-group. For $k \in \omega$, p_k is the kth prime number (under the standard ordering).

For a structure $\mathcal{S}$, $Th(\mathcal{S})$ denotes the first-order theory of $\mathcal{S}$. For a string $\sigma \in \omega^{<\omega}$, $len(\sigma)$ denotes the length of σ.

2.1. Decidable categoricity.

Let $\mathbf{d}$ be a Turing degree. A decidable structure $\mathcal{S}$ is *decidably* $\mathbf{d}$*-categorical* if for any decidable copy $\mathcal{A}$ of $\mathcal{S}$, there is a $\mathbf{d}$-computable isomorphism from $\mathcal{A}$ onto $\mathcal{S}$. If $\mathbf{d} = \mathbf{0}$, then we say that $\mathcal{S}$ is *decidably categorical*.

Decidably categorical structures have a nice model-theoretic characterization [36]. To state it we need a few more definitions. Suppose that L is a countable language, and $\mathcal{M}$ is an L-structure. A first-order L-formula $\psi(x_0, \dots, x_n)$ is a *complete formula* of the theory $Th(\mathcal{M})$ if $\mathcal{M} \models \exists \bar{x}\psi(\bar{x})$ and for any L-formula $\varphi(\bar{x})$, we have either $\mathcal{M} \models \forall \bar{x}[\psi(\bar{x}) \to \varphi(\bar{x})]$, or $\mathcal{M} \models \forall \bar{x}[\psi(\bar{x}) \to \neg\varphi(\bar{x})]$. A structure $\mathcal{M}$ is an *atomic model* if any tuple $\bar{a}$ from $\mathcal{M}$ satisfies a complete formula of $Th(\mathcal{M})$. Recall that a structure $\mathcal{M}$ is a *prime model* (of the theory $Th(\mathcal{M})$) if $\mathcal{M}$ is elementary embeddable into any model of $Th(\mathcal{M})$. It is well-known (see, e.g., [8]) that $\mathcal{M}$ is a prime model if and only if $\mathcal{M}$ is a countable atomic model. A structure $\mathcal{M}$ is called an *almost prime model* if there is a finite tuple $\bar{c}$ from $\mathcal{M}$ such that $(\mathcal{M}, \bar{c})$ is a prime model.

Theorem 2.1 (Nurtazin's criterion [36])**.** *Suppose that $\mathcal{M}$ is a decidable structure in a language L. Then $\mathcal{M}$ is decidably categorical if and only if there is a finite tuple $\bar{c}$ from $\mathcal{M}$ with the following properties:*

a) *$(\mathcal{M}, \bar{c})$ is a prime model;*
b) *given an $(L \cup \{\bar{c}\})$-formula $\psi(\bar{x})$, one can effectively check whether ψ is a complete formula of the theory $Th(\mathcal{M}, \bar{c})$.*

In particular, any decidably categorical structure is an almost prime model.

The *decidable categoricity spectrum* of a decidable structure $\mathcal{S}$ is the set

$$DecCatSpec(\mathcal{S}) = \{\mathbf{d} : \mathcal{S} \text{ is decidably } \mathbf{d}\text{-categorical}\}.$$

If $\mathbf{c}$ is the least degree in the spectrum $DecCatSpec(\mathcal{S})$, then $\mathbf{c}$ is called the *degree of decidable categoricity* of $\mathcal{S}$.

Goncharov [24] initiated the systematic investigation of decidable categoricity spectra. In particular, Goncharov [24] showed that any computably enumerable (c.e.) Turing degree is a degree of decidable categoricity for some prime model. In [5] this result was extended to hyperarithmetical degrees: If α is a computable successor ordinal and $\mathbf{d}$ is a degree such that $\mathbf{d} \geq \mathbf{0}^{(\alpha)}$ and $\mathbf{d}$ is c.e. in $\mathbf{0}^{(\alpha)}$, then $\mathbf{d}$ is a degree of decidable categoricity.

Note that decidable categoricity is closely connected to computable categoricity: For a Turing degree $\mathbf{d}$, a computable structure $\mathcal{S}$ is $\mathbf{d}$-*computably categorical* if for any computable presentation $\mathcal{A}$ of $\mathcal{S}$, there is a $\mathbf{d}$-computable isomorphism from $\mathcal{A}$ onto $\mathcal{S}$. The *degree of categoricity* of $\mathcal{S}$ is the least degree $\mathbf{d}$ such that $\mathcal{S}$ is $\mathbf{d}$-computably categorical. For various results on degrees of categoricity, see [6, 9, 19, 20].

2.2. **Decidable abelian groups.** For a non-zero $n \in \omega$, let $(n \,|\, \cdot)$ be the following unary predicate:

$$(n \mid x) \text{ iff } \exists y(ny = x).$$

A subgroup A of G is *pure* if for all $a \in A$ and $n \in \omega \setminus \{0\}$, $(n \mid a)$ in G implies that $(n \mid a)$ in A. If S is a subset of a torsion-free G, then $\langle S \rangle_*$ denotes the least pure subgroup of G that contains S. (If the group is not torsion-free then there may be no such least pure subgroup.)

Proposition 2.1 (Ershov [17], see also [27, Proposition 1.1])**.** *A countable abelian group G is decidable if and only if the theory $Th(G)$ is decidable and the structure $(G, p^k \,|\, \cdot)_{p\in\mathbb{P},\, k\in\omega}$ is computable.*

Following [17, Chap. 3, § 4] and [27, § 7.1], we recall the definitions of Szmielew invariants. Let G be an abelian group and $G\{p^n\} = \{x \in G : p^n x = 0\}$. For $p \in \mathbb{P}$ and non-zero $n, k \in \omega$, one can define the following first-order formulas:

(1) $G \models A_{p,n,k}$ iff G contains $\oplus_{1\leq i\leq k}\mathbb{Z}(p^n)$ as a pure subgroup. ($A_{p,n,k}$ is an $\exists\forall$-formula.)
(2) $G \models B_{p,n,k}$ iff $\oplus_{1\leq i\leq k}\mathbb{Z}(p^n)$ is a subgroup of G. ($B_{p,n,k}$ is an $\exists$-formula.)
(3) $G \models C'_{p,n,k}(g_1, \dots, g_k)$ iff the elements $g_1, \dots, g_k \in G$ satisfy the following: If we consider their images $g'_1, \dots, g'_k$ in the quotient group $\overline{G} = G/G\{p^n\}$, then $g'_1 + p\overline{G}, \dots, g'_k + p\overline{G}$ are linearly independent in $\overline{G}/p\overline{G}$. ($C'_{p,n,k}$ is an $\forall$-formula.)
(4) $C_{p,n,k} = \exists x_1 \dots \exists x_k C'_{p,n,k}(x_1, \dots, x_k)$. (This is an $\exists\forall$-formula.)

The *Szmielew invariants* of G are defined as follows.

a) $\alpha_{p,n}(G) = \sup\{k \in \omega : G \models A_{p,n,k}\}$.
b) $\beta_p(G) = \inf\{\sup\{k \in \omega : G \models B_{p,n,k}\} : n \in \omega\}$.
c) $\gamma_p(G) = \inf\{\sup\{k \in \omega : G \models C_{p,n,k}\} : n \in \omega\}$.

Szmielew [41] proved that abelian groups G and H are elementarily equivalent if and only if G and H have the same Szmielew invariants.

2.3. **Effective algebra for torsion groups.** Here we give a brief overview of effective algebraic techniques that will be used in the proofs of the results on torsion groups.

Let A be an abelian group. The *p-height* of an element $a \in A$ is defined as follows:

$$h_p^A(a) = \begin{cases} \text{maximal } k \in \omega \text{ such that } p^k \mid a \text{ in } A, & \text{if such a } k \text{ exists,} \\ \text{infinite,} & \text{otherwise.} \end{cases}$$

We say that the p-height h_p^A is *computable* if it is a computable function from the domain of A into $\omega \cup \{\infty\}$. If the group A is clear from the context, then we will omit the superscript A in h_p^A.

For a p-group A, the elements of infinite p-height generate a subgroup A' of A. Iterating this process define a subgroup $A^{(\alpha)}$ for every ordinal α. The non-zero elements of $A_\alpha = A^{(\alpha)}/A^{(\alpha+1)}$ have finite height. If A is countable, then there is a countable α such that

$$A^{(\alpha)} = A^{(\alpha+1)}.$$

The least such α is called the *Ulm type* of the p-group A.

One can show that the *Ulm factors* A_α are direct sums of cyclic p-groups. This leads to the following well-known classification of isomorphism types:

Theorem 2.2 (Ulm)**.** *The isomorphism type of a countable reduced p-group is completely determined by the isomorphism types of its Ulm factors.*

Call a countable p-group A *multicyclic* if it is isomorphic to a direct sum of cyclic and quasicyclic p-groups, i.e.

$$A \cong \bigoplus_{0<k<\omega} \Big(\bigoplus_{0\le i<n_k} \mathbb{Z}(p^k)\Big) \oplus \bigoplus_{0\le i<m} \mathbb{Z}(p^\infty),$$

where $0 \le n_k \le \omega$ and $0 \le m \le \omega$. The *character* of a multicyclic p-group A is the set

$$\chi(A) = \{(k,l) : 0 < k < \omega,\ 0 < l < 1 + n_k\}.$$

We say that the character $\chi(A)$ is *bounded* if there is a natural number K such that $k \le K$ for any pair (k,l) from $\chi(A)$. A set $X \subseteq \omega^2$ is called a *character* if $X = \chi(B)$ for some multicyclic p-group B.

Proposition 2.2 (obtained in the proof of [13, Theorem 3.1])**.** *Let A be a computable multicyclic p-group. Then one can find, effectively in h_p^A, a complete decomposition $\bigoplus_i V_i$ of A into cyclic and quasicyclic summands. Furthermore, the procedure is uniform in p, A, and h_p^A.*

We will use the technique of p-basic trees to work with p-groups.

Definition 2.1 ([39])**.** A *p-basic tree* is a set X together with a binary operation $p^n \cdot x$ of the sort $\{p^n : 0 < n < \omega\} \times X \to X$ such that:

(1) there is a unique element $0 \in X$ for which $p \cdot 0 = 0$,

(2) $p^k \cdot (p^m \cdot g) = p^{k+m} \cdot g$, for all $g \in X$ and $k, m \in \omega$, and
(3) for every element $x \in X$, there is a natural number n with $p^n \cdot x = 0$.

If a prime p is fixed, then one can think of a p-basic tree as a rooted tree with 0 being the root. Given a p-basic tree X, one obtains a p-group $G(X)$ as follows: The set $X \setminus \{0\}$ is treated as the set of generators for $G(X)$, and we add $px = y$ into the collection of relations if $p \cdot x = y$ in X. Every countable abelian p-group is generated by some p-basic tree [39]. Each element of the group $G(X)$ can be uniquely expressed as $\sum_{x \in X} m_x x$, where $m_x \in \{0, 1, \dots, p-1\}$.

Lemma 2.1. *Let T be a computable p-basic tree such that the relation*

$$TR(x, k) := (\textit{the tree rank of } x \textit{ is at least } k), \ \textit{where } x \in T, \ k \in \omega,$$

is computable. Then the p-group $G(T)$ is decidable if and only if the theory $Th(G(T))$ is decidable.

Proof. Suppose that $Th(G(T))$ is decidable. Given a non-zero element $g \in G(T)$, one can effectively find the elements $x_0, \dots, x_n \in T$ such that $g = \sum_{i \leq n} m_i x_i$, where $m_i \in \{1, 2, \dots, p-1\}$. Then the condition $(p^k \mid g)$ holds if and only if the tree rank of every x_i is at least k. Hence, by Proposition 2.1, the group $G(T)$ is decidable. □

Non-isomorphic trees can produce isomorphic p-groups. Here we will not give a complete description of the congruence relation $\sim$ on rooted trees which is defined by the rule: $T \sim X$ iff the groups $G(T)$ and $G(X)$ are isomorphic. The reader is referred to [39] for the detailed analysis of $\sim$. Instead, we will describe an elementary transformation of a tree T which does not change the isomorphism type of the corresponding p-group $G(T)$.

Suppose that T is a p-basic tree (treated as a rooted tree). Consider the following procedure:

Take a simple chain extending $v \in T$, detach it, and
attach this chain to the root of T.

The procedure is called *stripping*. If the tree rank of v does not change after stripping, then the stripped tree T_1 and the original tree T give rise to isomorphic p-groups: $G(T_1) \cong G(T)$. This process can be iterated: informally speaking, one can replace infinitely many simple chains at once (while preserving tree ranks), and obtain a *fully stripped tree* representing the same group. For example, a fully stripped tree for a reduced p-group of Ulm type 1 is just a collection of finite simple chains attached to 0.

In our proofs, we will also use a *more subtle* notion of the *p-height*, sometimes it is called the *ordinal p-height*: Consider a p-basic tree T which represents a p-group A. If $g = \sum_{v \in F} m_v v$, where F is a finite subset of T and $0 < m_v < p$, then the *p-height* of v is the minimum of the tree-ranks of $v \in F$. The definition does not depend on the choice of T, and it can be given by using a direct approach which does not refer to trees at all. Furthermore, if $h_p^A(g) = k < \omega$, then the ordinal p-height of g is equal to k. If $h_p^A(g)$ is infinite, then the ordinal p-height of g is either an infinite ordinal, or ∞. Hence, when it is clear from the context, we will not specify which particular notion of p-height we use.

Theorem 2.3 (Ash, Knight, and Oates [2]; Khisamiev [26]). *Suppose that A is a countable reduced p-group of Ulm type $N < \omega$. Then the following conditions are equivalent:*

(1) *A has a computable copy;*
(2) *A has a computable p-basic tree representing it;*
(3) (a) *for every $i < N$, the character $\chi(A_i)$ is a Σ^0_{2i+2} set, and*
    (b) *for every $i < N$, the set*

$$\#A_i := \{n : (n, 1) \in \chi(A_i)\}$$

is $\mathbf{0}^{(2i)}$-limitwise monotonic.

For a natural number n, a computable structure $\mathcal{S}$ is *relatively Δ^0_n categorical* if for any copy $\mathcal{A} \cong \mathcal{S}$ with domain subset of ω, there is a $\Delta^0_n(\mathcal{A})$-computable isomorphism from $\mathcal{A}$ onto $\mathcal{S}$.

Proposition 2.3 (Barker [4, Proposition 6.6]). *Let k be a natural number, and A be a computable reduced p-group.*

(1) *If $A^{(k+1)}$ is a finite group and the character $\chi(A_k)$ is unbounded, then A is relatively Δ^0_{2k+2} categorical.*
(2) *Suppose that A_k is isomorphic to*

$$F \oplus \bigoplus_{i\in\omega} \mathbb{Z}(p^m),$$

where $0 < m < \omega$ and F is a finite group. Then A is relatively Δ^0_{2k+1} categorical.
(3) *Suppose that $\chi(A_k)$ is bounded and A_k is isomorphic to*

$$B \oplus \bigoplus_{i\in\omega} \mathbb{Z}(p^l) \oplus \bigoplus_{j\in\omega} \mathbb{Z}(p^m),$$

where B is a group and $0 < l < m < \omega$. Then A is relatively Δ^0_{2k+2} categorical.

In particular, any computable reduced p-group of Ulm type $N < \omega$ is relatively Δ^0_{2N} categorical.

3. p-Groups vs. torsion groups

In this section, we show that any decidable reduced p-group has degree of decidable categoricity $\mathbf{c} \in \{\mathbf{0}^{(n)} : n \in \omega\}$ (Theorem 3.1). On the other hand, every Turing degree $\mathbf{d}$ which is d.c.e. in and above $\mathbf{0}^{(2k+1)}$, $k \in \omega$, is a degree of decidable categoricity for some reduced torsion group (Theorem 3.3). We start with the case of Ulm type 1, where we show that there is no difference between degrees of decidable categoricity of p-groups and torsion groups (Proposition 3.1).

3.1. **Ulm type 1.** We start from considering the simplest case — groups of Ulm type 1. The theorem below shows that these groups do not witness the differences that we aim to illuminate.

Proposition 3.1. *Any decidable torsion group of Ulm type 1 has degree of decidable categoricity $\mathbf{d} \in \{\mathbf{0}, \mathbf{0}'\}$.*

The *proof* consists of three parts: Lemma 3.1 establishes the statement of the theorem in the reduced case. Lemma 3.2 considers p-groups of Ulm type 1. In particular, it gives a criterion of decidable categoricity for multicyclic p-groups. At last, Lemma 3.3 uses the preceding two lemmas to complete the proof of the theorem.

Lemma 3.1. *Any decidable reduced torsion group of Ulm type 1 is decidably categorical.*

Proof. Suppose that A is a decidable reduced torsion group of Ulm type 1. For a prime p, consider the p-component A_p of A. Since A has Ulm type 1, A_p is a multicyclic p-group. Note that the structures $(A_p, p^k\,|)_{k\in\omega}$ are computable, uniformly in p.

Let B be a decidable copy of A. It is sufficient to build (uniformly in p) a computable isomorphism f_p from A_p onto B_p. Since A is reduced and decidable, the p-heights $h_p^{A_p}$ and $h_p^{B_p}$ are computable. Using Proposition 2.2, we construct effective complete decompositions $A_p = \bigoplus_{i\in I} U_i$ and $B_p = \bigoplus_{i\in I} V_i$ into cyclic summands. Furthermore, given i, one can effectively calculate the orders of U_i and V_i. Therefore, using a back-and-forth construction, it is not hard to build a computable isomorphism $f_p\colon A_p \cong B_p$. □

Lemma 3.2. *Suppose that A is a decidable p-group of Ulm type 1. Then the following hold:*

(a) *If A is reduced or the character $\chi(A)$ is bounded, then A is decidably categorical.*
(b) *If A is not reduced and $\chi(A)$ is unbounded, then A has degree of decidable categoricity $\mathbf{0}'$.*

In particular, the degree of decidable categoricity for A is either $\mathbf{0}$ or $\mathbf{0}'$.

Proof. (a) The reduced case was already discussed in Lemma 3.1. Suppose that A is a decidable non-reduced multicyclic p-group with a bounded character. Fix a natural number K such that the order of any cyclic summand in A is less than p^K. Then for an element $a \in A$, the condition $h_p(a) = \infty$ is equivalent to $(p^K \mid a)$. Hence, the p-height h_p^A is computable, and one can show, similarly to the proof of Lemma 3.1, that A is decidably categorical.

(b) Let A be a decidable non-reduced multicyclic p-group with an unbounded character. It is easy to show that the character $\chi(A)$ is computable (e.g., using the formulas $A_{p,n,k}$ from the definition of Szmielew invariants). Recall that $h_p(a)$ is infinite if and only if $\forall k(p^k \mid a)$. Therefore, the p-height h_p^A is $\mathbf{0}'$-computable, and again, using an argument similar to that of Lemma 3.1, it is not hard to prove that A is decidably $\mathbf{0}'$-categorical.

Since $\chi(A)$ is a computable set, it is easy to build a copy A_0 of A such that the structure $(A_0, q^k\,|)_{q\in\mathbb{P},k\in\omega}$ is computable: Here we only sketch a construction. For simplicity, assume that the reduced part of A is isomorphic to $\bigoplus_{k\in\omega}\mathbb{Z}(p^{m_k})$, where $0 < m_0 < m_1 < \dots$. Then we take a decidable copy D_0 of the divisible part of A (obtained, e.g, from Proposition 2.1), and set

$$A_0 = D_0 \oplus \bigoplus_{k\in\omega}\langle a_k\rangle,$$

where a_k is an element of the order p^{m_k}.

Since $Th(A_0) = Th(A)$ is decidable, by Proposition 2.1, A_0 is also decidable. Moreover, the p-height $h_p^{A_0}$ is computable. Now, in order to prove that $\mathbf{0}'$ is the degree of decidable categoricity for A, it is sufficient to build a decidable copy B of A with $h_p^B \geq_T \mathbf{0}'$.

For simplicity, we build B for the case when $A \cong \mathbb{Z}(p^\infty) \oplus \bigoplus_{k\in\omega} \mathbb{Z}(p^{m_k})$, where $0 < m_0 < m_1 < \dots$. The general case can be treated in a similar way.

We will construct a computable p-basic tree T for B. Lemma 2.1 shows that it is sufficient to guarantee that for any $g \in T$ and any $k \in \omega$, one can effectively check whether the tree rank of g is at least k. This condition will be ensured by the following property: Let $T[g,s]$ denote the tree with the root g which contains all descendants of g in the tree $T[s]$ (constructed at a stage s). Then any s and any $g \in T[s]$ will satisfy one of the two conditions:

(1) either $T[g,t] = T[g,s]$ for all $t \geq s$, or
(2) the rank of $T[g,s+1]$ is strictly greater than the rank of $T[g,s]$.

Fix a strongly computable sequence of finite sets $\{W^s\}_{s\in\omega}$ such that $\bigcup_s W^s = \emptyset'$, $W^s \subseteq W^{s+1}$, $card(W^{s+1} - W^s) \leq 1$, and $s \notin W^s$ for all s. W.l.o.g., we may assume that $W^{2t} = W^{2t+1}$ for any t.

The construction of T proceeds in stages. At a stage s, we build a finite tree $T[s] \subseteq \omega^{<\omega}$ with root Λ, and we choose a terminal node tn_s in $T[s]$ with the following properties: The length of tn_s is equal to $(s+1)$, and for any $i \leq s$, $tn_s(i)$ is even iff $i \notin W^s$.

Stage 0. Set $T[0] = \{\Lambda, \langle 0\rangle\}$ and $tn_0 = \langle 0\rangle$.

Stage $s+1$. If $W^{s+1} = W^s$, then find the least m_i which has not already been used in the construction. Attach a fresh simple chain of length m_i to the root of $T[s]$. Let $tn_{s+1} := tn_s\widehat{\ }0$, and put tn_{s+1} into $T[s+1]$.

Suppose that $W^{s+1} - W^s = \{k\}$. Note that $k \leq s$. Let $\sigma := tn_s \upharpoonright k$. Find the least unused $m_i > len(tn_s) - k$ with the following property: Consider a tree T_1 which is obtained from $T[s]$ by appending two simple chains — the first one has length $(m_i + k - len(tn_s))$ and is attached to tn_s, and the second one has length $L := m_i + k + 1$ and is attached to σ. Then the fully stripped version of T_1 gives a rise to a p-group $\mathbb{Z}(p^L) \oplus \mathbb{Z}(p^{m_i}) \oplus \bigoplus_{j\in F} \mathbb{Z}(p^{m_j})$ for some finite set F with $i \notin F$.

Attach a fresh simple chain of size $(m_i + k - len(tn_s))$ to tn_s. Define tn_{s+1} as follows:

$$tn_{s+1}(j) := \begin{cases} tn_s(j), & \text{if } j \neq k \text{ and } j \leq s,\\ \text{a fresh odd number}, & \text{if } j = k,\\ 0, & \text{if } j = s+1, \end{cases}$$

and add every $\tau \subseteq tn_{s+1}$ into $T[s+1]$.

This completes the description of the construction. Let $T := \bigcup_{s\in\omega} T[s]$, and consider the p-group $B := G(T)$. It is not difficult to establish the following properties of the construction.

Claim 3.1. (1) *If g is a terminal node in $T[s]$ and $g \neq tn_s$, then g is a terminal node in any $T[r]$, where $r \geq s$.*

(2) *For any $i \in \omega$, there is a limit $tn^*(i) = \lim_s tn_s(i)$. Moreover, if $i \notin \emptyset'$, then $tn^*(i) = 0$. If $i \in \emptyset'$, then $tn^*(i)$ is odd.*

(3) *The tree T has only one infinite path $P = (tn^*(0), tn^*(1), tn^*(2), \dots)$.*

(4) *Recall that $TR(x,k)$ is true iff the tree rank of x is at least k. Suppose that g is an element from $T[s]$. Let $s_1 \geq s$ be the least stage such that either tn_{s_1}*

is not a descendant of g*, or* $T[s_1] \models TR(g,k)$*. Then we have* $T \models TR(g,k)$ *if and only if* $T[s_1] \models TR(g,k)$*.*

The last item of the claim and Lemma 2.1 together imply the following: In order to prove the decidability of the group B, it is sufficient to show that B is isomorphic to A.

Consider $\widehat{T}$, the fully stripped version of the tree T. Roughly speaking, one can obtain the tree $\widehat{T}$ using the following procedure: Whenever we see that tn_{s+1} is not a child of tn_s, we find the least k such that $tn_s(k) \neq tn_{s+1}(k)$, take the node $g_s := \langle tn_s(0), tn_s(1), \dots, tn_s(k)\rangle$, and strip all the descendants of g_s (which have not already been stripped), while preserving the tree ranks. The procedure shows that $\widehat{T}$ consists of only one infinite chain and infinitely many finite chains that are attached to the root. Hence, the divisible part of B is isomorphic to $\mathbb{Z}(p^\infty)$, and B has Ulm type 1. Moreover, attaching fresh chains at non-zero stages ensures that the character $\chi(G(\widehat{T}))$ is equal to $\{(m_k, 1) : k \in \omega\}$. Thus, B is isomorphic to A.

Now note the following:

- If $i \not\in \emptyset'$, then there is a node $\sigma \in T$ such that $\sigma(i) = 0$ and the p-height of σ in B is infinite. Furthermore, for any $\tau \in T$, if $\tau(i) \neq 0$, then $h^B_p(\tau)$ is finite.
- Suppose that $i \in \emptyset'$. Then there is a node $\sigma \in T$ such that $\sigma(i)$ is odd and $h^B_p(\sigma) = \infty$. Moreover, for any $\tau \in T$ with $\tau(i)$ even, we have $h^B_p(\tau) < \infty$.

Hence, $h^B_p \geq_T \emptyset'$. Lemma 3.2 is proved. □

Lemma 3.3. *Suppose that* A *is a decidable torsion group of Ulm type 1.*

(a) *If there is a prime* p *such that the* p*-component* A_p *is non-reduced and has an unbounded character, then* A *has degree of decidable categoricity* $\mathbf{0}'$*.*
(b) *If every* A_p *is either reduced or has a bounded character, then* A *is decidably categorical.*

Proof. (a) First, note that for a prime q, the group A_q is computable, and the q-height $h^{A_q}_q$ is $\mathbf{0}'$-computable, uniformly in q. Therefore, for a decidable copy $B \cong A$, one can build a $\mathbf{0}'$-computable isomorphism f_q from A_q onto B_q, uniformly in q. The map $\bigcup_{q\in\mathbb{P}} f_q$ is easily extended to a $\mathbf{0}'$-computable isomorphism from A onto B. Thus, A is decidably $\mathbf{0}'$-categorical.

Fix a prime p such that the p-component A_p is non-reduced and has an unbounded character. Note that the theory $Th(A_p)$ is decidable: A_p is a p-group, and for any formula $\psi \in \{A_{p,n,k}, B_{p,n,k}, C_{p,n,k} : n,k \in \omega\}$, we have $A_p \models \psi$ iff $A \models \psi$. Thus, one can obtain a recursive axiomatization for $Th(A_p)$. By Proposition 2.1, A_p is a decidable structure.

The proof of Lemma 3.2 shows that there are two decidable copies B_0 and B_1 of A_p such that for any isomorphism f from B_0 onto B_1, we have $f \geq_T \mathbf{0}'$. Consider the groups

$$C_0 = B_0 \oplus \bigoplus_{q\in\mathbb{P}\setminus\{p\}} A_q, \quad C_1 = B_1 \oplus \bigoplus_{q\in\mathbb{P}\setminus\{p\}} A_q.$$

It is not hard to show that the structures C_0 and C_1 are decidable copies of A, and any isomorphism from C_0 onto C_1 computes $\mathbf{0}'$. Therefore, $\mathbf{0}'$ is the degree of decidable categoricity for A.

(b) Suppose that every p-component of A is either reduced or has a bounded character. We show that the p-height $h_p^{A_p}$ is computable, uniformly in p.

Given a non-zero element $a \in A_p$, we search for the least $k \in \omega$ such that one of the following two cases hold:

(i) $A \models (p^k \mid a)$ and $A \models (p^{k+1} \nmid a)$. Then $h_p^{A_p}(a) = k$.

(ii) $A \models (p^k \mid a)$ and $A \models \forall x[(p^k \mid x) \to (p^{k+1} \mid x)]$. Then $h_p^{A_p}(a) = \infty$.

If the p-height of a (inside A_p) is finite, then a will eventually satisfy case (i) and a cannot satisfy (ii). If the p-height of a is infinite, then the character of A_p is bounded and a will eventually satisfy (ii).

Since the group A is decidable, the described procedure computes the height $h_p^{A_p}$, uniformly in p. Therefore, using Proposition 2.2, one can build a computable isomorphism between any two decidable presentations of A. Lemma 3.3 and Proposition 3.1 are proved. □

3.2. **Jump inversions.** Our results on degrees of categoricity will heavily use the jump inversions that are based on the technique of computable p-basic trees. Here we give a brief overview of these inversions.

Ash, Knight, and Oates [2] introduced a procedure which allows to pass from a Π^0_2 p-basic tree C to a "nice" computable p-basic tree U (see the proposition below). The proof of the result can be found in [2], see also [12] for a discussion and an extended sketch of the proof.

Proposition 3.2 (Ash, Knight, and Oates [2])**.** *Let T be a computable p-basic tree of Ulm type 1, where the root 0 has tree-rank ω. Suppose that C is a Π^0_2 subtree of $\omega^{<\omega}$ (C is treated as a p-basic tree). Then there exists a computable p-basic tree U expanding C such that the p-group $G(U)_0$ is isomorphic to $G(T)$, and $G(U)'$ is isomorphic to $G(C)$.*

Suppose that A is a countable p-group. Roughly speaking, Proposition 3.2 allows us to pass from a Δ^0_3 presentation of A' and a computable presentation of $A_0 = A/A'$ to a computable copy of A: Consider $A_0 = G(T)$ and $A' = G(C)$, and recall that any $\Delta^0_2(X)$ subtree of $\omega^{<\omega}$ is isomorphic to a $\Pi^0_1(X)$ subtree.

We also introduce a new jump inversion which will be useful in working with decidable groups:

Proposition 3.3. *Let $X \subseteq \omega \times \omega$ be a computable unbounded character. Suppose that C is a Π^0_1 subtree of $\omega^{<\omega}$. Then there exists a computable p-basic tree U expanding C with the following properties:*

(1) *the p-group $G(U)_0$ has character X,*

(2) *$G(U)' \cong G(C)$, and*

(3) *given a node $v \in U$ and a natural number k, we can effectively check whether the tree-rank of v is at least k.*

Proof sketch. This is essentially a modified version of Proposition 3.2. Fix a computable sequence of trees $\{C[s]\}_{s\in\omega}$ such that $C = \bigcap_{s\in\omega} C[s]$ and $C[s] \supseteq C[s+1]$ for all s. W.l.o.g., we assume that for any $\sigma \in C[0]$ and any $i < len(\sigma)$, $\sigma(i)$ is an even number. For simplicity, we consider $X = \{(n_k, 1) : k \in \omega\}$, where $0 < n_0 < n_1 < n_2 < \dots$.

An informal idea of the construction is as follows. We want to copy the tree C, and for every node $v \in C$, we attach infinitely many fresh finite chains to v.

The size of each of the chains is equal to some n_i, and every n_i is used only once: in particular, any two different chains have different sizes. If this naive attempt is successful, then the resulting tree U will give a rise to a p-group which has the first two of the required properties: The nodes which have infinite tree-ranks are precisely the nodes from C (this is the second required property). Moreover, if we do the stripping procedure, we will see that any chain (from above) of size n_i can be represented as a simple chain which emanates from the root of U. Therefore, every $\mathbb{Z}(p^{n_i})$ detaches as a direct summand in $G(U)$. Furthermore, we ensured that $\chi(G(U)_0) = X$ (i.e. the first required property).

The main problem here is that the tree C is given by a Π^0_1 approximation, hence, sometimes we need to give up on some nodes v from $C[s]$. In order to guarantee that the third required property holds, we proceed as follows. Suppose that at some stage we want to give up on two nodes σ and τ, where σ is a child of τ. Then we first deal with σ by choosing a large fresh n_i and attaching a fresh chain of size $(n_i - 1)$ to σ. Only after that, we are allowed to abandon this σ and to start dealing with τ (in a similar way). The described procedure is justified by the following reasoning. Since σ "dies" forever (i.e. after that particular stage we will never add descendants of σ into U), all the descendants of σ, and σ itself, can be fully stripped by relocating a finite number of chains. The last chain that will be stripped is the longest chain which goes through σ, i.e. the chain $(\sigma, \sigma_1, \sigma_2, \dots, \sigma_{n_i - 1})$. The chain will be detached from τ and attached to the root of U. Since the size of this chain is equal to n_i, we will preserve the desired character X: the group $\mathbb{Z}(p^{n_i})$ still detaches as a direct summand in $G(U)$.

Now we present a more formal description of the construction. At a stage s, we build a finite tree $U[s]$. At stage 0, set $U[0] = \{\Lambda\}$.

At an odd stage $s = 2t + 1$, for every node v from $U[s] \cap C[t]$, choose a fresh n_j and attach a fresh simple chain of size n_j to this v. After that, find the least unused n_k and attach a fresh chain of size n_k to the root of the tree. The chains are chosen in such a way that for any τ from a chain, the last symbol of τ is an odd number.

Suppose that $s = 2t + 2$. If $U[s] \cap C[t] = U[s] \cap C[t + 1]$ (i.e. we do not see any changes in the considered nodes from $C[t]$), then choose (the least) σ from $C[t + 1] - U[s]$ and add all $\tau \subseteq \sigma$ into U.

If $U[s] \cap C[t] \neq U[s] \cap C[t + 1]$, then there is a node $\sigma \in U[s]$ such that $\sigma \in C[t+1] - C[t]$. Suppose that $U[s] \cap (C[t+1] - C[t]) = \{\sigma_0, \sigma_1, \dots, \sigma_k\}$. We assume that a condition $\sigma_i \subset \sigma_j$ implies that $i < j$. For every $i \leq k$, we choose a large fresh n_m and attach a fresh chain of size $(n_m - 1)$ to σ_i.

This concludes the description of the construction. It is not hard to show that the tree U has the desired properties. In particular, for any $v \in U[s]$, there is the following dichotomy: either the tree-rank of v in $U[s+1]$ is strictly greater then the tree-rank in $U[s]$; or the tree-rank of v does not grow after stage s. This ensures the third condition of the theorem. □

Proposition 3.3 will be used in the following way: Suppose that A is a countable p-group which has a decidable presentation. The proposition allows us to pass from a Δ^0_2 presentation of A' and a computable presentation of A_0 to a decidable copy of A (note that the decidability follows from Lemma 2.1).

3.3. **Degrees of decidable categoricity for reduced p-groups.** The main result of the subsection is the following

Theorem 3.1. *Let G be a decidable reduced p-group of a finite Ulm type. Then G has degree of decidable categoricity $\mathbf{d} \in \{\mathbf{0}^{(m)} : m < \omega\}$.*

In order to prove the theorem, we obtain a similar result for computable categoricity which we believe is new, at least in this form:

Theorem 3.2. *Let G be a computable reduced p-group of a finite Ulm type. Then G has degree of categoricity $\mathbf{d} \in \{\mathbf{0}^{(m)} : m < \omega\}$.*

The simulteneous *proof* of these two theorems will be split into two parts: The first part calculates degrees of categoricity for computable reduced p-groups of small Ulm type (to be explained below). The second part uses these calculations and jump inversions to finish the proof.

3.3.1. *Groups of small Ulm type.* The proofs of Theorems 3.1 and 3.2 will use induction on Ulm type N. Therefore, we need to establish the base of induction by considering some specific cases for p-groups of Ulm type $N \leq 2$:

Lemma 3.4. *Let A be a computable reduced p-group of Ulm type $N \leq 2$.*

(a) *Suppose that A has Ulm type 1, the character $\chi(A)$ is bounded, and*
$$A \cong B \oplus \bigoplus_{i<\omega} \mathbb{Z}(p^k) \oplus \bigoplus_{j<\omega} \mathbb{Z}(p^l),$$
where B is a group and $0 < k < l < \omega$. Then A has degree of categoricity $\mathbf{0}'$.

(b) *If A' is finite and the character $\chi(A_0)$ is unbounded, then A has degree of categoricity $\mathbf{0}'$.*

(c) *Suppose that A' is isomorphic to*
$$F \oplus \bigoplus_{i<\omega} \mathbb{Z}(p^k),$$
where F is a finite group and $0 < k < \omega$. Then A has degree of categoricity $\mathbf{0}''$.

Proof. (a) It is not hard to verify that A is relatively Δ^0_2 categorical (it follows, e.g., from part (3) of Proposition 2.3, or [7, Theorem 3.5]). Now it is sufficient to show that there are two computable copies C and D of A such that for any isomorphism $f\colon C \cong D$, we have $f \geq_T \mathbf{0}'$.

First, we build a "nice" p-basic tree T for C. If the character $X := \chi(A)$ contains $(n, 1)$ for a number n, then we find
$$r_n := card(\{m : (n, m) \in X\}),$$
and we attach exactly r_n simple chains of size n to the root of T. The procedure is effective, since X is a bounded character. This gives rise to a computable group $C := G(T)$ isomorphic to A. Moreover, the p-height h^C_p is a computable function.

A "bad" p-basic tree U is constructed as follows. Take T and attach to its root infinitely many new chains of size k. Let $\{\sigma_i\}_{i\in\omega}$ be an effective list of terminal nodes of these chains. If i belongs to $\emptyset'$, then we attach a fresh chain of size $(l-k)$ to the node σ_i. Define $D := G(U)$, and note that $h^D_p(\sigma_i) = l - 1$ iff $i \in \emptyset'$. Moreover, it is easy to show that D is a copy of A. Hence, $h_p^D \geq_T \mathbf{0}'$, and C, D are desired computable presentations of A.

(b) Part (1) of Proposition 2.3 implies that A is relatively Δ_2^0 categorical. As in the proof of (a), we will "encode" $\mathbf{0}'$ into isomorphisms between two computable presentations of A.

We build two computable p-basic trees T and U (treated as subtrees of $\omega^{<\omega}$). As usual, we may assume that the domains of T and U are disjoint. Moreover, when needed, we may think of an element $g \in G(T) \cup G(U)$ as a natural number (using some effective encoding from $G(\omega^{<\omega})$ into ω). For $V \in \{T, U\}$, 0_V denotes the root of V.

Trees $T[0]$ and $U[0]$ are defined as follows. Fix finite p-basic trees T' and U' such that $G(T') \cong G(U') \cong A'$. Since the group A' is finite, we may assume that any non-root node from T' has at most one child (consider the fully stripped version of T'). A similar convention holds for U'. For every node $v \in T' \cup U'$, attach a computable set of new children $H(v) := \{w_i(v) : i \in \omega\}$ to v. We say that $H(v)$ is the set of (v)-*witnesses*.

Suppose that $V \in \{T, U\}$. Our construction will guarantee that if a node u has more than one child in V, then $u \in V'$. Therefore, if $u \notin V'$ and $u \in V[s]$, then there is a well-defined notion of the *last descendant* of u in $V[s]$: There is only one (maximal) simple chain going through u inside $V[s]$, and the last descendant of u is the terminal node of this chain.

Our construction will satisfy the following series of requirements:

$\mathcal{R}_e$: If f is an isomorphism from $G(T)$ onto $G(U)$, then for the (0_T)-witness $w_e^0 := w_e(0_T)$, its image $f(w_e^0)$ *must be greater* (under the standard ordering of natural numbers) than

$$\psi(e) := 1 + \sum_{i \leq e,\, \varphi_i(e)\downarrow} \varphi_i(e).$$

We argue that satisfying all $\mathcal{R}_e$-s is enough: Indeed, consider a total function $\xi \colon e \mapsto f(w_e^0)$. If every $\mathcal{R}_e$ is satisfied, then ξ dominates every partial computable function. Therefore, we have $f \geq_T \xi \geq_T \mathbf{0}'$.

We choose limitwise monotonic functions $g(x)$ and $h(x)$ such that:

- For any non-zero n and k, we have $(n, k) \in \chi(A_0)$ iff $card(\{x : g(x) = n\}) \geq k$.
- For any x, $(h(x), 1) \in \chi(A_0)$ and $h(x) < h(x+1)$.

The existence of these functions follows, e.g., from [7, Lemma 2.13] (recall that the character $\chi(A_0)$ is unbounded). We fix limitwise monotonic approximations of the functions: $g(x)[s]$ and $h(x)[s]$. W.l.o.g., one can assume that $h(x)[s] < h(x+1)[s]$ for any x, s. We also choose a limitwise monotonic approximation $\psi(e)[s]$ of the function ψ from the $\mathcal{R}_e$-requirements.

Strategy for meeting $\mathcal{R}_e$. Choose a fresh number x_e. We attach a simple chain of size $(h(x_e) - 1)$ to the node w_e^0 in T. (Surely, at the current stage s_0, we do not know the real value of $h(x_e)$. Thus, we use the size $(h(x_e)[s_0] - 1)$ and after s_0, we will do chain updates, to be explained below.)

When the value $\psi(e)[s]$ grows, we say that $\mathcal{R}_e$ *requires attention* and proceed as follows. Find all elements v from the tree $U[s]$ with the following properties: $p^k v$ is a (u)-witness for some $u \in U'$ and $k \in \omega$, and there is an element $g \in G(U[s])$ such that $g \leq_\omega \psi(e)[s+1]$ and g is a linear combination (in $G(U[s])$) which includes v with a non-zero coefficient. For each such v, find its last descendant d_v (inside $U[s]$), choose a fresh number $m_v > x_e + 1 + len(d_v)$, and attach a fresh chain of size $(h(m_v) - len(d_v) + len(p^{k+1}v))$ to d_v. Thus, the length of the last descendant of v goes up from $len(d_v)$ to $(len(p^{k+1}v) + h(m_v))$. Declare that the requirement $\mathcal{R}_e$ does not require attention.

Note that $p^{k+1}v \in U'$ and (after all required chain updates) the stripping will ensure that the p-group $\mathbb{Z}(p^{h(m_v)})$ will detach as a direct summand in $G(U)$ — this is an explanation for the choice of the length of the chain attached to d_v.

Now we clarify the intuition behind the strategy. Given an element $g \in G(U)$, one can think of it as a (potential) image of $w_e^0 \in G(T)$ under an isomorphism. If $g \leq_\omega \psi(e)$, then at some stage s, we will see that $g \in G(U[s])$ and $g \leq_\omega \psi(e)[s+1]$. Suppose that $g = \sum_{v\in F} l_v v$, where F is a finite subset of $U[s]$ and $0 < l_v < p$. For every $v \in F - U'$, our strategy attaches a fresh chain to the last descendant d_v. The length of this chain is at least $(h(m_v) - len(d_v)) \geq h(m_v - len(d_v)) \geq h(x_e+1) > h(x_e)$. This ensures that $(p^{h(x_e)+1} \mid g)$ in $G(U)$. On the other hand, $(p^{h(x_e)+1} \nmid w_e^0)$ in $G(T)$. Therefore, for any isomorphism $f\colon G(T) \to G(U)$, we must have $f(w_e^0) > \psi(e)$, and the $\mathcal{R}_e$-requirement is satisfied.

Our main concern in the construction is that we need to ensure that both trees T and U give rise to p-groups isomorphic to A. This will be done via Ulm's theorem:

- the tree T' will contain precisely the nodes from T with an infinite tree-rank, and
- the character $\chi(G(T)')$ will be equal to $\chi(A)$.

Since T' is finite, these conditions imply that $G(T)$ is isomorphic to A.

Construction. During the construction, we will put labels on (v)-witnesses:

(1) Every (0_T)-witness w_e^0 will eventually obtain a permanent label $[h; z]$ for some $z \in \omega$.
(2) For $v \in (T' \cup U') - \{0_T\}$, a (v)-witness can obtain one of labels $[g; z]$ or $[h; z]$, where $z \in \omega$.

The intuition behind the labels is the following: A label on a (v)-witness w encodes the length of the last descendant of w. For example, if some $w \in T$ obtains a label $[g; z]$ and this label is never deleted, then the length of (the unique) chain attached to w in T will be equal to $(g(z) - 1)$.

For a tree $V \in \{T, U\}$ and $k, s \in \omega$, we introduce the *update number* $UN(V; k, s)$: this is the cardinality of the set

$$\{x : \text{there is } v \in V[s] \text{ such that } v \text{ is labelled by } [h; x] \text{ and } h(x)[s] = k\}.$$

Update numbers will help us to track down the character $\chi(A')$ and to encode it into the tree V: Essentially, $UN(V; k, s)$ keeps counting the number of direct summands $\mathbb{Z}(p^k)$, built inside $V[s]$ by following the approximation $h(x)[s]$.

At an odd stage $s + 1 = 2t + 1$, we find the least $z \leq s$ such that:

(i) no element from $T[s]$ is labelled by $[g; z]$, and
(ii) $UN(T; g(z)[s], s) < card(\{z_1 \leq s : g(z_1)[s] = g(z)[s]\})$.

If this z is found, then we choose the least (v)-witness y from $T[s]$ such that $v \neq 0_T$ and y does not have children. Put the label $[g; z]$ onto y and attach a fresh simple chain of size $(g(z)[s] - 1)$ to y.

After that, for every $v \in T'$, we choose a fresh number x_v and put the label $[h; x_v]$ onto the least (v)-witness w_v such that w_v never had any labels before. We call this w_v a *special* $(v; s)$*-witness* (at stage $s+1$).

At the end, we do the *chain updating* for T:

(1) If a (v)-witness w has a label $[h; x]$ and the length of the chain hanging from w (inside $T[s]$) is less than $L := (h(x)[s] - 1)$, then extend the chain to length L.

(2) Suppose that some (v)-witness w has a label $[g; z]$, and the (current value of the) update number $UN(T; g(z)[s], s)$ is greater than

$$card(\{z_1 \leq s : g(z_1)[s] = g(z)[s]\}).$$

Then delete the label $[g; z]$, find a fresh number x' such that $h(x') > g(z)[s]$, and put the label $[h; x']$ onto w. Repeat this procedure until there are no such "bad" witnesses w.

(3) If a (v)-witness w has a label $[g; x]$ and the length of the chain hanging from w is less than $M := (g(x)[s] - 1)$, then extend the chain to length M.

We also do similar actions for the tree U, i.e. searching for the least z with (i-ii) and chain updating.

Consider an even stage $s + 1 = 2t + 2$. First, for every $e < t$ with $\psi(e)[s-1] < \psi(e)[s+1]$, we declare that $\mathcal{R}_e$ requires attention. We also assume that $\mathcal{R}_t$ requires attention. After that for every $e \leq t$, if $\mathcal{R}_e$ requires attention, we follow the strategy for $\mathcal{R}_e$, described above.

This concludes the description of the construction. The verification is based on the following claim:

Claim 3.2. *Suppose that* $V \in \{T, U\}$.

(1) *The set of nodes from* V *with infinite tree-ranks is equal to* V'.

(2) *Let* n *be a non-zero natural number, and let*

$$r_n := card(\{k : (n, k) \in \chi(A')\}).$$

Then V *contains precisely* r_n *chains* P *of size* n *such that the beginning of* P *is a* (v)*-witness for some* $v \in V'$*, and the end of* P *is a terminal node in* V.

(3) *Every requirement* $\mathcal{R}_e$ *is satisfied.*

Proof. (1) Let $v \in V'$. Note that any (v)-witness w eventually obtains a label. If at some stage s, w was labelled by, say, $[g; z]$, then the chain hanging from w inside T has length at least $g(z)[s]$. Hence, the choice of special $(v; s)$-witnesses ensures that the tree-rank of v is infinite (recall that the function h is strictly increasing).

If $v \in V - V'$, then $p^k v$ is a (u)-witness for some $k \in \omega$ and $u \in V'$. Note that the labels of form $[h; x]$ are never removed, and if a label $[g; z]$ is removed, then it is immediately replaced by some $[h; x]$. Therefore, the chain emanating from u has a finite size, and the tree-rank of v is finite.

(2) First, assume that r_n is infinite. Then there is a stage s^* such that for any x and any $s \geq s^*$, we have either $h(x)[s] = h(x)[s^*] \leq n$ or $h(x)[s] > n$ (recall that $h(x)[s] < h(x+1)[s]$). Moreover, there are infinitely many numbers $y > n+1$ such that there is a stage $s_y \geq s^*$ with $g(y)[s_y] = g(y) = n$. After the stage s_y, the label

$[g;y]$ will be eventually put on some w_y, and it will never be removed. Thus, it will produce a required chain P_y of size n.

Suppose that r_n is finite. Consider a stage s^* with the following properties:

- If y is a number with $g(y) = n$, then we have $g(y)[s] = n$ for any $s \geq s^*$. We call the corresponding label $[g;y]$ a *nice n-label*.
- For any x and any $s \geq s^*$, either $h(x)[s] = h(x)[s^*] \leq n$ or $h(x)[s] > n$.

Working with update numbers guarantees one of the two final outcomes:

(1) If there is no x with $h(x) = n$, then there will be exactly r_n "permanent" nice n-labels (i.e. eventually these labels will stay forever).
(2) If there is (a unique) x^* with $h(x^*) = n$, then there will be exactly $(r_n - 1)$ "permanent" nice n-labels.

In any case, the construction produces exactly r_n required chains of size n.

(3) Note that the requirements do not injure each other. Since ψ is a limitwise monotonic function, $\mathcal{R}_e$ can require attention only finitely many times. Since the number x_e (which is chosen by $\mathcal{R}_e$) does not change, we will eventually satisfy $\mathcal{R}_e$, as it was argued in the paragraph after the strategy description. □

Recall that $V \in \{T, U\}$. The first item of the claim shows that $G(V)'$ is isomorphic to A'. The second item implies the following: the stripping procedure reveals that the character $\chi(G(V)')$ is equal to $\chi(A)$. Hence, by Ulm's Theorem, $G(V) \cong A$. The third item shows that for any isomorphism $f\colon G(T) \cong G(U)$, we have $f \geq_T \mathbf{0}'$.

(c) Relative Δ^0_3 categoricity of A follows from part (2) of Proposition 2.3. Now we will build two computable rooted trees T and U such that $G(T) \cong G(U) \cong A$ and the isomorphisms between $G(T)$ and $G(U)$ "encode" $\mathbf{0}''$.

Fix a limitwise monotonic function $g(x)$ such that for any non-zero n and k, we have $(n,k) \in \chi(A_0)$ iff there are at least k different numbers x with $g(x) = n$. Such a function g exists, since the character $\chi(A_0)$ is Σ^0_2 and the set $\#A_0$ is limitwise monotonic (recall Theorem 2.3). Let $g(x)[s]$ be a limitwise monotonic approximation of $g(x)$.

Choose a computable relation $R(x,y)$ such that $e \in \overline{\emptyset''}$ if and only if $\exists^\infty y R(e,y)$.

Fix a computable tree T' such that $G(T')$ is isomorphic to

$$A' \cong F \oplus \bigoplus_{i\in\omega} \mathbb{Z}(p^M),$$

where $0 < M < \omega$ and F is a finite group. Since F is finite, w.l.o.g., we may assume that in T', every non-root node has at most one child. Let $\{u_i : i \in \omega\}$ be an effective list of all terminal nodes from T'.

First, we give the construction of a "nice" tree T. For every i, we add a computable set of children $\{u(i,j) : j \in \omega\}$ to u_i. In the construction, we will put labels of form $[g;x]$ on nodes $u(i,j)$.

At a stage s, we do the following actions:

(1) Find the least x_0 such that no node is labelled by $[g;x_0]$. Choose the least pair (i,j) such that $u(i,j)$ is not labelled and put $[g;x_0]$ onto $u(i,j)$. Attach a chain of length $(g(x_0) - 1)$ to $u(i,j)$.
(2) For every $i \leq s$, choose the least j_i such that $u(i,j_i)$ is not labelled. Find a number x^i such that no node is labelled by $[g;x^i]$ and $g(x^i) > s$. Put the label $[g;x^i]$ onto $u(i,j_i)$ and attach a chain of length $(g(x^i) - 1)$ to $u(i,j)$.

It is not hard to verify the following

Claim 3.3. (1) *T' is precisely the set of nodes with infinite tree-rank in T. In particular, this implies that $G(T)' \cong A'$.*
(2) *The characters $\chi(G(T)_0)$ and $\chi(A_0)$ are equal. Thus, by Ulm's Theorem, $G(T)$ is isomorphic to A.*
(3) *Given an element $g \in G$, one can effectively check whether g has an infinite p-height.*

Second, we construct a "bad" tree U. We choose a limitwise monotonic function $h(x)$ such that $h(x) < h(x+1)$ and $(h(x), 1) \in \chi(A_0)$ for all x. Let $h(x)[s]$ be its limitwise monotonic approximation such that $h(x)[s] < h(x+1)[s]$ for all x, s.

Fix a computable tree U' such that $G(U') \cong A'$ and every non-root node of U' has at most one child. Consider two effective lists of vertices: $\{u_i\}_{i\in I}$ is the list of all terminal nodes from U' with length not equal to M, and $\{v_i\}_{i\in\omega}$ is the list of all terminal nodes of U' such that $len(v_i) = M$.

We want to guarantee the following condition: a node v_i has tree-rank ω if and only if $i \in \overline{\emptyset''}$. This fact and an isomorphism $G(U) \cong A$ together will ensure that any isomorphism from $G(U)$ onto $G(T)$ must compute $\mathbf{0}''$.

As in before, we consider computable sets of children: $\{u(i,j)\}_{j\in\omega}$ are children of u_i, and $\{v(i,j)\}_{j\in\omega}$ are children of v_i. The key difference here is that we add all $u(i,j)$ into our U, but we will add the nodes $v(i,j)$ one at a time. We will put labels of form $[g;x]$, $[h;x]$, and $\{h;x\}$ on nodes from U.

At a stage s, we proceed as follows:

(1) Find the least x_0 such that no node is labelled by $[g;x_0]$. Choose the least pair (i,j) such that $u(i,j)$ is not labelled and put $[g;x_0]$ onto $u(i,j)$. Attach a chain of length $(g(x_0)[s]-1)$ to $u(i,j)$.
(2) For every $i \in I$ with $i \leq s$, choose the least j_i such that $u(i,j_i)$ is not labelled. Find a number x^i such that no node is labelled by $[g;x^i]$ and $g(x^i) > s$. Put the label $[g;x^i]$ onto $u(i,j_i)$ and attach a chain of length $(g(x^i)[s]-1)$ to $u(i,j)$.
(3) For every $k \leq s$, if the predicate $R(k,s)$ is true, then do the following actions:
  (3.1) If a child of v_k has a label $\{h;x\}$, then replace it with the label $[h;x]$.
  (3.2) Choose a fresh number z such that $h(z)[s]$ is greater than $(1+len(v_k))$. Add a new child $v(k,j)$ into U, put the label $\{h;z\}$ onto $v(k,j)$, and attach a chain of size $(h(z)[s]-len(v_k)-1)$ to $v(k,j)$.
(4) For every $l \leq s$, we search for the least labels $[g;x]$ and $[h;z]$ (or $[g;x]$ and $\{h;z\}$) such that some elements are labelled by them and $g(x)[s] = h(z)[s] = l$. If there are such labels, then we choose a fresh number z' and replace $[g;x]$ with $[h;z']$.
(5) We do the *chain updating* in a natural way: E.g., if some node $\tau \in U$ has a label $[g;x]$ and $g(x)[s-1] < g(x)[s]$, then extend (if needed) the (unique) chain hanging from τ to length $(g(x)[s]-1)$. A similar procedure is applied to the labels $[h;x]$. We emphasize the following important feature which was not used in the proof of the (b)-part of the lemma: If τ has a label $\{h;x\}$, then we update the chain to the length $(h(x)[s]-len(\tau)-1)$.

We claim that the group $G(U)$ is isomorphic to A:

- If $k \in \overline{\emptyset''}$, then every child $v(k,j)$, $j \in \omega$, is added into U. Moreover, this child obtains a permanent label of form $[h;z]$ and grows a chain of length $h(z)$. Hence the tree-rank of v_k is exactly ω.
- If $k \not\in \overline{\emptyset''}$, then the node v_k has only finitely many children in U. Each of these children τ (except the last one) has a label $[h;x_\tau]$, and the last child σ has a label $\{h;y_\sigma\}$, where $y_\sigma > x_\tau$. If we apply stripping to v_k, then we will see that the groups $\mathbb{Z}(p^{h(x_\tau)})$ and $\mathbb{Z}(p^{h(y_\sigma)})$ detach as direct summands in $G(U)'$.
- The two previous items show that after stripping, we will have $G(U)' \cong G(U') \cong A'$.
- The chain updating and our dealing with labels (described above) ensure that $\chi(G(U)_0) = \chi(A_0)$. Hence, by Ulm's Theorem, we have $G(U) \cong A$.

Every isomorphism f from $G(U)$ onto $G(T)$ must compute $\mathbf{0}''$: In $G(T)$, we can decide whether the p-height of an element is equal to ω by a computable procedure. On the other hand, $\mathbf{0}''$ is Turing reducible to the set of elements from $G(U)$ with p-height ω. Lemma 3.4 is proved. □

We emphasize the following important feature of the proof of Lemma 3.4: Consider an arbitrary oracle X and go through the X-relativized version of the proof, i.e., we analyze the X-relativized constructions of the trees T and U. It is not hard to show that the *witnessing sets* (to be defined below) in all three items of the lemma can be kept computable, although the resulting trees T and U will be just X-computable.

Definition 3.1. Consider the three items of the proof of Lemma 3.4:

(a) In the first item, the *T-witnessing set* is the set of all terminal nodes from T with length k. The *U-witnessing set* is the set $\{\sigma_i : i \in \omega\}$ from U.
(b) The *T-witnessing set* is the set $\{w^0_e : e \in \omega\}$. The *U-witnessing set* is the set $\{w_i(v) : v \in U',\ i \in \omega\}$.
(c) The *T-witnessing set* is the set $\{u'_j : j \in \omega\}$, the set of all u_i from T' such that the length of u_i is equal to M. The *U-witnessing set* is the set $\{v_i : i \in \omega\} \subset U'$.

3.3.2. *Proofs of Theorems 3.1 and 3.2.* First, we obtain the result on computable categoricity:

Proof of Theorem 3.2. Suppose that A is a computable reduced p-group of a finite Ulm type M. Then there is a natural number N such that A satisfies exactly one of the following three cases:

(i) $A^{(N+1)}$ is a finite group, and the character $\chi(A_N)$ is unbounded. Furthermore, if $A^{(N+1)} = 0$, then $M = N+1$. If $A^{(N+1)} \neq 0$, then $M = N+2$.
(ii) A_N is isomorphic to
$$F \oplus \bigoplus_{i\in\omega} \mathbb{Z}(p^k),$$
where $0 < k < \omega$ and F is finite. Moreover, $M = N+1$ and $A^{(N)} = A_N$.
(iii) $\chi(A_N)$ is bounded and A_N is isomorphic to
$$B \oplus \bigoplus_{i\in\omega} \mathbb{Z}(p^k) \oplus \bigoplus_{j\in\omega} \mathbb{Z}(p^l),$$

where $0 < k < l < \omega$ and B is a group. Furthermore, $M = N + 1$ and $A^{(N)} = A_N$.

Formally, the proof should involve induction on N (or Ulm type M). Note that the base of induction is already given by Lemma 3.4. Now we outline a more informal proof for the first of the three cases above.

By the first part of Proposition 2.3, the group A is relatively Δ^0_{2N+2} categorical. Hence, we want to encode the Turing degree $\mathbf{0}^{(2N+1)}$ into the isomorphisms between two copies $G(T)$ and $G(U)$ of A. These copies will be represented by computable p-basic trees T and U, respectively.

Recall that by Theorem 2.3, the character $\chi(A_N)$ is a Σ^0_{2N+2} set and $\#A_N$ is $\mathbf{0}^{(2N)}$-limitwise monotonic. Thus, the relativized version of Lemma 3.4.(b) allows us to build two $\mathbf{0}^{(2N)}$-computable trees $T^{(N)}$ and $U^{(N)}$ with $G(T^{(N)}) \cong G(U^{(N)}) \cong A^{(N)}$ such that

$$(\forall f\colon G(T^{(N)}) \cong G(U^{(N)}))[f \geq_T \mathbf{0}^{(2N+1)}]. \tag{1}$$

Without loss of generality, we may assume that the trees $T^{(N)}$ and $U^{(N)}$ are Π^0_{2N} subtrees of $\omega^{<\omega}$: Our only concern here is that the property (1) is preserved. This is ensured by the following: Since the witnessing sets (see the discussion after Lemma 3.4) can be chosen computable, we can preserve their computability while transferring from a $\Delta^0_2(\mathbf{0}^{(2N-1)})$ subtree of $\omega^{<\omega}$ to a $\Pi^0_1(\mathbf{0}^{(2N-1)})$ subtree of $\omega^{<\omega}$.

Now we use the jump inversion from (relativized) Proposition 3.2 for $C = T^{(N)}$ and a $\mathbf{0}^{(2N-2)}$-computable tree $\widehat{T}$ representing the group A_{N-1} (recall that $\chi(A_{N-1})$ is $\Sigma^0_2(\mathbf{0}^{(2N-2)})$ and $\#A_{N-1}$ is $\mathbf{0}^{(2N-2)}$-limitwise monotonic). We obtain two $\mathbf{0}^{(2N-2)}$-computable trees $T^{(N-1)}$ and $U^{(N-1)}$ such that $G(T^{(N-1)}) \cong G(U^{(N-1)}) \cong A^{(N-1)}$.

Suppose that h is an isomorphism from $G(T^{(N-1)})$ onto $G(U^{(N-1)})$. Then $T^{(N)}$ is a subtree of $T^{(N-1)}$ and $h_1 := h \restriction G(T^{(N)})$ is an isomorphism from $G(T^{(N)})$ onto $G(U^{(N)})$, since $U^{(N)}$ is precisely the set of all nodes with infinite rank inside $U^{(N-1)}$. Hence, by (1) we have

$$h \geq_T h_1 \geq_T \mathbf{0}^{(2N+1)}.$$

Moreover, again, our witnessing sets are kept computable inside $T^{(N-1)}$ and $U^{(N-1)}$.

Thus, we can iterate the jump inversion from Proposition 3.2, until we obtain computable trees $T^{(0)}$ and $U^{(0)}$ such that $G(T^{(0)}) \cong G(U^{(0)}) \cong A$ and for any isomorphism h from $G(T^{(0)})$ onto $G(U^{(0)})$, we have $h \geq_T \mathbf{0}^{(2N+1)}$.

The proof of the cases (ii) and (iii) can be arranged in a similar way: For (ii), we encode the degree $\mathbf{0}^{(2N)}$ into the isomorphisms between $G(T)$ and $G(U)$. For (iii), we use $\mathbf{0}^{(2N+1)}$. Theorem 3.2 is proved. □

Proof of Theorem 3.1. This is obtained similarly to the previous proof. If we deal with the case (i), then we proceed as before, modulo the following two key modifications:

(1) Note that the structure $(A, p^k \mid)_{k\in\omega}$ is computable, thus, in A, the condition "the p-height of an element g is at least ω" is equivalent to a Π^c_1 condition $\forall k(p^k \mid g)$. Therefore, an analysis of the proof of Proposition 2.3 reveals that A is Δ^0_{2N+1} categorical relative to decidable presentations. Thus, we will encode $\mathbf{0}^{(2N)}$ into the isomorphisms between $G(T)$ and $G(U)$.

(2) Using the jump inversion of Ash, Knight, and Oates, we obtain a sequence of trees $\{T^{(N)}, U^{(N)}, T^{(N-1)}, U^{(N-1)}, \dots, T^{(1)}, U^{(1)}\}$ such that for $V \in \{T, U\}$ and a non-zero $i \leq N$, we have:

- $V^{(i)}$ is $\mathbf{0}^{(2i-1)}$-computable and $G(V^{(i)}) \cong A^{(i)}$,
- for any isomorphism $f\colon G(T^{(i)}) \to G(U^{(i)})$, we have $f \geq_T \mathbf{0}^{(2N)}$.

This can be done, since the decidability of A implies that $\chi(A_i)$ is a Σ^0_{2i+1} set and $\#A_i$ is $\mathbf{0}^{(2i-1)}$-limitwise monotonic. We may assume that $T^{(1)}$ and $U^{(1)}$ are Π^0_1 subtrees of $\omega^{<\omega}$.

After that, we use the jump inversion from Proposition 3.3 (we apply it to a tree $V \in \{T^{(1)}, U^{(1)}\}$ and the computable character $\chi(A_0)$) and obtain two computable trees T and U representing p-groups isomorphic to A. Lemma 2.1 shows that the groups $G(T)$ and $G(U)$ are decidable. Moreover, any isomorphism from $G(T)$ onto $G(U)$ computes $\mathbf{0}^{(2N)}$. □

3.4. **Examples of degrees of decidable categoricity for torsion groups.**

Theorem 3.3. *Let k be a natural number. Suppose that $\mathbf{d}$ is a Turing degree such that $\mathbf{d}$ is d.c.e. in and above $\mathbf{0}^{(2k+1)}$. Then there is a decidable, reduced torsion group with degree of decidable categoricity $\mathbf{d}$.*

Proof. Essentially, we follow the lines of the proof of Theorem 3.1: First, we show that any d.c.e. degree $\mathbf{d}$ is a degree of categoricity for a computable torsion group (Lemma 3.5). Then, we use this fact and jump inversions to establish the theorem.

Lemma 3.5. *Let X be an oracle. Suppose that $\mathbf{d}$ is a Turing degree d.c.e. in and above X. Then there is an X-computable torsion group A with the following properties:*

(i) *any two X-computable presentations of A are $\mathbf{d}$-computably isomorphic, and*
(ii) *there is an X-computable $\tilde{A} \cong A$ such that any isomorphism $f\colon \tilde{A} \cong A$ computes $\mathbf{d}$.*

Proof. A relativized version of the argument from [20, Theorem 3.1] shows that one can choose a set $D \in \mathbf{d}$ such that D is d.c.e. in X and for any oracle Y, we have:

$$(\overline{D} \text{ is c.e. in } Y) \Rightarrow D \leq_T Y \oplus X. \tag{2}$$

Suppose that $D = W - V$, where $V \subseteq W$ are X-c.e. sets.

The desired group A is built as a sum $\bigoplus_{p\in\mathbb{P}} A_p$. Recall that for $k \in \omega$, p_k denotes the k-th prime number. We define $A_{p_k} = B_k \oplus C_k$, where:

(a) If $k = 2m$, then

$$B_k \cong \mathbb{Z}(p_k^2), \quad C_k \cong \begin{cases} \mathbb{Z}(p_k), & \text{if } m \notin X, \\ \mathbb{Z}(p_k^3), & \text{if } m \in X. \end{cases}$$

(b) If $k = 2m+1$, then

$$B_k \cong \begin{cases} \mathbb{Z}(p_k^2), & \text{if } m \notin V, \\ \mathbb{Z}(p_k^3), & \text{if } m \in V; \end{cases} \qquad C_k \cong \begin{cases} \mathbb{Z}(p_k), & \text{if } m \notin W, \\ \mathbb{Z}(p_k^3), & \text{if } m \in W. \end{cases}$$

It is easy to show that the structure A is X-computable. Suppose that $\widehat{A}$ is an X-computable copy of A. Recall that $D \geq_T X$, thus, one can D-computably recover an isomorphism from the p_{2m}-component of A onto the p_{2m}-component of $\widehat{A}$, uniformly in m. Now we want to build an isomorphism f between the p_k-components A_{p_k} and $\widehat{A}_{p_k}$, where $k = 2m+1$. If $m \in D$, then we know the isomorphism type of A_{p_k}, and hence, we can build f in a natural way. If $m \notin D$, then first, we find subgroups of A and $\widehat{A}$ isomorphic to $\mathbb{Z}(p_k) \oplus \mathbb{Z}(p_k^2)$, and establish an isomorphism f_0 between them. If after that we find out that $m \in V$, then we extend f_0 to an isomorphism $f\colon A_{p_k} \cong \widehat{A}_{p_k}$, where $A_{p_k} \cong \mathbb{Z}(p_k^3) \oplus \mathbb{Z}(p_k^3)$. Indeed, such an extension is possible: Suppose that $\bar{a} = a_0, a_1, \dots, a_l$ is a tuple consisting of all elements from $dom(f_0)$, and $\bar{b} = f_0(a_0), f_0(a_1), \dots, f_0(a_l)$. Then $\bar{a}$ and $\bar{b}$ satisfy the same addition tables, and furthermore, for each i, the p-height of a_i is equal to the p-height of b_i. This implies that $(A, \bar{a}) \cong (\widehat{A}, \bar{b})$.

Therefore, A satisfies the property (i). An X-computable copy $\tilde{A} \cong A$ is constructed similarly to A, but we use the following finite groups:

(a') If $k = 2m$, then

$$\tilde{B}_k \cong \begin{cases} \mathbb{Z}(p_k^2), & \text{if } m \notin X, \\ \mathbb{Z}(p_k^3), & \text{if } m \in X; \end{cases} \qquad \tilde{C}_k \cong \begin{cases} \mathbb{Z}(p_k), & \text{if } m \notin X, \\ \mathbb{Z}(p_k^2), & \text{if } m \in X. \end{cases}$$

(b') If $k = 2m+1$, then

$$\tilde{B}_k \cong \begin{cases} \mathbb{Z}(p_k^2), & \text{if } m \notin W, \\ \mathbb{Z}(p_k^3), & \text{if } m \in W; \end{cases} \qquad \tilde{C}_k \cong \begin{cases} \mathbb{Z}(p_k), & \text{if } m \notin W, \\ \mathbb{Z}(p_k^2), & \text{if } m \in W - V, \\ \mathbb{Z}(p_k^3), & \text{if } m \in V. \end{cases}$$

Recall that for a group G, $G\{p\} = \{x \in G : px = 0\}$. Let f be an isomorphism from $\tilde{A}$ onto A. For $m \in \omega$, choose a non-zero element $b_m \in \tilde{B}_{2m}\{p\}$. Then $m \in X$ if and only if $f(b_m) \in C_{2m}\{p\}$. Thus, $f \geq_T X$.

Let d_m be a non-zero element from $\tilde{B}_{2m+1}\{p\}$. Then $m \notin D$ iff either $f(d_m) \in B_{2m+1}\{p\}$, or $m \in V$. Hence, $\overline{D}$ is c.e. in $(f \oplus X)$. By (2), we obtain that $D \leq_T (f \oplus X) \equiv_T f$. Lemma 3.5 is proved. □

Now suppose that $\mathbf{d}$ is a degree d.c.e. in and above $\mathbf{0}^{(2k+1)}$. By Lemma 3.5, we obtain two $\mathbf{0}^{(2k+1)}$-computable groups $A^{(k+1)}$ and $B^{(k+1)}$ which satisfy the following:

- any $\mathbf{0}^{(2k+1)}$-computable copies of $A^{(k+1)}$ are $\mathbf{d}$-computably isomorphic,
- $A^{(k+1)}$ and $B^{(k+1)}$ are isomorphic, and every $f\colon A^{(k+1)} \cong B^{(k+1)}$ computes $\mathbf{d}$.

For simplicity, we give a proof for the case $k = 2$. Notice that the proof of Lemma 3.5 implies that there are uniformly $\mathbf{0}^{(5)}$-computable sequences of trees $\{T_m^{(3)}\}_{m\in\omega}$ and $\{U_m^{(3)}\}_{m\in\omega}$ such that $T_m^{(3)}$ is a p_m-basic tree giving rise to the p_m-component $(A^{(3)})_{p_m}$, and $U_m^{(3)}$ gives rise to $(B^{(3)})_{p_m}$.

Fix a computable rooted tree V such that for each non-zero $n \in \omega$, V contains a unique chain of size n attached to the root. It is easy to show that V represents a reduced multicyclic p-group with the character $X_0 = \{(n,1) : 0 < n < \omega\}$.

For each $m \in \omega$, we apply the jump inversion of Ash, Knight, and Oates (Proposition 3.2) to the trees $T_m^{(3)}$ and V, and we obtain a $\mathbf{0}^{(3)}$-computable sequence of trees $\{T_m^{(2)}\}_{m\in\omega}$. Applying the jump inversion to $U_m^{(3)}$ and V, we obtain a tree $U_m^{(2)}$. Consider $\mathbf{0}^{(3)}$-computable groups

$$A^{(2)} = \bigoplus_{m\in\omega} G_m(T_m^{(2)}), \quad B^{(2)} = \bigoplus_{m\in\omega} G_m(U_m^{(2)}),$$

where $G_m(S)$ is the p_m-group corresponding to a rooted tree S. It is not hard to show that for $C \in \{A^{(2)}, B^{(2)}\}$, the group

$$(3) \quad C^{[1]} = \bigoplus_{p\in\mathbb{P}} \{g\colon g \text{ is from the } p\text{-component of } C, \text{ and } g \text{ has an infinite } p\text{-height}\}$$

is isomorphic to $A^{(3)}$. This implies that any $\mathbf{0}^{(3)}$-computable copies of $A^{(2)}$ are $\mathbf{d}$-computably isomorphic. Moreover, $A^{(2)} \cong B^{(2)}$, and every $f\colon A^{(2)} \cong B^{(2)}$ computes $\mathbf{d}$.

Applying the inversion from Proposition 3.2 to $T^{(2)}_m$ and V (also, to $U^{(2)}_m$ and V), we obtain $\mathbf{0}'$-computable sequences of trees $\{T^{(1)}_m\}_{m\in\omega}$ and $\{U^{(1)}_m\}_{m\in\omega}$. The $\mathbf{0}'$-computable groups

$$A^{(1)} = \bigoplus_{m\in\omega} G_m(T^{(1)}_m), \quad B^{(1)} = \bigoplus_{m\in\omega} G_m(U^{(1)}_m)$$

have the following properties:

- any $\mathbf{0}'$-computable copies of $A^{(1)}$ are $\mathbf{d}$-computably isomorphic,
- $A^{(1)} \cong B^{(1)}$, and any $f\colon A^{(1)} \cong B^{(1)}$ computes $\mathbf{d}$.

At last, we apply the jump inversion from Proposition 3.3 (using a tree $V \in \{T^{(1)}_m, U^{(1)}_m\}$, $m \in \omega$, and the character X_0) and obtain two computable sequences of trees $\{T_m\}_{m\in\omega}$ and $\{U_m\}_{m\in\omega}$. Define

$$A = \bigoplus_{m\in\omega} G_m(T_m), \quad B = \bigoplus_{m\in\omega} G_m(U_m).$$

The groups A and B are isomorphic, and every isomorphism from A onto B computes $\mathbf{d}$.

We show that A is a decidable group with degree of decidable categoricity $\mathbf{d}$. Note that the structure $(A, p^k \mid)_{p\in\mathbb{P},k\in\omega}$ is computable: by Lemma 2.1, for $x \in G_m(T_m)$, we can effectively check whether $(p^k_m \mid x)$, and this is uniform in $m \in \omega$. Now we consider the formulas from the definition of Szmielew invariants: for every $p \in \mathbb{P}$,

- $A \models A_{p,n,k}$ if and only if $k = 1$;
- $A \models B_{p,n,k}$ for all non-zero n and k;
- $A \models C_{p,n,k}$ for all non-zero n and k.

Thus, the theory $Th(A)$ is recursively axiomatizable and hence, decidable. By Proposition 2.1, A is decidable. A similar argument shows that B is also decidable.

If C is a decidable copy of A, then the group $C^{[1]}$ from (3) is a $\mathbf{0}'$-computable copy of $A^{(1)}$. Thus, any two decidable copies of A are $\mathbf{d}$-computably isomorphic. Therefore, $\mathbf{d}$ is the degree of decidable categoricity for A. Theorem 3.3 is proved. □

Similarly to Theorem 3.3, one can prove the following

Corollary 3.1. *Let k be a natural number. Suppose that $\mathbf{d}$ is a Turing degree such that $\mathbf{d}$ is d.c.e. in and above $\mathbf{0}^{(2k)}$. Then there is a computable, reduced torsion group with degree of categoricity $\mathbf{d}$.*

4. Homogeneous completely decomposable groups

4.1. **Effective algebra for completely decomposable groups.** Following [11], we give preliminaries on effective algebraic techniques that will be used for working with completely decomposable groups. Here $\mathbb{Q}$ denotes the additive group of rationals.

Let G be a torsion-free group. Elements $g_0, g_1, \dots, g_n$ from G are *linearly independent* if for any $c_0, c_1, \dots, c_n \in \mathbb{Z}$, the equality $c_0 g_0 + c_1 g_1 + \dots + c_n g_n = 0$ implies that $c_0 = c_1 = \dots = c_n = 0$. An infinite set of elements is *linearly independent* if every its finite subset is linearly independent. A maximal linearly independent subset is a *basis*. All bases of G have the same cardinality, and this cardinality is called the *rank* of G.

For $k \in \omega$, consider the p_k-height $h^G_{p_k} \colon G \to \omega \cup \{\infty\}$. For a non-zero $g \in G$, the sequence
$$\chi(g) = \left(h^G_{p_0}(g), h^G_{p_1}(g), h^G_{p_2}(g), \dots\right)$$
is called the *characteristic* of the element g in G.

Suppose that $\alpha = (k_0, k_1, k_2, \dots)$ and $\beta = (l_0, l_1, l_2, \dots)$ are two characteristics. We say that $\alpha \leq \beta$ if $k_i \leq l_i$ for all $i \in \omega$ (here $m < \infty$ for all $m \in \omega$).

Characteristics α and β are *equivalent* (denoted by $\alpha \simeq \beta$) if there are only finitely many n with $k_n \neq l_n$, and k_n, l_n are finite for these n. The $\simeq$-equivalence classes are called *types*.

We write $\mathbf{t}(g)$ for the type of an element g. If $G \leq \mathbb{Q}$ is non-zero (equivalently, $rk\, G = 1$) then all non-zero elements of G have equivalent types. Therefore, for a group G of rank 1, there is a well-defined notion of the *type* of G, written $\mathbf{t}(G)$, which is equal to the type of any non-zero element in G.

Theorem 4.1 ([3])**.** *Suppose that G and H are countable torsion-free groups of rank 1. Then G and H are isomorphic if and only if $\mathbf{t}(G) = \mathbf{t}(H)$.*

A torsion-free group G is *completely decomposable* if G is a direct sum of groups each having rank 1. A completely decomposable group is *homogeneous* if all its elementary summands are isomorphic.

Definition 4.1 ([11])**.** Let S be a set of primes, and let G be a torsion-free group. If $S \neq \emptyset$, then elements $b_1, \dots, b_k$ from G are called *S-independent* in G if for all integers $m_1, \dots, m_k$ and all $p \in S$, we have:
$$\left(p \mid \sum_{i=1}^{k} m_i b_i\right) \;\Rightarrow\; \bigwedge_{i=1}^{k} (p \mid m_i).$$
If $S = \emptyset$, then elements are *S-independent* if they are linearly independent. Note that S-independence always implies linear independence.

Every maximal S-independent set of G is called an *S-basis* of G. An S-basis is *excellent* if it is a maximal linearly independent subset of G.

In this section, we always assume that P is a set of primes, and we use the notation $\widehat{P} := \mathbb{P} \setminus P$. By $Q^{(P)}$ we denote the localization of integers by the set P, i.e., the additive subgroup of rationals generated by the set
$$\{1/p^m : p \in P,\ m \in \omega\}.$$
We define $G_P := \bigoplus_{i\in\omega} Q^{(P)}$.

For a given characteristic $\alpha = (k_0, k_1, k_2, \dots)$ and a torsion-free group G, we consider a subgroup

$$G[\alpha] = \{g \in G : \chi(g) \geq \alpha\}.$$

We also define the group $Q(\alpha)$ as the subgroup of $\mathbb{Q}$ generated by the set

$$\{1/p_m^t : m \in \omega,\ t < 1 + k_m\}.$$

Theorem 4.2 ([11, Theorem 4.10])**.** *Suppose that $G = \bigoplus_{i\in\omega} H$, where $H \leq \mathbb{Q}$ and $\mathbf{t}(H) = \mathbf{f}$. Let $\alpha = (\alpha_0, \alpha_1, \alpha_2, \dots)$ be a characteristic of type $\mathbf{f}$. Then the group $G[\alpha]$ is isomorphic to G_P, where*

$$P = \{p_i : i \in \omega,\ \alpha_i = \infty\}.$$

Moreover, if B is an excellent $\widehat{P}$-basis of $G[\alpha]$, then G is generated by B over $Q(\alpha)$.

Proposition 4.1 ([11, Lemma 5.2])**.** *Suppose that $G = \bigoplus_{i\in\omega} H$, where $H \leq \mathbb{Q}$ and $\mathbf{t}(H) = \mathbf{f}$. Consider a characteristic $\alpha = (\alpha_0, \alpha_1, \alpha_2, \dots)$ of type $\mathbf{f}$ and the set $P = \{p_i : \alpha_i = \infty\}$. Let G_1 and G_2 be computable copies of G. If both $G_1[\alpha]$ and $G_2[\alpha]$ have Σ^0_n excellent $\widehat{P}$-bases, then there is a Δ^0_n computable isomorphism from G_1 onto G_2.*

Definition 4.2 ([11, Definition 3.4])**.** A structure C is a computable presentation of a (left) module M over a ring R if:

(1) the ring R is isomorphic to a c.e. subring R_1 of a computable ring R_2;
(2) C is a computable presentation of M as an abelian group; and
(3) there is a computable function $f\colon R_1 \times C \to C$ which maps (r, m) to $r \cdot m \in C$, for every $r \in R_1$ and $m \in C$.

In [11], it was shown that for a set of primes P, the group G_P is computably presentable as an abelian group if and only if it is computably presentable as a module over $Q^{(P)}$.

Suppose that M is a module over a ring R. If S is a subset of M, then $span_R(S)$ denotes the R-span of S.

Lemma 4.1 ([11, Lemma 4.6])**.** *Let P be a set of primes. Suppose that $B = \{b_0, \dots, b_k\} \subseteq \bigoplus_{i\leq n} Q^{(P)} e_i$, and B is a linearly independent set. Then there exists a set $C = \{c_0, \dots, c_n\} \subseteq \bigoplus_{i\leq n} Q^{(P)} e_i$, and coefficients $r_0, \dots, r_k \in Q^{(P)}$ such that:*

(1) $\bigoplus_{i\leq n} Q^{(P)} c_i = \bigoplus_{i\leq n} Q^{(P)} e_i$, and
(2) $span_{Q^{(P)}}(\{r_0c_0, \dots, r_kc_k\}) = span_{Q^{(P)}}(B)$.

4.2. **Decidable presentations.** In this subsection, we obtain a criterion of decidable presentability for homogeneous completely decomposable groups.

Let $\alpha = (k_0, k_1, k_2, \dots)$ be a characteristic. We say that α is *computable* if the function $f\colon i \mapsto k_i$ is a total computable function from ω to $\omega \cup \{\infty\}$. Note that the computability of a characteristic is a type-invariant property. Hence, a type $\mathbf{f}$ is called *computable* if every $\alpha \in \mathbf{f}$ is computable.

Proposition 4.2. *Suppose that $G = \bigoplus_{i\in\omega} H$, where $H \leq \mathbb{Q}$ and the type of H is equal to $\mathbf{f}$. Then G has a decidable copy if and only if $\mathbf{f}$ is computable (as a total computable function from ω to $\omega \cup \{\infty\}$).*

Proof. ($\Rightarrow$). Suppose that G is decidable. Fix $p \in \mathbb{P}$. Choose a non-zero element $g \in H$, and note the following:

a) If $h_p^H(g) < \infty$, then w.l.o.g., we may assume that $(p \nmid g)$. This implies that for any non-zero n and k, we have $G \models C_{p,n,k}$, where $C_{p,n,k}$ is the formula from the definition of Szmielew invariants. In particular, $\gamma_p(G) = \omega$.

b) If $h_p^H(g) = \infty$, then for all n and k, we have $G \not\models C_{p,n,k}$. Hence, $\gamma_p(G) = 0$.

Therefore, since the theory $Th(G)$ is decidable, the set $\{p : h_p^H(g) = \infty\}$ is computable. Furthermore, if $h_p^H(g) < \infty$, then the p-height of g inside H can be computed effectively. Thus, the type $\mathbf{f}$ is computable.

($\Leftarrow$). Suppose that $\mathbf{f}$ is computable. First, it is not difficult to show that the theory $Th(G)$ is decidable: G is a torsion-free abelian group, and checking the truth of the formulas $C_{p,n,k}$ is effective in $\mathbf{f}$ (as explained above). Using the Szmielew invariants, it is easy to write down a recursive axiomatization for $Th(G)$.

Now it is sufficient to build a copy of G such that its relations $(p^k \mid \cdot)$, where $p \in \mathbb{P}$ and $k \in \omega$, are uniformly computable. A "nice" copy of H is constructed as follows: Let $\tilde{H}$ be the group $Q(\chi(g))$, where g is a fixed non-zero element from H. It is clear that $\mathbf{t}(\tilde{H}) = \mathbf{f}$, hence, by Theorem 4.1, $\tilde{H}$ is isomorphic to H.

Consider an irreducible fraction m/n from $\tilde{H}$, where $m \in \mathbb{Z} \setminus \{0\}$ and $n \in \omega \setminus \{0\}$. Suppose that $m = p^l \cdot m_1$, where $(p \nmid m_1)$. Then the fraction m/n is divisible by p^k in $\tilde{H}$ if and only if one of the following conditions holds:

(1) $h_p^H(g) = \infty$, or
(2) $h_p^H(g) = t < \infty$ and $k \leq l + t$.

Therefore, w.l.o.g., one may assume that the structure $(\tilde{H}, p^k \mid)_{p \in \mathbb{P}, k \in \omega}$ is computable.

Let $\tilde{G} = \bigoplus_{i \in \omega} \tilde{H}$. Suppose that a non-zero $x \in \tilde{G}$ is equal to $g_0 + g_1 + \cdots + g_n$, where $g_i \neq 0$ and for $i \neq j$, g_i and g_j belong to different copies of $\tilde{H}$. Then x is divisible by p^k if and only if $(p^k \mid g_i)$ inside $\tilde{H}$, for every $i \leq n$. Thus, $(\tilde{G}, p^k \mid)_{p \in \mathbb{P}, k \in \omega}$ is a computable structure, and by Proposition 2.1, the group $\tilde{G}$ is a decidable copy of G. Proposition 4.2 is proved. □

4.3. **Decidable categoricity.** We describe degrees of decidable categoricity of homogeneous completely decomposable groups:

Theorem 4.3. *If G is a decidable, homogeneous completely decomposable group, then G has degree of decidable categoricity* $\mathbf{d} \in \{\mathbf{0}, \mathbf{0}'\}$.

Proof. Suppose that $G = \bigoplus_{i \in \omega} H$, where $H \leq \mathbb{Q}$ and $\mathbf{t}(H) = \mathbf{f}$. By Proposition 4.2, the type $\mathbf{f}$ is computable. Fix a characteristic $\alpha = (\alpha_0, \alpha_1, \alpha_2, \dots)$ of type $\mathbf{f}$. Let $P = \{p_k : \alpha_k = \infty\}$.

First, we prove that G is always decidably $\mathbf{0}'$-categorical. By Proposition 4.1, it is sufficient to show that for any decidable copy A of G, the subgroup $A[\alpha]$ has a Σ^0_2 excellent $\widehat{P}$-basis. The proof of this fact essentially follows that of [11, Theorem 5.1], hence, some of the details are omitted.

If $\widehat{P} = \emptyset$, then $A[\alpha] = A$. In order to build an excellent $\emptyset$-basis C of A, we only need to know the linear dependency relation in A, and this relation is Σ^0_1. Hence, it is easy to construct the required Σ^0_2 basis C.

For $\widehat{P} \neq \emptyset$, a procedure of finding an excellent $\widehat{P}$-basis C is arranged as follows. We build C as $\bigcup_{n \in \omega} C_n$, where $C_n \subseteq C_{n+1}$. At a non-zero stage n, we do the following actions:

(1) Take the n-th element g_n from the subgroup $A[\alpha]$.

(2) Find an extension C_n of C_{n-1} in $A[\alpha]$ such that C_n is a finite $\widehat{P}$-independent set, and $C_n \cup \{g_n\}$ is linearly dependent. In [11], it was shown that such an extension C_n can always be found.

It is clear that the constructed C is an excellent $\widehat{P}$-basis of $A[\alpha]$. Now it is sufficient to establish the following:

Lemma 4.2. *The procedure described above is effective in* $\mathbf{0}'$.

Proof. Choose an element $h_0 \in A$ with $\chi(h_0) = \alpha$. Then the property "$\chi(g) \geq \alpha$" is equivalent to an infinitary formula

$$\bigwedge_{p \in \widehat{P}} \bigwedge_{k \in \omega} [(p^k \mid h_0) \to (p^k \mid g)].$$

Thus, since A is decidable, $A[\alpha]$ is a Π^0_1 subgroup in A.

We need to show that for a finite set $B \subseteq A[\alpha]$, one can, effectively in $\mathbf{0}'$ and uniformly in B, check whether B is $\widehat{P}$-independent.

The condition "a set $\{b_0, b_1, \dots, b_n\}$ is $\widehat{P}$-independent in $A[\alpha]$" can be written via a formula

$$\bigwedge_{\overline{m} \in \mathbb{Z}^{n+1}} \bigwedge_{p \in \mathbb{P}} \left[\left((p \in \widehat{P}) \,\&\, \exists x \left(x \in A[\alpha] \,\&\, px = \sum_{i=0}^{n} m_i b_i \right) \right) \to \bigwedge_{i=0}^{n} p \mid m_i \right]. \tag{4}$$

Since $A[\alpha]$ is a Π^0_1 subgroup, the formula (4) is Π^c_2. Nevertheless, in [11], it was shown that it can be rewritten as an equivalent Π^c_1-formula:

a. Note that the condition $(\alpha_j < k)$, where α_j is the j-th coordinate of the characteristic α, is equivalent to a computable condition $\neg(p_j^k \mid h_0)$.

b. For every $p_j \in \widehat{P}$, the Σ^c_2 formula

$$\exists x \left(x \in A[\alpha] \,\&\, p_j x = \sum_{i=0}^{n} m_i b_i \right)$$

is equivalent to a Σ^c_1 formula

$$\exists k (\exists y \in A) \left[(\alpha_j < k) \,\&\, p_j^k y = \sum_{i=0}^{n} m_i b_i \right] \tag{5}$$

Since (4) is equivalent to a Π^c_1 condition, the procedure for finding an excellent $\widehat{P}$-basis C is effective in $\mathbf{0}'$. Lemma 4.2 is proved. □

Lemma 4.2 and Proposition 4.1 together imply that the group G is decidably $\mathbf{0}'$-categorical. We consider three cases depending on the set $\widehat{P}$:

Case 1. Suppose that $\widehat{P} = \emptyset$. This implies that H is isomorphic to the group of rationals and $G \cong \bigoplus_{i \in \omega} \mathbb{Q}$. Following a classical paper of Mal'tsev [31], we sketch a proof for the fact that $\mathbf{0}'$ is the degree of decidable categoricity for $G = \bigoplus_{i \in \omega} \mathbb{Q}$.

It is well-known that the standard copy G_{st} of G is decidable (e.g., use Proposition 2.1). Moreover, in G_{st}, we can effectively check whether a finite set B is linearly dependent. It is sufficient to build a decidable copy A of G and two computable sequences of elements $\{a_e\}_{e \in \omega}$ and $\{b_e\}_{e \in \omega}$ such that the set $\{a_e, b_e\}$ is linearly dependent in A if and only if $e \in \emptyset'$. We construct A as a sum $\bigoplus_{e \in \omega} A_e$.

Fix an effective enumeration of the group $H_e := \mathbb{Q} a_e + \mathbb{Q} b_e$. At a stage s, if $e \notin \emptyset'[s]$, then set $A_e[s] := H_e[s]$. If s_0 is the first stage such that $e \in \emptyset'[s_0]$, then find a fresh natural number N such that N is greater than absolute value of any product $(m \cdot n \cdot k \cdot l)$, where $m, n, k, l \in \mathbb{Z}$ and

$$\frac{m}{n} a_e + \frac{k}{l} b_e \in H_e[s_0].$$

Declare $b_e := a_e / N$ and after stage s_0, construct A_e as the group $\mathbb{Q} a_e$. This concludes Case 1.

Case 2. Suppose that $\widehat{P}$ is a finite non-empty set. An analysis of the proof of Lemma 4.2 reveals that for any decidable copy G_1 of G, one can find a c.e. excellent $\widehat{P}$-basis of $G_1[\alpha]$:

- Since $\widehat{P}$ is finite, $G_1[\alpha]$ is a computable subgroup of G_1.
- For $p_j \in \widehat{P}$, the formula (5) is equivalent to a computable condition

$$p_j^{\alpha_j + 1} \mid \sum_{i=0}^{n} m_i b_i.$$

Hence, by Proposition 4.1, the group G is decidably categorical.

Case 3. Suppose that $\widehat{P}$ is an infinite set. W.l.o.g., we may identify G with its standard decidable copy which was built in Proposition 4.2. Recall that the copy was arranged as a sum $\sum_{i \in \omega} Q(\alpha)$. One may assume that the group $Q(\alpha)$ is decidable.

We define a subgroup $G[> \alpha]$ as follows:

$$G[> \alpha] := \{g \in G : \chi(G) \geq \alpha \ \&\ \chi(G) \neq \alpha\}.$$

It is easy to show that the groups $G[\alpha]$ and $G[> \alpha]$ are computable: e.g., if

$$h = \frac{p_{i_0}^{l_0} p_{i_1}^{l_1} \dots p_{i_m}^{l_m}}{n}, \text{ where } 0 < l_t < \omega,$$

is an irreducible fraction from $Q(\alpha)$, then $h \in G[> \alpha]$ if and only if there is a number i_t, $t \leq n$, with $\alpha_{i_t} < \infty$.

A "bad" copy G_1 of G is constructed as a sum $\bigoplus_{e \in \omega} H_e$. Fix a computable sequence of generators $\{a_e\}_{e \in \omega}$. Choose an effective enumeration of the group $Q(\alpha) a_e$. At a stage s, if $e \notin \emptyset'[s]$, then define $H_e[s] := (Q(\alpha) a_e)[s]$. If s_0 is the first stage such that $e \in \emptyset'[s_0]$, then choose a fresh prime $p \in \widehat{P}$ such that $p > card(H_e[s_0 - 1])$ and

$$\Big(\frac{m}{n} a_e \in H_e[s_0 - 1]\Big) \Rightarrow p > \max\{|m|, |n|\}.$$

We pick a new generator b_e and declare $p \cdot b_e = a_e$. After the stage s_0, we construct H_e as the group $Q(\alpha) b_e$.

It is easy to show that $\chi(a_e) \simeq \alpha$, hence, $H_e \cong Q(\alpha)$ and G_1 is a computable copy of G. Since G is a decidable structure, the theory $Th(G_1)$ is decidable. Suppose that $k \in \omega$, p is a prime, and $h \in H_e$. Then $(p^k \mid h)$ inside H_e if and only if one of the following conditions holds:

- $p \notin \widehat{P}$;
- $p \in \widehat{P}$, and there is a stage s such that $card(H_e[s]) > p$, $b_e \notin H_e[s]$, and $(p^k \mid h)$ inside $Q(\alpha) a_e$;
- $p \in \widehat{P}$, and there is a stage s such that $b_e \in H_e[s]$, and $(p^k \mid h)$ inside $Q(\alpha) b_e$.

Therefore, by Proposition 2.1, the group G_1 is decidable. The construction ensures that (the domain of) the subgroup $G_1[> \alpha]$ computes $\mathbf{0}'$: Indeed, if $e \in \emptyset'$, then $\chi(a_e) > \alpha$. If $e \notin \emptyset'$, then $\chi(a_e) = \alpha$. Theorem 4.3 is proved. □

4.4. **Linear independence.** For homogeneous completely decomposable groups G, we establish an unexpected connection between the decidability of G (which is essentially a first-order property) and the decidability of the linear independence in G (which is described by a computable infinitary formula).

Theorem 4.4. *Suppose that G is a homogeneous completely decomposable group which is not divisible. Then every decidable copy of G has an algorithm for linear independence.*

The *proof* of the theorem consists of two parts: In the first part (Subsection 4.4.1), we give a proof for the case when G is a free group, i.e. $G \cong \bigoplus_{i\in\omega} \mathbb{Z}$. This proof is simpler, but it already contains all the key ideas. The second part (Subsection 4.4.2) proves the general case.

Note that the theorem substantially uses the non-divisibility of G: If G is divisible, then it is easy to construct a decidable copy of G such that its linear independence relation is Turing equivalent to $\mathbf{0}'$ (e.g., see the proof of Theorem 4.3).

4.4.1. *The free group.*

Lemma 4.3. *For every decidable copy G of the free abelian group $\oplus_{i\in\omega}\mathbb{Z}$, there is an algorithm for linear independence.*

Proof. Fix a prime number p. We give a description of an algorithm for linear independence.

Given a tuple $\bar{a} = a_0, \ldots, a_k$ from G, we simultaneously run two procedures:

(i) Search for linear dependencies for $\{a_0, \ldots, a_k\}$ in a usual way.

(ii) The second procedure (we call it the *generalized Euclidean Algorithm*) works as follows. We check whether there exist $m_0, \ldots, m_k \in \mathbb{Z}_p$ such that not all of m_i-s are zeros, and the formal sum $\sum_{i\leq k} m_i a_i$ is divisible by p in the group G. Consider two cases:

Case 1. If such $m_0, \ldots, m_k$ do not exist, then declare the tuple $\bar{a}$ independent and stop.

Case 2. Once such $m_0, \ldots, m_k$ are found, consider a formal abelian group
$$F := \left\langle a_0, \ldots, a_k, x \;\middle|\; px = \sum_{i\leq k} m_i a_i \right\rangle_{ab},$$
where $a_0, \ldots, a_k, x$ are viewed as symbols. The group F is freely generated abelian. We effectively calculate its generators $b_0, \ldots, b_k$. Then the generalized Euclidean Algorithm repeats with the tuple $b_0, \ldots, b_k$ from G in place of $a_0, \ldots, a_k$.

The correctness of the algorithm described above is verified in the next two claims.

Claim 4.1. *Suppose that $\bar{a} = a_0, \ldots, a_k$ are linearly independent. Then the generalized Euclidean Algorithm eventually halts on $\bar{a}$.*

Proof. Consider a free abelian group
$$A := \langle a_0, \ldots, a_k \rangle = \mathbb{Z}a_0 \oplus \cdots \oplus \mathbb{Z}a_k.$$

The pure subgroup $B := \langle a_0, \dots, a_k\rangle_*$ is also free, and the rank of B is equal to $k+1$.

Every iteration of the generalized Euclidean Algorithm replaces the current group by a strictly larger subgroup inside B:

$$A = \langle a_0, \dots, a_k\rangle \subsetneq \langle b_0, \dots, b_k\rangle \subsetneq \langle b'_0, \dots, b'_k\rangle \subsetneq \dots \subseteq B. \tag{6}$$

Indeed, since the set $\{a_0, \dots, a_k\}$ is $\mathbb{P}$-independent, the combination $v := \sum_{i\leq k} m_i a_i$ is not divisible by p inside A, but $(p \mid v)$ inside $\langle b_0, \dots, b_k\rangle$.

One can show (e.g., by applying Lemma 4.1 with $P = \emptyset$) that there exist (free) bases of A and B — $\xi_0, \dots, \xi_k$ and $\eta_0, \dots, \eta_k$, respectively, such that

$$\xi_0 = l_0\eta_0,\ \dots,\ \xi_k = l_k\eta_k,$$

where $l_0, \dots, l_k$ are integers. In particular, this implies that the cardinality of the quotient group B/A is finite. Therefore, only finitely many inclusions in (6) can be strict. On the other hand, if the generalized Euclidean Algorithm takes another step, then it must produce a strictly larger subgroup of B. Therefore, the algorithm eventually halts. □

Claim 4.2. *If the generalized Euclidean Algorithm halts on $\bar{a} = a_0, \dots, a_k$, then $\bar{a}$ are linearly independent.*

Proof. Note that at every stage of the algorithm, we have

$$span_{\mathbb{Q}}(b_0, \dots, b_k) = span_{\mathbb{Q}}(a_0, \dots, a_k)$$

in the formal $\mathbb{Q}$-module $\sum_{i\leq k} \mathbb{Q}b_i$. This implies that the $\mathbb{Z}$-ranks of $\{a_0, \dots, a_k\}$ and $\{b_0, \dots, b_k\}$ must also be equal. Hence, w.l.o.g., we may assume that the algorithm halts on the tuple $a_0, \dots, a_k$ itself.

Assume that $\bar{a}$ are linearly dependent, i.e. we have

$$n_0a_0 + \dots + n_ka_k = 0$$

for some integers $n_0, \dots, n_k$, where not all n_i are zeros. If every n_i is divisible by p, then

$$pn'_0a_0 + \dots + pn'_ka_k = 0, \text{ where } n_i = pn'_i,\ i \leq k.$$

Therefore, $n'_0a_0 + \dots + n'_ka_k = 0$, since the group G is torsion-free.

Hence, one may assume that $n_i = pl_i + r_i$, where $0 \leq r_i < p$, and there is at least one non-zero r_i. Then we conclude that

$$\sum_{i\leq k}(pl_i + r_i)a_i = 0 \ \Rightarrow\ \sum_{i\leq k} r_ia_i = -p\sum_{i\leq k} l_ia_i,$$

and $\sum_{i\leq k} r_ia_i$ is a nonzero p-divisible element in G. This contradicts our assumption that the algorithm halts on the tuple $\bar{a}$. □

Lemma 4.3 is proved. □

4.4.2. *General case.* Suppose that G is a decidable group such that $G \cong \bigoplus_{i\in\omega} H$, where $H \leq \mathbb{Q}$ and $\mathbf{t}(H) = \mathbf{f}$. As usual, we choose a computable characteristic $\alpha = (\alpha_0, \alpha_1, \alpha_2, \dots)$ of type $\mathbf{f}$. Fix a prime $p = p_{i_0}$ such that $\alpha_{i_0} < \infty$. W.l.o.g., we assume that $\alpha_{i_0} = 0$ (if needed, one can replace α with $\alpha' \simeq \alpha$). Let $P = \{p_i : \alpha_i = \infty\}$.

For a tuple $\bar{a} = a_0, \dots, a_n$ from G, we simultaneously run the same pair of algorithms as in the proof of Lemma 4.3. We verify the work of generalized Euclidean Algorithm.

Lemma 4.4. *Suppose that $\bar{a}$ is a linearly independent tuple. Then the generalized Euclidean Algorithm halts on $\bar{a}$ after finitely many steps.*

Proof. Note that there exists a non-zero integer t such that $ta_0, \dots, ta_k \in G[\alpha]$. Since $\alpha_{i_0} = 0$, we may assume that $p \not| t$. We fix such an integer t, and we will work with the elements $ta_0, \dots, ta_k$.

Observe that every integer linear combination of $ta_0, \dots, ta_k$ belongs to $G[\alpha]$. Moreover, every p-root (if it exists) of an element $b \in G[\alpha]$ is itself in $G[\alpha]$ (recall again that $\alpha_{i_0} = 0$).

Whenever we adjoin an element x to the group $\langle a_0, \dots, a_k\rangle$ such that $px = \sum_i m_i a_i$, $m_i \in \mathbb{Z}_p$, we also adjoin tx to $\langle ta_0, \dots, ta_k\rangle$, and $p(tx) = \sum_i m_i(ta_i)$. If the former action properly extends $\langle a_0, \dots, a_k\rangle$ under inclusion, then the latter also properly extends $\langle ta_0, \dots, ta_k\rangle$ (since $p \not| t$).

Note that $B := \langle ta_0, \dots, ta_k\rangle_* \cap G[\alpha]$ is a free module of a finite rank over the localization $Q^{(P)}$: Indeed, by Theorem 4.2, the group $G[\alpha]$ is isomorphic to $\bigoplus_{i\in\omega} Q^{(P)}$, and hence it can be treated as a free $Q^{(P)}$-module of infinite rank. Since $Q^{(P)}$ is a principal ideal domain and B is a submodule of $G[\alpha]$, B is also a free $Q^{(P)}$-module (see, e.g., [29, Theorem 7.1 in Chap. 3]).

Consider a $Q^{(P)}$-module

$$A := span_{Q^{(P)}}(ta_0, \dots, ta_k) \cap G[\alpha] \subseteq G[\alpha].$$

The $Q^{(P)}$-rank of the module is equal to the $Q^{(P)}$-rank of $\{ta_0, \dots, ta_k\}$ which is itself equal to the rank of $\{a_0, \dots, a_k\}$.

Now we have $Q^{(P)}$-modules $A \subseteq B$. Extending A by a new p-root gives a submodule of B properly extending A: Indeed, if the generalized Euclidean Algorithm adjoins a new root x, then the set $\{ta_0, \dots, ta_k\}$ is $\{p\}$-independent in A, but it is $\{p\}$-dependent in $\langle ta_0, \dots, ta_k, tx\rangle \cap G[\alpha]$.

Consider a chain of $Q^{(P)}$-modules:

$$A \subsetneq C_0 \subsetneq C_1 \subsetneq C_2 \subsetneq \dots \subseteq B. \tag{7}$$

We show that any such proper chain between A and B must be finite.

By Lemma 4.1, we can choose $Q^{(P)}$-free bases $\xi_0, \dots, \xi_k$ and $\eta_0, \dots, \eta_k$ of A and B, respectively, such that for some non-zero integers $l_0, \dots, l_k$, we have $\xi_i = l_i \eta_i$. Every submodule C with $A \subseteq C \subseteq B$ is also k-generated, with generators being linear combinations of $\eta_0, \dots, \eta_k$ with integer coefficients.

Suppose that elements

$$c_j = u_{j,0}\eta_0 + \dots + u_{j,k}\eta_k, \; j = 0, \dots, k, \; u_{j,i} \in \mathbb{Z},$$

generate C over $Q^{(P)}$. Any such C is completely described by the classes $c_j + A$, $j = 0, \dots, k$. In other words, C can be described by a tuple $(\xi_0, \dots, \xi_k, c_0, \dots, c_k)$ which consists of generators over $Q^{(P)}$, but these generators *are* dependent over $Q^{(P)}$.

If we proceed with coefficient reductions $u'_{j,i} := (u_{j,i} \mod l_i)$, $i, j \le k$, and define

$$c'_j = u'_{j,0}\eta_0 + \dots + u'_{j,k}\eta_k,$$

then the tuples $(\xi_0, \dots, \xi_k, c'_0, \dots, c'_k)$ and $(\xi_0, \dots, \xi_k, c_0, \dots, c_k)$ will describe the same module C. Since there exist only finitely many such tuples $(\xi_0, \dots, \xi_k, c'_0, \dots, c'_k)$, any strictly increasing chain (7) must be finite. Therefore, for a linearly independent tuple $\bar{a}$, the generalized Euclidean Algorithm must eventually halt. Lemma 4.4 is proved. □

Using the same proof as in Claim 4.2 (since it only requires that G is torsion-free), one can show the following: If the generalized Euclidean Algorithm halts on a tuple $\bar{a}$, then the tuple is linearly independent. This concludes the proof of Theorem 4.4.

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